Smith's Hill High School

Promoting excellence in a spirit of trust and co-operation

STUDENT NAME / BOS NUMBER:___________________________

TEACHER:___________________________

Year 12 2022

HSC Mathematics Extension 2

Assessment Task 4

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen
- NESA-approved Calculators may be used
- A reference sheet is provided.
- In Questions 11 – 16 show relevant mathematical reasoning and/or calculations

Total Marks – **100**

Section I Questions 1 – 10 **10 marks**

Allow about 15 minutes for this section

Section II Questions 11 – 16 **90 marks**

Allow about 2 hour and 45 minutes for this section

I undertake this assessment with the knowledge that I am not hindered or impaired by illness/misadventure.

Signature: ____________________________

This paper MUST NOT be removed from the exam room

Section 1

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet provided for Questions 1 – 10.

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1 The angle between the vectors $\underset{\sim}{\boldsymbol{u}} = 2\underset{\sim}{\boldsymbol{\imath}} + 3\underset{\sim}{\boldsymbol{\jmath}} + \underset{\sim}{\boldsymbol{k}}$ and $\underset{\sim}{\boldsymbol{v}} = 2\underset{\sim}{\boldsymbol{\imath}} + \underset{\sim}{\boldsymbol{\jmath}} - 3\underset{\sim}{\boldsymbol{k}}$ is equal to:

A. $16° \, 36'$

B. $44° \, 24'$

C. $57° \, 41'$

D. $73° \, 24'$

2 Consider the following statement:

For all integers $a \ and \ b$ where b is non-zero, there exist rational numbers $p \ and \ q$ such that

$$\frac{5}{\mathrm{a} - \mathrm{b}\sqrt{7}} = \mathrm{p} + \mathrm{q}\sqrt{7}.$$

Which of the following is equivalent to the statement above?

A. $\forall \, a, b \ \in \mathbb{N}, \ b \neq 0 \ \ni p, q \ \in \mathbb{R}$ such that $\frac{5}{a-b\sqrt{7}} = p + q\sqrt{7}$

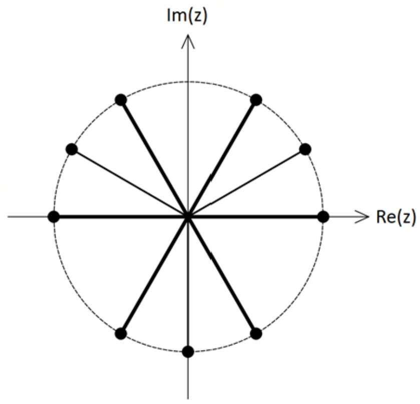
B. $\forall \, a, b \ \in \mathbb{Z}, \ b \neq 0 \ \ni p, q \ \in \mathbb{Q}$ such that $\frac{5}{a-b\sqrt{7}} = p + q\sqrt{7}$

C. $\forall \, a, b \ \in \mathbb{Z}, \ b = 0 \ \ni p, q \ \in \mathbb{Q}$ such that $\frac{5}{a-b\sqrt{7}} = p + q\sqrt{7}$

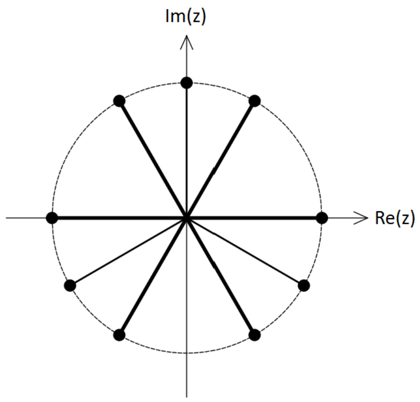
D. $\forall \, a, b \ \in \mathbb{C}, \ b \neq 0 \ \ni p, q \ \in \mathbb{R}$ such that $\frac{5}{a-b\sqrt{7}} = p + q\sqrt{7}$

3 Which Argand diagram shows the solutions to $(z^3 - i)(z^6 - 1) = 0$?

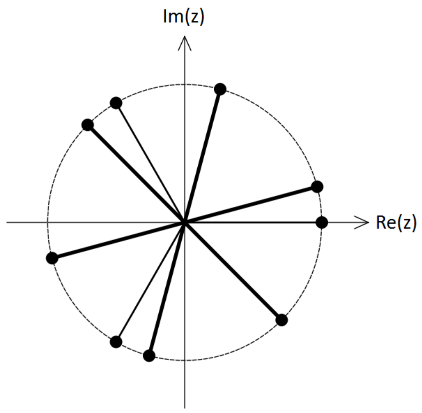
A.



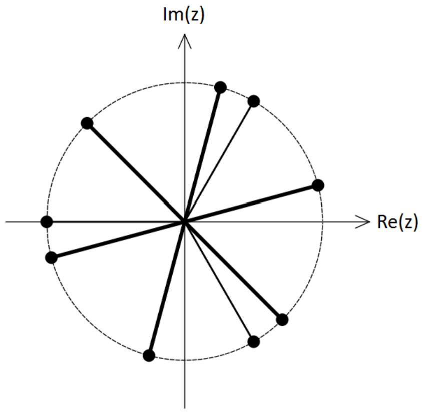
B.



C.



D.



4 OABC is a rectangle with $\overrightarrow{OA} = 3\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 2\underset{\sim}{k}$ and $\overrightarrow{OC} = 6\underset{\sim}{\imath} + 4\underset{\sim}{\jmath} + \mathrm{a}\underset{\sim}{k}$ for some constant a. What is the value of a ?

A. -2

B. -3

C. -5

D. -6

5 A particle is moving in a line executing Simple Harmonic Motion. At any time t seconds it has displacement x metres from a fixed point on the line and velocity $v\ \mathrm{ms}^{-1}$ given by $v^2 = 9 - 4(x-1)^2$. What is the amplitude of the motion?

A. 1.5 metres

B. 2 metres

C. 2.5 metres

D. 3 metres

6 Which of the following is an expression for $e^{i3\theta} + e^{i\theta}$?

A. $2\sin\theta\, e^{i2\theta}$

B. $2\cos\theta\, e^{i2\theta}$

C. $2\sin 2\theta\, e^{i\theta}$

D. $2\cos 2\theta\, e^{i\theta}$

7 Consider the statement

$For\ any\ function\ f(x),$

$f(x)\ is\ not\ continuous\ at\ x = c \implies f(x) is\ not\ differentiable\ at\ x = c.$

Which of the following is correct?

A. The contrapositive and converse staements are both false.

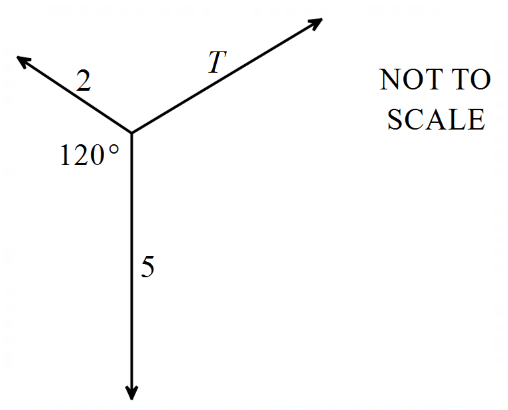
B. The contrapositive statement is false and the converse statement is true.

C. The contrapositive staement is true and the converse statement is false.

D. The contrapositive statement is true and the converse statement is true.

8 A particle is acted on by three forces as shown in the diagram below.

For the particle to be in equilibrium, the value of T is?



A. 3

B. $\sqrt{19}$

C. $\sqrt{29}$

D. $\sqrt{39}$

9 Which expression is equal to $\int tan^{-1} x\, dx$?

A. $x\, tan^{-1} x - \int \frac{x}{1+x^2} dx$

B. $x^2\, tan^{-1} x - \int \frac{x^2}{1+x^2} dx$

C. $\frac{1}{2}(tan^{-1} x)^2$

D. $\frac{x}{2}(tan^{-1} x)^2 - \int \frac{x}{tan\, x} dx$

10 The velocity, $v\text{ms}^{-1}$, of a particle at time $t \geq 0$ seconds and at position $x\ m$ from the origin is $v = cos^2 x$.

The acceleration of particle, in ms^{-2}, when $x = \frac{\pi}{6}$ is

A. $-\frac{\sqrt{3}}{2}$

B. $-\frac{3\sqrt{3}}{8}$

C. $\frac{3\sqrt{3}}{4}$

D. $\frac{\sqrt{3}}{4}$

END OF SECTION I

Go to next page for Section II

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer the questions in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Evaluate $\int_{-1}^{2} 2x\sqrt{x+2}\,dx$. 3

(b) Sketch the region on the Argand diagram defined by $|z-2i| > |z+1|$. 3

(c) A particle, initially at rest at position $x = 1$, accelerates along a horizontal line according to the rule $a = e^{-v^2}$.

Find the displacement of the particle, x, in terms of its velocity v. 3

(d) Evaluate $\int_{0}^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$ 4

(e) Find $\int xe^{x}\,dx$ 2

END OF QUESTION 11 – Go to next page for Question 12

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Consider the three vectors

$\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + m\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ where $m \in \mathbb{R}$.

(i) Find the value(s) for m for which $|\underset{\sim}{b}| = 3\sqrt{2}$. **2**

(ii) For what values of m is $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$? **1**

(iii) Evaluate $3\underset{\sim}{a} - 2\underset{\sim}{c}$. **1**

(b) If $\alpha\beta\delta$ represents a three-digit number and $\alpha + \beta + \delta$ is divisible by three, show that this number is divisible by three. **3**

(c) $OABC$ is a right pyramid in which

$|\overrightarrow{OA}| = 10, |\overrightarrow{OB}| = 20, |\overrightarrow{OC}| = 10$ and $\angle AOB = \angle AOC = \angle BOC = 90^\circ$.

The unit vectors $\underset{\sim}{i}, \underset{\sim}{j}$, and $\underset{\sim}{k}$ are parallel to $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$.

Points R and S are the midpoints of sides AB and BC, respectively.

(i) Determine the vectors $\overrightarrow{OR}$ and $\overrightarrow{OS}$ in terms of $\underset{\sim}{i}, \underset{\sim}{j}$, and $\underset{\sim}{k}$. **2**

(ii) Calculate, to the nearest minute, the angle between $\overrightarrow{OR}$ and $\overrightarrow{OS}$. **1**

(iii) Show that the length of the perpendicular from R to $\overrightarrow{OS}$ is $3\sqrt{5}$. **2**

(d) (i) Points A and B represent the complex numbers $\cos\theta + i\sin\theta$ and $\cos(\theta + \alpha) + i\sin(\theta + \alpha)$ respectively, and O is the origin and $0 < \alpha < \pi$. Show that the area of triangle OAB is $\frac{1}{2}\sin\alpha$. **1**

(ii) Deduce that, if $\alpha_i \geq 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} \alpha_i < \pi$, then : $\sin(\sum_{i=1}^{n} \alpha_i) \leq \sum_{i=1}^{n} \sin\alpha_i$. **2**

END OF QUESTION 12 – Go to next page for Question 13

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) (i) Find the values of *A*, *B*, and *C* such that: 3

$$\frac{6x^2+19x-10}{x^3+7x^2+10x} = \frac{A}{x+5} - \frac{B}{x} + \frac{C}{x+2}.$$

(ii) Hence, find and simplify $\int \frac{6x^2+19x-10}{x^3+7x^2+10x}\,dx$. 2

(b) The acceleration of a particle moving in simple harmonic motion is given by $\ddot{x} = -n^2x$ where x is the displacement of the particle from the origin and n is a constant.

(i) Show that the velocity *v* of the particle is given by $v^2 = n^2(a^2 - x^2)$ where a is the amplitude of the motion. 2

(ii) Given that the speed of the particle is *V* m/s when it is k metres from the origin and that its speed is $\frac{V}{3}$ m/s when it is $3k$ metres from the origin, show that: 3

(α) the particle's amplitude is $\sqrt{10}k$ metres.

(β) the period of the motion is $\frac{6\pi k}{V}$ seconds.

(c) (i) Using de Moivre's theorem, show that 3

$$\tan 5\boldsymbol{\theta} = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

(ii) Hence show that $\tan^2\left(\frac{\pi}{5}\right)$ and $\tan^2\left(\frac{2\pi}{5}\right)$ are the roots of the equation $x^2 - 10x + 5 = 0$. 2

END OF QUESTION 13 – Go to next page for Question 14

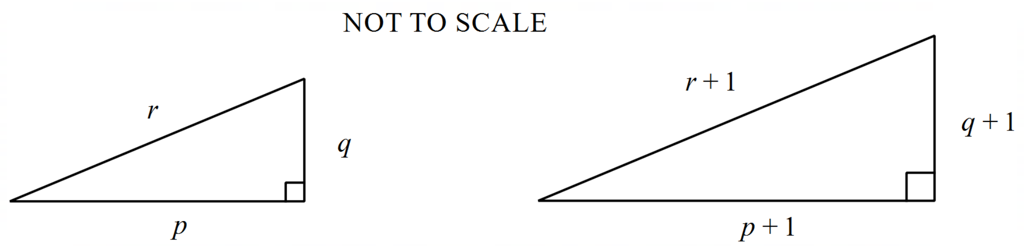
Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) It is given that $\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{c} + \underset{\sim}{d}$ and $\underset{\sim}{b} + \underset{\sim}{c}$ is perpendicular to $\underset{\sim}{a} + \underset{\sim}{d}$, where $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ and $\underset{\sim}{d}$ are non-zero vectors.

Show that $\underset{\sim}{b} - \underset{\sim}{d}$ is perpendicular to $\underset{\sim}{c} - \underset{\sim}{a}$. 2

(b) Find the vector equation of the line joining the points *M* (6, 4, –5) and *N* (–1, 3, –2). 1

(c) Consider the figure below, which shows two right-angled triangles.



Prove by contradiction that p, q and r cannot all be integers. 3

(d) Let $\omega = e^{i\left(\frac{\pi}{n}\right)}$.

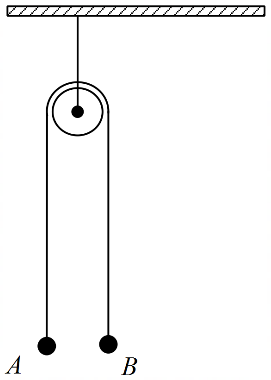
(i) Show that ω is a solution of $z^{2n} = 1$. 1

(ii) Deduce that $1 + \omega + \cdots + \omega^{2n-1} = 0$. 1

(iii) Show that $1 + \omega + \cdots + \omega^{n-1} = 1 + i\,cot\left(\frac{\pi}{2n}\right)$. 3

Question 14 continued the next page

(e) Two particles are connected by a light inextensible string which passes over a smooth pulley as shown in the diagram.



Particles A and B, respectively, have masses $m\text{kg}$ and 6kg, and are initially held in position with the string taut and the hanging parts of the string vertical.

The system is released from rest and particle A moves upwards.

Show that the tension in the string, as A ascends, is $\frac{12mg}{m+6}$. 4

END OF QUESTION 14 – Go to next page for question 15

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) (i) Derive the reduction formula :

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$ 2

(ii) Hence or otherwise, find $\int_0^2 x^3 e^x \, dx$. 2

(b) Evaluate $\int_2^5 \frac{dx}{\sqrt{5+4x-x^2}}$ 3

(c) A particle P of mass m kg is released from rest and falls under its own weight.

The particle is subject to air resistance of magnitude $kmv^2\ N$, where $v\ \mathrm{ms}^{-1}$ is the speed of the particle after it has travelled $x\,\mathrm{m}$ and k is a positive constant.

(i) Show that $v^2 = (1 - e^{-2kx})\frac{g}{k}$. 3

(ii) Hence determine the terminal velocity, V_T , of the particle. 1

(iii) Show that the particle attains a speed of $\frac{1}{2}V_T$ after it has fallen a distance of $\frac{1}{2k} ln\left(\frac{4}{3}\right)$ metres. 2

(d) If $\int_1^4 f(x)dx = k$ for some constant k, what is the value of $\int_1^4 f(5-\ x)dx$? 2

END OF QUESTION 15 – Go to next page for Question 16

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) By considering $\frac{d}{dx}\left(\frac{\sin x}{x}\right)$, or otherwise, determine 2

$\int \frac{\boldsymbol{x\cos x - \sin x}}{\boldsymbol{x^2 + \sin^2 x}}\,dx.$

(b) Sarah, a ski jumper comes off the end of the jump horizontally and falls $90\ m$ vertically before contacting the slope a record $180\ m$ horizontally from the end of the jump.
What was the initial speed of this jumper as Sarah left the jump?
Answer in exact form. Assume there is no resistance and gravity is $10\ ms^2$ 4

(c) Let $f_n(x) = x^n + x^{-n}$, where $n \in \square$ and $x \neq 0$.

(i) Show that $f_{n+1}(x) = \left(x + x^{-1}\right) f_n(x) - f_{n-1}(x)$. 2

(ii) Using the recurrence from part (i), prove by mathematical induction that all positive integers n

if $x + x^{-1}$ is rational, then $x^n + x^{-n}$ is rational. 4

(iii) Deduce that, for all integers n, $\sqrt[n]{\sqrt{3}+\sqrt{2}} + \sqrt[n]{\sqrt{3}-\sqrt{2}}$ is irrational. 3

End of paper

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STUDENT NAME / BOS NUMBER:___________________________

TEACHER:___________________________

Year 12 2022

HSC Mathematics Extension 2

Assessment Task 4

Section I – multiple answer sheet

Select the alternative A, B, C, or D that best answers the question by placing a **X** in the box.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

EXT – II Trial 2022 M/C

1.	Angle between Vectors $$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert}$$ $\underset{\sim}{u}\cdot\underset{\sim}{v} = (2)(2)+(3)(1)+(1)(-3) = 4$ $\lvert\underset{\sim}{u}\rvert = \sqrt{2^2+3^2+1^2} = \sqrt{14}$ $\lvert\underset{\sim}{v}\rvert = \sqrt{2^2+1^2+(-3)^2} = \sqrt{14}$ $$\cos\theta = \frac{4}{\sqrt{14}\sqrt{14}} = \frac{4}{14}$$ $$\theta = 73^\circ\,24'$$	D
2.	$\mathbb{C}$ is the set of complex numbers $\mathbb{N}$ is the set of natural numbers $\mathbb{R}$ is the set of real numbers $\mathbb{Q}$ is the set of rational numbers $\mathbb{Z}$ is the set of integers $\forall$ is the symbol meaning, "for all" $\in$ is the symbol meaning, "is an element of" $\ni$ is the symbol meaning, "there exist" "non zero" implies that $b \neq 0$ $\forall\, a,b \in \mathbb{Z},\ b\neq 0 \ni p,q \in \mathbb{Q}$ such that $\frac{5}{a-b\sqrt{7}} = p+q\sqrt{7}$	B

3. A.

4.	C	$\angle AOC = 90^\circ \quad \therefore \left(3\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}\right).\left(6\underset{\sim}{i}+4\underset{\sim}{j}+a\underset{\sim}{k}\right)=0 \quad \therefore a=-5$	MEX12-3
5.	A	$v^2 = 9-4(x-1)^2$. $\therefore v=0$ for $x-1=\pm\frac{3}{2}$, $x=-\frac{1}{2}$ or $x=\frac{5}{2}$. Particle moves 3 m between its extreme positions. Amplitude is 1.5 m.	MEX12-6
6.	B	$e^{i3\theta}+e^{i\theta} = \left(1+e^{i2\theta}\right)e^{i\theta}$ $\left(1+e^{i2\theta}\right) = 1+\cos 2\theta + i\sin 2\theta = 2\cos\theta\left(\cos\theta+i\sin\theta\right) = 2\cos\theta\, e^{i\theta}$ $\therefore e^{i3\theta}+e^{i\theta} = 2\cos\theta\, e^{i2\theta}$	MEX12-4
7.	C	Contrapositive: *f(x) is differentiable at* $x=c \Rightarrow$ *f(x) is continuous at* $x=c$ True Converse: *f(x) is not differentiable at* $x=c \Rightarrow$ *f(x) is not continuous at* $x=c$ False	MEX12-2

8. B

9. A.

10. B

Q 10. $v = \cos^2 x$

$$a = v\frac{dv}{dx}$$

$$= \cos^2 x \times 2\cos x \times (-\sin x)$$

$$a = -2\sin x \cos^3 x$$

when $x = \frac{\pi}{6}$

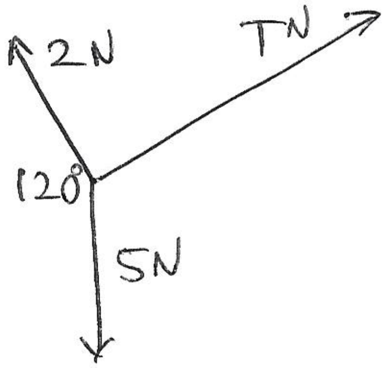
$$a = -2\sin\left(\frac{\pi}{6}\right)\cos^3\left(\frac{\pi}{6}\right)$$

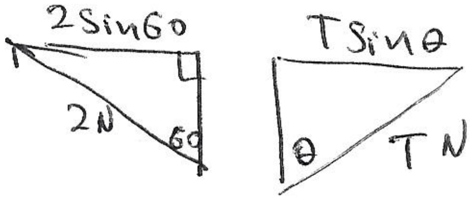
$$= -2 \times \frac{1}{2} \times \left(\frac{\sqrt{3}}{2}\right)^3$$

$$= -\frac{3\sqrt{3}}{8}$$

B

Q8.

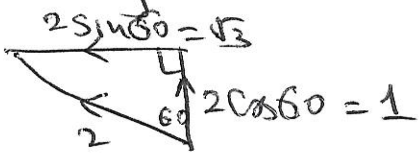




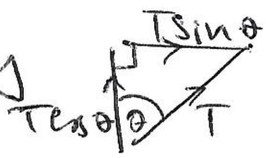
Equilibrium

$T = \sqrt{19}$

Resolving Vertically



Horizontally



$$1 + T\cos\theta = 5$$

$$T\cos\theta = 4$$

$$T^2\cos^2\theta = 16$$

$$T\sin\theta = \sqrt{3}$$

$$T^2\sin^2\theta = 3$$

$$T(\cos^2\theta + \sin^2\theta) = 16 + 3 = 19$$

$$\therefore T = \sqrt{19} \quad T > 0$$

B

12 (a)

(i)	$\lvert b \rvert = \sqrt{2^2+3^2+m^2} = 3\sqrt{2}$ Squaring both sides $4+9+m^2=18$ $m^2=5$ $m = \pm\sqrt{5}$	2	**1 for working** **1 for answer**
(ii)	Perpendicular if $a \cdot b = 0$ $i.e.\ (2\times 2)+(-2)(3)+4m=0$ $4-6+4m=0$ $4m=2$ $m=\frac{1}{2}$	1	**Correct Answer**
(iii)	$3a-2c = 3(2i-2j+4k)-2(i+j-k)$ $= 4i-8j+14k$	1	**Correct Answer**

(b)

	$\alpha\beta\gamma$ represents $100\alpha+10\beta+\delta$ *As* $\alpha+\beta+\delta$ is divisible by 3 $\alpha+\beta+\delta = 3k$ *where k s an integer* Now, $100\alpha+10\beta+\delta = 99\alpha+9\beta+\alpha+\beta+\delta$ $= 3(33\alpha+3\beta)+3k$ $= 3(33\alpha+3\beta+k)$ which is divisible by 3.	2	**2 for any correct proof** **1 for working towards correct proof.**

(c) (i)

Criteria	Marks
• Provides correct solution	2
• Makes some progress towards a solution	1

Sample answer:

$\overrightarrow{OA} = 10\mathbf{i}, \overrightarrow{OB} = 20\mathbf{j}$ and $\overrightarrow{OC} = 10\mathbf{k}$

$\therefore \overrightarrow{OR} = 5\mathbf{i}+10\mathbf{j}$ and $\overrightarrow{OS} = 10\mathbf{j}+5\mathbf{k}$

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\cos\theta = \frac{(5\mathbf{i}+10\mathbf{j})\cdot(10\mathbf{j}+5\mathbf{k})}{\sqrt{5^2+10^2}\sqrt{10^2+5^2}}$$

$$= \frac{5\times 0+10\times 10+0\times 5}{125}$$

$$= \frac{4}{5}$$

36°52'

$$\therefore \theta = \cos^{-1}\left(\frac{4}{5}\right) \approx 36°52' \text{ (to nearest minute)}$$

2 –

(c)(iii)

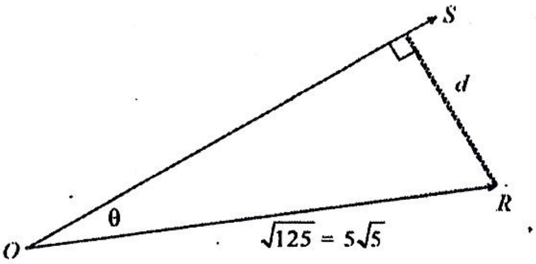
Criteria	Marks
• Provides correct solution	2
• Makes some progress towards a solution	1

Sample answer:

Let d be the perpendicular distance from R to $\overrightarrow{OS}$.

We want the projection onto $\overrightarrow{OS}$.

$$d = 5\sqrt{5}\sin\theta = 5\sqrt{5}\times\frac{3}{5} = 3\sqrt{5}$$

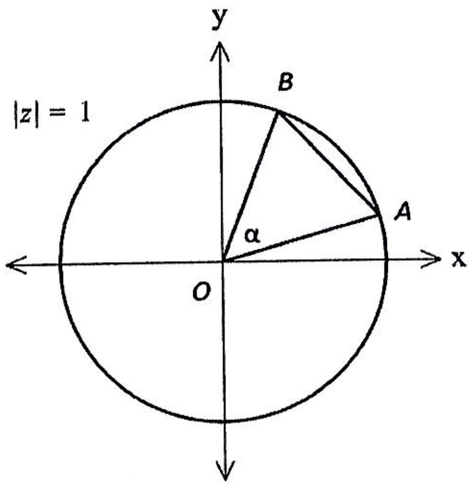


(d)(i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\text{Area of}\,\square AOB = \frac{1}{2}\times 1\times 1\times\sin\left[(\theta+\alpha)-\theta\right]$$
$$= \frac{1}{2}\sin\alpha$$



(ii)

Criteria	Marks
• Provides correct solution	2
• Makes some progress towards the solution	1

Sample answer:

Let $A_0, A_1, A_2, \ldots, A_n$ be a sequence of points on the upper half of the unit circle in the Argand diagram, where $\angle A_0OA_1 = \alpha_1, \angle A_1OA_2 = \alpha_2, \ldots, \angle A_{n-1}OA_n = \alpha_n$ and Area of $\square A_{i-1}OA_i = \frac{1}{2}\sin\alpha_i$.

Since $\sum_{i=1}^{n}\alpha_i < \pi$, area of triangle AOA_n is less than the $\sum_{i=1}^{n}$ Area of $\square A_{i-1}OA_i$

i.e. $\frac{1}{2}\sin\left(\sum_{i=1}^{n}\alpha_i\right) < \sum_{i=1}^{n}\frac{1}{2}\sin\alpha_i$

$\Rightarrow \sin\left(\sum_{i=1}^{n}\alpha_i\right) < \sum_{i=1}^{n}\sin\alpha_i$

Q14
(a)

Criteria	Marks
• Provides correct solution	2
• Makes some progress towards the solution	1

Sample answer:

$$\begin{aligned}(\mathbf{b}-\mathbf{d})\cdot(\mathbf{c}-\mathbf{a}) &= \mathbf{b}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{a}-\mathbf{d}\cdot\mathbf{c}+\mathbf{d}\cdot\mathbf{a}\\ &= (\mathbf{b}\cdot\mathbf{c}+\mathbf{d}\cdot\mathbf{a})-(\mathbf{b}\cdot\mathbf{a}+\mathbf{d}\cdot\mathbf{c})\\ &= (\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}+\mathbf{d}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{d})\\ &= (\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{d}+\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{d})\\ &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d})-(\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{d})\\ &= 0-0 \qquad \text{since } (\mathbf{a}+\mathbf{b})\perp(\mathbf{c}+\mathbf{d}) \text{ and } (\mathbf{b}+\mathbf{c})\perp(\mathbf{a}+\mathbf{d})\\ &= 0\\ &\therefore (\mathbf{b}-\mathbf{d})\perp(\mathbf{c}-\mathbf{a})\end{aligned}$$

(b)

(6, 4, –5) and (–1, 3, –2)

Let $a=\begin{pmatrix}6\\4\\-5\end{pmatrix}$ and $b=\begin{pmatrix}-1\\3\\-2\end{pmatrix}$

$$\therefore \overrightarrow{AB}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}-\begin{pmatrix}6\\4\\-5\end{pmatrix}=\begin{pmatrix}-7\\-1\\3\end{pmatrix}$$

$$\therefore r=\begin{pmatrix}6\\4\\-5\end{pmatrix}+\lambda\begin{pmatrix}-7\\-1\\3\end{pmatrix}$$

1

(c)

Criteria	Marks
• Provides correct solution	3
• Makes some progress towards the solution	2
• Makes little progress towards the solution	1

Sample answer:

Assume that p, q, r are all integers.

Then $p^2+q^2=r^2$ in the smaller triangle, and $(p+1)^2+(q+1)^2=(r+1)^2$ in the larger triangle.

Expanding the second equation, and substituting $p^2+q^2=r^2$ gives $2p+2q+2=2r+1$, a contradiction, since $2(p+q+1)$ is even and $2r+1$ is odd. Hence the assumption is incorrect and p, q, r cannot all be integers.

4 –

(d)

(i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\omega^{2n} = \left(e^{\pi i/n}\right)^{2n}$$
$$= e^{2\pi i}$$
$$= 1$$

$\therefore \omega$ is a solution of $z^{2n} = 1$

(ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\omega^{2n} - 1 = 0$$
$$\Rightarrow (\omega - 1)\left(\omega^{2n-1} + \cdots + \omega + 1\right) = 0$$

Since $\omega \neq 1$, $\omega^{2n-1} + \cdots + \omega + 1 = 0$

(iii)

Criteria	Marks
• Provides correct solution	3
• Makes significant progress towards the calculation	2
• Makes some progress towards the calculation	1

Sample answer:

$$1 + \omega + \cdots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega}$$

$$= \frac{1-e^{\pi i}}{1-\cos\left(\frac{\pi}{n}\right) - i\sin\left(\frac{\pi}{n}\right)}$$

$$= \frac{1-(-1)}{1-\cos\left(\frac{\pi}{n}\right) - i\sin\left(\frac{\pi}{n}\right)} \times \frac{1-\cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)}{1-\cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)}$$

$$= \frac{2\left(1-\cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)\right)}{\left(1-\cos\left(\frac{\pi}{n}\right)\right)^2 + \left(\sin\left(\frac{\pi}{n}\right)\right)^2}$$

$$= \frac{2\left(1-\cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)\right)}{2-2\cos\left(\frac{\pi}{n}\right)}$$

$$= \frac{1-\cos\left(\frac{\pi}{n}\right)}{1-\cos\left(\frac{\pi}{n}\right)} + i\frac{\sin\left(\frac{\pi}{n}\right)}{1-\cos\left(\frac{\pi}{n}\right)}$$

$$= 1 + i\frac{2\sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)}{2\sin^2\left(\frac{\pi}{2n}\right)}$$

$$= 1 + i\cot\left(\frac{\pi}{2n}\right)$$

14 (e)

Criteria	Marks
• Provides correct solution	4
• Correctly determines a	3
• Attempts to solve simultaneous equations	2
• Correctly resolves forces for at least one mass	1

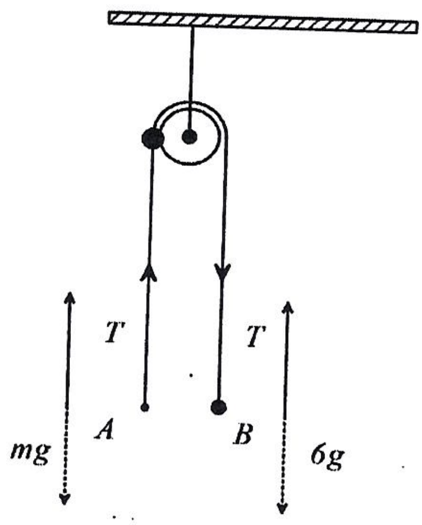
Sample answer:

Resolving for A: $T - mg = ma$

Resolving for B: $6g - T = 6a$

Adding gives: $6g - mg = (m+6)a$

$$\therefore \ a = \frac{(6-m)g}{m+6}$$



Substituting into the force equation for A gives:

$$T = mg + m\frac{(6-m)g}{m+6}$$

$$= mg\left(1 + \frac{6-m}{m+6}\right)$$

$$= \frac{12mg}{m+6}$$

Q16

(a)

Criteria	Marks
• Provides a correct solution	~~3~~ 2
• Makes some progress towards a solution	~~2~~ 1
• ~~Makes little progress towards a solution~~	~~1~~

Sample answer:

$$\int \frac{x\cos x-\sin x}{x^2+\sin^2 x}\,dx = \int \frac{\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}}{1+\dfrac{\sin^2 x}{x^2}}\,dx$$

$$= \int \frac{1}{1+\dfrac{\sin^2 x}{x^2}}\left(\frac{x\cos x-\sin x}{x^2}\right)dx$$

$$= \int \frac{1}{1+\left(\dfrac{\sin x}{x}\right)^2}\,\frac{d}{dx}\left(\frac{\sin x}{x}\right)dx$$

$$= \int \frac{1}{1+u^2}\,du,\ \ where\, u=\frac{\sin x}{x}$$

$$= \tan^{-1}u+c$$

$$= \tan^{-1}\left(\frac{\sin x}{x}\right)+c$$

(b)

$\ddot{x}=0$

$\dot{x}=c$

When $t=0,\ \dot{x}=V\cos 0°=V$

$\dot{x}=V$

$x=Vt+d$

When $t=0,\ x=0$

$0=0+d$

$d=0$ 1

$x=Vt$ [1]

$\ddot{y}=-g=-10$

$\dot{y}=-10t+e$

When $t=0,\ \dot{y}=V\sin 0°=0$

$\dot{y}=-10t$

$y=-5t^2+f$

When $t=0,\ y=90$

$90=0+f$

$f=90$ 1

$y=-5t^2+90$ [2]

Landing at $x=180$:

$180=Vt$ [from1]

$t=\dfrac{180}{V}$.

Landing at $y=0$: 1

$0=-5t^2+90$ [from2]

$0=-5\left(\dfrac{180}{V}\right)^2+90$

$5\left(\dfrac{180^2}{V^2}\right)=90$

$\dfrac{180^2}{V^2}=18$

$V^2=\dfrac{180^2}{18}=1800$

$V=\sqrt{1800}$ 1

$=30\sqrt{2}\ \text{ms}^{-1}$

8

1.6
(c)

Question 16(c) (i)

Criteria	Marks
• Provides correct solution	2

Sample answer:

$$f_{n+1}(x) = x^{n+1} + x^{-(n+1)}$$
$$= x^{n+1} + \frac{1}{x^{n+1}}$$
$$= \left(x + \frac{1}{x}\right)\left(x^n + \frac{1}{x^n}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)$$
$$= \left(x + x^{-1}\right) f_n(x) - f_{n-1}(x)$$

Question 16(c) (ii)

Criteria	Marks
• Provides correct solution	4
• Makes significant progress towards a solution	3
• Makes some progress towards a solution	2
• Correctly shows the base cases are true, or correctly assumes for more than one case	1

Sample answer:

Let P_n be the proposition that f_n is rational, where $n \in \square^+$

Then P_1 is true, as $f_1(x) = x + x^{-1}$ is given as rational, and

P_2 is true, as $f_2(x) = x^2 + x^{-2} = \left(x + x^{-1}\right)^2 - 2$ and the set of rational numbers are closed under multiplication and subtraction.

Assume that P_{n-1} and P_n are true, for $n \geq 2$

i.e. that $f_{n-1}(x)$ and $f_n(x)$ are rational

Then $f_{n+1}(x) = \left(x + x^{-1}\right) f_n(x) -_{n-1}(x)$ is the product and subtraction of rational numbers

Hence $f_{n+1}(x)$ is rational, i.e. P_{n+1} is true whenever P_{n-1} and P_n are true

Since P_1 and P_2 are true, then P_n is true for all positive integers n

Question 16(c) (iii)

Criteria	Marks
• Provides correct solution	3
• Makes some progress towards a solution	
• Attempts to argue indirectly	1

Sample answer:

If $x = \sqrt[n]{\sqrt{3}+\sqrt{2}}$, then

$$
\begin{aligned}
x^{-1} &= \frac{1}{\sqrt[n]{\sqrt{3}+\sqrt{2}}} \\
&= \frac{1}{\sqrt[n]{\sqrt{3}+\sqrt{2}}} \times \frac{\sqrt[n]{\sqrt{3}-\sqrt{2}}}{\sqrt[n]{\sqrt{3}-\sqrt{2}}} \\
&= \frac{\sqrt[n]{\sqrt{3}-\sqrt{2}}}{\sqrt[n]{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}} \\
&= \frac{\sqrt[n]{\sqrt{3}-\sqrt{2}}}{\sqrt[n]{3-2}} \\
&= \sqrt[n]{\sqrt{3}-\sqrt{2}}
\end{aligned}
$$

Hence, $x^n + x^{-n} = \left(\sqrt{3}+\sqrt{2}\right) + \left(\sqrt{3}-\sqrt{2}\right)$

$= 2\sqrt{3}$, which is irrational

By contrapositive, if $f_n(x)$ is irrational, then $f_1(x)$ is irrational.

Hence $\sqrt[n]{\sqrt{3}+\sqrt{2}} + \sqrt[n]{\sqrt{3}-\sqrt{2}}$ is irrational