Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for questions 1-10

---

1. The point $(-1, a)$ lies on the line described by the vector equation

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

Which of the following is the correct value for a?

(A) $-\frac{2}{3}$

(B) $\frac{2}{3}$

(C) $\frac{5}{4}$

(D) 2

2. The vectors $\underset{\sim}{u} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + p\underset{\sim}{j} - 3\underset{\sim}{k}$ are perpendicular.

What is the value of p?

(A) -1

(B) $-\frac{3}{4}$

(C) $\frac{3}{4}$

(D) 1

3. Which of the following is a solution to the equation $\left|e^{i\theta} - 1\right| = 2$?

(A) $\theta = -\frac{\pi}{2}$

(B) $\theta = 0$

(C) $\theta = \frac{\pi}{2}$

(D) $\theta = \pi$

4. A particle is moving in SHM along a straight line with amplitude of 8 *m*. The speed of the particle is 12 *m/s* when it is 4 *m* from the centre of its motion.

What is the period of the motion?

(A) $\frac{2\pi}{3}\ seconds$

(B) $\frac{2\sqrt{3}\pi}{3}\ seconds$

(C) $\sqrt{3}\ seconds$

(D) $3\ seconds$

5. What is the contrapositive of the statement below?

If you're happy and you know it, then you will clap your hands.

(A) If you don't clap your hands, then you're happy and you know it.

(B) If you clap your hands, then you're either not happy or you don't know it.

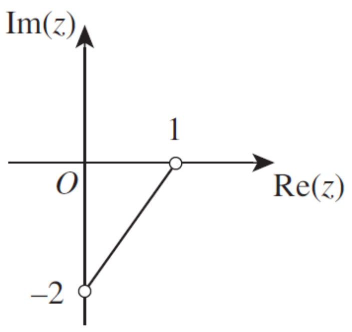
(C) If you don't clap your hands, then you're not happy and you don't know it.

(D) If you don't clap your hands, then you're either not happy or you don't know it.

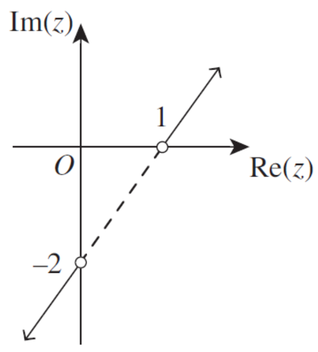
6. Which of the following diagrams best represents the solution to the equation

$$arg\left(\frac{z+2i}{z-1}\right) = \pi?$$

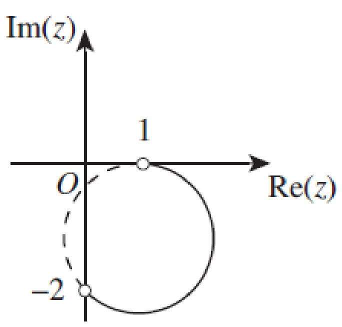
(A)



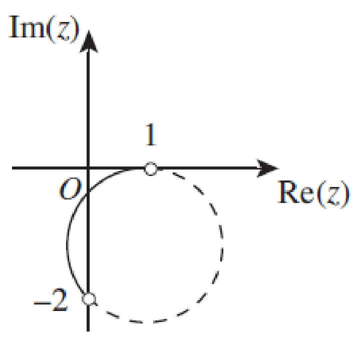
(B)



(C)



(D)



7. On an Argand diagram, the points $z,\ -\bar{z},\ z^{-1}$ and $-\overline{(z^{-1})}$ form the vertices of a quadrilateral.

The quadrilateral must be a:

(A) Square

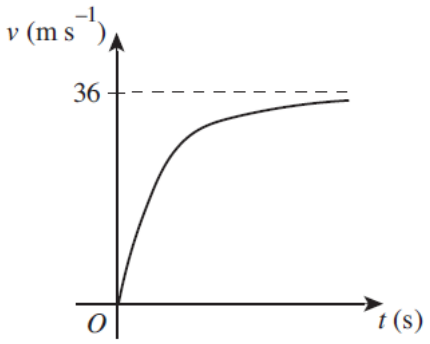
(B) Rectangle

(C) Rhombus

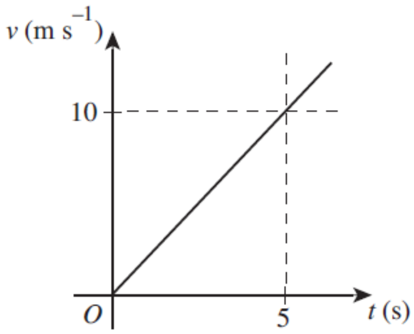
(D) Trapezium

8. A mass of 1 kg is dropped from a height in a resisting medium under a constant gravitational acceleration of 10 m/s^2. The resistive force is directly proportional to the speed v. If the constant of proportionality is 0.5, which of the following best represents the velocity-time graph for the mass?

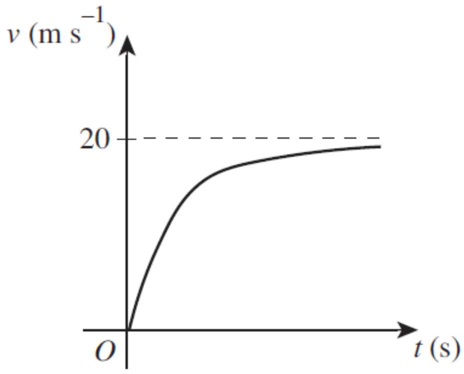
(A)



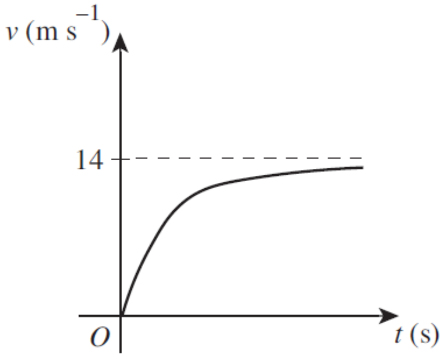
(B)



(C)



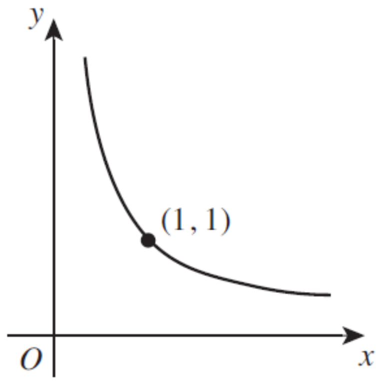
(D)



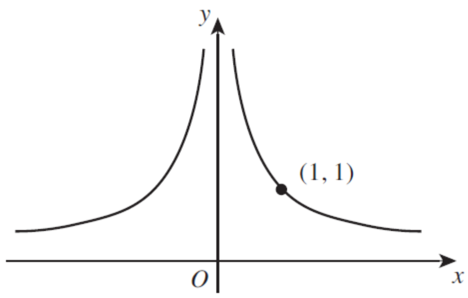
9. Which of the following shows the graph represented by the parametric equations:

$$x = \sqrt{t-2}\,,\ y = \frac{1}{2-t}$$

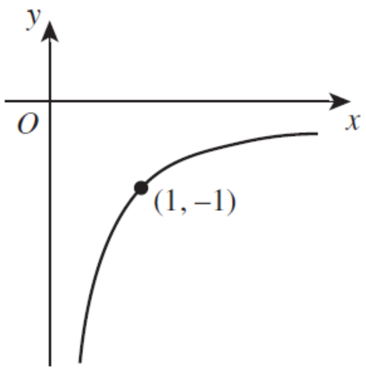
(A)



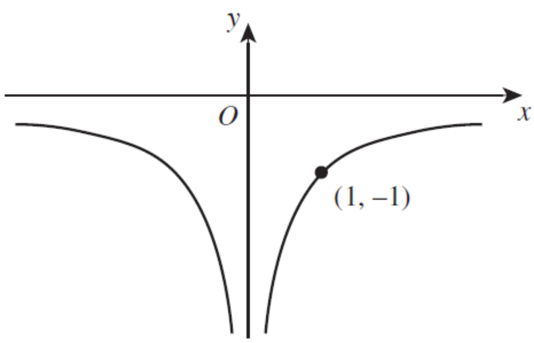
(B)



(C)



(D)



10. It is given that, for any continuous function *h(x)*,

$$\int_0^1 h(x)\,dx = \int_0^1 h(1-x)\,dx$$

f(x) and *g(x)* are continuous functions such that:

$$f(x) = f(1-x) \quad \text{and}$$

$$g(x) + g(1-x) = 2$$

Which of the following is equivalent to $\int_0^1 f(x)g(x)\,dx$?

(A) $\int_0^1 f(x)\,dx$

(B) $2\int_0^1 f(x)\,dx$

(C) $3\int_0^1 f(x)\,dx$

(D) $4\int_0^1 f(x)\,dx$

End of Section I

Section II

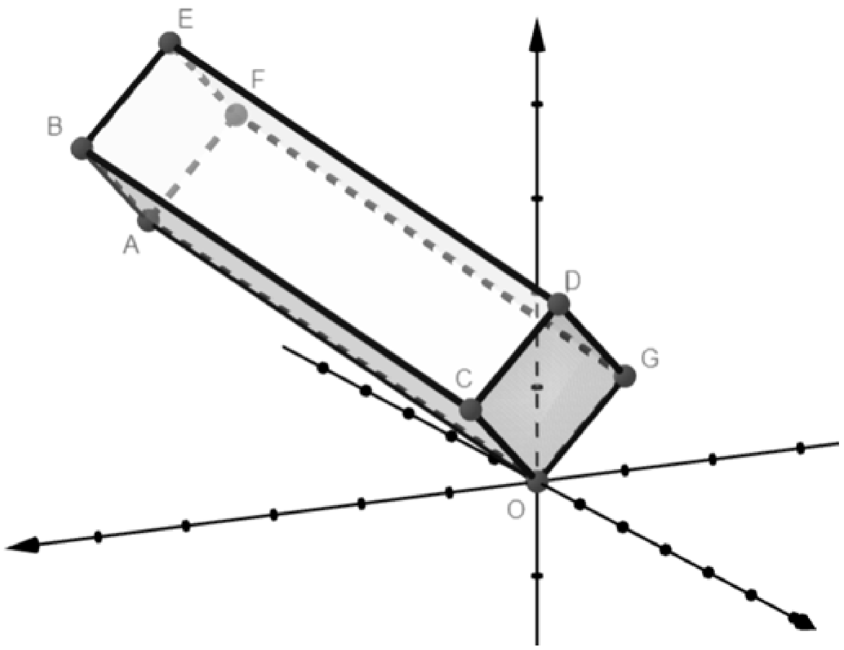
90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) (Start a new writing booklet.) **Marks**

a.



The diagram above shows a rectangular prism.

Let $\overrightarrow{OA} = 3\underset{\sim}{i} - 12\underset{\sim}{j} + 3\underset{\sim}{k}$, $\overrightarrow{OC} = 2\underset{\sim}{i} + \underset{\sim}{j} + a\underset{\sim}{k}$ and $\overrightarrow{OG} = -2\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

i) Show that $a = 2$ 1

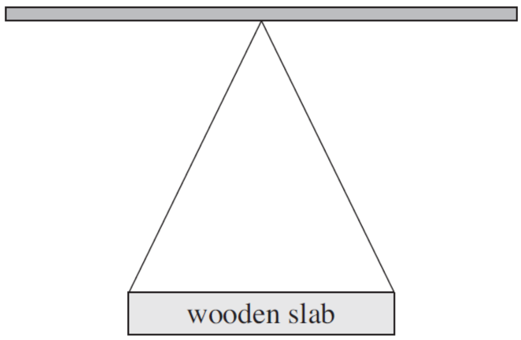
ii) Hence, or otherwise, show that $y = 0$ and $z = 2$ 3

iii) Calculate the height of point E above the $x - y$ plane. 2

Question 11 continued on next page.

Question 11 continued.

b. A swing is being designed so that two inelastic ropes of negligible weight and equal length will hang from the same point on an iron bar to support a horizontal wooden seat of negligible weight, as shown in the diagram below:



The swing is designed to support a mass of 50 *kg* and each rope has a breaking strain of 326 *N*. Calculate the maximum angle between the two ropes, correct to the nearest degree. Assume that acceleration due to gravity is 10 m/s^2. 2

c. Find $\int \frac{1}{\sqrt{13+36x-9x^2}}\,dx$ 3

d. Evaluate $\int \frac{3}{x^2\sqrt{x^2-9}}\,dx$, using the substitution $x = 3sec\theta$ 4

End of Question 11

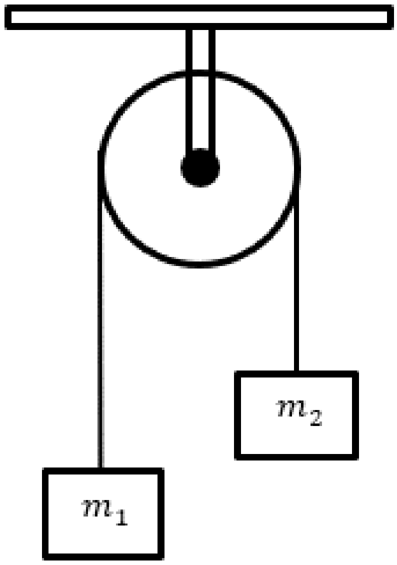
Question 12 (16 marks) (Start a new writing booklet.) **Marks**

a. i) When a, b and c are real and $a \neq b \neq c$, prove that

$$a^2 + b^2 + c^2 > ab + bc + ca$$ 1

ii) Hence, prove that if $a + b + c = 1$, then $ab + bc + ca < \frac{1}{3}$ 2

b. Two masses m_1 and m_2 are connected by a light, inextensible string which passes over a frictionless pulley as shown in the diagram below. The acceleration due to gravity is g.



If the mass m_1 descends with acceleration $\frac{2g}{5}$, find the ratio of the masses, $\frac{m_1}{m_2}$ 3

c. Evaluate $\int_0^1 (2x + 1)e^x dx$ 3

Question 12 continued on next page.

Question 12 continued.

d. The vector $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ makes angles α, β and γ with the X, Y and Z axes respectively.

Show that $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$ 2

e. Consider $I_n = \int x^n cosx\, dx$

i) Find a recurrence formula for I_n 3

ii) Hence, evaluate $\int_0^{\pi} x^4 cosx\, dx$ 2

End of Question 12

Question 13 (15 marks) (Start a new writing booklet.) **Marks**

a. Let z be a complex number such that $2|z-1| = |z-4|$

Find $|z|$ 2

b. Find $\int \frac{4x^2-2x-33}{9-x^2}\,dx$ 4

c. i) Use De Moivre's Theorem to show that $cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$ 2

ii) Hence, solve $8x^4 - 8x^2 + 1 = 0$ and deduce that

$$cos^2\left(\frac{\pi}{8}\right)cos^2\left(\frac{3\pi}{8}\right) = \frac{1}{8}$$ 3

d. i) Prove that, if a is an integer, then $a^2 + a$ is even. 2

ii) Hence, or otherwise, prove that:

If p is odd, then $x^2 + x - p^2 = 0$ has no integer solution. 2

End of Question 13

Question 14 (15 marks) (Start a new writing booklet.) **Marks**

a. Consider the complex number $N = \frac{z-2}{z-i}$ where z is also a complex number.

If N is purely imaginary, show that z lies on a circle. State the centre and radius of the circle and give the coordinates of any points which must be excluded. 5

b. Consider two lines $L_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ and $L_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

i) Show that the lines L_1 and L_2 do not intersect. 1

Points P and Q lie on lines L_1 and L_2 respectively.

ii) Find the vector equation of $\overrightarrow{PQ}$ in terms of λ and μ. 1

iii) Hence, or otherwise, find the shortest distance between lines L_1 and L_2. 3

c. A sphere has parametric equations $x = 3sin\theta + 2$, $y = 3cos^2\theta + 1$ and

$z = 3cos\theta sin\theta + 5$

i) Show that the Cartesian equation of the sphere is

$$(x-2)^2 + (y-1)^2 + (z-5)^2 = 9$$ 2

The vector equation of line L is $L = (-2 + 2\lambda)\underset{\sim}{i} + (-3 + 2\lambda)\underset{\sim}{j} + (3 + \lambda)\underset{\sim}{k}$

ii) Show that line L passes through the centre of the sphere. 1

iii) Find the coordinates of the points of intersection of line L and the surface of the sphere. 2

End of Question 14

Question 15 (15 marks) (Start a new writing booklet.) **Marks**

a. A sequence is defined by the recursive formula $T_n = \frac{T_{n-1} \times (2n+1)}{2n-3}$,

where $T_1 = 3$ and $n > 1$

i) Prove by mathematical induction that $T_n = 4n^2 - 1$ 3

ii) Using the formula

$$\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$$

prove that the sum of n terms of the sequence from part i) is

$$\frac{n(4n^2+6n-1)}{3}$$ 2

b. By considering the series

$$\sum_{k=1}^{n} k$$

and the AM-GM inequality

$$\frac{x_1+x_2+x_3+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \ldots x_n} \ ,$$

prove that $\left(\frac{n+1}{2}\right)^n \geq n!$ for integers $n \geq 1$. 3

Question 15 continued on next page.

Question 15 continued.

c. A particle is projected from a point O on a horizontal plane with an initial velocity of 40 *m/s* at an angle of 30° to the horizontal.

Assume that acceleration due to gravity is $10\ m/s^2$ and ignore air resistance.

i) Write down the vectors in exact form which describe the velocity and displacement of the particle. 2

ii) Find the range of the particle. 2

iii) At the same time as the first particle is launched, a second particle is projected in the opposite direction with an initial velocity of 30 *m/s* from a point on the same horizontal level as O. Find the angle of projection, to the nearest degree, of the second particle, if the particles are to collide. 3

End of Question 15

Question 16 (14 marks) (Start a new writing booklet.) **Marks**

a. A particle of mass *m* falls from rest in a medium where the resistance to motion has magnitude *mkv*, where *v* is the velocity after falling a distance of *x* and *k* is a positive constant. The particle has a terminal velocity *V* and acceleration due to gravity is *g*.

i) Show that the acceleration of the particle is given by

$$a = g - kv$$ 1

ii) Show that $x = \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}$ 3

iii) Show that the particle has fallen a distance of $\frac{V^2}{2g}(ln4 - 1)$ in the time that it takes to reach half of its terminal velocity. 2

b. i) If $z = cis\theta$ show that $z^n + z^{-n} = 2cosn\theta$ 1

ii) Find the 4 solutions, in $x + iy$ form, of the equation

$$2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$$ 4

c. Show that the limiting sum of the series $\frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \cdots$ is $\frac{4}{9}$ 3

End of Question 16

End of Paper

Start here for Question Number: 14 2023 12 Ext 2 Trial Sol^ns

1. $2-3\lambda = -1$ DCDBD

$3 = 3\lambda$ ADCCA

$\lambda = 1$

$a = -2+4$

$= 2$ D

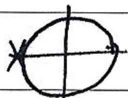
2. $3.2 - 4p - 3 = 0$

$3 = 4p$

$p = \frac{3}{4}$ C

3. $e^{i\theta} = 3$ or $e^{i\theta} = -1$

not valid



$\therefore \theta = \pi$ D

4. $v^2 = n^2(a^2 - x^2)$

$12^2 = n^2(8^2 - 4^2)$

$144 = 48n^2$

$n^2 = 3$

$n = \sqrt{3}$ $n > 0$

$\therefore T = \frac{2\pi}{\sqrt{3}}$

$= \frac{2\sqrt{3}\pi}{3}$ B

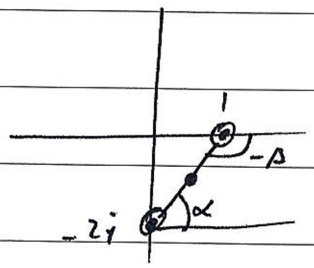
5. If you dont clap hands, then not (happy <u>and</u> know it)

→ then either not happy or dont know it

D

6. $\arg\left(\dfrac{z+2i}{z-1}\right) = \pi$

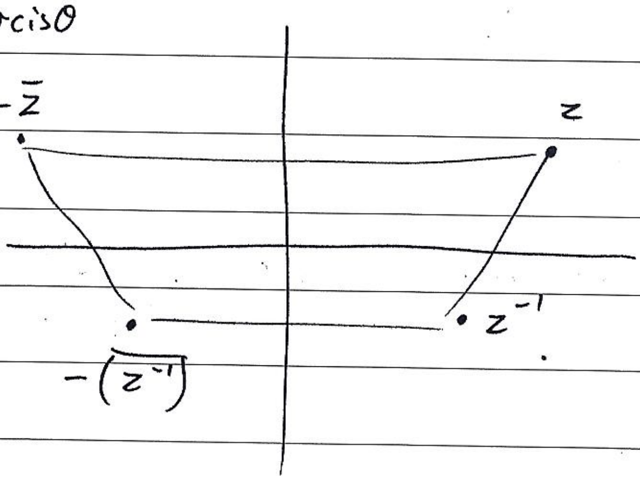
$\arg(z-(-2i)) - \arg(z-1) = \pi$



$\alpha - (-\beta) = \pi$

A

7. $z = r\text{cis}\theta$

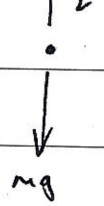


$z^{-1} = \dfrac{1}{r\text{cis}\theta}$

$= \frac{1}{r}\text{cis}(-\theta)$

D

8.



$m\ddot{x} = mg - \frac{v}{2}$

$\ddot{x} = g - \frac{v}{2}$

$\ddot{x} = \dfrac{20-v}{2}$

Additional writing space on back page.

$$\frac{dv}{dt} = \frac{20-v}{2}$$

$$\frac{dv}{20-v} = \frac{dt}{2}$$

$$-\int_0^v \frac{-1}{20-v}\,dv = \frac{1}{2}\int_0^t dt$$

$$-\left[\ln|20-v|\right]_0^v = \frac{t}{2}$$

$$\ln|20-v| - \ln 20 = -\frac{t}{2}$$

$$\frac{20-v}{20} = e^{-\frac{t}{2}}$$

$$20 - v = 20e^{-\frac{t}{2}}$$

$$v = 20\left(1 - e^{-\frac{t}{2}}\right)$$

C

9. $x = \sqrt{t-2}$ Note $x \geq 0$

$x^2 = t-2$

$2 - t = -x^2$

$\therefore y = \frac{-1}{x^2}$ and since $x \geq 0$, $y < 0$ with $x > 0$

C

10. $\int_0^1 f(x)g(x)\,dx = \int_0^1 f(1-x)\left[2 - g(1-x)\right]dx$

$= 2\int_0^1 f(1-x)\,dx - \int_0^1 f(1-x)g(1-x)\,dx$

$= 2\int_0^1 f(x)\,dx - \int_0^1 f(x)g(x)\,dx$

$2\int_0^1 f(x)g(x)\,dx = 2\int_0^1 f(x)\,dx$

A

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

Start here for Question Number: **15**

11a. i) $\vec{OA} \perp \vec{OC}$

$\therefore \vec{OA} \cdot \vec{OC} = 0$

ie $3.2 - 12 + 3a = 0$ (1)

$3a = 6$

$a = 2$

ii) $\vec{OA} \perp \vec{OG}$ and $\vec{OC} \perp \vec{OG}$

$\therefore 3x - 2 - 12y + 3z = 0$ and $2x - 2 + y + 2z = 0$ (1)

$-12y + 3z = 6$ $\qquad y + 2z = 4$

$-4y + z = 2 \quad \ldots (1)$ $\qquad 4y + 8z = 16 \quad \ldots (2)$

$(1) + (2) \Rightarrow \quad 9z = 18$ (1)

$z = 2$

Subst $z = 2$ into $(2) \Rightarrow \quad 4y + 16 = 16$ (1)

$y = 0$

iii) $\vec{OE} = \vec{OA} + \vec{AB} + \vec{BE}$

$= \vec{OA} + \vec{OC} + \vec{OG}$

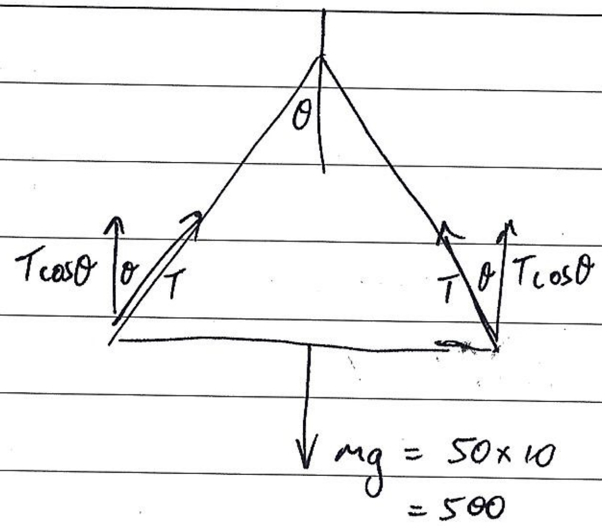
$= 3\underset{\sim}{i} - 12\underset{\sim}{j} + 3\underset{\sim}{k} + 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} - 2\underset{\sim}{i} + 2\underset{\sim}{k}$ (1)

$= 3\underset{\sim}{i} - 11\underset{\sim}{j} + 7\underset{\sim}{k}$

$\therefore$ E is 7 units above the x-y plane. (1)

6

b.



Vertically: $2T\cos\theta = 500$ 1

$2 \times 326 \cos\theta = 500$

$\cos\theta = \frac{250}{326}$

$\theta = 39.92625234$

$\therefore$ angle between ropes is $80°$ (nearest deg) 1

c. $\displaystyle\int \frac{1}{\sqrt{13 + 36x - 9x^2}}\,dx = \int \frac{1}{\sqrt{13 - 9(x^2 - 4x + 4) + 36}}\,dx$ 1

$$= \int \frac{1}{\sqrt{49 - 9(x-2)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{3}{\sqrt{7^2 - (3(x-2))^2}}\,dx$$ 1

$$= \frac{1}{3}\sin^{-1}\left(\frac{3(x-2)}{7}\right) + c$$ 1

Additional writing space on back page.

/5

d. $\int \frac{3}{x^2\sqrt{x^2-9}}\,dx$ $\qquad x = 3\sec\theta$

$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$

$= \int \frac{3 \times 3\sec\theta \cancel{\tan\theta}\, d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta - 9}}$ (1)

$1 + t^2 = s^2$

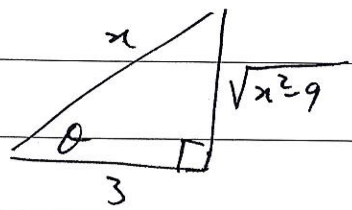
$t^2 = s^2 - 1$

$= \int \frac{\cancel{\tan}\theta\, d\theta}{\sec\theta \times 3\sqrt{\cancel{\tan}^2\theta}}$

$= \frac{1}{3}\int \cos\theta\, d\theta$ (1)

$\sec\theta = \frac{x}{3}$

$= \frac{\sin\theta}{3} + c$ (1)



$= \frac{\sqrt{x^2-9}}{3x} + c$ (1)

12 a. 1) $(a-b)^2 > 0$ $\quad$ for $a \neq b$

$a^2 - 2ab + b^2 > 0$

$a^2 + b^2 > 2ab$

Similarly, $b^2 + c^2 > 2bc$

$c^2 + a^2 > 2ca$

$\therefore\ 2(a^2+b^2+c^2) > 2ab + 2bc + 2ca$ (1)

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

8

Start here for Question Number: 16

$\therefore \quad a^2+b^2+c^2 > ab+bc+ca$

ii) $\quad (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

$\therefore \quad a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca)$ 1

$\therefore \quad (a+b+c)^2 - 2(ab+bc+ca) > ab+bc+ca$

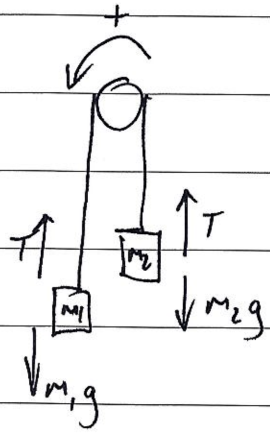
from (1)

$(a+b+c)^2 > 3(ab+bc+ca)$

$1 > 3(ab+bc+ca)$ 1

$\frac{1}{3} > ab+bc+ca$

b. $\quad \Sigma F = m\ddot{x} = m_1 g - T + T - m_2 g$



$(m_1+m_2) \times \frac{2g}{5} = g(m_1-m_2)$ 1

$2(m_1+m_2) = 5(m_1-m_2)$ 1

$2m_1+2m_2 = 5m_1-5m_2$

$7m_2 = 3m_1$

$\frac{7}{3} = \frac{m_1}{m_2}$

ratio of masses is $\frac{7}{3}$ 1

5

c. $\int_0^1 (2x+1)\, e^x \, dx = \int_0^1 (2x+1) \times \frac{d(e^x)}{dx}\, dx$

$= \left[e^x (2x+1) \right]_0^1 - \int_0^1 e^x \times 2 \, dx$ (1)

$= 3e - 1 - \left[2e^x \right]_0^1$ (1)

$= 3e - 1 - (2e - 2)$

$= e + 1$ (1)

d. $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$

X axis is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\overrightarrow{OP}$. X axis $= x$ $\quad |\overrightarrow{OP}| = \sqrt{x^2+y^2+z^2}$

$\therefore \; x = \sqrt{x^2+y^2+z^2} \cos\alpha$ (1)

Similarly $y = \sqrt{x^2+y^2+z^2} \cos\beta$
$z = \sqrt{x^2+y^2+z^2} \cos\gamma$

Squaring each: $x^2 = (x^2+y^2+z^2) \cos^2\alpha$
$y^2 = (x^2+y^2+z^2) \cos^2\beta$
$z^2 = (x^2+y^2+z^2) \cos^2\gamma$ (1)

Additional writing space on back page.

$$\therefore \quad x^2+y^2+z^2 = (x^2+y^2+z^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

e. i) $I_n = \int x^n \cos x \, dx$

$$= \int x^n \times \frac{d(\sin x)}{dx} \, dx$$

$$= x^n \sin x - \int \sin x \cdot n x^{n-1} \, dx \quad 1$$

$$= x^n \sin x - n \int x^{n-1} \frac{d(-\cos x)}{dx} \, dx \quad 1$$

$$= x^n \sin x - n \left[-x^{n-1} \cos x + \int \cos x \times (n-1) x^{n-2} \, dx \right]$$

$$= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx$$

$$= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2} \quad 1$$

ii) $I_4 = \int_0^{\pi} x^4 \cos x \, dx$

$$= \left[x^4 \sin x + 4x^3 \cos x \right]_0^{\pi} - 4 \times 3 I_2$$

$$= \pi^4 \sin \pi + 4\pi^3 \cos \pi - (0+0) - 12 I_2$$

$$= -4\pi^3 - 12 I_2 \quad 1$$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

4

Office Use Only - Do NOT write anything, or make any marks below this line.

Start here for Question Number: 11

$$I_2 = \left[x^2 \sin x + 2x \cos x\right]_0^{\pi} - 2 \times 1\, I_0$$

$$= \left[\pi^2 \sin\pi + 2\pi\cos\pi - (0+0)\right] - 2I_0$$

$$= -2\pi - 2I_0$$

$$I_0 = \int_0^{\pi} \cos x \, dx$$

$$= \left[\sin x\right]_0^{\pi}$$

$$= \sin\pi - \sin 0$$

$$= 0$$

$$\therefore I_2 = -2\pi$$

$$\therefore I_4 = -4\pi^3 - 12(-2\pi)$$

$$= 24\pi - 4\pi^3$$ 1

13a. $2|z-1| = |z-4|$

$$2\sqrt{(x-1)^2 + y^2} = \sqrt{(x-4)^2 + y^2}$$ 1

$$4(x-1)^2 + 4y^2 = (x-4)^2 + y^2$$

$$4(x^2-2x+1) + 4y^2 = x^2 - 8x + 16 + y^2$$

$$4x^2 - 8x + 4 + 4y^2 = x^2 - 8x + 16 + y^2$$

$$3x^2 + 3y^2 = 12$$

$$x^2 + y^2 = 4$$

2

$\therefore$ z lies on the circle $x^2+y^2=4$

$\therefore |z| = 2$ 1

b. Let $\dfrac{4x^2-2x-33}{9-x^2} = a + \dfrac{b}{3-x} + \dfrac{c}{3+x}$ 1

$\therefore 4x^2-2x-33 = a(9-x^2) + b(3+x) + c(3-x)$

$x=3$, $4\times9-2\times3-33 = 0 + 6b + 0$

$-3 = 6b$

$b = -\frac{1}{2}$ 1

$x=-3$, $4\times9+6-33 = 0 + 0 + 6c$

$9 = 6c$

$c = \frac{3}{2}$

$x=0$, $-33 = 9a - \frac{3}{2} + \frac{9}{2}$

$-36 = 9a$

$a = -4$

$\therefore \displaystyle\int \frac{4x^2-2x-33}{9-x^2}\,dx = \int -4\,dx + \frac{1}{2}\int \frac{-1}{3-x}\,dx + \frac{3}{2}\int \frac{1}{3+x}\,dx$ 1

4

Additional writing space on back page.

$$= -4x + \frac{1}{2}\ln|3-x| + \frac{3}{2}\ln|3+x| + c$$ 1

c. i) $(\text{cis}\,\theta)^4 = (\cos\theta + i\sin\theta)^4$

$\text{cis}\,4\theta = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$ 1

$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

Equating Reals:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$ 1
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

ii) let $x = \cos\theta$ in $8x^4 - 8x^2 + 1 = 0$

where solutions will be those of $\cos 4\theta = 0$

$\cos 4\theta = 0$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \ldots$

$\therefore$ Sol'ns to $8x^4 - 8x^2 + 1 = 0$ are 1

$x = \cos\frac{\pi}{8}$, $x = \cos\frac{3\pi}{8}$, $x = \cos\frac{5\pi}{8}$ and $x = \cos\frac{7\pi}{8}$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

4

Start here for Question Number: 12

Product of roots:

$$\cos\frac{\pi}{8} \times \cos\frac{3\pi}{8} \times \cos\frac{5\pi}{8} \times \cos\frac{7\pi}{8} = \frac{1}{8}$$ (1)

Now, $\cos\frac{7\pi}{8} = -\cos\frac{\pi}{8}$ and $\cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$

$\therefore \cos\frac{\pi}{8} \times \cos\frac{3\pi}{8} \times -\cos\frac{3\pi}{8} \times -\cos\frac{\pi}{8} = \frac{1}{8}$ (1)

ie $\cos^2\frac{\pi}{8} \times \cos^2\frac{3\pi}{8} = \frac{1}{8}$

d. 1) If a is even, let $a = 2M$ where $M \in \mathbb{Z}$

$$\begin{aligned}\therefore a^2 + a &= (2M)^2 + 2M \\ &= 4M^2 + 2M \\ &= 2(2M^2 + M)\end{aligned}$$ (1)

which is even

If a is odd, let $a = 2N+1$ where $N \in \mathbb{Z}$

$$\begin{aligned}\therefore a^2 + a &= (2N+1)^2 + 2N+1 \\ &= 4N^2 + 4N + 1 + 2N + 1 \\ &= 4N^2 + 6N + 2 \\ &= 2(2N^2 + 3N + 1)\end{aligned}$$

which is even (1)

$\therefore$ if a is an integer, then $a^2 + a$ is even.

4

11) Consider x^2+x is even for all $x \in \mathbb{Z}$
from (1)

Now, if p is odd then p^2 is odd (1)

Solving $x^2+x-p^2=0$
$x^2+x=p^2$

but x^2+x is even and p^2 is odd
which is a contradiction (1)

$\therefore$ $x^2+x-p^2=0$ has no integer solution.

14a. let $z = x+iy$

$\therefore$ $N = \dfrac{(x-2)+iy}{x+i(y-1)} \times \dfrac{x-i(y-1)}{x-i(y-1)}$ (1)

$= \dfrac{x(x-2)+y(y-1) + i\left(xy-(x-2)(y-1)\right)}{x^2+(y-1)^2}$

Since N is purely imaginary, $Re(N)=0$

ie $\dfrac{x(x-2)+y(y-1)}{x^2+(y-1)^2} = 0$ (1)

$x^2-2x+y^2-y=0$

$x^2-2x+1+y^2-y+\left(\frac{1}{2}\right)^2 = 1+\frac{1}{4}$

4

Additional writing space on back page.

$$(x-1)^2 + \left(y - \tfrac{1}{2}\right)^2 = \frac{5}{4} \quad 1$$

which is a circle with centre $(1, \frac{1}{2})$ 1
and radius $\frac{\sqrt{5}}{2}$

Since $N = \frac{z-2}{z-i}$, $z = 2$ and $z = i$

should be excluded, ie pts, $(2,0)$ and $(0,1)$. 1

6. 1) If L_1 and L_2 intersect

$2 - \lambda = -1 + \mu$ ... 1

$1 + \lambda = 1 - 2\mu$ ... 2

$1 - \lambda = \mu$ ... 3

Subst 3 into 1 and 2 $\Rightarrow$

1 $\Rightarrow$ $2 - \lambda = -1 + 1 - \lambda$

$2 = 0$?? not a sensible result

2 $\Rightarrow$ $1 + \lambda = 1 - 2(1 - \lambda)$

$1 + \lambda = 1 - 2 + 2\lambda$

$2 = \lambda$

If $\lambda = 2$, 3 $\Rightarrow$ $\mu = -1$

and, if $\mu = -1$

1 $\Rightarrow$ $2 - \lambda = -1 - 1$

$4 = \lambda$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

3

Office Use Only - Do NOT write anything, or make any marks below this line.

Start here for Question Number: 13

Hence, there are no consistent ~~values~~ values for λ and μ which satisfy the 3 equations. 1

$\therefore$ L_1 and L_2 do not intersect.

ii) For $P\begin{pmatrix}2-\lambda\\1+\lambda\\1-\lambda\end{pmatrix}$ and $Q\begin{pmatrix}-1+\mu\\1-2\mu\\0+\mu\end{pmatrix}$

$$\overrightarrow{PQ} = \begin{pmatrix}-1+\mu-(2-\lambda)\\1-2\mu-(1+\lambda)\\\mu-(1-\lambda)\end{pmatrix}$$

$$= \begin{pmatrix}-3+\mu+\lambda\\-2\mu-\lambda\\-1+\mu+\lambda\end{pmatrix}$$ 1

iii) The shortest distance between any point and a line is the $\perp$ distance.

for $\perp$ dist, dot product is 0

Find dot product from $\overrightarrow{PQ}$ to:

L_1: $\begin{pmatrix}-3+\mu+\lambda\\-2\mu-\lambda\\-1+\mu+\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix} = 0$ 1

ie $3-\mu-\lambda-2\mu-\lambda+1-\mu-\lambda = 0$

$4-4\mu-3\lambda = 0$

$4\mu+3\lambda = 4$ ... (1)

L_2: $\begin{pmatrix}-3+\mu+\lambda\\-2\mu-\lambda\\-1+\mu+\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} = 0$

/3

$$-3+\mu+\lambda+4\mu+2\lambda-1+\mu+\lambda=0$$

$$-4+6\mu+4\lambda=0$$

$$6\mu+4\lambda=4$$

$$3\mu+2\lambda=2 \quad \ldots (2)$$

$(1) \times 2 \Rightarrow \quad 8\mu+6\lambda=8 \quad \ldots (3)$

$(2) \times 3 \Rightarrow \quad 9\mu+6\lambda=6 \quad \ldots (4)$

$(4)-(3) \Rightarrow \quad \mu=-2$

Subst. $\mu=-2$ into $(2) \Rightarrow$

$$-6+2\lambda=2$$

$$2\lambda=8$$

$$\lambda=4$$

$\therefore$ P is $\begin{pmatrix}-2\\5\\-3\end{pmatrix}$ and Q is $\begin{pmatrix}-3\\5\\-2\end{pmatrix}$ 1

$\therefore |\overrightarrow{PQ}| = \sqrt{(-3--2)^2+(5-5)^2+(-2--3)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$ 1

$\therefore$ shortest dist. between L_1 and L_2 is $\sqrt{2}$

2

Additional writing space on back page.

Office Use Only - Do NOT write anything, or make any marks below this line.

680253 15250 641 13 439 2792110091

c. i) $x-2 = 3\sin\theta$

$y-1 = 3\cos^2\theta$

$z-5 = 3\cos\theta\sin\theta$

$(x-2)^2+(y-1)^2+(z-5)^2 = 9\sin^2\theta + 9\cos^4\theta + 9\cos^2\theta\sin^2\theta$ (1)

$= 9(\cos^4\theta + \cos^2\theta\sin^2\theta + \sin^2\theta)$

$= 9(\cos^2\theta(\cos^2\theta+\sin^2\theta) + \sin^2\theta)$ (1)

$= 9(\cos^2\theta + \sin^2\theta)$

$= 9$

ii) Centre of sphere is $\begin{pmatrix}2\\1\\5\end{pmatrix}$

If L passes through centre,

$-2+2\lambda = 2$ --- ①

$-3+2\lambda = 1$ --- ②

$3+\lambda = 5$ --- ③

from ③ $\Rightarrow$ $\lambda = 2$

~~$\lambda = 2$ in ① $\Rightarrow$ $-2+1$~~

from ① $\Rightarrow$ $2\lambda = 4$

$\lambda = 2$

from ② $\Rightarrow$ $2\lambda = 4$

$\lambda = 2$ (1)

Since $\lambda = 2$ for all, line L passes through the centre.

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

3

Start here for Question Number: **14**

iii) At pts of int :

$$(-2+2\lambda-2)^2 + (-3+2\lambda-1)^2 + (3+\lambda-5)^2 = 9$$

$$(2\lambda-4)^2 + (2\lambda-4)^2 + (\lambda-2)^2 = 9$$

$$4\lambda^2 - 16\lambda + 16 + 4\lambda^2 - 16\lambda + 16 + \lambda^2 - 4\lambda + 4 = 9$$

$$9\lambda^2 - 36\lambda + 27 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1 \text{ or } \lambda = 3$$ 1

If $\lambda = 1$, pt of int is $(-2+2)\underset{\sim}{i} + (-3+2)\underset{\sim}{j} + (3+1)\underset{\sim}{k}$

ie $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

If $\lambda = 3$, pt of int is $(-2+6)\underset{\sim}{i} + (-3+6)\underset{\sim}{j} + (3+3)\underset{\sim}{k}$

ie $\begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$

$\therefore$ Pts of int are $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$ 1

15 a. i) when $n = 2$

$$T_2 = \frac{T_1 \times (2\times2+1)}{2\times2-3}$$

$$= \frac{3\times5}{1}$$

2

$= 15$

and $T_2 = 4 \times 2^2 - 1$

$= 16 - 1$

$= 15$

$\therefore$ true for $n = 2$ 1

Assume true for $n = k$

ie $T_k = 4k^2 - 1$

Prove true for $n = k+1$

ie $T_{k+1} = 4(k+1)^2 - 1$

Now, $T_{k+1} = \dfrac{T_k\left(2(k+1)+1\right)}{2(k+1)-3}$

$= \dfrac{(4k^2-1)(2k+3)}{2k-1}$ from assumption 1

$= \dfrac{(2k-1)(2k+1)(2k+3)}{(2k-1)}$

$= (2k+1)(2k+3)$

$= 4k^2 + 8k + 3$

$= 4(k^2 + 2k + 1) - 1$

$= 4(k+1)^2 - 1$ 1

$=$ RHS

$\therefore$ true for $n = k+1$

3

Additional writing space on back page.

11) $T_n = 4n^2 - 1$

$S_n = 4\times 1^2 - 1 + 4\times 2^2 - 1 + 4\times 3^2 - 1 + \dots + 4\times n^2 - 1$

$= 4(1^2 + 2^2 + 3^2 + \dots + n^2) - n$ 1

$= 4 \times \dfrac{n(n+1)(2n+1)}{6} - n$

$= \dfrac{2n(n+1)(2n+1) - 3n}{3}$

$= \dfrac{n}{3}\left[2(n+1)(2n+1) - 3\right]$

$= \dfrac{n}{3}\left[2(2n^2 + 3n + 1) - 3\right]$

$= \dfrac{n}{3}\left[4n^2 + 6n + 2 - 3\right]$ 1

$= \dfrac{n}{3}\left[4n^2 + 6n - 1\right]$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

$\therefore$ by induction, the statement is true for all integers $n \geq 2$.

b. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n$

$= \frac{n(n+1)}{2}$ since it is an AP 1

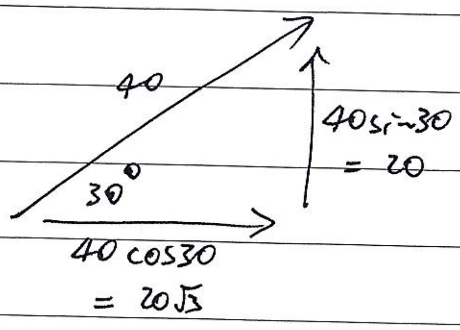
Now, $\frac{1+2+3+\ldots+n}{n} \geq \sqrt[n]{1 \times 2 \times 3 \times \ldots \times n}$

ie $\frac{n(n+1)}{2n} \geq \sqrt[n]{n!}$ from above 1

$\frac{n+1}{2} \geq \sqrt[n]{n!}$ 1

$\left(\frac{n+1}{2}\right)^n \geq n!$

c. 1)



$a = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

3

Start here for Question Number: 11

$$v = \begin{pmatrix} c \\ -10t + c \end{pmatrix}$$

when $t = 0$, $v = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}$

$$\therefore \quad v = \begin{pmatrix} 20\sqrt{3} \\ -10t + 20 \end{pmatrix}$$ 1

$$s = \begin{pmatrix} 20\sqrt{3}t + c \\ -5t^2 + 20t + c \end{pmatrix}$$

when $t = 0$, $s = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \therefore \quad s = \begin{pmatrix} 20\sqrt{3}t \\ -5t^2 + 20t \end{pmatrix}$ 1

ii) particle lands when $-5t^2 + 20t = 0$

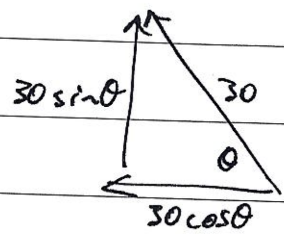
$-5t(t - 4) = 0$

$t = 0$ (launch) or $t = 4$ 1

when $t = 4$, range $= 20\sqrt{3} \times 4$

$= 80\sqrt{3}$ m 1

iii) Particle 2:



$$v = \begin{pmatrix} 30\cos\theta \\ -10t + 30\sin\theta \end{pmatrix}$$

4

$$s = \begin{pmatrix} 30t\cos\theta \\ -5t^2 + 30t\sin\theta \end{pmatrix}$$ 1

If the particles are to collide; their heights will be equivalent (Note: can't use displacement as $\rightarrow$ $\leftarrow$ they are different)

$\therefore \quad -5t^2 + 20t = -5t^2 + 30t\sin\theta$ 1

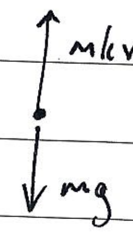
$20 = 30\sin\theta$

$\sin\theta = \frac{2}{3}$

$\theta = 41.8103149$

$= 42^\circ$ (nearest deg) 1

16 a. i)



$\Sigma F = m\ddot{x} = mg - mkv$ 1

$\ddot{x} = g - kv$

ie $a = g - kv$

ii) $v\dfrac{dv}{dx} = g - kv$

$\dfrac{dv}{dx} = \dfrac{g - kv}{v}$

$\dfrac{v}{g - kv}\,dv = dx$ 1

5

Additional writing space on back page.

$$\frac{-1}{k} \times \frac{-kv}{g-kv}\, dv = dx$$

$$\frac{-1}{k}\left(\frac{g-kv-g}{g-kv}\right) dv = dx$$

$$dx = \frac{-1}{k}\left(1 - \frac{g}{g-kv}\right) dv$$

$$\int_0^x dx = \frac{-1}{k}\int_0^v 1\, dv + \frac{g}{k}\int_0^v \frac{1}{g-kv}\, dv \quad 1$$

$$x = \frac{-v}{k} - \frac{g}{k^2}\int_0^v \frac{-k}{g-kv}\, dv$$

$$= \frac{-v}{k} - \frac{g}{k^2}\Big[\ln|g-kv|\Big]_0^v$$

$$= \frac{-v}{k} - \frac{g}{k^2}\Big[\ln|g-kv| - \ln g\Big]$$

$$= \frac{-v}{k} + \frac{g}{k^2}\Big[\ln g - \ln(g-kv)\Big]$$

$g > kv$ since initial $v = 0$

$$= \frac{-v}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) \quad 1$$

iii) Terminal vel when $\ddot{x} = 0$

ie $g = kV$

$$V = \frac{g}{k}$$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

2

Start here for Question Number: **12**

need x when $v = \frac{V}{2}$

$$= \frac{g}{2k}$$ 1

ie $x = -\frac{g}{2k^2} + \frac{g}{k^2}\ln\left(\frac{g}{g - k \times \frac{g}{2k}}\right)$

$$= \frac{-g}{2k^2} + \frac{g}{k^2}\ln 2$$

$$= \frac{-g}{2k^2} + \frac{g}{2k^2} \times 2\ln 2$$

$$\frac{V^2}{2g} = \frac{g^2}{k^2} \times \frac{1}{2g}$$

$$= \frac{g}{2k^2} \times \ln 4 - \frac{g}{2k^2}$$

$$= \frac{g}{2k^2}$$

$$= \frac{g}{2k^2}(\ln 4 - 1)$$ 1

$$= \frac{V^2}{2g}(\ln 4 - 1)$$

b. i) $z^n + z^{-n} = (\mathrm{cis}\,\theta)^n + (\mathrm{cis}\,\theta)^{-n}$

$$= \mathrm{cis}\,n\theta + \mathrm{cis}(-n\theta)$$

$$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$$ 1

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta$$

ii) $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$

$$z^2\left(2z^2 + 3z + 5 + \frac{3}{z} + \frac{2}{z^2}\right) = 0$$

Note $z \neq 0$ due to original eqn

3

$$\therefore \quad 2z^2 + 3z + 5 + \frac{3}{z} + \frac{2}{z^2} = 0$$

$$2\left(z^2 + \frac{1}{z^2}\right) + 3\left(z + \frac{1}{z}\right) + 5 = 0 \qquad 1$$

$$2 \times 2\cos 2\theta + 3 \times 2\cos\theta + 5 = 0 \qquad \text{from (1)}$$

$$4\cos 2\theta + 6\cos\theta + 5 = 0$$

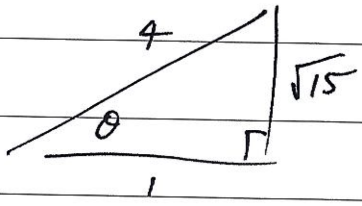
$$4(2\cos^2\theta - 1) + 6\cos\theta + 5 = 0$$

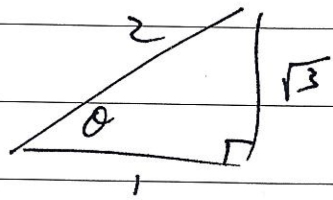
$$8\cos^2\theta + 6\cos\theta + 1 = 0$$

$$\cos\theta = \frac{-6 \pm \sqrt{36 - 4 \times 8 \times 1}}{2 \times 8}$$

$$= \frac{-6 \pm 2}{16}$$

$$\cos\theta = \frac{-1}{4} \quad \text{or} \quad \cos\theta = \frac{-1}{2} \qquad 1$$





$$\therefore \sin\theta = \pm\frac{\sqrt{15}}{4} \qquad \sin\theta = \pm\frac{\sqrt{3}}{2} \qquad 1$$

$\therefore$ Sol'ns are $z = \frac{-1}{4} + \frac{\sqrt{15}}{4}i$, $z = -\frac{1}{4} - \frac{\sqrt{15}}{4}i$ 1

$z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$, $z = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$

4

Additional writing space on back page.

c. $\frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \cdots = \frac{1}{4} + \frac{1+1}{4^2} + \frac{1+1+1}{4^3} + \cdots$

$= \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) + \left(\frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) +$ 1

$\left(\frac{1}{4^3} + \frac{1}{4^4} + \cdots\right) + \cdots$

$= \frac{\frac{1}{4}}{1-\frac{1}{4}} + \frac{\frac{1}{4^2}}{1-\frac{1}{4}} + \frac{\frac{1}{4^3}}{1-\frac{1}{4}} + \cdots$

$= \frac{1}{3} + \frac{1}{3\times4} + \frac{1}{3\times4^2} + \cdots$ 1

$= \frac{\frac{1}{3}}{1-\frac{1}{4}}$

$= \frac{1}{3} \div \frac{3}{4}$

$= \frac{1}{3} \times \frac{4}{3}$

$= \frac{4}{9}$ 1

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.

3

Office Use Only - Do NOT write anything, or make any marks below this line.