SHORE

Examination Number:

2023

YEAR 12 ASSESSMENT TASK 4

Mathematics Extension 2

General Instructions

- Reading Time - 10 minutes
- Working time - 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For Questions in Section 2, show relevant mathematical reasoning and/or calculations

Total marks 100

Section I - 10 marks (pages 1 - 6)

- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II - 90 marks (pages 7 - 13)

- Attempt Questions 11 - 16
- Allow about 2 hours 45 minutes for this section

Question	Marks	Total
1	1	
2	1	
3	1	
4	1	
5	1	
6	1	
7	1	
8	1	
9	1	
10	1	
11	15	
12	15	
13	15	
14	15	
15	15	
16	15	
Total:	100	

Section 1

10 marks

Attempt Questions 1 - 10.
Allow about 15 minutes for this section.
Use the Multiple-Choice answer sheet for Questions 1 - 10.

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1. Consider the following statement for all real numbers x and y: **1**

if the sum $x + y$ is rational, then x is rational and y is rational.

Which of the following is the contrapositive of this statement for $x, y \in \mathbb{R}$?

A. If x is irrational or y is irrational then the sum $x + y$ is irrational.

B. If the sum $x + y$ is irrational then x is rational or y is rational.

C. If x is rational and y is rational then the sum $x + y$ is irrational.

D. If x is rational or y is rational then the sum $x + y$ is irrational.

2. A primitive of $x^{17}\left(-2xe^{-x^2}\right)+17x^{16}e^{-x^2}$ is? **1**

A. $2x^{16}e^{-x^2}$.

B. $2x^{17}e^{-x^2}-\int 2x^{16}e^{-x^2}\,dx$.

C. $-17x^{16}e^{-x^2}+\int x^{16}e^{-x^2}\,dx$.

D. $x^{17}e^{-x^2}$.

3. What is the Cartesian equation of the line $\underset{\sim}{r}=\begin{pmatrix}-5\\6\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\end{pmatrix}$? **1**

A. $2y+3x=-3$.

B. $2x-3y=-8$.

C. $3y+2x=8$.

D. $3x-2y=28$.

4. In the complex plane, a circle $\mathbb{C}$ has diameter AB, where the points A and B represent the complex numbers $1-5i$ and $3-i$ respectively. **1**

What is the equation of $\mathbb{C}$?

A. $|z-2+3i|=2\sqrt{5}$.

B. $|z-2+3i|=\sqrt{5}$.

C. $|z+2-3i|=2\sqrt{5}$.

D. $|z+2-3i|=\sqrt{5}$.

5. Given the points $A(-1,-2,0)$, $B(1,b,3)$ and $C(-4,2,8)$. $\angle ABC$ will equal 90° when b has the values? **1**

A. ± 1.

B. ± 2.

C. ± 3.

D. ± 4.

6. Consider the statement: **1**

"For all odd primes $p < q$ there exists positive non-primes $r < s$ such that $p^2 + q^2 = r^2 + s^2$."

Which of the following is the negation of this statement?

A. For all odd primes $p < q$ there exists positive non-primes $r < s$ such that $p^2+q^2 = r^2+s^2$.

B. For all odd primes $p < q$ and for all positive non-primes $r < s$ $p^2 + q^2 \neq r^2 + s^2$.

C. There exists odd primes $p < q$ such that for all positive non-primes $r < s$, $p^2+q^2 = r^2+s^2$.

D. There exists odd primes $p < q$ such that for all positive non-primes $r < s$, $p^2+q^2 \neq r^2+s^2$.

7. When using the substitution $t = \tan\left(\dfrac{\theta}{2}\right)$, the indefinite integral $\displaystyle\int \frac{d\theta}{4\sin\theta - 3\cos\theta}$ can be expressed as? **1**

A. $\displaystyle\int \frac{2dt}{3t^2 + 8t - 3}$.

B. $\displaystyle 2\int \frac{dt}{8t + 3 - 3t^2}$.

C. $\displaystyle 2\int \frac{dt}{3t^2 - 8t + 3}$.

D. $\displaystyle\int \frac{dt}{3t^2 + 8t - 3}$.

8. The displacement $\underset{\sim}{r}$ of a particle at time t is represented parametrically by; $\underset{\sim}{r} = \cos t\underset{\sim}{i} + \sin(2t)\underset{\sim}{j} + 2t\underset{\sim}{k}$. The Cartesian equation in the $x - y$ plane is? **1**

A. $y = 2x\sqrt{x^2 - 1}$.

B. $y = 2x\sqrt{1 - x^2}$.

C. $y = x\sqrt{x^2 - 1}$.

D. $y = x\sqrt{1 - x^2}$.

9. A particle is moving along a straight line. At time t, its velocity is v and its displacement from a fixed origin is x. **1**

If $v^2 = \dfrac{1}{2}(x - 3)$, which of the following best describes the particle's acceleration and velocity?

A. increasing acceleration and constant velocity.

B. constant acceleration and decreasing velocity.

C. constant acceleration and increasing velocity.

D. increasing acceleration and increasing velocity.

10. Consider the sphere $\mathbb{S} : x^2 + y^2 + z^2 = 26$. A line $\mathbb{L}$ passes through the points $(3, -1, -2)$ **1**
and $(5, 3, -4)$. The line $\mathbb{L}$ will intersect the sphere S at the point?

A. $(-5, 0, 1)$ and $(7, 7, -6)$.

B. $(4, -1, 3)$ and $(2, -3, -1)$.

C. $(2, 3, -4)$ and $(4, 1, -3)$.

D. $(1, -5, 0)$ and $(4, 1, -3)$.

Section 2

90 marks

Attempt Questions 11 - 16

Allow about 2 hours 45 minutes for this section.

Answer the questions in the space provided. Show all of your working.

Question 11. (15 marks) Use a SEPARATE writing booklet.

(a) Find the real part of z if $z = \dfrac{1+e^{i\theta}}{e^{i\frac{\theta}{2}}}$. **2**

(b) In an Argand diagram, the point P representing the complex number z is described by $|z-(1+i)| = 1$.

(i) Sketch the Cartesian equation representing z. **1**

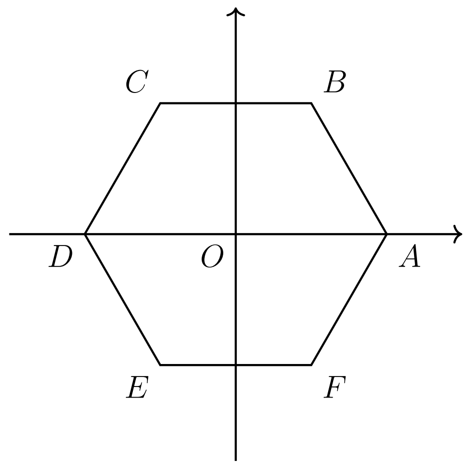
(ii) Shade the region where $|z-(1+i)| \leq 1$ and $0 < \operatorname{Arg}(z-i) < \dfrac{\pi}{4}$. **3**

(c) Write the complex number $-1 + i\tan(3)$ in the form $ae^{i\theta}$ where $a, \theta \in \mathbb{R}$. **3**

(d) Find all the complex numbers $z = a + ib$, where a, b are real, such that $|z|^2 - iz = 16 - 2i$. **3**

Question 11 continues...

(e) In an Argand diagram, a regular hexagon $ABCDEF$, with the vertices taken in anticlockwise order, has its centre at the origin O and vertex A at $z = 2$.



If z is a complex number that lies on the hexagon;

(i) Find the set of values of $\mathrm{Im}\,(z)$. **1**

(ii) Find the set of values of $|z|$. **1**

(iii) If the hexagon is rotated in a clockwise direction about the origin through an angle of $45°$, find the complex number which is represented by the new position of vertex C. Write your answer in modulus-argument form. **1**

Question 12. **(15 marks) Use a SEPARATE writing booklet.**

(a) It is asserted that **1**

$$|2x+1| \leq 5 \implies |x| \leq 2.$$

Use a counter-example to disprove the assertion.

(b) Consider the vector $\underset{\sim}{r}$, where $\underset{\sim}{r} = 2a\underset{\sim}{i} + b\underset{\sim}{j}$, $\forall\, a, b \in \mathbb{R}$.

(i) Write an expression for $\left|\underset{\sim}{r}\right|$ in terms of a and b. **1**

(ii) By choosing another appropriate vector for $\underset{\sim}{v}$, use the triangle inequality to prove **3**

$$\sqrt{a^2+b^2} \leq \frac{\sqrt{4a^2+b^2}+\sqrt{a^2+4b^2}}{3}.$$

(c) Use contradiction to prove the following; **3**
There do **not** exist positive integers m and n such that $m^2 - n^2 = 1$.

(d) Given the recurrence relation $T_n = T_{n-1} + 2^n$ for $n \geq 2$ and $T_1 = 3$, prove by Mathematical Induction that $T_n = 2^{n+1} - 1 \quad \forall n \geq 1$. **3**

(e) By considering $f(x) = x - \ln\left(1 + x + \frac{1}{2}x^2\right)$, show that $e^x < 1 + x + \frac{1}{2}x^2$ for $x < 0$. **4**

Question 13. **(15 marks) Use a SEPARATE writing booklet.**

(a) If $z = \cos\theta + i\sin\theta$,

(i) By using expressions for $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$, where n is a positive integer, expand $\left(z+\dfrac{1}{z}\right)^4 + \left(z-\dfrac{1}{z}\right)^4$ to show that $\cos^4\theta + \sin^4\theta = \dfrac{1}{4}\left(\cos(4\theta) + 3\right)$. **3**

(ii) By letting $x = \cos\theta$, show that the equation $8x^4 + 8\left(1-x^2\right)^2 = 7$ has roots $\pm\cos\dfrac{\pi}{12}, \pm\cos\dfrac{5\pi}{12}$. **2**

(b) Evaluate $\displaystyle\int_1^{e^2} \frac{\ln x}{x^3}\,dx$. **3**

(c) Consider the complex number $z = e^{i\theta}$.

(i) Show that $\cos i = \dfrac{e^1 + e^{-1}}{2}$. **1**

(ii) Find a similar expression for $\sin i$. **1**

(iii) Hence solve $e^w = \cos i - i\sin i$, where $w \in \mathbb{C}$. **2**

(d) Find the range of values of p, for $p \in \mathbb{R}$ such that $x^4 - px^3 + px - 1 = 0$ has 1 pair of complex conjugate roots. **3**

Question 14. **(15 marks) Use a SEPARATE writing booklet.**

(a) Find $\displaystyle\int \frac{\cos^2 x}{1-\sin x}\,dx$. **2**

(b) Find a primitive of $\sec^6 x$. **3**

(c) By using the substitution $p = 1 + x^{\frac{3}{2}}$, find $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+\sqrt{x^3}}}\,dx$. **3**

(d) If $I_n = \displaystyle\int_0^t \frac{1}{\left(1+x^2\right)^n}\,dx$ for $n = 1, 2, 3, \ldots$

(i) Show that $2nI_{n+1} = (2n-1)\,I_n + \dfrac{t}{\left(1+t^2\right)^n}$ for $n = 1, 2, 3, \ldots$ **4**

(ii) Hence find I_3 in terms of t. **3**

Question 15. (15 marks) Use a SEPARATE writing booklet.

(a) Given $\underset{\sim}{u} = \begin{pmatrix} -9 \\ 8 \\ 1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$, and $\underset{\sim}{u} - 6\underset{\sim}{v} + 4\underset{\sim}{w} = 0$, find vector $\underset{\sim}{w}$. **2**

(b) Given $\left|\underset{\sim}{v}\right| = 2$, $\left|\underset{\sim}{w}\right| = 1$ and $\underset{\sim}{v} \cdot \underset{\sim}{w} = 2$, find the value of $\left|\underset{\sim}{v} - 3\underset{\sim}{w}\right|$. **3**

(c) Show that if $\underset{\sim}{c} = \left|\underset{\sim}{a}\right| \underset{\sim}{b} + \left|\underset{\sim}{b}\right| \underset{\sim}{a}$ where $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are all non-zero vectors then $\underset{\sim}{c}$ bisects the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **3**

(d) The points A, B, C and D have position vectors $\left(0, \frac{\sqrt{3}}{6}x, \frac{\sqrt{2}}{\sqrt{3}}x\right)$, $\left(0, \frac{\sqrt{3}}{6}x, 0\right)$, $\left(-\frac{x}{2}, 0, 0\right)$ and $\left(\frac{x}{2}, 0, 0\right)$ respectively. If M is the midpoint of CD, show that $\angle AMB \approx 70.5^\circ$. **3**

(e) Two particles travel along the curves $\underset{\sim}{r}(t)$ and $\underset{\sim}{w}(t)$, defined parametrically by **4**

$\underset{\sim}{r}(t) = t\underset{\sim}{i} + t^2\underset{\sim}{j} + t^3\underset{\sim}{k}$ and $\underset{\sim}{w}(t) = (1-2t)\underset{\sim}{i} + (1+6t)\underset{\sim}{j} + (1+14t)\underset{\sim}{k}$.

Show that the particle's paths intersect but the particles do not collide.

Question 16. **(15 marks) Use a SEPARATE writing booklet.**

(a) A particle is travelling in a straight line. Its displacement, x cm, from O at a given time, t seconds after the start of the motion, is given by $x = 2 + \cos^2 t$.

(i) Show that the particle is undergoing simple harmonic motion. **2**

(ii) Find the period of the motion. **1**

(iii) Determine the distance travelled by the particle in the first π seconds. **1**

(b) A particle of mass m kg, moving in a straight line, has acceleration of $x \tan^{-1} x$ ms^{-2}; where x is measured in metres and time t is in seconds. Given that $v = 3$ when $x = 0$, find the velocity of the particle at position x. **4**

(c) A particle of mass m kilograms is dropped from rest in a medium where the resistance to motion has magnitude $0.1mv^2$ Newtons when the velocity of the particle is v ms^{-1}. After t seconds the particle has fallen x metres, and has velocity v ms^{-1} and acceleration a ms^{-2}. The particle hits the ground $\ln\left(1+\sqrt{2}\right)$ seconds after it is dropped. Take $g = 10$ ms^{-2}.

(i) By drawing a diagram showing the forces on the particle, deduce that $a = \dfrac{1}{10}\left(100 - v^2\right)$. **1**

(ii) Show that the particle hits the ground with a speed $5\sqrt{2}$ ms^{-1}. **4**

(iii) Show that the position of the particle when it reaches half of its terminal velocity is given by $x = 10\ln\left(\dfrac{2\sqrt{3}}{3}\right)$. **2**

End of Paper.

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YEAR 12 ASSESSMENT TASK 4

Mathematics Extension 2

General Instructions

- Reading Time - 10 minutes
- Working time - 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For Questions in Section 2, show relevant mathematical reasoning and/or calculations

Total marks 100

Section I - 10 marks (pages 1 - 7)

- Attempt Questions 1 - 10
- Allow about 15 minutes for this section

Section II - 90 marks (pages 8 - 33)

- Attempt Questions 11 - 16
- Allow about 2 hours 45 minutes for this section

Question	Marks	Total
1	1	
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10	1	
11	15	
12	15	
13	15	
14	15	
15	15	
16	15	
Total:	100	

Section 1

10 marks

Attempt Questions 1 - 10.
Allow about 15 minutes for this section.
Use the Multiple-Choice answer sheet for Questions 1 - 10.

---

1. Consider the following statement for all real numbers x and y: **1**

if the sum $x + y$ is rational, then x is rational and y is rational.

Which of the following is the contrapositive of this statement for $x, y \in \mathbb{R}$?

A. If x is irrational or y is irrational then the sum $x + y$ is irrational.

B. If the sum $x + y$ is irrational then x is rational or y is rational.

C. If x is rational and y is rational then the sum $x + y$ is irrational.

D. If x is rational or y is rational then the sum $x + y$ is irrational.

Solution: P : sum of $x + y$ is rational.
Q : x is rational and y is rational
$\sim P$: sum of $x + y$ is irational
$\sim Q$: x is irrational **or** y is irational.

The contrapositive of $P \implies Q$ is $\sim Q \implies \sim P$

Answer: A

2. A primitive of $x^{17}\left(-2xe^{-x^2}\right)+17x^{16}e^{-x^2}$ is? **1**

A. $2x^{16}e^{-x^2}$.

B. $2x^{17}e^{-x^2}-\int 2x^{16}e^{-x^2}\,dx$.

C. $-17x^{16}e^{-x^2}+\int x^{16}e^{-x^2}\,dx$.

D. $x^{17}e^{-x^2}$**.**

Solution: Answer D.

3. What is the Cartesian equation of the line $\underset{\sim}{r}=\begin{pmatrix}-5\\6\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\end{pmatrix}$? **1**

A. $2y+3x=-3$**.**

B. $2x-3y=-8$.

C. $3y+2x=8$.

D. $3x-2y=28$.

Solution: Answer A.

4. In the complex plane, a circle $\mathbb{C}$ has diameter AB, where the points A and B represent the complex numbers $1 - 5i$ and $3 - i$ respectively. **1**

What is the equation of $\mathbb{C}$?

A. $|z - 2 + 3i| = 2\sqrt{5}$.

B. $|z - 2 + 3i| = \sqrt{5}$**.**

C. $|z + 2 - 3i| = 2\sqrt{5}$.

D. $|z + 2 - 3i| = \sqrt{5}$.

Solution: Answer B.

5. Given the points $A(-1, -2, 0)$, $B(1, b, 3)$ and $C(-4, 2, 8)$. $\angle ABC$ will equal $90°$ when b has the values? **1**

A. ± 1.

B. ± 2.

C. ± 3**.**

D. ± 4.

Solution: Answer D

6. Consider the statement: **1**

"For all odd primes $p < q$ there exists positive non-primes $r < s$ such that $p^2 + q^2 = r^2 + s^2$."

Which of the following is the negation of this statement?

A. For all odd primes $p < q$ there exists positive non-primes $r < s$ such that $p^2+q^2 = r^2+s^2$.

B. For all odd primes $p < q$ and for all positive non-primes $r < s$ $p^2 + q^2 \neq r^2 + s^2$.

C. There exists odd primes $p < q$ such that for all positive non-primes $r < s$, $p^2+q^2 = r^2+s^2$.

D. There exists odd primes $p < q$ such that for all positive non-primes $r < s$, $p^2 + q^2 \neq r^2 + s^2$.

Solution: Answer D. Statement has been negated.

7. When using the substitution $t = \tan\left(\frac{\theta}{2}\right)$, the indefinite integral $\displaystyle\int \frac{d\theta}{4\sin\theta - 3\cos\theta}$ can be expressed as? **1**

A. $\displaystyle\int \frac{2dt}{3t^2 + 8t - 3}$ **.**

B. $\displaystyle 2\int \frac{dt}{8t + 3 - 3t^2}$.

C. $\displaystyle 2\int \frac{dt}{3t^2 - 8t + 3}$.

D. $\displaystyle\int \frac{dt}{3t^2 + 8t - 3}$.

Question 7 continues...

Question 7 continues…

Solution: Answer A

8. The displacement $\underset{\sim}{r}$ of a particle at time t is represented parametrically by; $\underset{\sim}{r} = \cos t\underset{\sim}{i} + \sin(2t)\,\underset{\sim}{j} + 2t\underset{\sim}{k}$. The Cartesian equation in the $x - y$ plane is? **1**

A. $y = 2x\sqrt{x^2 - 1}$.

B. $y = 2x\sqrt{1 - x^2}$**.**

C. $y = x\sqrt{x^2 - 1}$.

D. $y = x\sqrt{1 - x^2}$.

Solution: Answer A

9. A particle is moving along a straight line. At time t, its velocity is v and its displacement from a fixed origin is x. **1**

If $v^2 = \dfrac{1}{2}(x - 3)$, which of the following best describes the particle's acceleration and velocity?

A. increasing acceleration and constant velocity.

B. constant acceleration and decreasing velocity.

C. constant acceleration and increasing velocity.

D. increasing acceleration and increasing velocity.

Solution: Answer C.

10. Consider the sphere $\mathbb{S} : x^2 + y^2 + z^2 = 26$. A line $\mathbb{L}$ passes through the points $(3, -1, -2)$ and $(5, 3, -4)$. The line $\mathbb{L}$ will intersect the sphere S at the point? **1**

A. $(-5, 0, 1)$ and $(7, 7, -6)$.

B. $(4, -1, 3)$ and $(2, -3, -1)$.

C. $(2, 3, -4)$ and $(4, 1, -3)$.

D. $(1, -5, 0)$ **and** $(4, 1, -3)$**.**

Solution: Answer D.

Section 2

90 marks

Attempt Questions 11 - 16

Allow about 2 hours 45 minutes for this section.

Answer the questions in the space provided. Show all of your working.

Question 11. **(15 marks) Use a SEPARATE writing booklet.**

(a) Find the real part of z if $z = \dfrac{1+e^{i\theta}}{e^{i\frac{\theta}{2}}}$. **2**

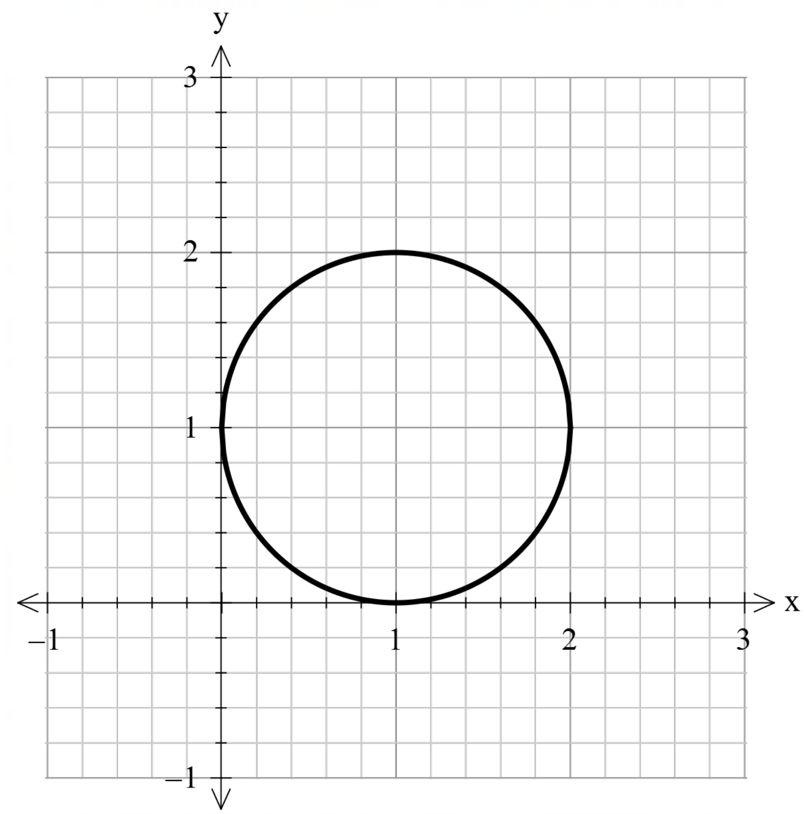
Solution:

$$\begin{aligned}\frac{1+e^{i\theta}}{e^{i\frac{\theta}{2}}} &= e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}\\ &= \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} + \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\\ &= 2\cos\frac{\theta}{2} + 0i\\ \text{Real part} &= 2\cos\frac{\theta}{2}\end{aligned}$$

(b) In an Argand diagram, the point P representing the complex number z is described by $|z-(1+i)| = 1$.

(i) Sketch the Cartesian equation representing z. **1**

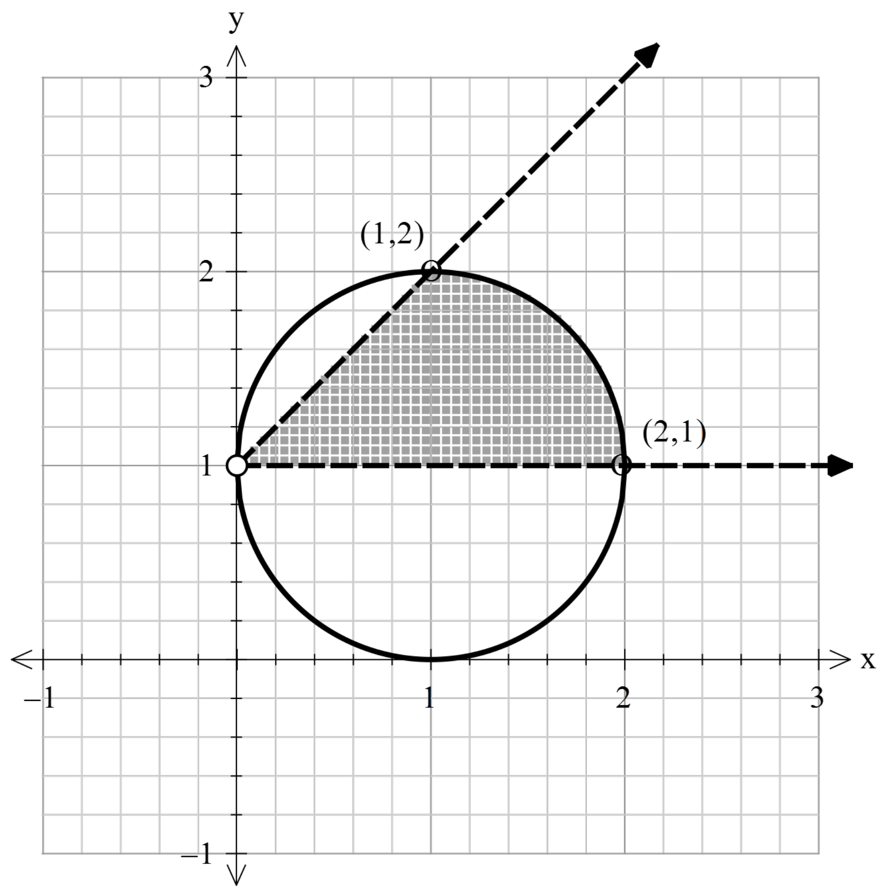
Question 11 continues...



Solution:

(ii) Shade the region where $|z-(1+i)| \leq 1$ and $0 < \operatorname{Arg}(z-i) < \dfrac{\pi}{4}$. **3**

A



Solution:

(c) Write the complex number $-1 + i\tan(3)$ in the form $ae^{i\theta}$ where $a, \theta \in \mathbb{R}$. **3**

Solution: This complex number is in the 3rd quadrant.

$$\begin{aligned}
-1 + i\tan 3 &= \frac{-1}{\cos 3}\left(\cos 3 - i\sin 3\right)\\
&= \frac{-1}{\cos 3}\left(\cos\left(-3\right) + i\sin\left(-3\right)\right)\\
&= \frac{-1}{\cos 3} \times e^{-3i}\\
\text{OR } &= \frac{1}{\cos 3} \times -1 \times e^{-3i}\\
&= \frac{1}{\cos 3} \times e^{i\pi} \times e^{-3i}\\
&= \frac{1}{\cos 3} e^{i(\pi-3)}
\end{aligned}$$

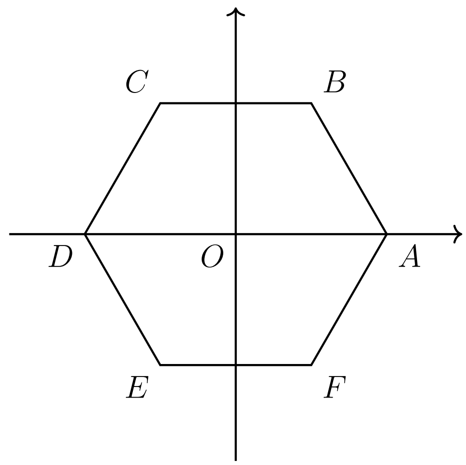
(d) Find all the complex numbers $z = a + ib$, where a, b are real, such that $|z|^2 - iz = 16 - 2i$. **3**

Solution: Let $z = x + iy$

$$\begin{aligned}
x^2 + y^2 - i(x + iy) &= 16 - 2i\\
x^2 + y^2 + y - ix &= 16 - 2i\\
\text{Imaginary part: } x &= 2\\
\text{Real part: } x^2 + y^2 + y &= 16\\
4 + y^2 + y &= 16\\
y^2 + y - 12 &= 0\\
\left(y+4\right)\left(y-3\right) &= 0\\
y &= -4, 3
\end{aligned}$$

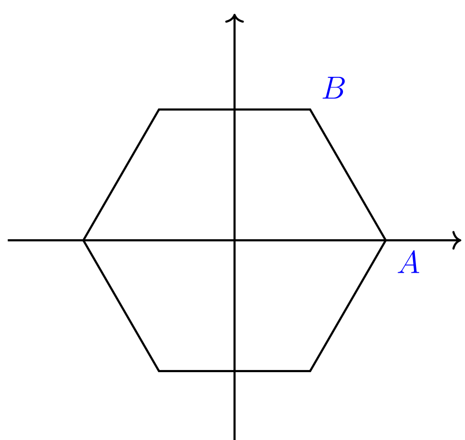
$\therefore z = 2 - 4i$ and $2 + 3i$

(e) In an Argand diagram, a regular hexagon $ABCDEF$, with the vertices taken in anticlockwise order, has its centre at the origin O and vertex A at $z = 2$.



If z is a complex number that lies on the hexagon;

(i) Find the set of values of $\text{Im}\,(z)$. **1**



Solution: Point B can be obtained by rotating point A anti-clockwise through $\frac{\pi}{3}$. i.e. Point B represents the complex number $(2 + 0i) \times \text{cis}\,(\frac{\pi}{3}) = 1 + \sqrt{3}i$

$$-\sqrt{3} \leq \text{Im}(z) \leq \sqrt{3}$$

(ii) Find the set of values of $|z|$. **1**

Solution: The largest value of $|z|$ is 2, since point A is furthest from the origin. The smallest distance from the origin occurs at the y-intercept of the hexagon .i.e. $\sqrt{3}$

$$\sqrt{3} \leq |z| \leq 2$$

(iii) If the hexagon is rotated in a clockwise direction about the origin through an angle of 45°, find the complex number which is represented by the new position of vertex C. Write your answer in modulus-argument form. **1**

Solution: Vertex C represents $-1+\sqrt{3}i$. When it is rotated **clockwise** through 45°, it will become:

$$\begin{aligned}\left(-1+\sqrt{3}i\right) \times \text{cis}\left(\frac{-\pi}{4}\right) &= 2\text{cis}\left(\frac{2\pi}{3}\right) \times \text{cis}\left(\frac{-\pi}{4}\right) \\ &= 2\text{cis}\left(\frac{5\pi}{12}\right)\end{aligned}$$

Question 12. **(15 marks) Use a SEPARATE writing booklet.**

(a) It is asserted that **1**

$$|2x+1| \le 5 \implies |x| \le 2.$$

Use a counter-example to disprove the assertion.

Solution: $x = -3$ satisfies the given absolute value equation but does not satisfy $|x| \le 2$.

(b) Consider the vector $\underset{\sim}{r}$, where $\underset{\sim}{r} = 2a\underset{\sim}{i} + b\underset{\sim}{j}$, $\forall\, a, b \in \mathbb{R}$.

(i) Write an expression for $\left|\underset{\sim}{r}\right|$ in terms of a and b. **1**

Solution:

$$\begin{aligned} |\underset{\sim}{r}|^2 &= (2a)^2 + b^2 \\ &= 4a^2 + b^2 \\ |\underset{\sim}{r}| &= \sqrt{4a^2+b^2} \end{aligned}$$

(ii) By choosing another appropriate vector for $\underset{\sim}{v}$, use the triangle inequality to prove **3**

$$\sqrt{a^2+b^2} \le \frac{\sqrt{4a^2+b^2} + \sqrt{a^2+4b^2}}{3}.$$

Solution: Consider the vector $\underset{\sim}{v} = a\underset{\sim}{i} + 2b\underset{\sim}{j}$.
So $|\underset{\sim}{v}| = \sqrt{a^2+4b^2}$
The triangle inequality is: $|\underset{\sim}{r} + \underset{\sim}{v}| \le |\underset{\sim}{r}| + |\underset{\sim}{v}|$

Question 12 continues...

Now,

$$\begin{aligned}
|\underset{\sim}{r} + \underset{\sim}{v}|^2 &= |2a\underset{\sim}{i} + b\underset{\sim}{j} + a\underset{\sim}{i} + 2b\underset{\sim}{j}|^2 \\
&= |3a\underset{\sim}{i} + 3b\underset{\sim}{j}|^2 \\
|\underset{\sim}{r} + \underset{\sim}{v}| &= \sqrt{9a^2 + 9b^2} \\
&= 3\sqrt{a^2 + b^2}
\end{aligned}$$

i.e.

$$\begin{aligned}
3\sqrt{a^2+b^2} &\leq \sqrt{4a^2+b^2} + \sqrt{a^2+4b^2} \\
\sqrt{a^2+b^2} &\leq \frac{\sqrt{4a^2+b^2} + \sqrt{a^2+4b^2}}{3}
\end{aligned}$$

(c) Use contradiction to prove the following; **3**
There do **not** exist positive integers m and n such that $m^2 - n^2 = 1$.

Solution: Assume that there are positive integers m and n such that $m^2 - n^2 = 1$.

$$\begin{aligned}
m^2 - n^2 &= 1 \\
(m+n)(m-n) &= 1
\end{aligned}$$

$m+n$ and $m-n$ are integers and both are factors of 1.
This can only happen if $m = 1$ and $n = 0$
which contradicts the assumption that both m and n are **positive** integers.
Thus, there do **not** exist positive integers m and n such that $m^2 - n^2 = 1$.

(d) Given the recurrence relation $T_n = T_{n-1} + 2^n$ for $n \geq 2$ and $T_1 = 3$, prove by Mathematical Induction that $T_n = 2^{n+1} - 1 \qquad \forall n \geq 1$. **3**

Solution: Let $S(n)$ be the statement, $T_n = 2^{n+1} - 1 \; \forall n \geq 1$.

$S(1) : T_1 = 2^{1+1} - 1 = 3$ which is true.
Assume $S(k)$ is true.

$$T_k = 2^{k+1} - 1 \text{ for } k \geq 1, k \in \mathbb{N} \tag{1}$$

Prove $S(k+1)$ is true. $\{$RTP: $T_{k+1} = 2^{k+2} - 1$ for $k \geq 1, k \in \mathbb{N}\}$

$$\begin{aligned} T_{k+1} &= T_k + 2^{k+1} \text{ using the given defn of } T_n \\ &= 2^{k+1} - 1 + 2^{k+1} \text{ using (1).} \\ &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2 \times 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

as required.

Therefore, by the Principle of Mathematical Induction, $S(n)$ is true.

(e) By considering $f(x) = x - \ln\left(1 + x + \dfrac{1}{2}x^2\right)$, show that $e^x < 1 + x + \dfrac{1}{2}x^2$ for $x < 0$. **4**

Solution: Begin by showing that $f(x) < 0$ for $x < 0$

$$\begin{aligned} f'(x) &= 1 - \frac{1+x}{1+x+\frac{1}{2}x^2} \\ &= \frac{1+x+\frac{1}{2}x^2 - 1 - x}{1+x+\frac{1}{2}x^2} \\ &= \frac{\frac{1}{2}x^2}{1+x+\frac{1}{2}x^2} \end{aligned}$$

Now, a stationary point occurs at $x = 0$. Also $f(0) = 0$ and for $x < 0, f'(x) > 0$, thus the curve $y = f(x)$ is increasing for $x < 0$ with a stat. pt at $x = 0$. In fact, $f(x)$ is increasing for all value of x.
Thus, $f(x) < 0$ for $x < 0$.
i.e. $x - \ln\left(1 + x + \frac{1}{2}x^2\right) < 0$ for $x < 0$

$$x - \ln\left(1 + x + \frac{1}{2}x^2\right) < 0$$
$$x < \ln\left(1 + x + \frac{1}{2}x^2\right)$$
$$\therefore e^x < 1 + x + \frac{1}{2}x^2$$

after taking exponentials of each side. Note that since the exponential function is itself monotonically increasing then the inequality sign does not change.

Question 13. **(15 marks) Use a SEPARATE writing booklet.**

(a) If $z = \cos\theta + i\sin\theta$,

(i) By using expressions for $z^n + \dfrac{1}{z^n}$ and $z^n - \dfrac{1}{z^n}$, where n is a positive integer, expand $\left(z + \dfrac{1}{z}\right)^4 + \left(z - \dfrac{1}{z}\right)^4$ to show that $\cos^4\theta + \sin^4\theta = \dfrac{1}{4}\left(\cos(4\theta) + 3\right)$. **3**

Solution: Now $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ and $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^4 + \left(z - \frac{1}{z}\right)^4 &= (2\cos\theta)^4 + (2i\sin\theta)^4 \\
z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4} + z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} &= 16\cos^4\theta + 16\sin^4\theta \\
2z^4 + \frac{2}{z^4} + 12 &= 16\cos^4\theta + 16\sin^4\theta \\
2\left(z^4 + \frac{1}{z^4}\right) + 12 &= 16\cos^4\theta + 16\sin^4\theta \\
2 \times 2\cos 4\theta + 12 &= 16\cos^4\theta + 16\sin^4\theta \\
\frac{1}{4}\left(\cos(4\theta) + 3\right) &= \cos^4\theta + \sin^4\theta
\end{aligned}$$

(ii) By letting $x = \cos\theta$, show that the equation $8x^4 + 8\left(1 - x^2\right)^2 = 7$ has roots $\pm\cos\dfrac{\pi}{12}, \pm\cos\dfrac{5\pi}{12}$. **2**

Solution:

$$\begin{aligned}
8x^4 + 8\left(1 - x^2\right)^2 &= 7 \\
8\cos^4\theta + 8\left(1 - \cos^2\theta\right)^2 &= 7 \\
8\cos^4\theta + 8\sin^4\theta &= 7 \\
\cos^4\theta + \sin^4\theta &= \frac{7}{8} \\
\frac{1}{4}\left(\cos(4\theta) + 3\right) &= \frac{7}{8} \\
\cos(4\theta) + 3 &= \frac{7}{2} \\
\cos(4\theta) &= \frac{1}{2} \\
4\theta &= \pm\frac{\pi}{3}, \pm\frac{5\pi}{3} \\
\theta &= \pm\frac{\pi}{12}, \pm\frac{5\pi}{12}
\end{aligned}$$

(b) Evaluate $\displaystyle\int_1^{e^2} \frac{\ln x}{x^3} dx$. **3**

Solution: Using integration by parts with $u = \ln x, v' = x^{-3}$.

$$\begin{aligned}
\int_1^{e^2} \frac{\ln x}{x^3} dx &= \left[\ln x \times \frac{-1}{2x^2}\right]_1^{e^2} + \frac{1}{2}\int_1^{e^2} x^{-3} dx \\
&= \left\{\ln e^2 \times \frac{-1}{2e^4}\right\} - \left\{\ln 1 \times \frac{-1}{2 \times (1)^2}\right\} - \frac{1}{4}\left[x^{-2}\right]_1^{e^2} \\
&= \frac{-1}{e^4} - 0 - \frac{1}{4}\left\{e^{-4} - 1^{-2}\right\} \\
&= \frac{-1}{e^4} - \frac{1}{4e^4} + \frac{1}{4} \\
&= \frac{-5}{4e^4} + \frac{1}{4}
\end{aligned}$$

(c) Consider the complex number $z = e^{i\theta}$.

(i) Show that $\cos i = \dfrac{e^1 + e^{-1}}{2}$. **1**

Solution: Now $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$
So,

$$\begin{aligned}\cos i &= \frac{e^{i\times i} + e^{-i\times i}}{2}\\ &= \frac{e^{-1} + e^{1}}{2}\end{aligned}$$

(ii) Find a similar expression for $\sin i$. **1**

Solution: Similarly, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

$$\begin{aligned}\sin i &= \frac{e^{i\times i} - e^{-i\times i}}{2i}\\ &= \frac{e^{-1} - e^{1}}{2i}\end{aligned}$$

(iii) Hence solve $e^w = \cos i - i\sin i$, where $w \in \mathbb{C}$. **2**

Solution: Now;

$$\begin{aligned}\cos i - i\sin i &= \frac{e^{-1} + e^{1}}{2} - i \times \frac{e^{-1} - e^{1}}{2i}\\ \cos i - i\sin i &= e^{1}\\ e^{w} &= e^{1}\\ w &= 1 + 2in\pi, n \in \mathbb{Z}\end{aligned}$$

(d) Find the range of values of p, for $p \in \mathbb{R}$ such that $x^4 - px^3 + px - 1 = 0$ has 1 pair of complex conjugate roots. **3**

Solution:

$$\begin{aligned} x^4 - px^3 + px - 1 &= 0 \\ x^4 - 1 - p\left(x^3 - x\right) &= 0 \\ (x^2+1)(x^2-1) - px(x^2-1) &= 0 \\ (x^2-1)(x^2+1-px) &= 0 \\ x &= \pm 1, x^2 - px + 1 = 0 \end{aligned}$$

We now require the discriminant of the quadratic to be less than zero to achieve a conjugate pair of complex roots.

$$\begin{aligned} \Delta &< 0 \\ (-p)^2 - 4 \times 1 \times 1 &< 0 \\ p^2 - 4 &< 0 \end{aligned}$$

i.e. $-2 < p < 2$

Question 14. **(15 marks) Use a SEPARATE writing booklet.**

(a) Find $\displaystyle\int \frac{\cos^2 x}{1-\sin x} dx$. **2**

Solution:

$$\begin{aligned}\int \frac{\cos^2 x}{1-\sin x} dx &= \int \frac{1-\sin^2 x}{1-\sin x} dx\\ &= \int \frac{(1-\sin x)(1+\sin x)}{1-\sin x} dx\\ &= \int (1+\sin x) dx\\ &= x - \cos x + C\end{aligned}$$

(b) Find a primitive of $\sec^6 x$. **3**

Solution:

$$\begin{aligned}\int \sec^6 x dx &= \int \sec^4 x \sec^2 x dx\\ &= \int (1+\tan^2 x)^2 \sec^2 x dx\\ \text{Let } u &= \tan x\\ \int \sec^6 x dx &= \int (1+u^2)^2 du\\ &= \int (1+2u^2+u^4) du\\ &= u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C\\ &= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C\end{aligned}$$

(c) By using the substitution $p = 1 + x^{\frac{3}{2}}$, find $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+\sqrt{x^3}}}dx$. **3**

Solution:

$$\begin{aligned}
\text{Let } I &= \int \frac{\sqrt{x}}{\sqrt{1+\sqrt{x^3}}}dx \\
\text{Let } p &= 1 + x^{\frac{3}{2}} \\
dp &= \frac{3}{2}x^{\frac{1}{2}}dx \\
\frac{2}{3}dp &= \sqrt{x}dx \\
I &= \frac{2}{3}\int \frac{1}{\sqrt{p}}dp \\
&= \frac{2}{3} \times 2p^{\frac{1}{2}} + C \\
&= \frac{4}{3}\sqrt{1+x^{\frac{3}{2}}} + C
\end{aligned}$$

(d) If $I_n = \displaystyle\int_0^t \frac{1}{(1+x^2)^n}dx$ for $n = 1, 2, 3, \ldots$

(i) Show that $2nI_{n+1} = (2n-1)\,I_n + \dfrac{t}{(1+t^2)^n}$ for $n = 1, 2, 3, \ldots$ **4**

Solution: Note:

$$\begin{aligned}
(1+x^2)^{-n} &= (1+x^2)^1(1+x^2)^{-(n+1)} \\
&= (1+x^2)^{-(n+1)} + x^2(1+x^2)^{-(n+1)} \\
I_n &= I_{n+1} + \int_0^t x^2(1+x^2)^{-(n+1)}dx \\
&= I_{n+1} + \int_0^t x \times x(1+x^2)^{-(n+1)}dx
\end{aligned}$$

by parts with $u = x, v' = x(1+x^2)^{-(n+1)}$

$$\begin{aligned}
\therefore I_n &= I_{n+1} + \left[\frac{-x}{2n}(1+x^2)^{-n}\right]_0^t + \frac{1}{2n}\int(1+x^2)^{-n}dx \\
I_n &= I_{n+1} - \frac{1}{2n} \times \frac{t}{(1+t^2)^n} + \frac{1}{2n}I_n \\
2nI_n &= 2nI_{n+1} - \frac{t}{(1+t^2)^n} + I_n \\
\therefore 2nI_{n+1} &= (2n-1)I_n + \frac{t}{(1+t^2)^n}
\end{aligned}$$

(ii) Hence find I_3 in terms of t. **3**

Solution: Applying the above result with $n = 2$ gives:

$$\begin{aligned}
I_{n+1} &= \frac{(2n-1)}{2n}I_n + \frac{t}{2n\,(1+t^2)^n} \\
I_3 &= \frac{3}{4}I_2 + \frac{t}{4(1+t^2)^2} \\
&= \frac{3}{4}\left\{\frac{1}{2}I_1 + \frac{t}{2(1+t^2)^1}\right\} + \frac{t}{4(1+t^2)^2} \\
&= \frac{3}{8}I_1 + \frac{3t}{8(1+t^2)} + \frac{t}{4(1+t^2)^2} \\
&= \frac{3}{8}\left[\tan^{-1}x\right]_0^t + \frac{3t}{8(1+t^2)} + \frac{t}{4(1+t^2)^2} \\
&= \frac{3}{8}\tan^{-1}t + \frac{3t}{8(1+t^2)} + \frac{t}{4(1+t^2)^2}
\end{aligned}$$

Question 15. **(15 marks) Use a SEPARATE writing booklet.**

(a) Given $\underset{\sim}{u} = \begin{pmatrix} -9 \\ 8 \\ 1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}$, and $\underset{\sim}{u} - 6\underset{\sim}{v} + 4\underset{\sim}{w} = 0$, find vector $\underset{\sim}{w}$. **2**

Solution:

$$\begin{aligned} \underset{\sim}{u} - 6\underset{\sim}{v} + 4\underset{\sim}{w} &= 0 \\ 4\underset{\sim}{w} &= 6\underset{\sim}{v} - \underset{\sim}{u} \\ 4\underset{\sim}{w} &= 6 \times \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} -9 \\ 8 \\ 1 \end{pmatrix} \\ 4\underset{\sim}{w} &= \begin{pmatrix} 9 \\ 34 \\ -19 \end{pmatrix} \\ \underset{\sim}{w} &= \frac{1}{4}\begin{pmatrix} 9 \\ 34 \\ -19 \end{pmatrix} \end{aligned}$$

(b) Given $\left|\underset{\sim}{v}\right| = 2$, $\left|\underset{\sim}{w}\right| = 1$ and $\underset{\sim}{v} \cdot \underset{\sim}{w} = 2$, find the value of $\left|\underset{\sim}{v} - 3\underset{\sim}{w}\right|$. **3**

Solution:

$$\begin{aligned} \left|\underset{\sim}{v} - 3\underset{\sim}{w}\right|^2 &= \left(\underset{\sim}{v} - 3\underset{\sim}{w}\right) \cdot \left(\underset{\sim}{v} - 3\underset{\sim}{w}\right) \\ &= \underset{\sim}{v} \cdot \underset{\sim}{v} - 6\underset{\sim}{v} \cdot \underset{\sim}{w} + 9\underset{\sim}{w} \cdot \underset{\sim}{w} \\ &= |\underset{\sim}{v}|^2 - 6 \times 2 + 9|\underset{\sim}{w}|^2 \\ &= 4 - 12 + 9 \\ &= 1 \\ \therefore \left|\underset{\sim}{v} - 3\underset{\sim}{w}\right| &= 1 \end{aligned}$$

Question 15 continues...

(c) Show that if $\underset{\sim}{c} = \left|\underset{\sim}{a}\right| \underset{\sim}{b} + \left|\underset{\sim}{b}\right| \underset{\sim}{a}$ where $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are all non-zero vectors then $\underset{\sim}{c}$ bisects the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **3**

Solution: Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{c}$:

$$\begin{aligned}
\cos\theta &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{c}}{|\underset{\sim}{a}||\underset{\sim}{c}|} \\
&= \frac{\underset{\sim}{a} \cdot \left(\left|\underset{\sim}{a}\right| \underset{\sim}{b} + \left|\underset{\sim}{b}\right| \underset{\sim}{a}\right)}{|\underset{\sim}{a}||\underset{\sim}{c}|} \\
&= \frac{|\underset{\sim}{a}|\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{a}|^2|\underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{c}|} \\
&= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{a}||\underset{\sim}{b}|}{|\underset{\sim}{c}|}
\end{aligned}$$

similarly the angle between $\underset{\sim}{b}$ and $\underset{\sim}{c}$ is:

$$\begin{aligned}
\cos\theta &= \frac{\underset{\sim}{b} \cdot \underset{\sim}{c}}{|\underset{\sim}{b}||\underset{\sim}{c}|} \\
&= \frac{\underset{\sim}{b} \cdot \left(\left|\underset{\sim}{a}\right| \underset{\sim}{b} + \left|\underset{\sim}{b}\right| \underset{\sim}{a}\right)}{|\underset{\sim}{b}||\underset{\sim}{c}|} \\
&= \frac{|\underset{\sim}{a}||\underset{\sim}{b}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{b}|\underset{\sim}{b}|}{|\underset{\sim}{a}||\underset{\sim}{c}|} \\
&= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{a}||\underset{\sim}{b}|}{|\underset{\sim}{c}|}
\end{aligned}$$

The angles have the same expression, so they are equal, thus vector $\underset{\sim}{c}$ bisects the angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(d) The points A, B, C and D have position vectors $\left(0, \frac{\sqrt{3}}{6}x, \frac{\sqrt{2}}{\sqrt{3}}x\right)$, $\left(0, \frac{\sqrt{3}}{6}x, 0\right)$, $\left(-\frac{x}{2}, 0, 0\right)$ and $\left(\frac{x}{2}, 0, 0\right)$ respectively. If M is the midpoint of CD, show that $\angle AMB \approx 70.5°$. **3**

Solution: M has coordinates $(0,0,0)$.
$\overrightarrow{MA} = \left(0, \frac{\sqrt{3}}{6}x, \frac{\sqrt{2}}{\sqrt{3}}x\right)$ and $\overrightarrow{MB} = \left(0, \frac{\sqrt{3}}{6}x, 0\right)$
Angle between the two vector is:

$$\begin{aligned}
\cos\theta &= \frac{\overrightarrow{MA}\cdot\overrightarrow{MB}}{\left|\overrightarrow{MA}\right|\left|\overrightarrow{MB}\right|} \\
&= \frac{\left(0, \frac{\sqrt{3}}{6}x, \frac{\sqrt{2}}{\sqrt{3}}x\right)\cdot\left(0, \frac{\sqrt{3}}{6}x, 0\right)}{\sqrt{\frac{3}{36}x^2 + \frac{2}{3}x^2} \times \sqrt{\frac{3}{36}x^2}} \\
&= \frac{\frac{3}{36}x^2}{\frac{\sqrt{81}}{36}x^2} \\
&= \frac{1}{3} \\
\theta &= 70.5^\circ
\end{aligned}$$

(e) Two particles travel along the curves $\underset{\sim}{r}(t)$ and $\underset{\sim}{w}(t)$, defined parametrically by **4**

$\underset{\sim}{r}(t) = t\underset{\sim}{i} + t^2\underset{\sim}{j} + t^3\underset{\sim}{k}$ and $\underset{\sim}{w}(t) = (1-2t)\underset{\sim}{i} + (1+6t)\underset{\sim}{j} + (1+14t)\underset{\sim}{k}$.

Show that the particle's paths intersect but the particles do not collide.

Solution: Particles will collide when they have the same position at the same time.
x direction:

$$\begin{aligned}
t &= 1-2t \\
t &= \frac{1}{3}
\end{aligned}$$

y direction:

$$\begin{aligned} t^2 &= 1 + 6t \\ t^2 - 6t &= 1 \\ t &= 3 + \sqrt{10} \\ &\neq \frac{1}{3} \end{aligned}$$

Therefore the particles do NOT collide.
For intersecting paths:

$\underset{\sim}{r}(t) = t\underset{\sim}{i} + t^2\underset{\sim}{j} + t^3\underset{\sim}{k}$ and $\underset{\sim}{w}(s) = (1-2s)\underset{\sim}{i} + (1+6s)\underset{\sim}{j} + (1+14s)\underset{\sim}{k}$

x direction: $t = 1 - 2s$
y direction: $t^2 = 1 + 6s$
Solving simultaneously gives:

$$\begin{aligned} t^2 + 3t - 4 &= 0 \\ (t+4)(t-1) &= 0 \\ t &= 1 \text{ since } t \geq 0 \\ \therefore s &= 0 \end{aligned}$$

substituting $t = 1$ and $s = 0$ into the z-direction:

$$\begin{aligned} t^3 &= 1^3 = 1 \\ \text{and } 1 + 14s &= 1 + 14 \times 0 = 1 \end{aligned}$$

Thus the particle's paths intersect at $\underset{\sim}{r}(1)$ and $\underset{\sim}{w}(0)$. i.e. at $(1, 1, 1)$

Question 16. **(15 marks) Use a SEPARATE writing booklet.**

(a) A particle is travelling in a straight line. Its displacement, x cm, from O at a given time, t seconds after the start of the motion, is given by $x = 2 + \cos^2 t$.

(i) Show that the particle is undergoing simple harmonic motion. **2**

Solution: We are required to show $\ddot{x} = -n^2(x - c)$

$$\begin{aligned} x &= 2 + \cos^2 t \\ \dot{x} &= -2\cos t \sin t \\ \ddot{x} &= -2(\cos^2 t - \sin^2 t) \\ &= -2(2\cos^2 t - 1) \\ &= -4\cos^2 t + 2 \\ &= -4(x - 2) + 2 \\ &= -4(x - \frac{5}{2}) \end{aligned}$$

which is in the form required; with $n = 2$ and $c = \frac{5}{2}$. Therefore the particle is undergoing SHM.

(ii) Find the period of the motion. **1**

Solution:

$$\begin{aligned} \text{Period } &= \frac{2\pi}{n} \\ &= \frac{2\pi}{2} \\ &= \pi \end{aligned}$$

(iii) Determine the distance travelled by the particle in the first π seconds. **1**

Question 16 continues...

Solution: The distance travelled is double the distance travelled in the first $\frac{\pi}{2}$ seconds.

$$\begin{aligned}\text{Distance} &= 2 \times \left| \int_0^{\frac{\pi}{2}} v dt \right| \\ &= 2 \times \left| \int_0^{\frac{\pi}{2}} -2 \cos t \sin t dt \right| \\ &= 4 \times \left| \left[\frac{1}{2} \cos^2 t \right]_0^{\frac{\pi}{2}} \right| \\ &= 2 \times \left| \left(\cos^2 \frac{\pi}{2} - \cos^2 0 \right) \right| \\ &= 2 \text{ metres}\end{aligned}$$

or
the amplitude is 0.5 and centre of motion is 2.5.
The distance travelled will be $4 \times 0.5 = 2$ metres.

(b) A particle of mass m kg, moving in a straight line, has acceleration of $x \tan^{-1} x$ ms^{-2}; where x is measured in metres and time t is in seconds. Given that $v = 3$ when $x = 0$, find the velocity of the particle at position x. **4**

Solution:

$$v \frac{dv}{dx} = x \tan^{-1} x$$
$$\int_3^v v dv = \int_0^x x \tan^{-1} x dx$$

The integral on the RHS can be done using integration by parts.

$$u = \tan^{-1} x, v' = x$$
$$u' = \frac{1}{1+x^2}, v = \frac{x^2}{2}$$

Now;

$$\begin{aligned}
\int_3^v v\,dv &= \left[\frac{x^2}{2}\tan^{-1}x\right]_0^x - \frac{1}{2}\int_0^x \left(\frac{x^2}{x^2+1}\right)dx \\
\left[\frac{v^2}{2}\right]_3^v &= \left(\frac{x^2}{2}\tan^{-1}x\right) - \left(\frac{0^2}{2}\tan^{-1}0\right) - \frac{1}{2}\int_0^x \left(\frac{x^2+1-1}{x^2+1}\right)dx \\
\frac{v^2}{2} - \frac{9}{2} &= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int_0^x \left(1 - \frac{1}{x^2+1}\right)dx \\
\frac{v^2}{2} - \frac{9}{2} &= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right]_0^x \\
v^2 - 9 &= x^2\tan^{-1}x - \left\{\left(x - \tan^{-1}x\right) - \left(0 - \tan^{-1}0\right)\right\} \\
v^2 &= x^2\tan^{-1}x - x + \tan^{-1}x + 9 \\
v &= \sqrt{x^2\tan^{-1}x - x + \tan^{-1}x + 9}
\end{aligned}$$

Take the positive square root since $v = 3 > 0$ when $x = 0$.

(c) A particle of mass m kilograms is dropped from rest in a medium where the resistance to motion has magnitude $0.1mv^2$ Newtons when the velocity of the particle is v ms^{-1}. After t seconds the particle has fallen x metres, and has velocity v ms^{-1} and acceleration a ms^{-2}. The particle hits the ground $\ln\left(1+\sqrt{2}\right)$ seconds after it is dropped. Take $g = 10$ ms^{-2}.

(i) By drawing a diagram showing the forces on the particle, deduce that $a = \frac{1}{10}\left(100 - v^2\right)$. **1**

Solution: Take down as positive.
Force 1: Weight is acting in a positive direction. Weight $= mg = m \times 10 = 10m$
Force 2: Resistance is acting in a negative direction. Resistance $= 0.1mv^2$
Net Force = sum of all the forces
$ma = 10m - 0.1mv^2$
$a = \frac{1}{10}\left(100 - v^2\right)$

(ii) Show that the particle hits the ground with a speed $5\sqrt{2}$ ms^{-1}. **4**

Solution:

$$\begin{aligned}
\frac{dv}{dt} &= \frac{1}{10}\left(100 - v^2\right)\\
\int_0^v \frac{dv}{100 - v^2} &= \int_0^{\ln\left(1+\sqrt{2}\right)} 0.1dt
\end{aligned}$$

Now use partial fractions on the LHS

$$\begin{aligned}
\frac{1}{20}\int_0^v \left(\frac{1}{10+v} + \frac{1}{10-v}\right) dv &= 0.1\left[t\right]_0^{\ln\left(1+\sqrt{2}\right)}\\
\left[\ln|10+v| - \ln|10-v|\right]_0^v &= 2\ln\left(1+\sqrt{2}\right)\\
\ln\left(\frac{10+v}{10-v}\right) - \ln\left(\frac{10+0}{10-0}\right) &= \ln\left(1+\sqrt{2}\right)^2\\
\frac{10+v}{10-v} &= 1 + 2\sqrt{2} + 2\\
10 + v &= (10-v)(3+2\sqrt{2})\\
10 + v &= 30 + 20\sqrt{2} - 3v - 2v\sqrt{2}\\
4v + 2v\sqrt{2} &= 20 + 20\sqrt{2}\\
v\left(2+\sqrt{2}\right) &= 10 + 10\sqrt{2}\\
v &= \frac{10+10\sqrt{2}}{2+2\sqrt{2}}\\
&= \frac{10+10\sqrt{2}}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}}\\
&= \frac{10\sqrt{2}}{2}\\
&= 5\sqrt{2}\text{ms}^{-1}
\end{aligned}$$

(iii) Show that the position of the particle when it reaches half of its terminal velocity is given by $x = 10\ln\left(\frac{2\sqrt{3}}{3}\right)$. **2**

Solution: Terminal velcoity occurs when the net force is zero. i.e. when the acceleration is zero.$100 - v_T^2 = 0 \implies v_T = 10$.

$$\begin{aligned}
\int_0^5 \frac{v}{100 - v^2} dv &= \frac{1}{10} \int_0^x dx \\
-\frac{1}{2} \left[\ln \left|100 - v^2\right|\right]_0^5 &= \frac{x}{10} \\
-\frac{1}{2} \ln \left(\frac{75}{100}\right) &= \frac{x}{10} \\
\ln \left(\frac{3}{4}\right)^{-\frac{1}{2}} &= \frac{x}{10}
\end{aligned}$$

$$\begin{aligned}
x &= 10 \ln \left(\frac{4}{3}\right)^{\frac{1}{2}} \\
&= 10 \ln \left(\frac{2}{\sqrt{3}}\right) \\
&= 10 \ln \left(\frac{2\sqrt{3}}{3}\right)
\end{aligned}$$

End of Paper.