Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10

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1. A particle is moving in Simple Harmonic Motion in a straight line. In one minute of its motion it completes 15 oscillations and travels 120 metres. What is the amplitude of the motion?

(A) 2 metres

(B) 4 metres

(C) 8 metres

(D) 16 metres

2. If $z = \frac{1+2i}{3+4i}$, what is the modulus of z?

(A) $\frac{\sqrt{13}}{\sqrt{7}}$

(B) $\frac{\sqrt{13}}{5}$

(C) $\frac{\sqrt{5}}{5}$

(D) $\frac{5}{\sqrt{5}}$

3. If $\frac{1}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1}$, what are the values of a and b ?

(A) $a = -1, b = -1$

(B) $a = 1, b = -1$

(C) $a = -1, b = 1$

(D) $a = 1, b = 1$

4. Which of the following is a true statement about the lines l_1 and l_2?

$$l_1:\ \underset{\sim}{r} = \binom{3}{2} - \lambda\binom{1}{2} \qquad l_2:\ y = -\frac{x}{2} + \frac{3}{2}$$

(A) l_1 and l_2 are parallel

(B) l_1 and l_2 are perpendicular

(C) l_1 and l_2 are neither parallel nor perpendicular

(D) l_1 and l_2 are the same line

5. Consider the following statement:

$\forall n \in \mathbb{N}, if\ n^2\ is\ odd, then\ n\ is\ odd.$

Which of the following is the negation of the statement?

(A) $\forall n \in \mathbb{N}, if\ n^2\ is\ odd, then\ n\ is\ not\ odd$

(B) $\exists n \in \mathbb{N}, if\ n^2\ is\ odd, then\ n\ is\ not\ odd$

(C) $\forall n \in \mathbb{N}, if\ n\ is\ not\ odd, then\ n^2\ is\ also\ not\ odd$

(D) $\exists n \in \mathbb{N}, if\ n\ is\ not\ odd, then\ n^2\ is\ also\ not\ odd$

6. Consider the position vector of a particle $\underset{\sim}{r} = -3sint\,\underset{\sim}{i} + 3cost\,\underset{\sim}{j} + t\underset{\sim}{k}$.

Which of the following statements best describes the motion of the particle?

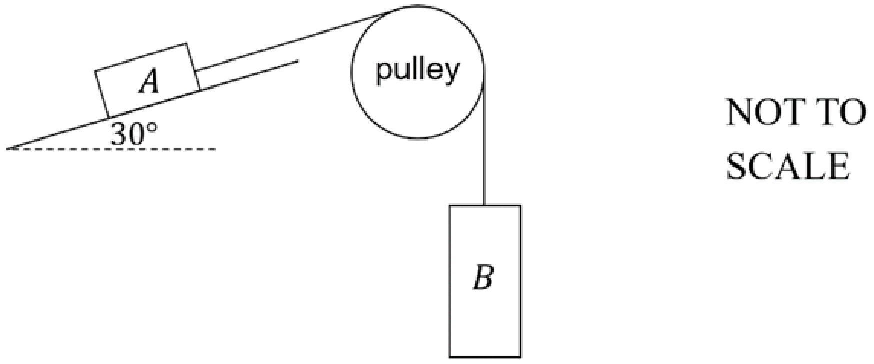
(A) A helix about the z axis in an anticlockwise direction.

(B) A helix about the z axis in a clockwise direction.

(C) A helix about the x axis in an anticlockwise direction.

(D) A helix about the x axis in a clockwise direction.

7. Particles A and B, of masses M_A and M_B respectively, are connected by a light, inextensible string which passes over a frictionless pulley, as shown below:



Particle A lies on a frictionless surface inclined at 30° to the horizontal whilst particle B hangs vertically from the string.

The particles are initially at rest and, when they are released, neither particle moves.

Which expression shows the relationship between M_A and M_B?

(A) $M_A = M_B$

(B) $M_A = \sqrt{3}M_B$

(C) $M_A = 2M_B$

(D) $M_A = 2\sqrt{3}M_B$

8. Given the proposition:

$\forall n \in \mathbb{Z}$, if z^n is not purely imaginary, then z is not purely imaginary.

Given that each of the following statements is true, which statement disproves the proposition?

(A) $(1+i)^2 = 2i$

(B) $(0+i)^3 = -i$

(C) $(1+i)^3 = -2+2i$

(D) $(0+i)^4 = 1$

9. It is given that $f(a) = -2,\ f(b) = 8,\ g(a) = 1$ and $g(b) = 2$.

What is the value of $\int_a^b \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}\,dx$?

(A) 6

(B) 2

(C) – 6

(D) – 18

10. Which of the following equations is equivalent to the intersection of

$|z| = |z - 2k - 2ki|$ and $|z - k - ki| < \sqrt{2}k$, where k is a positive constant?

(A) $\arg(z-k) = \arg(z-ki)$

(B) $\arg(z-k) = \arg(z-ki) + \pi$

(C) $\arg(z-2k) = \arg(z-2ki)$

(D) $\arg(z-2k) = \arg(z-2ki) + \pi$

End of Section I

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) (Start a new writing booklet.) **Marks**

a. On the Argand diagram provided in the Question 11 writing booklet sketch the region represented by the intersection of the inequalities: 3

$$1 \leq |z+1| \leq 2 \quad \text{and} \quad \frac{\pi}{4} < arg(z+1) < \frac{2\pi}{3}$$

b. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

i) Find $\left|\underset{\sim}{a}\right|$ 1

ii) Find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, in degrees correct to 1 decimal place. 2

c. Find the equation of the vector which passes through the points $(5, 4, 3)$ and $(-2, 1, -1)$. 1

Question 11 continued on next page.

Question 11 continued.

d. Evaluate $\int_0^{\frac{\pi}{6}} sin3x\; cos^4 3x\; dx$ 3

e. i) If $\frac{x^2+1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, find the values of A, B and C. 3

ii) Hence, find $\int \frac{x^2+1}{x^3+2x^2+x}\, dx$ 2

End of Question 11

Question 12 (15 marks) (Start a new writing booklet.) **Marks**

a. $PQRS$ is a parallelogram with vertices $P(2,4,3)$, $Q(4,3,1)$, $R(3,2,5)$ and $S(x,y,z)$. 2

Find the coordinates of point S.

b. A line l has vector equation $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$ 2

Show that the point $P(2,-5,\ 3)$ lies on line l.

c. For $x \geq 0$, show that $\frac{x}{x^2+9} \leq \frac{1}{6}$ 2

d. i) Given that $z = cos\theta + isin\theta$, show that $z^n + z^{-n} = 2cosn\theta$ 1

ii) Hence, express $cos^4\theta$ in terms of $cosn\theta$. 3

e. Using the substitution $u = sinx + 1$, or otherwise, find 2

$$\int \frac{cosx}{sin^2x+2sinx+5}\,dx$$

Question 12 continued on next page.

f. Consider the statement P in the set of integers:

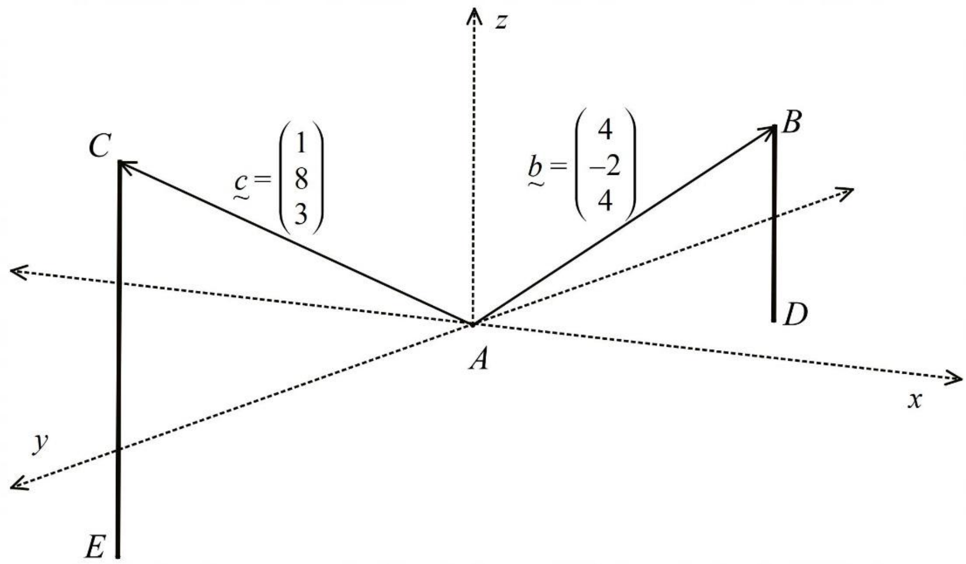
If $5n^4 + 8n$ is a multiple of 16, then n is even.

Prove that the converse of P is true. 3

End of Question 12

Question 13 (14 marks) (Start a new writing booklet.) **Marks**

a. Two vertical posts CE and BD stand on a horizontal plane ADE. Using A as the origin, the vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ represent the locations of the top of each post, B and C respectively. 2



Show that $\angle CAB$ is a right angle.

b. Find $\int \frac{1}{\sqrt{6-5x-x^2}}\,dx$ 3

c. Let l_1 be the straight line segment with equation 3

$$\underset{\sim}{r_1} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for } -3 \leq \lambda \leq 1.$$

The straight line segment l_2 is perpendicular to l_1 and the ends of l_2 are also its x and y intercepts.

Given that the endpoint of the position vector $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ lies on both l_1 and l_2, find the vector equation of l_2.

Question 13 continued on next page.

Question 13 continued.

d. Find the 4th roots of $1+\sqrt{3}i$, leaving your answer in exponential form. 3

e. Find the points of intersection of the line 3

$\underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$ with the sphere

$(x-3)^2 + (y-2)^2 + (z+1)^2 = 36$

End of Question 13

Question 14 (15 marks) (Start a new writing booklet.) **Marks**

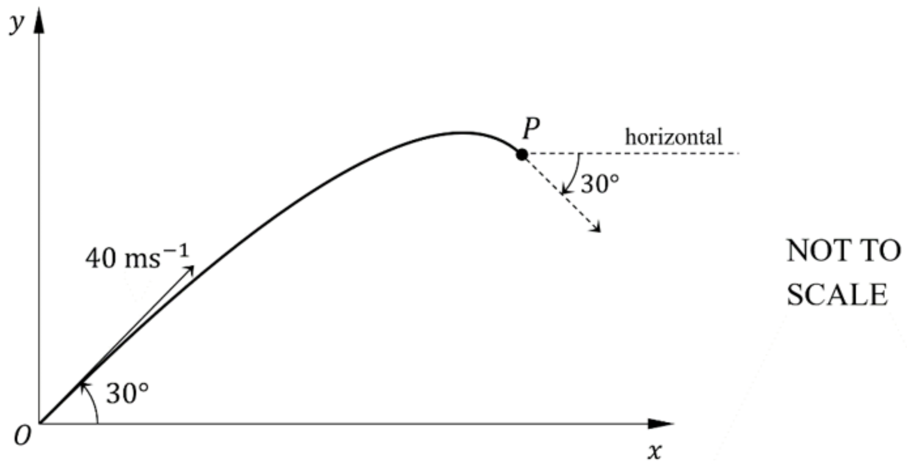
a. A particle of mass 1 kilogram is projected from the origin with speed 40 m/s at an angle of 30° to the horizontal plane as shown in the diagram for part ii) below.

The position vector of the particle at time t seconds after the particle was projected is given by:

$$\underset{\sim}{r}(t) = \begin{pmatrix} 5\sqrt{3}(1 - e^{-4t}) \\ \frac{45}{8}(1 - e^{-4t}) - \frac{5}{2}t \end{pmatrix}$$ (Do NOT prove this)

i) Find the velocity, $\underset{\sim}{v}(t)$, of the particle t seconds after it was projected. 1

ii) At point P the particle is descending at 30° to the horizontal. 2



Show that the time taken for the particle to reach P is $log_e\sqrt[4]{17}$ seconds.

Question 14 continued on next page.

Question 14 continued.

b. The equations of two lines l_1 and l_2 are: 2

$$\underset{\sim}{r_1} = \begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2^k \\ 6 - 2^k \\ 7 - 2^k \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 8 + 2^k \\ 6 \\ 5 + 2^k \end{pmatrix}$$

where λ is a real number and k is a constant.

For what value of k are the two lines parallel?

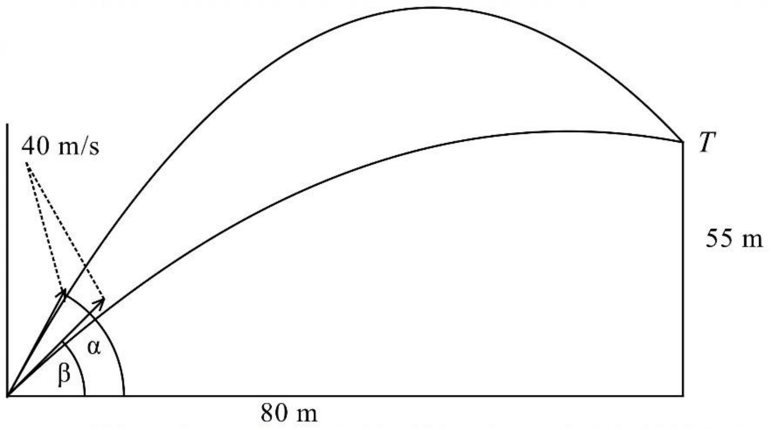
c. The polynomial $P(z) = z^4 - 8z^3 + pz^2 + qz - 80$ has root $3 + i$, where p and q are real numbers.

i) Find all of the roots of $P(z)$. 3

ii) Write $P(z)$ as a product of real linear and quadratic factors. 1

Question 14 continued on next page.

d. A missile is fired into the air with velocity of 40 m/s at an angle of α above the horizontal. Shortly afterward a second missile is fired from the same point with the same velocity but at an angle of β above the horizontal, as shown below.



NOT TO SCALE

The missiles both simultaneously hit their target, T, which is 80m horizontally and 55 m vertically from the launch point.

The equations of motion for the first particle are given by:

$$x = 40t\cos\alpha \quad \text{and} \quad y = 40t\sin\alpha - \frac{gt^2}{2}$$

i) Show that the equations of the paths of the missiles are given by: **2**

$$y = x\tan\alpha - \frac{gx^2}{3200}\sec^2\alpha \quad \text{and}$$

$$y = x\tan\beta - \frac{gx^2}{3200}\sec^2\beta \quad \text{respectively.}$$

ii) Use $g = 10\ m/s^2$ to show that $\tan\alpha = \frac{5}{2}$ and $\tan\beta = \frac{3}{2}$ **2**

iii) Find the time, in exact form, which elapses between the firing of the two missiles. **2**

End of Question 14

Question 15 (15 marks) (Start a new writing booklet.) **Marks**

a. Consider the integrals: 2

$$I_n = \int_0^1 x^2(1-x^2)^n\,dx \quad \text{and} \quad J_n = \int_0^1 (1-x^2)^n\,dx$$

Show that $I_n = \frac{1}{2(n+1)} J_{n+1}$

b. A body of mass one kilogram is projected vertically upwards from the ground at a speed of 40 m/s. The body is under the effects of both gravity and an air resistance which, at any time, has a magnitude of $\frac{v^2}{20}$, where v is the magnitude of the body's velocity at that time. Use $g = 10\ m/s^2$.

i) Show that, while the body is travelling upwards, its equation of motion is: 1

$$\ddot{x} = -\left(10 + \frac{v^2}{20}\right)$$

ii) Calculate the greatest height, in exact form, reached by the body. 2

iii) Having reached its greatest height, the body falls back to the ground. The body remains under the effects of the same air resistance and gravity. 3

Show that the body is travelling at only one third of its initial launch speed when it hits the ground.

Question 15 continued on next page.

c. i) Show that $\frac{2k}{k+2} < \frac{2k+2}{k+3}$ for $k > 0$ 1

ii) Use mathematical induction to prove that: 3

$$\frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \cdots + \frac{n}{(n+2)!} < \frac{2n}{n+2} - \frac{1}{(n+2)!} \text{ for } n \geq 1$$

d. A particle of mass M kilograms is moving up a slope which is at an angle of $27°$ to the horizontal. 3

The particle has initial velocity of u m/s up the slope and comes to a stop after 6 seconds.

It is subject to acceleration due to gravity of 10 m/s^2 and a resistive force due to friction of $0.2vM$ Newtons, where v is the velocity of the particle in m/s.

Find the value of u, correct to 2 decimal places.

End of Question 15

Question 16 (16 marks) (Start a new writing booklet.) **Marks**

a. Prove by contradiction that, for $a \geq 2$: 3

$$\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$$

b. Find $\int \frac{(2tan\theta+3)sec^2\theta}{sec^2\theta+tan\theta}\,d\theta$ 3

c. It is known that the arithmetic mean of a set of numbers is greater than the geometric mean of the numbers.

That is: $$\frac{x_1+x_2+x_3+\cdots+x_n}{n} \geq \sqrt[n]{x_1x_2x_3\ldots x_n}$$

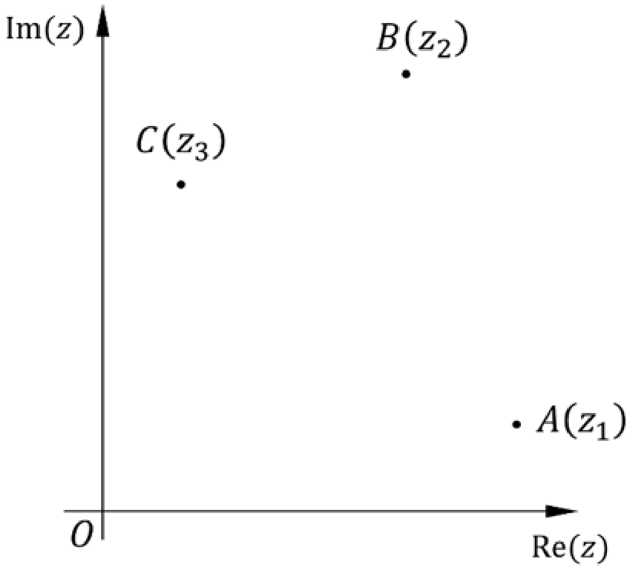
i) Given that $a+b+c=k$ and $a,b,c>0$, show that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{k}$ 3

ii) Hence, given that $k=1$, prove that $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8$ 3

Question 16 continued on next page.

Question 16 continued.

d. The complex numbers z_1, z_2 and z_3 are represented by the points A, B and C respectively. 4



NOT TO SCALE

The complex numbers satisfy $z_2 - z_1 = iz_1$ and $z_3 = e^{i\frac{\pi}{3}}z_1$

By considering $\angle BOC$ prove that $cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$

End of Question 16

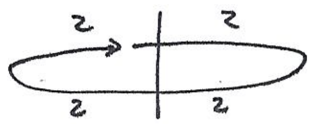
End of Paper

2024 Ext 2 Trial Sol'ns

A C B B B

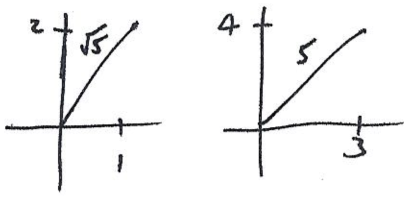
A C D A D

1. $\frac{120}{15} = 8$



A

2.

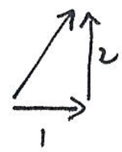


$|z| = \frac{\sqrt{5}}{5}$ C

3. $\frac{1}{x(x+1)} = \frac{x+1}{(x)} \times \frac{-x}{(x+1)}$

$a = 1 \quad , \quad b = -1$ B

4. $m_1 = 2$



$m_2 = -\frac{1}{2}$ B

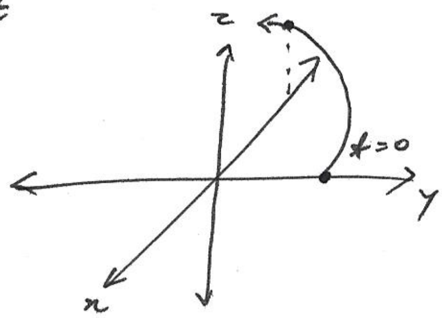
5. $\forall n \in N$, if P then Q

negates to

$\exists n \in N$, if P then not Q

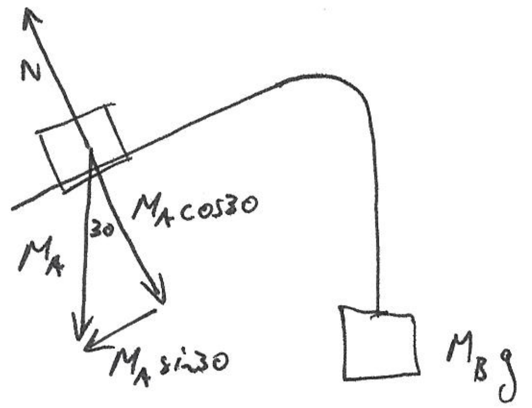
B

6. $t=0, \quad \underset{\sim}{r} = 0\underset{\sim}{i} + 3\underset{\sim}{j} + 0\underset{\sim}{k}$



$t=\frac{\pi}{2}, \quad \underset{\sim}{r} = -3\underset{\sim}{i} + 0\underset{\sim}{j} + \frac{\pi}{2}\underset{\sim}{k}$ A

7.



$$M_A \sin 30 = M_B$$

$$\frac{M_A}{2} = M_B$$

$$M_A = 2M_B \qquad C$$

8. By obs $\Rightarrow$ D using contrapositive.

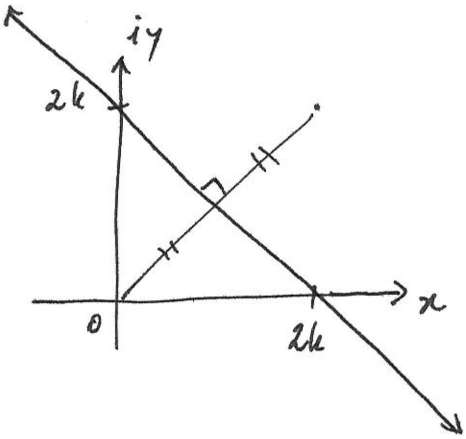
9. $$\int_a^b \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,dx = \left[\frac{f(x)}{g(x)}\right]_a^b$$

$$= \frac{f(b)}{g(b)} - \frac{f(a)}{g(a)}$$

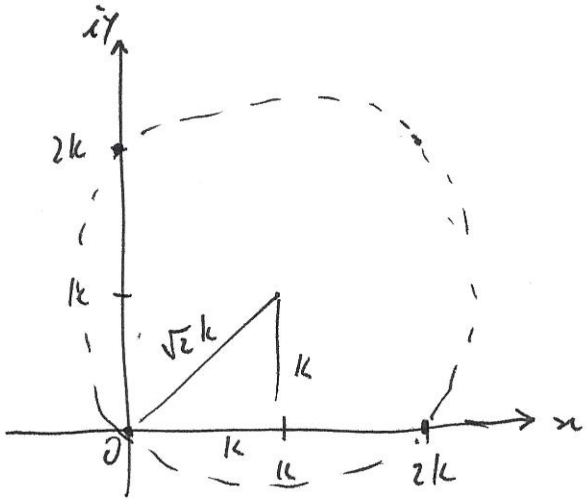
$$= \frac{8}{2} - \frac{-2}{1}$$

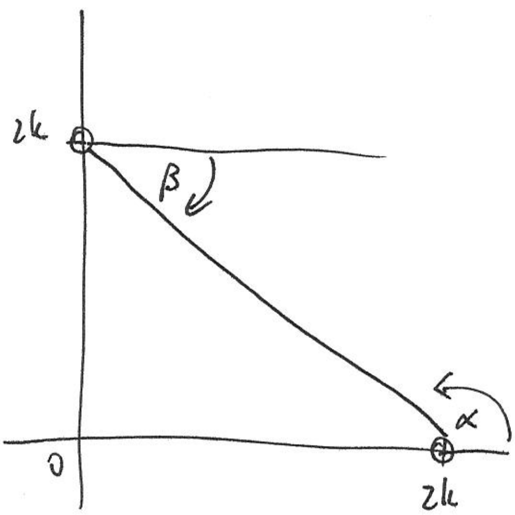
$$= 6 \qquad A$$

10. $|z| = |z - (2k + 2ki)|$



$|z - (k + ki)| \leq \sqrt{2}k$





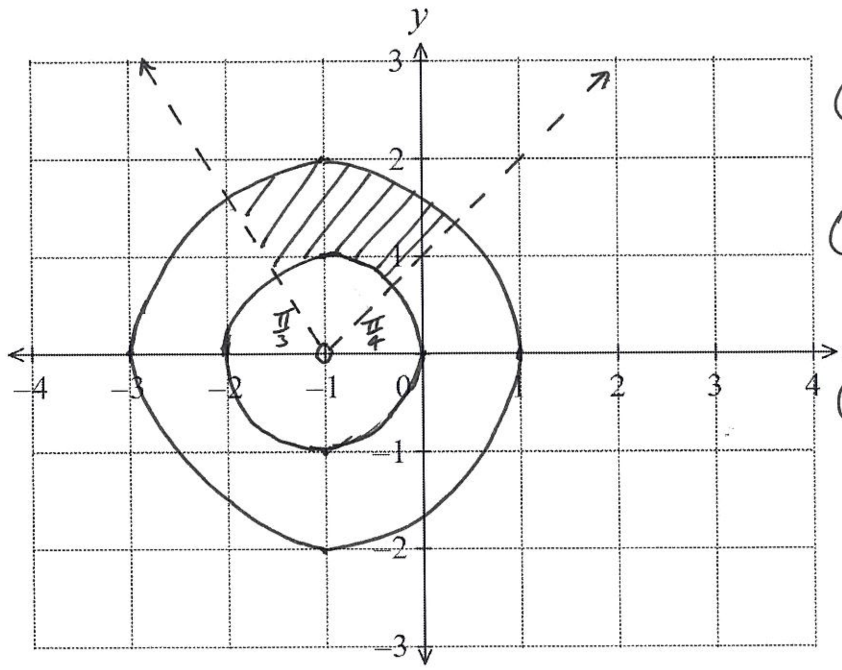
$\alpha - \beta = \pi$

ie $\arg(z - 2k) - \arg(z - 2ki) = \pi$ D

Student Number

Start Here:

a. Use the diagram below for your answer to Question 11a.



(1) one correct

(2) minor error

(3) correct.

b. i) $|\underset{\sim}{a}| = \sqrt{2^2+3^2+1^2}$

$= \sqrt{14}$ (1)

ii) $|\underset{\sim}{b}| = \sqrt{(-1)^2+1^2+4^2}$

$= \sqrt{18}$

$= 3\sqrt{2}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\therefore \cos\theta = \dfrac{2\times -1 + 3\times 1 + 1\times 4}{\sqrt{14} \times 3\sqrt{2}}$ (1) dot product.

$= 0.3149703942$

$\theta = 71.64097347$

$= 71.6^\circ$ (1dp) (1)

Student Number

c. $\underset{\sim}{r} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -3 \\ -4 \end{pmatrix}$ or other correct (1)

d. $\displaystyle\int_0^{\frac{\pi}{6}} \sin 3x (\cos 3x)^4 \, dx = -\frac{1}{3}\int_0^{\frac{\pi}{6}} -3\sin 3x (\cos 3x)^4 \, dx$ (1)

$= -\frac{1}{3}\left[\frac{(\cos 3x)^5}{5}\right]_0^{\frac{\pi}{6}}$ (1)

$= -\frac{1}{3}\left[\frac{\cos^5(\frac{\pi}{2})}{5} - \frac{\cos^5 0}{5}\right]$

$= -\frac{1}{3}\left(0 - \frac{1}{5}\right)$

$= \frac{1}{15}$ (1)

e. 1) $\dfrac{x^2+1}{x^3+2x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$x^2+1 = A(x^2+2x+1) + Bx(x+1) + Cx$ (1)

when $x=0$, $1 = A$ (1)

$x=-1$, $2 = -C$

$C = -2$

$x=1$, $2 = 4A + 2B + C$

$2 = 4 + 2B - 2$

$2B = 0$

$B = 0$

or other correct method.

ie. $A=1$, $B=0$, $C=-2$ (1)

Additional writing space on the back page

Student Number

11) $\int \frac{x^2+1}{x^3+2x^2+x}\,dx = \int \frac{1}{x} - \frac{2}{(x+1)^2}\,dx$

$= \ln|x| - 2\int (x+1)^{-2}\,dx$ ①

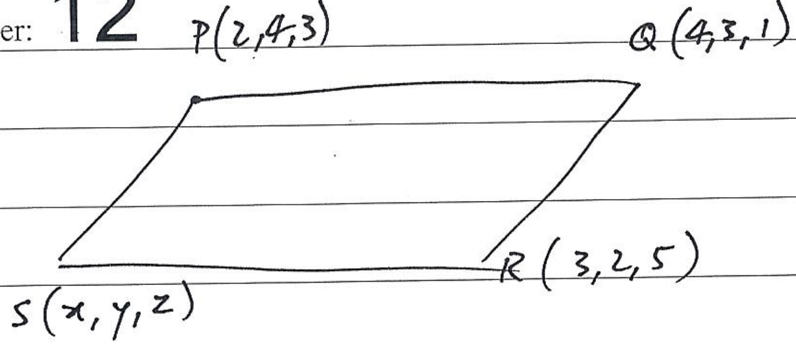
$= \ln|x| - 2\frac{(x+1)^{-1}}{-1} + c$

$= \ln|x| + \frac{2}{x+1} + c$ ①

Tick this box if you have continued this answer in another writing booklet

Start here for Question Number: 12

a.



① some sensible method.

$x = 3-2$ $\quad$ $y = 2+1$ $\quad$ $z = 5+2$

$= 1$ $\quad$ $= 3$ $\quad$ $= 7$

$\therefore$ S is $(1, 3, 7)$ ①

b. $3 + 2\lambda = 2$

$2\lambda = -1$

$\lambda = -\frac{1}{2}$ ① find λ

Confirm that $\lambda = -\frac{1}{2}$ works for y, z coords

$-2 + -\frac{1}{2} \times 6 = -2 - 3$

$= -5$

$\therefore$ correct for y coord

① confirmation or, find λ for all 3.

$5 - \frac{1}{2} \times 4 = 5 - 2$

$= 3$

$\therefore$ correct for z coord

$\therefore (2, -5, 3)$ lies on line l.

c. Consider $\frac{x}{x^2+9} - \frac{1}{6} = \frac{6x - (x^2+9)}{6(x^2+9)}$ (1)

$= \frac{6x - x^2 - 9}{6(x^2+9)}$

$= \frac{-(x^2 - 6x + 9)}{6(x^2+9)}$

$= \frac{-(x-3)^2}{6(x^2+9)}$

≤ 0 since $(x-3)^2 \geq 0$ (1)

and $6(x^2+9) > 0$

ie $\frac{x}{x^2+9} - \frac{1}{6} \leq 0$

$\therefore \frac{x}{x^2+9} \leq \frac{1}{6}$

OR

$(x-3)^2 \geq 0$ (1)

$x^2 - 6x + 9 \geq 0$

$x^2 + 9 \geq 6x$

$\frac{1}{6} \geq \frac{x}{x^2+9}$ since $x^2+9 > 0$ (1)

d. 1) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ (1)

$= 2\cos n\theta$

Additional writing space on back page.

11) Consider $\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$ (1)

ie $(2\cos\theta)^4 = \left(z^4+\frac{1}{z^4}\right) + 4\left(z^2+\frac{1}{z^2}\right) + 6$ (1)

$16\cos^4\theta = 2\cos 4\theta + 4 \times 2\cos 2\theta + 6$

$\cos^4\theta = \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}$ (1)

$= \frac{\cos 4\theta + 4\cos 2\theta + 3}{8}$

e. $\int \frac{\cos x}{\sin^2 x + 2\sin x + 5}\,dx = \int \frac{\cos x}{(\sin x+1)^2 + 4}\,dx$

let $u = \sin x + 1$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$\therefore I = \int \frac{1}{u^2+4}\,du$ (1)

$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{\sin x+1}{2}\right) + c$ (1)

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Start here for Question Number: 12

f. If P then Q

Converse: If Q then P ①

ie. if n is even then $5n^4+8n$ is a mult. of 16

let $n = 2P$ since n is even, where $P \in J$

$\therefore 5n^4 + 8n = 5(2P)^4 + 8(2P)$ ①

$= 5 \times 16P^4 + 16P$

$= 16(5P^4 + P)$

$= 16Q$ where $Q = 5P^4 + P$ and

$Q \in J$ since $P \in J$ ①

$\therefore$ If n is even then $5n^4 + 8n$ is a multiple of 16

Start here for Question Number: 13

a. $\underset{\sim}{b} \cdot \underset{\sim}{c} = |\underset{\sim}{b}||\underset{\sim}{c}| \cos\theta$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = 1 \times 4 + 8 \times -2 + 3 \times 4$

$= 4 - 16 + 12$

$= 0$ (1)

ie $|\underset{\sim}{b}||\underset{\sim}{c}| \cos\theta = 0$

$\therefore \cos\theta = 0$ since $|\underset{\sim}{b}|, |\underset{\sim}{c}| \neq 0$

$\therefore \theta = \frac{\pi}{2}$

(1)

ie $\angle CAB$ is a right angle

b. $\displaystyle\int \frac{1}{\sqrt{6-5x-x^2}}\,dx = \int \frac{1}{\sqrt{6-\left(x^2+5x+\left(\frac{5}{2}\right)^2\right)+\frac{25}{4}}}\,dx$ (1)

$\displaystyle = \int \frac{1}{\sqrt{\frac{49}{4}-\left(x+\frac{5}{2}\right)^2}}\,dx$

$\displaystyle = \int \frac{1}{\sqrt{\left(\frac{7}{2}\right)^2-\left(x+\frac{5}{2}\right)^2}}\,dx$ (1)

$\displaystyle = \sin^{-1}\left(\frac{x+\frac{5}{2}}{\frac{7}{2}}\right) + c$

$\displaystyle = \sin^{-1}\left(\frac{2x+5}{7}\right) + c$ (1)

c. since $l_1 \perp l_2$, direction vector of l_2 will be $\begin{pmatrix}1\\-2\end{pmatrix}$ (1)

Also, l_2 must pass through $\begin{pmatrix}2\\2\end{pmatrix}$

$\therefore \ \underset{\sim}{r}_2 = \begin{pmatrix}2\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-2\end{pmatrix}$ (1)

find endpoints:

when $x=0$, $2+\mu = 0$

$\mu = -2$

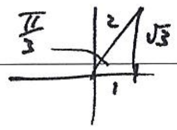
$y=0$, $2-2\mu = 0$

$2\mu = 2$

$\mu = 1$

$\therefore \ \underset{\sim}{r}_2 = \begin{pmatrix}2\\2\end{pmatrix} + \mu\begin{pmatrix}1\\-2\end{pmatrix}$ for $-2 \le \mu \le 1$ (1)

d. let $z^4 = 1+\sqrt{3}\,i$



$r^4 \operatorname{cis} 4\theta = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$ (1)

$\therefore \ r^4 = 2$ and $4\theta = \frac{\pi}{3} + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

$r = \sqrt[4]{2}$

$= \frac{\pi + 6k\pi}{3}$

$\theta = \frac{\pi(1+6k)}{12}$ (1)

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when $k=0$, $z_1 = \sqrt[4]{2}\, e^{i\frac{\pi}{12}}$ spacing $\frac{2\pi}{4} = \frac{\pi}{2} = \frac{6\pi}{12}$

$z_2 = \sqrt[4]{2}\, e^{i\frac{7\pi}{12}}$

$z_3 = \sqrt[4]{2}\, e^{-i\frac{11\pi}{12}}$

$z_4 = \sqrt[4]{2}\, e^{-i\frac{5\pi}{12}}$ 1

e. for $\underset{\sim}{r}$, $x = 3+2\lambda$, $y = 2-\lambda$, $z = -1-2\lambda$

$\therefore (3+2\lambda-3)^2 + (2-\lambda-2)^2 + (-1-2\lambda+1)^2 = 36$ 1

$4\lambda^2 + \lambda^2 + 4\lambda^2 = 36$

$9\lambda^2 = 36$

$\lambda = \pm 2$ 1

when $\lambda = 2$, $x = 7$, $y = 0$, $z = -5$

$\lambda = -2$, $x = -1$, $y = 4$, $z = 3$

ie Pts of intersection are

$(7, 0, -5)$ and $(-1, 4, 3)$ 1

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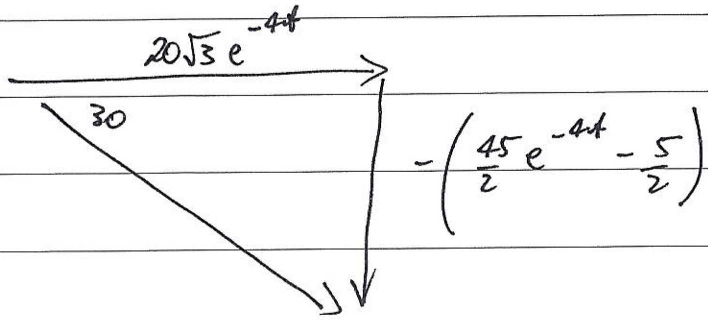
Start here for Question Number: **14**

a. i) $\underset{\sim}{r}(t) = \begin{pmatrix} 5\sqrt{3} - 5\sqrt{3}e^{-4t} \\ \frac{45}{8} - \frac{45}{8}e^{-4t} - \frac{5t}{2} \end{pmatrix}$

$\therefore \quad \underset{\sim}{v}(t) = \begin{pmatrix} 20\sqrt{3}e^{-4t} \\ \frac{45}{2}e^{-4t} - \frac{5}{2} \end{pmatrix}$ (1)

ii) 

$\therefore \tan 30 = \dfrac{\frac{5}{2} - \frac{45}{2}e^{-4t}}{20\sqrt{3}e^{-4t}}$ (1)

$\dfrac{20\sqrt{3}e^{-4t}}{\sqrt{3}} = \frac{5}{2} - \frac{45}{2}e^{-4t}$

$40e^{-4t} = 5 - 45e^{-4t}$

$85e^{-4t} = 5$

$e^{-4t} = \frac{1}{17}$

$-4t = \ln\left(\frac{1}{17}\right)$

$t = \dfrac{\ln\left(\frac{1}{17}\right)}{-4}$

$= \dfrac{-\ln(17)}{-4}$

$= \frac{1}{4}\ln(17)$

$= \ln(17)^{\frac{1}{4}}$

$= \ln\left(\sqrt[4]{17}\right)$ 1

b. l_1 and l_2 are parallel if direction vectors are in proportion.

ie $\begin{pmatrix} 2^k \\ 6-2^k \\ 7-2^k \end{pmatrix} = c\begin{pmatrix} 8+2^k \\ 6 \\ 5+2^k \end{pmatrix}$ where c is a constant 1

$\therefore\ 2^k = c(8+2^k)$ and $6-2^k = 6c$

$c = \dfrac{2^k}{8+2^k}$ $\qquad c = \dfrac{6-2^k}{6}$

$\therefore\ \dfrac{2^k}{8+2^k} = \dfrac{6-2^k}{6}$

$6\times2^k = 48 - 8\times2^k + 6\times2^k - (2^k)^2$

$(2^k)^2 + 8\times2^k - 48 = 0$

$(2^k-4)(2^k+12) = 0$

$\therefore\ 2^k = 4$ or $2^k = -12$

$k = 2$ $\qquad$ which is not possible

$\therefore\ k = 2$ only 1

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c. i) Since coefficients are real, $3-i$ is also a root. (1)

Let the roots be $3+i$, $3-i$, α, β

Now, $\alpha + \beta + 3+i + 3-i = 8$
$\alpha+\beta = 2$

and $\alpha\beta(3+i)(3-i) = -80$
$\alpha\beta(9+1) = -80$
$\alpha\beta = -8$ (1)

Form a quadratic in z with roots, α, β

$z^2 - 2z - 8 = 0$
$(z+2)(z-4) = 0$
$z = -2$ or $z = 4$

$\therefore$ The roots are $4, -2, 3+i, 3-i$ (1)

ii) $\therefore P(z) = (z+2)(z-4)(z-(3+i))(z-(3-i))$
$= (z+2)(z-4)(z^2-6z+10)$ (1)

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Start here for Question Number: **14**

d. i) First missile:

$$t = \frac{x}{40\cos\alpha}$$

$$\therefore y = \frac{-g}{2}\left(\frac{x^2}{1600\cos^2\alpha}\right) + 40\sin\alpha \times \frac{x}{40\cos\alpha} \quad 1$$

$$= \frac{-gx^2}{3200\cos^2\alpha} + x\tan\alpha$$

$$= x\tan\alpha - \frac{gx^2}{3200}\sec^2\alpha \quad 1$$

Similarly, for 2nd missile

$$y = x\tan\beta - \frac{gx^2}{3200}\sec^2\beta$$

ii) When missiles hit target, $x = 80, y = 55$

$$\therefore 55 = 80\tan\alpha - \frac{10 \times 6400}{3200}(1 + \tan^2\alpha) \quad 1$$

$$55 = 80\tan\alpha - 20 - 20\tan^2\alpha$$

$$20\tan^2\alpha - 80\tan\alpha + 75 = 0$$

$$4\tan^2\alpha - 16\tan\alpha + 15 = 0$$

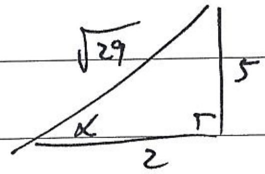
$$(2\tan\alpha - 3)(2\tan\alpha - 5) = 0$$

$$\tan\alpha = \frac{3}{2} \quad \text{or} \quad \tan\alpha = \frac{5}{2}$$

and, since $\alpha > \beta$

$\tan\alpha = \frac{5}{2}$ and $\tan\beta = \frac{3}{2}$ ①

iii) $\tan\alpha = \frac{5}{2}$



$\therefore \cos\alpha = \frac{2}{\sqrt{29}}$ ①

Similarly, $\cos\beta = \frac{2}{\sqrt{13}}$

Missile 1:

Time of flight

$t = \dfrac{80}{40 \times \frac{2}{\sqrt{29}}}$

$= \sqrt{29}$

Missile 2:

Time of flight

$t = \dfrac{80}{40 \times \frac{2}{\sqrt{13}}}$

$= \sqrt{13}$

$\therefore$ time between firing is $\sqrt{29} - \sqrt{13}$ sec. ①

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Start here for Question Number: **15**

a. $$I_n = \int_0^1 x^2(1-x^2)^n\,dx$$

$$= -\frac{1}{2}\int_0^1 x \times -2x(1-x^2)^n\,dx$$

$$= -\frac{1}{2}\int_0^1 x \times \frac{d\left(\frac{(1-x^2)^{n+1}}{n+1}\right)}{dx}\,dx \quad (1)$$

$$= -\frac{1}{2}\left[\frac{x(1-x^2)^{n+1}}{n+1}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{(1-x^2)^{n+1}}{n+1}\,dx$$

$$= 0 + \frac{1}{2(n+1)}\int_0^1 (1-x^2)^{n+1}\,dx \quad (1)$$

$$= \frac{1}{2(n+1)} J_{n+1}$$

OR

$$J_{n+1} = \int_0^1 (1-x^2)^{n+1}\,dx$$

$$= \int_0^1 \frac{d(x)}{dx}(1-x^2)^{n+1}\,dx \quad (1)$$

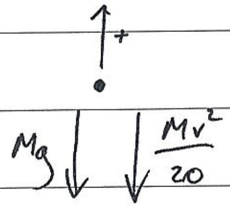
$$= \left[x(1-x^2)^{n+1}\right]_0^1 - \int_0^1 x(n+1)(1-x^2)^n \times -2x\,dx$$

$$= 0 + 2(n+1)\int_0^1 x^2(1-x^2)^n\,dx$$

$$= 2(n+1)I_n \quad (1)$$

$$\therefore \frac{1}{2(n+1)}J_{n+1} = I_n$$

b. i)



$$\sum F = m\ddot{x} = -Mg - \frac{Mv^2}{20}$$

$$\therefore 1 \times \ddot{x} = -1 \times 10 - \frac{1v^2}{20}$$

$$\ddot{x} = -10 - \frac{v^2}{20} \quad 1$$

$$= -\left(10 + \frac{v^2}{20}\right)$$

ii)

$$v\frac{dv}{dx} = -\left(\frac{200 + v^2}{20}\right)$$

$$\frac{dv}{dx} = \frac{-(200 + v^2)}{20v}$$

$$\frac{20v}{200 + v^2}\,dv = -dx$$

$$\int_{40}^{0} \frac{2v}{200 + v^2}\,dv = \frac{-1}{10}\int_{0}^{H} dx \quad 1$$

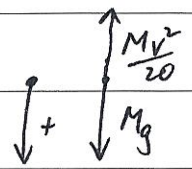
$$\Big[\ln|200 + v^2|\Big]_{40}^{0} = \frac{-1}{10}\Big[x\Big]_{0}^{H}$$

$$\ln 200 - \ln 1800 = \frac{-1}{10} \times H$$

$$\ln\left(\frac{1800}{200}\right) = \frac{H}{10}$$

$$H = 10 \ln 9 \quad (1)$$

iii)



$$\ddot{x} = 10 - \frac{v^2}{20}$$

$$v\frac{dv}{dx} = \frac{200 - v^2}{20} \quad (1)$$

$$\frac{20v}{200 - v^2}\,dv = dx$$

$$\int_0^v \frac{-2v}{200 - v^2}\,dv = \frac{-1}{10}\int_0^H dx$$

$$\Big[\ln|200 - v^2|\Big]_0^v = \frac{-1}{10}\Big[x\Big]_0^H \quad (1)$$

$$\ln|200 - v^2| - \ln 200 = -\frac{H}{10}$$

$$\ln|200 - v^2| = \ln 200 - \frac{10\ln 9}{10}$$

$$= \ln\left(\frac{200}{9}\right)$$

$$\therefore \quad 200 - v^2 = \frac{200}{9}$$

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Start here for Question Number: 11

$$v^2 = 200 - \frac{200}{9}$$

$$= \frac{1600}{9}$$

$$v = \frac{40}{3} \quad \text{since } v > 0 \quad 1$$

ie hits ground at $\frac{1}{3}$ of launch speed.

c. i) Consider $\dfrac{2k}{k+2} - \dfrac{2k+2}{k+3} = \dfrac{2k(k+3) - (2k+2)(k+2)}{(k+2)(k+3)}$

$$= \frac{2k^2 + 6k - (2k^2 + 6k + 4)}{(k+2)(k+3)}$$

$$= \frac{-4}{(k+2)(k+3)}$$

$$< 0 \quad \text{since } k > 0 \quad 1$$

ie $\dfrac{2k}{k+2} - \dfrac{2k+2}{k+3} < 0$

$$\therefore \frac{2k}{k+2} < \frac{2k+2}{k+3}$$

ii) Prove true when $n = 1$

$LHS = \dfrac{1}{3!}$ $\qquad$ $RHS = \dfrac{2 \times 1}{1+2} - \dfrac{1}{(1+2)!}$

$= \dfrac{1}{6}$ $\qquad$ $= \dfrac{2}{3} - \dfrac{1}{6}$

$= \frac{1}{2}$

$\frac{1}{6} < \frac{1}{2}$ (1)

$\therefore$ true for $n=1$

Assume true for $n=k$

ie. $\frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \dots + \frac{k}{(k+2)!} \leq \frac{2k}{k+2} - \frac{1}{(k+2)!}$

Prove true for $n=k+1$

ie. RTP: $\frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \dots + \frac{k}{(k+2)!} + \frac{(k+1)}{(k+3)!} < \frac{2k+2}{(k+3)} - \frac{1}{(k+3)!}$

$LHS = \frac{1}{3!} + \frac{2}{4!} + \frac{3}{5!} + \dots + \frac{k}{(k+2)!} + \frac{(k+1)}{(k+3)!}$

$< \frac{2k}{k+2} - \frac{1}{(k+2)!} + \frac{(k+1)}{(k+3)!}$ from assumption (1)

$< \frac{2k}{k+2} + \frac{(k+1)}{(k+3)!} - \frac{1}{(k+2)!} \times \frac{k+3}{k+3}$

$< \frac{2k}{k+2} + \frac{k+1-(k+3)}{(k+3)!}$

$< \frac{2k+2}{k+3} - \frac{2}{(k+3)!}$ from part (1)

$< \frac{2k+2}{k+3} - \frac{1}{(k+3)!}$ since $\frac{2}{(k+3)!} > \frac{1}{(k+3)!}$ (1)

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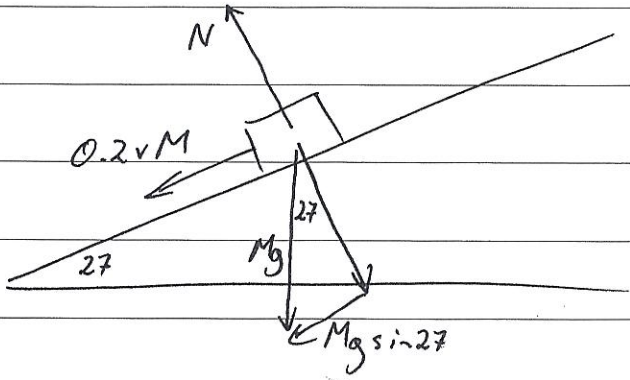
$\therefore$ true for $n=k+1$

$\therefore$ by induction, the statement is true.

d.



In direction of slope:

$$\sum F = M\ddot{x} = -0.2vM - Mg\sin 27$$

$$\ddot{x} = -0.2v - 10\sin 27$$

$$\frac{dv}{dt} = \frac{-v}{5} - 10\sin 27 \quad (1)$$

$$= \frac{-(v+50\sin 27)}{5}$$

$$\frac{dv}{v+50\sin 27} = \frac{-1}{5}\,dt$$

$$\int_v^0 \frac{1}{v+50\sin 27}\,dv = \frac{-1}{5}\int_0^6 dt$$

$$\left[\ln|v+50\sin 27|\right]_v^0 = \frac{-1}{5}\left[t\right]_0^6 \quad (1)$$

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Start here for Question Number: 13

$$\ln(50\sin 27) - \ln(u + 50\sin 27) = \frac{-6}{5}$$

$$\ln\left(\frac{u + 50\sin 27}{50\sin 27}\right) = \frac{6}{5}$$

$$\frac{u + 50\sin 27}{50\sin 27} = e^{\frac{6}{5}}$$

$$u = 50\sin 27 \times e^{\frac{6}{5}} - 50\sin 27$$

$$= 52.66555206$$

$$= 52.67 \text{ m/s } (2dp) \qquad 1$$

Start here for Question Number: 16

a. Assume that

$$\sqrt{a} + \sqrt{a+2} \leq \sqrt{a+8}$$

$$\left(\sqrt{a} + \sqrt{a+2}\right)^2 \leq a+8 \quad (1)$$

$$a + 2\sqrt{a^2+2a} + a+2 \leq a+8$$

$$2a + 2\sqrt{a^2+2a} + 2 \leq a+8$$

$$a + 2\sqrt{a^2+2a} \leq 6$$

Now, proof is for all $a \geq 2$ and smallest value of $a + 2\sqrt{a^2+2a}$ will occur when $a=2$ (1)

When $a=2$, $a + 2\sqrt{a^2+2a} = 2 + 4\sqrt{2}$

which leads to $2 + 4\sqrt{2} \leq 6$

$$4\sqrt{2} \leq 4$$

$$\sqrt{2} \leq 1 \quad (1)$$

Which is a contradiction

$$\therefore \sqrt{a} + \sqrt{a+2} \not\leq \sqrt{a+8}$$

$$\therefore \sqrt{a} + \sqrt{a+2} > \sqrt{a+8}$$

b. $$\int \frac{(2\tan\theta + 3)\sec^2\theta}{\sec^2\theta + \tan\theta}\,d\theta$$

$$= \int \frac{2\tan\theta\sec^2\theta + 3\sec^2\theta}{\sec^2\theta + \tan\theta}\,d\theta$$

$$= \int \frac{2\tan\theta\sec^2\theta + \sec^2\theta}{\sec^2\theta + \tan\theta}\,d\theta + 2\int \frac{\sec^2\theta}{\sec^2\theta + \tan\theta}\,d\theta \quad (1)$$

$$= \ln\left|\sec^2\theta + \tan\theta\right| + 2\int \frac{\sec^2\theta}{\tan^2\theta + \tan\theta + 1}\, d\theta$$

For $I = 2\int \dfrac{\sec^2\theta}{\tan^2\theta + \tan\theta + 1}\, d\theta$ let $u = \tan\theta$

$$\frac{du}{d\theta} = \sec^2\theta$$

$$du = \sec^2\theta\, d\theta$$

$$\therefore I = 2\int \frac{1}{u^2 + u + 1}\, du \qquad 1$$

$$= 2\int \frac{1}{\left(u^2 + u + \frac{1}{4}\right) + \frac{3}{4}}\, du$$

$$= 2\int \frac{1}{\left(u + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, du$$

$$= 2 \times \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{4\sqrt{3}}{3} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) + C$$

$$= \frac{4\sqrt{3}}{3} \tan^{-1}\left(\frac{2\tan\theta + 1}{\sqrt{3}}\right) + C$$

$$\therefore \int \frac{(2\tan\theta + 3)\sec^2\theta}{\sec^2\theta + \tan\theta}\, d\theta = \ln\left|\sec^2\theta + \tan\theta\right|$$

$$+ \frac{4\sqrt{3}}{3} \tan^{-1}\left(\frac{2\tan\theta + 1}{\sqrt{3}}\right) + C \qquad 1$$

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c. i) Using AM-GM

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geqslant \sqrt[3]{\frac{1}{a}\times\frac{1}{b}\times\frac{1}{c}}$$ (1)

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant \frac{3}{\sqrt[3]{abc}}$$

Also, $\frac{k}{3} \geqslant \sqrt[3]{abc}$ (1)

$\frac{3}{k} \leq \frac{1}{\sqrt[3]{abc}}$ since all terms are positive

$\frac{9}{k} \leq \frac{3}{\sqrt[3]{abc}}$ (1)

$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant \frac{9}{k}$

ii) since $k=1$, $a+b+c=1$

and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant 9$

Now, $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) = \frac{(1-a)}{a} \times \frac{(1-b)}{b} \times \frac{(1-c)}{c}$ (1)

$$= \frac{(1-a)(1-b-c+bc)}{abc}$$

$$= \frac{1-b-c+bc-a+ab+ca-abc}{abc}$$

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$$= \frac{1-(a+b+c)+(ab+bc+ca)-abc}{abc} \qquad 1$$

$$= \frac{1-1+ab+bc+ca-abc}{abc}$$

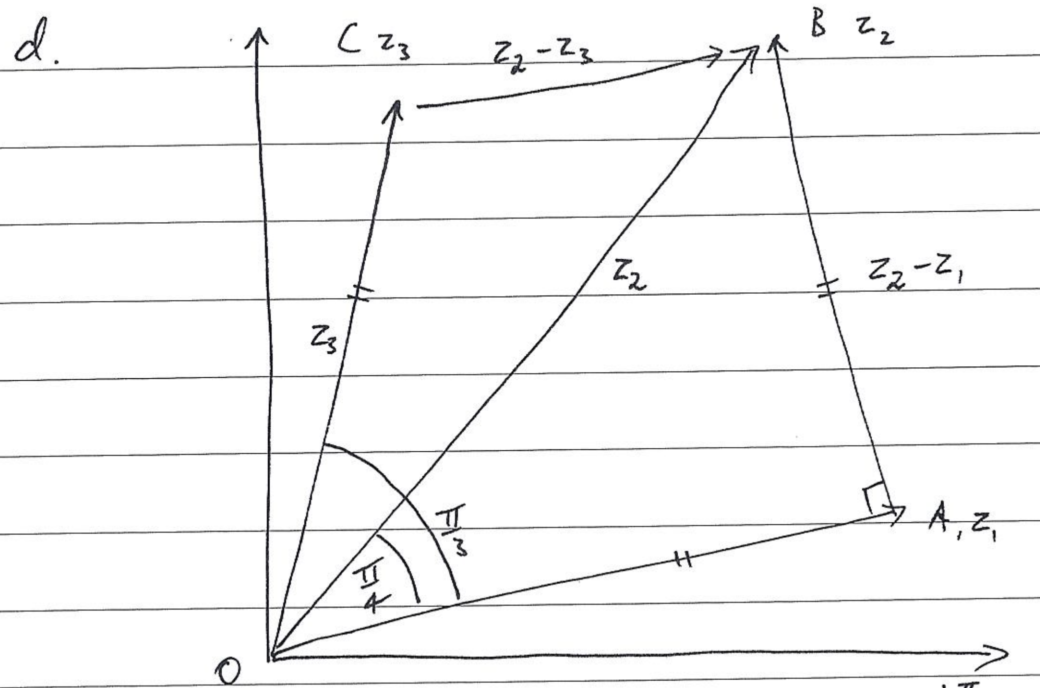
$$= \frac{ab+bc+ca}{abc} - \frac{abc}{abc}$$

$$= \frac{1}{c}+\frac{1}{a}+\frac{1}{b} - 1$$

$$\geqslant 9-1 \qquad \text{since } \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant 9 \qquad 1$$

$$\geqslant 8$$

d.



$$z_2 - z_1 = i z_1 \qquad\qquad z_3 = e^{i\frac{\pi}{3}} z_1$$

$$z_2 = z_1 + i z_1 \qquad\qquad = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) z_1$$

$$= (1+i) z_1 \qquad\qquad = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) z_1$$

Since $z_2 - z_1 = iz_1$,

$|z_2 - z_1| = |z_1|$

and so ΔOAB is right angled at A and isosceles.

$\therefore \angle AOB = \frac{\pi}{4}$

Since $z_3 = e^{i\frac{\pi}{3}} z_1$

$\angle AOC = \frac{\pi}{3}$

and $|z_3| = |z_1|$

① either

$\therefore \angle BOC = \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ ①

Using the cosine rule in ΔOBC

$$\cos\frac{\pi}{12} = \frac{|z_3|^2 + |z_2|^2 - |z_2 - z_3|^2}{2 \times |z_3| \times |z_2|}$$

$$= \frac{|z_1|^2 + (\sqrt{2}|z_1|)^2 - |z_2 - z_3|^2}{2 \times |z_1| \times \sqrt{2}|z_1|}$$

Now, $z_2 - z_3 = (1+i)z_1 - \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)z_1$

$= \left(\frac{1}{2} + \left(1 - \frac{\sqrt{3}}{2}\right)i\right)z_1$ ①

$$|z_2 - z_3| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(1 - \tfrac{\sqrt{3}}{2}\right)^2} \times |z_1|$$

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$$= \sqrt{\frac{1}{4} + 1 - \sqrt{3} + \frac{3}{4}} \times |z_1|$$

$$= \sqrt{2 - \sqrt{3}} \times |z_1|$$

$$\therefore \cos\frac{\pi}{12} = \frac{|z_1|^2\left[1 + 2 - (2 - \sqrt{3})\right]}{|z_1|^2 \times 2\sqrt{2}} \quad (1)$$

$$= \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

☐ ⇦ Tick this box if you have continued this answer in another writing booklet.