Section I

5 marks
Attempt Questions 1-5
Allow about 5 minutes for this section

Use the multiple choice answer sheet for questions 1-5

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1. Which of the following points lies on the line described by the vector equation

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

(A) $\begin{pmatrix} -3 \\ 9 \\ 1 \end{pmatrix}$

(B) $\begin{pmatrix} -3 \\ -8 \\ -3 \end{pmatrix}$

(C) $\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

(D) $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$

2. If the vectors $\underset{\sim}{u} = \lambda \underset{\sim}{i} + \lambda \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{v} = \lambda \underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular, then:

(A) $\lambda = -2 \ or \ \lambda = 4$

(B) $\lambda = -2 \ or \ \lambda = -4$

(C) $\lambda = 2 \ or \ \lambda = -4$

(D) $\lambda = 2 \ or \ \lambda = 4$

3. *P, Q* and *R* are points with position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively.

The points are collinear and *P* lies between *R* and Q. If $|QR| = 3|PR|$, then the position vector of $\underset{\sim}{r}$ is:

(A) $\underset{\sim}{r} = \frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

(B) $\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

(C) $\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{q}$

(D) $\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

4. A particle is moving in SHM along a straight line. At time *t* seconds it has displacement *x* metres from a fixed point on the line and velocity *v* metres/second given by $v^2 = -0.5x^2 + 2x + 2.5$.

What is the period of the motion?

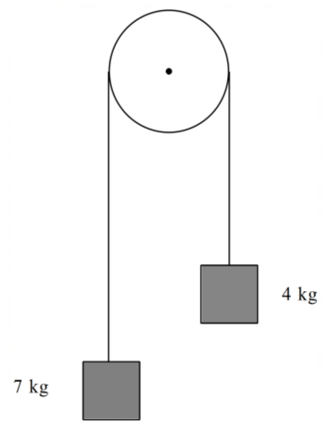
(A) $\pi\ seconds$

(B) $\pi\sqrt{2}\ seconds$

(C) $2\pi\ seconds$

(D) $2\pi\sqrt{2}\ seconds$

5. A light, inextensible string passes over a smooth pulley. Attached to the ends of the string are masses of 4 kg and 7 kg, as shown.



The acceleration in the downwards direction of the 7 kg mass is:

(A) $\frac{3g}{11}$

(B) $\frac{11g}{3}$

(C) $\frac{7g}{11}$

(D) $3g$

End of Section I

Section II

36 marks
Attempt Questions 6-8
Allow about 55 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 6-8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (Start a new writing booklet.) **Marks**

a. Find the centre and radius of the sphere with equation: 3

$$x^2 + y^2 + z^2 - x + 2y + 3z = 1$$

b. Relative to the origin, O, the points *A, B, C* and *D* have position vectors given by: 3

$A: -4\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

$B: 4\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$C: 4\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

$D: 2\underset{\sim}{j} - 6\underset{\sim}{k}$

Find the position vector of the point of intersection of the lines *AC* and *BD.*

c. Given that the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ satisfy $\underset{\sim}{u} = 9\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{v} = 8\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$, find the acute angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. 3

Question 6 continued on next page.

Question 6 continued.

d. Consider the points:

$$A = (4, 3, -2),\ B = (-3, -6, 10) \text{ and } C = (-38, -51, -49)$$

i) Find the equation of the line, $\underset{\sim}{r}$, which passes through *A* and *B*. 1

ii) Show that *A, B* and *C* are not collinear. 2

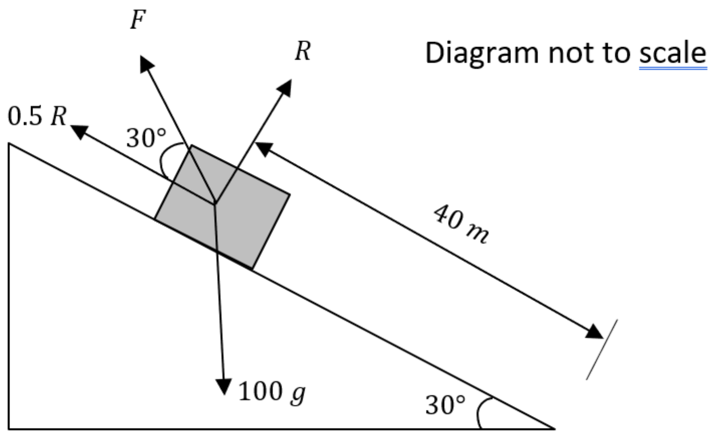
End of Question 6

Question 7 (Start a new writing booklet.) **Marks**

a. An object with mass 100 kg begins at rest and slides down a rough surface which is inclined at 30° to the horizontal.

The object experiences a resistive force of $0.5R$ Newtons, where R is the normal reaction force exerted by the surface on the object. In order to slow the object down, a force F is applied to the object at an angle of 30° to the plane, as shown in the diagram.

Take the acceleration due to gravity, g, to be 10 m/s^2.



i) By resolving forces perpendicular to the plane, show that **2**

$$R = 500\sqrt{3} - \frac{F}{2}$$

ii) Show that the acceleration of the object **down** the plane is given by: **2**

$$a = \frac{1000(2-\sqrt{3}) - F(2\sqrt{3}-1)}{400}$$

Question 7 continued on next page.

Question 7 continued.

b. A particle is projected from the Origin with an initial speed of V at an angle of θ above the horizontal. The particle is affected by both gravity and an air resistance proportional to the velocity such that the components of acceleration are:

$$\ddot{x} = -k\dot{x}$$
$$\ddot{y} = -k\dot{y} - g$$

Where k is a constant and $g \ m/s^2$ is the acceleration due to gravity.

i) Show that: 3

$$\dot{x} = Vcos\theta e^{-kt} \qquad \text{and,}$$

$$\dot{y} = \frac{1}{k}(g + kVsin\theta)e^{-kt} - \frac{g}{k}$$

ii) Show that the particle reaches maximum height when 2

$$t = \frac{1}{k}ln\left(\frac{g+kVsin\theta}{g}\right)$$

c. A mass of 1 *kg* moves along a straight line with velocity *v m/s.* The object has initial velocity *U m/s*, where $U > 0$, and starts at the Origin. It encounters a resistance of $v + v^3 \ m/s^2$ such that $\ddot{x} = -v(1 + v^2)$.

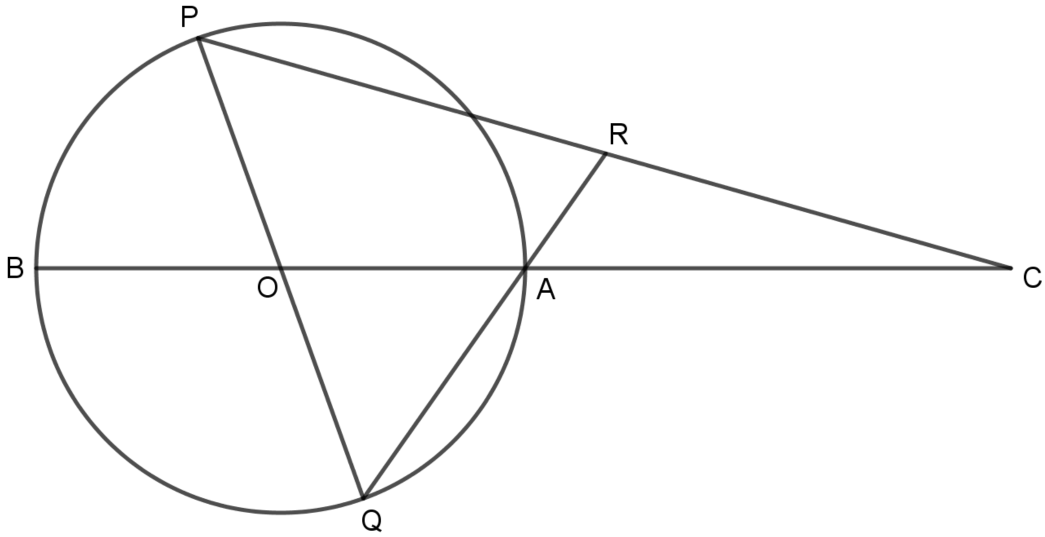
Show that $x = tan^{-1}\left(\frac{U-v}{1+Uv}\right)$ 3

End of Question 7

Question 8 (Start a new writing booklet.) **Marks**

a. Calculate the length of the vector vectors $2\underset{\sim}{a} - 3\underset{\sim}{b}$, given that $\left|\underset{\sim}{a}\right| = 3$, $\left|\underset{\sim}{b}\right| = 2$ and $\underset{\sim}{a}.\underset{\sim}{b} = 4$ 2

b.



In the diagram, *PQ* and *AB* are diameters of a circle with centre *O*. *BA* is produced to *C* so that $BA = AC$. *QA* produced meets *PC* in *R*. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OP} = \underset{\sim}{p}$.
Given $\overrightarrow{PR} = \lambda\overrightarrow{PC}$ and $\overrightarrow{QR} = \mu\overrightarrow{QA}$ for some scalars λ and μ.

i) Find expressions, in terms of $\underset{\sim}{a}$ and $\underset{\sim}{p}$, for $\overrightarrow{PR}$ and $\overrightarrow{QR}$. 2

ii) Show that *R* is the midpoint of *PC*. 2

Question 8 continued on next page.

Question 8 continued.

c. A particle of mass *m kg* falls vertically from rest under gravity in a medium where the resistance to motion has magnitude $\frac{mv^2}{g}$ Newtons when the speed of the particle is *v m/s* and the acceleration due to gravity is $g\ m/s^2$, At time *t* seconds the particle has fallen *x* metres and has velocity *v m/s*.

i) Show that $\ddot{x} = \frac{1}{g}(g^2 - v^2)$ 1

ii) Show that $v = g\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$. 3

You may wish to use the result $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} ln\left(\frac{a+x}{a-x}\right) + c$

iii) Find in simplest exact form, the time taken for the particle to reach half of its terminal velocity. 2

End of Question 8

End of Paper

2021 12 Ext2 Task 4 Sol'ns

CADDA

1. $\lambda = -1 \Rightarrow \underset{\sim}{r} = \begin{pmatrix} 1+2 \\ 2-3 \\ -1-1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} \Rightarrow$ C

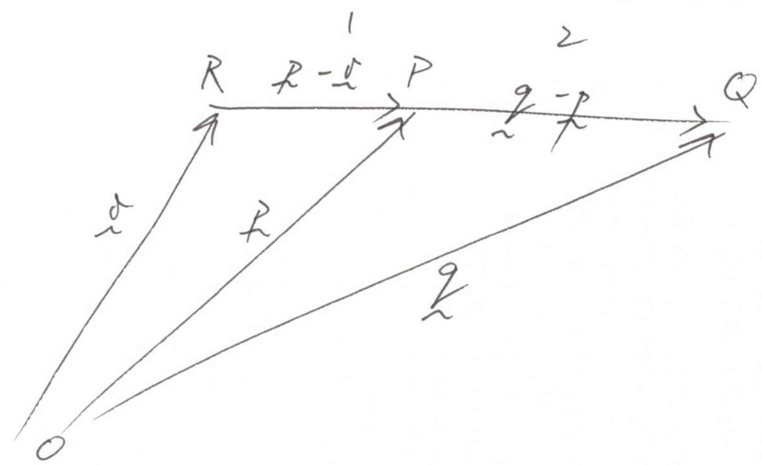
2. $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

ie $\lambda^2 - 2\lambda - 8 = 0$

$(\lambda + 2)(\lambda - 4) = 0$

$\lambda = -2$ or $\lambda = 4$ A

3.



$\underset{\sim}{q} - \underset{\sim}{p} = 2(\underset{\sim}{p} - \underset{\sim}{r})$

$\underset{\sim}{q} - \underset{\sim}{p} = 2\underset{\sim}{p} - 2\underset{\sim}{r}$

$2\underset{\sim}{r} = 3\underset{\sim}{p} - \underset{\sim}{q}$

$\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$ D

4. $$v^2 = -\frac{x^2}{2} + 2x + \frac{5}{2}$$

$$= \frac{1}{2}\left(5 - (x^2 - 4x)\right)$$

$$= \frac{1}{2}\left(5 - (x^2 - 4x + 4) + 4\right)$$

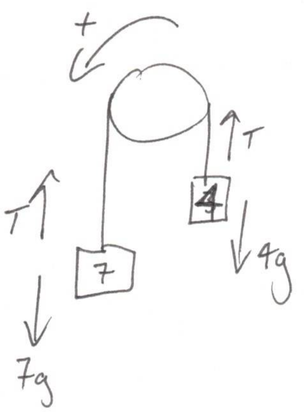
$$= \frac{1}{2}\left(9 - (x-2)^2\right)$$

$$v^2 = n^2(a^2 - x^2) \qquad \therefore \; n = \frac{1}{\sqrt{2}}$$

$$T = \frac{2\pi}{n}$$

$$= 2\sqrt{2}\,\pi \qquad \text{D}$$

5.



$$\Sigma F = M\ddot{x} = 7g - T + T - 4g$$

$$11\ddot{x} = 3g$$

$$\ddot{x} = \frac{3g}{11} \qquad \text{A}$$

6. a. $$x^2 + y^2 + z^2 - x + 2y + 3z = 1$$

Some attempt. 1

$$x^2 - x + \left(\tfrac{1}{2}\right)^2 + y^2 + 2y + 1 + z^2 + 3z + \left(\tfrac{3}{2}\right)^2 = 1 + \tfrac{1}{4} + 1 + \tfrac{9}{4}$$

$$\left(x - \tfrac{1}{2}\right)^2 + (y+1)^2 + \left(z + \tfrac{3}{2}\right)^2 = \tfrac{9}{2}$$

$\therefore$ Centre is $\left(\frac{1}{2}, -1, -\frac{3}{2}\right)$ 1

Radius is $\frac{3\sqrt{2}}{2}$ units 1

b. $\mathbf{r}_{AC} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -4 \\ -4 \end{pmatrix}$

① Some progress

$\mathbf{r}_{BD} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 4 \\ -12 \end{pmatrix}$

At pt. of int.

$$-4 + 8\lambda = 4 - 4\mu \quad ①$$
$$3 - 4\lambda = -2 + 4\mu \quad ②$$
$$3 - 4\lambda = 6 - 12\mu \quad ③$$

① Evidence

From ②, ③ $\quad -2 + 4\mu = 6 - 12\mu$

$$16\mu = 8$$
$$\mu = \frac{1}{2}$$

$\therefore$ Pt of int is $\begin{pmatrix} 4 - \frac{1}{2} \times 4 \\ -2 + \frac{1}{2} \times 4 \\ 6 - \frac{1}{2} \times 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ ①

c. $\mathbf{u} + \mathbf{v} = 17\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ ①

$\mathbf{u} - \mathbf{v} = \mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$ ②

① + ② $\Rightarrow$ $2\mathbf{u} = 18\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$

$\mathbf{u} = 9\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

① − ② $\Rightarrow$ $2\mathbf{v} = 16\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$

$\mathbf{v} = 8\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

$$|\underset{\sim}{u}| = \sqrt{9^2+4^2+1^2}$$
$$= \sqrt{98} \qquad (1)$$
$$|\underset{\sim}{v}| = \sqrt{8^2+5^2+3^2}$$
$$= \sqrt{98}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

$$(1) \quad 9.8 - 4.5 - 1\times 3 = 98\cos\theta$$
$$49 = 98\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3} \qquad (1)$$

d. i) $$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -9 \\ 12 \end{pmatrix} \qquad (1)$$

ii) If C lies on $\underset{\sim}{r}$ then
$$\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -9 \\ 12 \end{pmatrix} = \begin{pmatrix} -38 \\ -51 \\ -49 \end{pmatrix} \qquad (1)$$

ie $4 - 7\lambda = -38$ $3 - 9\lambda = -51$ $-2 + 12\lambda = -49$

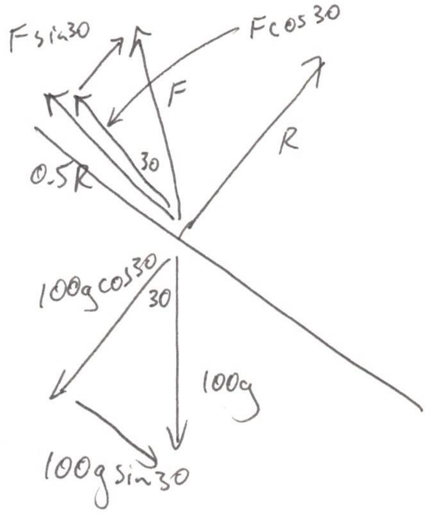
$42 = 7\lambda$ $54 = 9\lambda$ $12\lambda = -47$

$\lambda = 6$ $\lambda = 6$ $\lambda = -\frac{47}{12}$ (1)

$\neq 6$

Since the λ value is not the same for all components, C does not lie on $\overrightarrow{AB}$.

7a. i)



$\perp$ to plane $\quad R + F\sin 30 = 100g\cos 30$ (1)

$$R = 100 \times 10 \times \frac{\sqrt{3}}{2} - F \times \frac{1}{2}$$ (1)

$$= 500\sqrt{3} - \frac{F}{2}$$

ii) $\quad \sum F = m\ddot{x} = 100g\sin 30 - \frac{R}{2} - F\cos 30$ (1)

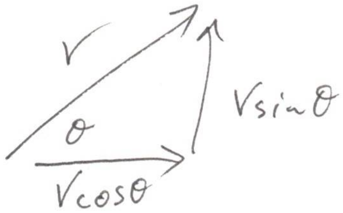
$$100a = 1000 \times \frac{1}{2} - \left(\frac{500\sqrt{3}}{2} - \frac{F}{4}\right) - F \times \frac{\sqrt{3}}{2}$$

$$= 500 - \frac{500\sqrt{3}}{2} + \frac{F}{4} - \frac{F\sqrt{3}}{2}$$

$$= \frac{2000 - 1000\sqrt{3} + F - 2F\sqrt{3}}{4}$$ (1)

$$a = \frac{1000(2-\sqrt{3}) - F(2\sqrt{3}-1)}{400}$$

b.



initially

1) $\ddot{x} = -k\dot{x}$

$$\frac{d\dot{x}}{dt} = -k\dot{x}$$

$$\frac{dt}{d\dot{x}} = \frac{-1}{k} \times \frac{1}{\dot{x}}$$

$$\int_0^t dt = -\frac{1}{k}\int_{V\cos\theta}^{\dot{x}} \frac{1}{\dot{x}}\, d\dot{x}$$

$$t = -\frac{1}{k}\left[\ln \dot{x}\right]_{V\cos\theta}^{\dot{x}}$$

$$-kt = \ln\left(\frac{\dot{x}}{V\cos\theta}\right)$$

$$e^{-kt} = \frac{\dot{x}}{V\cos\theta} \qquad 1$$

$$\dot{x} = V\cos\theta\, e^{-kt}$$

$$\ddot{y} = -k\dot{y} - g$$

$$\frac{d\dot{y}}{dt} = -k\dot{y} - g$$

$$\frac{dt}{d\dot{y}} = \frac{-1}{k\dot{y}+g}$$

$$\int_0^t dt = -\frac{1}{k}\int_{V\sin\theta}^{\dot{y}} \frac{k}{k\dot{y}+g}\, d\dot{y} \qquad 1$$

$$t = -\frac{1}{k}\Big[\ln(k\dot{y}+g)\Big]_{V\sin\theta}^{\dot{y}}$$

$$-kt = \ln\left(\frac{k\dot{y}+g}{kV\sin\theta+g}\right)$$

$$e^{-kt} = \frac{k\dot{y}+g}{kV\sin\theta+g}$$

$$(g+kV\sin\theta)e^{-kt} = k\dot{y}+g \qquad 1$$

$$\dot{y} = \frac{1}{k}(g+kV\sin\theta)e^{-kt} - \frac{g}{k}$$

ii) At max height, $\dot{y} = 0$

$$\therefore \frac{g}{k} = \frac{1}{k}(g+kV\sin\theta)e^{-kt} \qquad 1$$

$$\frac{g}{g+kV\sin\theta} = e^{-kt}$$

$$\ln\left(\frac{g}{g+kV\sin\theta}\right) = -kt$$

$$t = -\frac{1}{k}\ln\left(\frac{g}{g+kV\sin\theta}\right)$$

$$= \frac{1}{k}\ln\left(\frac{g}{g+kV\sin\theta}\right)^{-1}$$

$$= \frac{1}{k}\ln\left(\frac{g+kV\sin\theta}{g}\right) \qquad 1$$

c.

$$\ddot{x} = -v(1+v^2)$$

$$v\frac{dv}{dx} = -v(1+v^2)$$

$$\frac{dx}{dv} = \frac{-1}{1+v^2}$$

$$\int_0^x dx = -\int_U^v \frac{1}{1+v^2}\,dv \qquad 1$$

$$x = -\left[\tan^{-1} v\right]_U^v$$

$$= -\left(\tan^{-1} v - \tan^{-1} U\right)$$

$$= \tan^{-1} U - \tan^{-1} v \qquad 1$$

let $\tan^{-1} U = \alpha \qquad \tan^{-1} v = \beta$

$U = \tan\alpha \qquad v = \tan\beta$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{U-v}{1+Uv}$$

$$\therefore \alpha-\beta = \tan^{-1}\left(\frac{U-v}{1+Uv}\right)$$

ie $\tan^{-1} U - \tan^{-1} v = \tan^{-1}\left(\frac{U-v}{1+Uv}\right)$

ie $x = \tan^{-1}\left(\frac{U-v}{1+Uv}\right) \qquad 1$

8a. $\left|2\underset{\sim}{a}-3\underset{\sim}{b}\right|^2 = (2\underset{\sim}{a}-3\underset{\sim}{b})\cdot(2\underset{\sim}{a}-3\underset{\sim}{b})$

$= 4\underset{\sim}{a}\cdot\underset{\sim}{a} - 12\underset{\sim}{a}\cdot\underset{\sim}{b} + 9\underset{\sim}{b}\cdot\underset{\sim}{b}$ 1

$= 4|\underset{\sim}{a}|^2 - 12\times 4 + 9|\underset{\sim}{b}|^2$

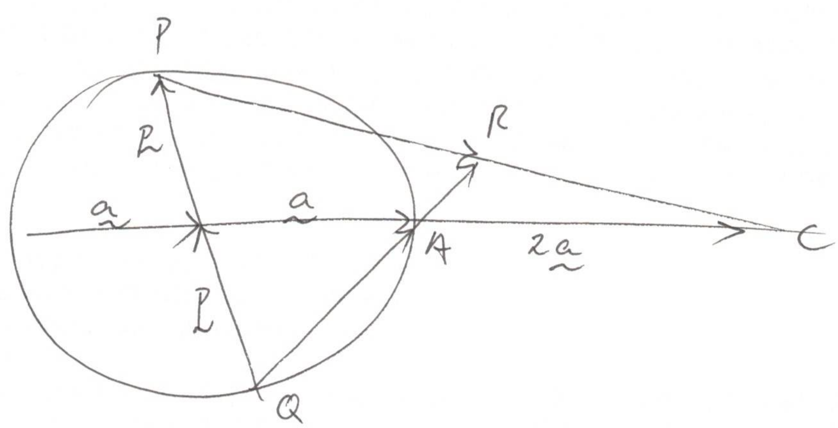
$= 4\times 9 - 48 + 9\times 4$

$= 24$

$\therefore \left|2\underset{\sim}{a}-3\underset{\sim}{b}\right| = \sqrt{24}$

$= 2\sqrt{6}$ 1

b.



i) $\overrightarrow{PC} = 3\underset{\sim}{a} - \underset{\sim}{p}$

$\overrightarrow{PR} = \lambda(3\underset{\sim}{a} - \underset{\sim}{p})$ 1

$\overrightarrow{QA} = \underset{\sim}{a} + \underset{\sim}{p}$

$\overrightarrow{QR} = \mu(\underset{\sim}{a} + \underset{\sim}{p})$ 1

ii) Now, $2\underset{\sim}{p} + \overrightarrow{PR} = \overrightarrow{QR}$

$2\underset{\sim}{p} + \lambda(3\underset{\sim}{a} - \underset{\sim}{p}) = \mu(\underset{\sim}{a} + \underset{\sim}{p})$ 1

$2\underset{\sim}{p} - \lambda\underset{\sim}{p} + 3\lambda\underset{\sim}{a} = \mu\underset{\sim}{a} + \mu\underset{\sim}{p}$

$\therefore \; 3\lambda = \mu \quad$ and $\quad 2 - \lambda = \mu$

$$\therefore \; 2 - \lambda = 3\lambda$$

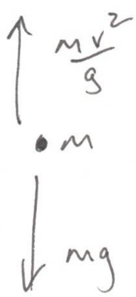
$$2 = 4\lambda$$

$$\lambda = \frac{1}{2} \qquad 1$$

$$\therefore \; \overrightarrow{PR} = \frac{1}{2} \times \overrightarrow{PC}$$

ie R is the midpoint of PC

c.



i)
$$m\ddot{x} = mg - \frac{mv^2}{g}$$

$$\ddot{x} = g - \frac{v^2}{g}$$

$$= \frac{1}{g}\left(g^2 - v^2\right) \qquad 1$$

ii)
$$\frac{dv}{dt} = \frac{1}{g}\left(g^2 - v^2\right)$$

$$\frac{dt}{dv} = \frac{g}{g^2 - v^2}$$

$$\int_0^t dt = g \int_0^v \frac{1}{g^2 - v^2} \, dv \qquad 1$$

$$t = g\left[\frac{1}{2g} \ln\left(\frac{g + v}{g - v}\right)\right]_0^v$$

$$t = \frac{1}{2}\left[\ln\left(\frac{g+v}{g-v}\right) - \ln\left(\frac{g}{g}\right)\right]$$

$$= \frac{1}{2}\ln\left(\frac{g+v}{g-v}\right)$$

$$e^{2t} = \frac{g+v}{g-v} \qquad (1)$$

$$(g-v)e^{2t} = g+v$$

$$ge^{2t} - ve^{2t} = g+v$$

$$ge^{2t} - g = ve^{2t} + v$$

$$g(e^{2t}-1) = v(e^{2t}+1) \qquad (1)$$

$$v = g\frac{(e^{2t}-1)}{(e^{2t}+1)}$$

iii) Terminal vel, v_T when $\ddot{x} = 0$

ie $g = \frac{v_T^2}{g}$

$$v_T = g \qquad (1)$$

Half $v_T = \frac{g}{2}$

$$\therefore t = \frac{1}{2}\ln\left(\frac{g+\frac{g}{2}}{g-\frac{g}{2}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{3g}{2} \div \frac{g}{2}\right)$$

$$= \frac{1}{2} \ln\left(\frac{3g}{2} \times \frac{2}{g}\right)$$

$$= \frac{1}{2} \ln 3 \qquad (1)$$

ie time to $\frac{V_T}{2}$ is $\frac{1}{2} \ln 3$ sec.