Student Number:
Teacher:

St George Girls High School

Mathematics Extension 2

2025 HSC Trial Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I**, use the multiple-choice answer sheet provided

  For questions in **Section II:**
  - Answer the questions in the booklets provided
  - Start each question in a new booklet
  - Show relevant mathematical reasoning and/or calculations
  - Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks : 100

Section I – 10 marks (pages 3 – 7)

- Attempt Questions 1– 10.
- Allow about 15 minutes for this section.

Section II – 90 marks (pages 8 – 12)

- Attempt Questions 11 – 16.
- Allow about 2 hours and 45 minutes for this section.

Q1 - 10	/10
Q11	/15
Q12	/15
Q13	/15
Q14	/15
Q15	/15
Q16	/15
TOTAL	**/100**

Section I - 10 marks

Attempt Questions 1 – 10
Allow about 15 minutes for this section.
Use the multiple-choice answer sheet provided for Questions 1 to 10.

1. Which complex number is a square root of $7 + 24i$?

   A. $-3 - 4i$

   B. $4 - 3i$

   C. $-4 - 3i$

   D. $-4 + 3i$

2. The sign on a fence reads:

   "If you are not authorised, you may not enter."

   Which of the following statements is logically equivalent to the one above?

   A. If you may enter, then you are authorised

   B. If you are authorised, then you may enter

   C. If you are not authorised, then you may enter

   D. If you may not enter, then you are not authorised

3. Evaluate $\left(\sqrt{3} + i\right)^8$.

   A. $128 - 128i\sqrt{3}$

   B. $128 + 128i\sqrt{3}$

   C. $256(\cos 120° + i \sin 120°)$

   D. $256\left(\cos\left(\frac{-2\pi}{3}\right) + i \sin\left(\frac{-2\pi}{3}\right)\right)$

4. Which of these algebraic fractions **cannot** be written as partial fractions in the form

$$\frac{a}{x+b}+\frac{c}{x-d}$$

where a, b, c, and d are real numbers?

A. $\dfrac{2x+6}{x^2+6x+5}$

B. $\dfrac{2x+4}{x^2-4x+8}$

C. $\dfrac{2x+2}{x^2+2x-3}$

D. $\dfrac{2x-8}{x^2-8x+16}$

5. Consider the complex number $z = a + ib$ (where $a, b \in \mathbb{R}, a \neq 0, b \neq 0$), and its complex conjugate $\bar{z}$.

Which row of the table correctly classifies the sum, difference, product and quotient of z and $\bar{z}$?

	$z+\bar{z}$	$z-\bar{z}$	$z\bar{z}$	$z \div \bar{z}$
A.	Real	Imaginary	Real	Imaginary
B.	Complex	Real	Complex	Real
C.	Real	Imaginary	Real	Complex
D.	Real	Real	Imaginary	Complex

6. The equation of the sphere with centre $(4, b, -1)$ is $x^2 - 8x + y^2 + 4y + z^2 + 2z = 4$. What is the value of b?

A. 2

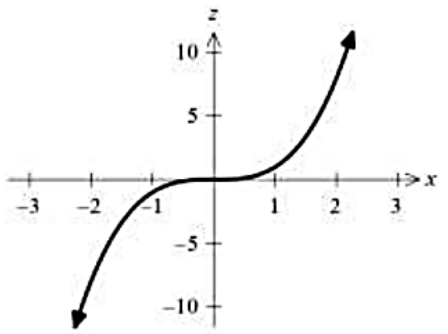
B. -2

C. -4

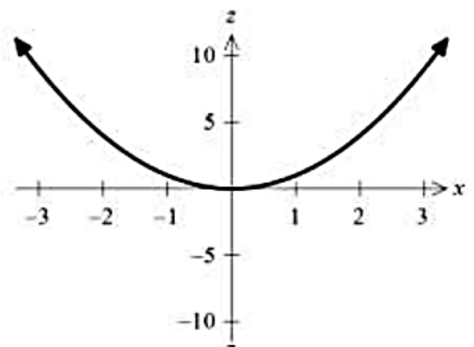
D. -8

7. The curve $r(t) = (t^2, t^4, t^6)$ is defined for all real values of t. Which of the following graphs best represents its projection onto the xz-plane?

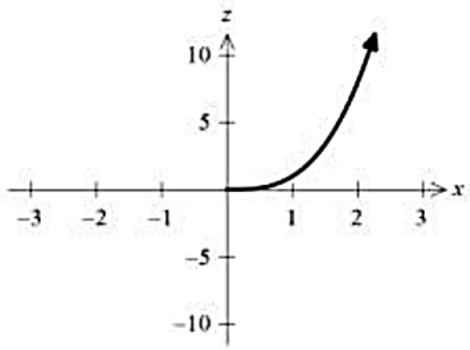
A.



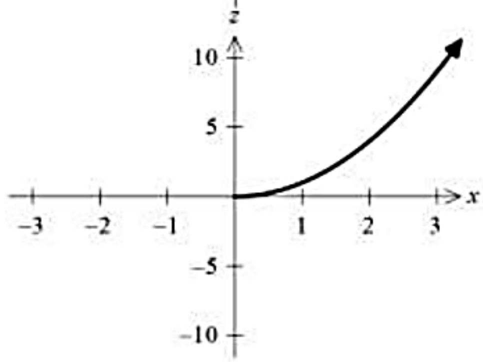
B.



C.



D.



8. A vector makes an angle of $\frac{\pi}{4}$ with the positive direction of the x-axis, and an angle of $\frac{\pi}{4}$ with the positive direction of the y-axis.

What angle does the vector make with the positive direction of the z-axis?

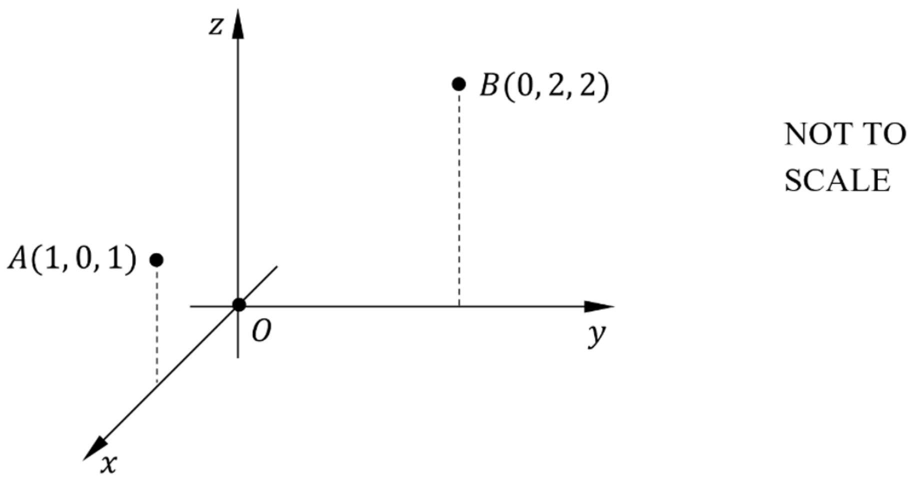
A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{4}$

9. Consider ΔAOB, where A is the point $(1,0,1)$ and B is the point $(0,2,2)$



Which of the following is true?

A. $\angle AOB$ is a right angle.

B. ΔAOB is isosceles.

C. $\angle OAB$ is twice the size of $\angle OBA$.

D. $\angle AOB$ is twice the size of $\angle OBA$.

10. Which of the following is equivalent to the expression $\int \frac{x^2}{(25-x^2)\sqrt{25-x^2}}\,dx$?

A. $\int \frac{\sin^2\theta}{\cos^3\theta}\,d\theta$

B. $\int 5\tan^2\theta\,d\theta$

C. $\int \sqrt{5}\tan^2\theta\,d\theta$

D. $\int \tan^2\theta\,d\theta$

End of Section I

Section II - 90 marks

Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Marks

Question 11 (15 marks) START A NEW WRITING BOOKLET.

(a) (i) Express $\dfrac{3x+11}{(x-3)(x+2)}$ in the form $\dfrac{A}{x-3}-\dfrac{B}{x+2}$. **2**

(ii) Hence find $\displaystyle\int \frac{3x+11}{(x-3)(x+2)}\,dx$. **2**

(b) Find the equation of the line joining $A=\begin{pmatrix}4\\2\\6\end{pmatrix}$ and $B=\begin{pmatrix}-1\\2\\3\end{pmatrix}$. **2**

(c) (i) Show that $(1-2i)^2=-3-4i$. **1**

(ii) Hence solve the equation $z^2-5z+(7+i)=0$. **2**

(d) Consider the complex number $z=x+iy$. Sketch the region in the complex plane defined by $|z-2|\le 3$ and $z+\bar{z}\ge 0$. **3**

(e) Find the vector projection of $\underset{\sim}{v}=4\underset{\sim}{i}+5\underset{\sim}{j}-\underset{\sim}{k}$ onto $\underset{\sim}{u}=\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}$. **3**

Question 12 (15 marks) START A NEW WRITING BOOKLET.

(a) Find the fourth roots of $8 + 8\sqrt{3}i$. **3**

(b) Find $\displaystyle\int \frac{1}{\sqrt{6x - 5 - x^2}}\,dx$. **3**

(c) Use mathematical induction to prove that $n^2 \geq 2n + 1$, for all integers $n \geq 3$. **3**

(d) Find $\displaystyle\int x(1 - x)^7\,dx$. **3**

(e) The lines ℓ_1 and ℓ_2 are given by the equations, **3**

$$\ell_1 : \vec{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ \ln a \\ \ln a \end{pmatrix} \qquad \ell_2 : \vec{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} -3 \\ \ln a \\ 1 \end{pmatrix}$$

where $s, t \in \mathbb{R}$ and a is a constant. Given that ℓ_1 and ℓ_2 are orthogonal (their direction vectors are perpendicular), find the value(s) of a.

Question 13 (15 marks) START A NEW WRITING BOOKLET.

(a) Consider the line $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -7 \\ 3 \end{pmatrix}$. Find the equation of a line that meets $\underset{\sim}{r}$ at right angles. **2**

(b) Find $\displaystyle\int_3^4 \frac{x}{(x - 2)^2(x - 1)}\,dx$. **4**

(c) (i) For any complex numbers $\omega_1 = a + bi$ and $\omega_2 = c + di$, $(a, b, c, d \in \mathbb{R})$, prove that $|\omega_1 + \omega_2|^2 + |\omega_1 - \omega_2|^2 = 2(|\omega_1|^2 + |\omega_2|^2)$ **3**

(ii) What does this result mean geometrically? **1**

(d) Find $\displaystyle\int e^x \sin x\,dx$. **3**

(e) Prove or disprove the following statement: **2**

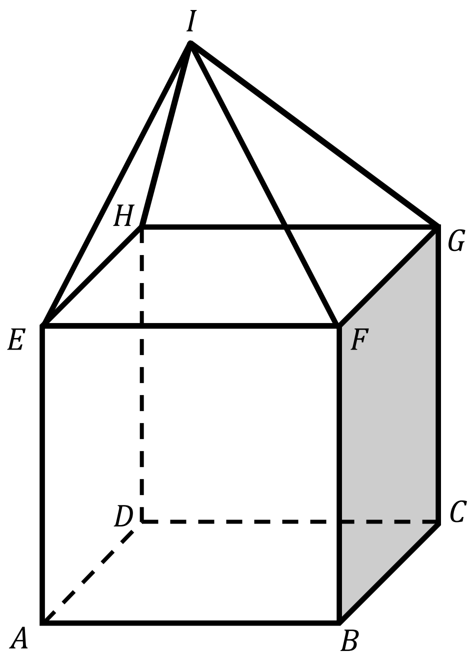
$$\forall x : x = 2k + 1\ (k \in \mathbb{Z}), \exists a, b \in \mathbb{Z} : a^2 - b^2 = x$$

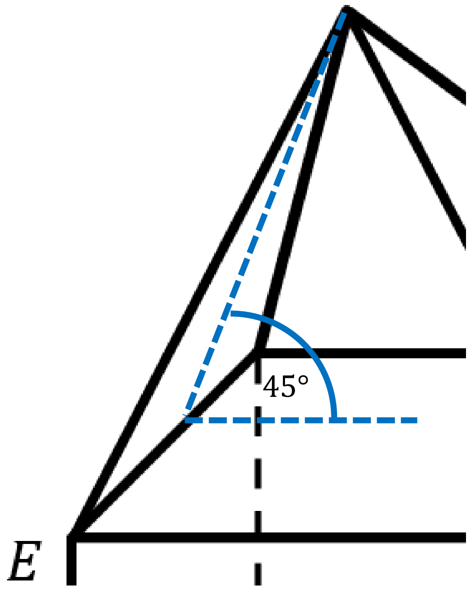
Question 14 (15 marks) START A NEW WRITING BOOKLET.

(a) Factorise the equation $P(z) = z^4 - 6z^3 + 18z^2 - 14z - 39$, given that one of the zeroes of $P(z)$ is $(2 - 3i)$. **3**

(b) Given a series defined by $T_0 = 0$, and $T_n = 2T_{n-1} + 1$ for $n \geq 1$, use mathematical induction to prove that $T_n = 2^n - 1$, for all positive integers n. **3**

(c) The new music building at SGGHS is to be made in the shape of a cube of side length one unit, surmounted by a right square pyramid, as shown. The roof is pitched at 45° – that is, the triangular roof panels meet the flat ceiling at 45°.





Let $\overrightarrow{AB} = \underset{\sim}{b}$, $\overrightarrow{BC} = \underset{\sim}{c}$, and $\overrightarrow{AE} = \underset{\sim}{e}$.

(i) Express $\overrightarrow{FI}$ in terms of $\underset{\sim}{b}$, $\underset{\sim}{c}$, and/or $\underset{\sim}{e}$. **2**

(ii) Hence or otherwise find the size of $\angle FIH$. **2**

(iii) A vertical flagpole is mounted at I, extending up to a point T. The size of the flagpole is chosen such that it is a third of the total height of the building, including the flagpole. Find $\overrightarrow{IT}$ in terms of $\underset{\sim}{e}$. **1**

(d) Find $\displaystyle\int \frac{\sqrt{(x^2-1)^3}}{x}\,dx$, using the substitution $x = sec\theta$, or otherwise. **4**

Question 15 (15 marks) START A NEW WRITING BOOKLET.

(a) State the negation of "if the road is wet, it must have rained". **1**

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, where n is an integer, and $n \geq 3$.

(i) Show that $I_n + I_{n-2} = \frac{1}{n-1}$. **3**

(ii) Hence evaluate I_7. **3**

(c) The points $z_1 = 2(3+i)$, $z_2 = 2(1+i)$ and $\arg\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{6}$.

(i) Let $z = x + iy$, where $x, y \in \mathbb{R}$.

Realise the denominator in the expression $\left(\frac{z-z_1}{z-z_2}\right)$. **2**

(ii) Hence, or otherwise, show that $\left|z - \{4 + 2(1+\sqrt{3})i\}\right| = 4$. **3**

(iii) Sketch the locus on an Argand diagram. **3**

Question 16 (15 marks) START A NEW WRITING BOOKLET.

(a) Let $f(x) = \frac{1+x-x^2}{1-x+x^2}$, where $0 < x < 1$. **3**

Without using calculus, prove that $1 < f(x) \leq \frac{5}{3}$.

(b) Use mathematical induction to prove $3^{2n^2} - 1$ is divisible by 8 ($n \in \mathbb{Z}, n \geq 1$). **4**

(c) The three positive numbers r, s, and t are such that $(r+n)(s+n)(t+n) = n^6$. **3**

Given $(a+b+c)^3 \geq 27abc$ for any three real positive numbers a, b, and c,

show that $rst \leq (n^2 - n)^3$

(d) Consider the equation $S_n = 1 + z + z^2 + z^3 + \cdots + z^n$, where $z = e^{i\theta}$.

(i) Show that $S_n = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$. **1**

(ii) Hence show that **4**

$$\sin(\theta) + \sin(2\theta) + \sin(3\theta) + \cdots + \sin(n\theta) = \frac{\sin\left(\frac{n\theta}{2}\right) \times \sin\left(\frac{n+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$

END OF EXAMINATION

12MXX AT4 2025 MULTIPLE CHOICE MARKER'S COMMENTS

1. $(-4-3i)^2 = 7+24i \quad \therefore C$

2. If P, then Q where P = you are not authorised
Q = you may not enter

Contrapositive is equivalent: If $\neg Q$, then $\neg P$
i.e. If you may enter, you are authorised
$\therefore A$

3. $(\sqrt{3}+i)^8 = 256\,\text{cis}\left(\frac{-2\pi}{3}\right) \quad \therefore D$

4. For option C, $x^2-4x+8 = (x-2)^2+2^2$
cannot be written as the product of two real linear factors $\therefore C$

5. For $z = a+ib$,

$z+\bar{z} = a+ib+a-ib$
$= 2a \quad \therefore$ real

$z-\bar{z} = a+ib-(a-ib)$
$= 2ib \quad \therefore$ imaginary

$z\bar{z} = (a+ib)(a-ib)$
$= a^2+b^2 \quad \therefore$ real

$z \div \bar{z} = \frac{a+ib}{a-ib} \times \frac{a+ib}{a+ib}$
$= \frac{a^2-b^2+2abi}{a^2+b^2} \quad \therefore$ complex $\quad \therefore C$

12MXX AT4 2025 MULTIPLE CHOICE MARKER'S COMMENTS (Continued)

6)

$$x^2-8x+y^2+4y+z^2+2z=4$$

$$x^2-8x+16+y^2+4y+4+z^2+2z+1 = 4+16+4+1$$

$$(x-4)^2+(y+2)^2+(z+1)^2=25$$

$\therefore b=-2$ $\qquad \therefore$ B

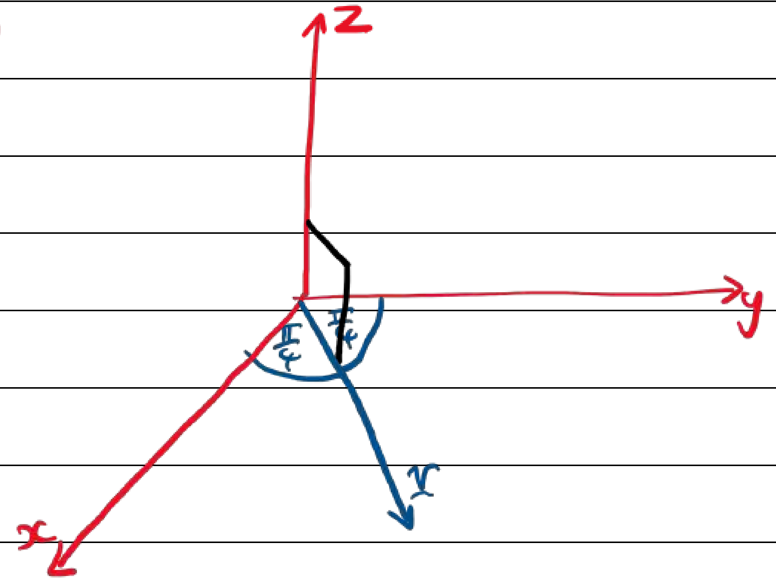
7) For all t, $x \geq 0$, $z \geq 0$ (since $x=t^2$, $z=t^6$)

also $x=t^2$

$z=t^6$

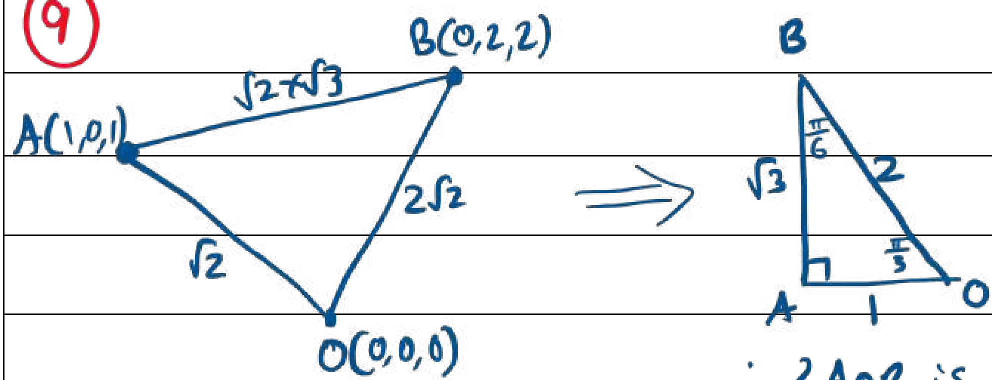
$\therefore z=x^3$ when $x=2$, $z=8$ $\quad \therefore$ C

8)



$\therefore \frac{\pi}{2}$ $\quad \therefore$ C

9)



$\therefore \angle AOB$ is twice $\angle OBA$

$\therefore$ D

12MXX AT4 2025 MULTIPLE CHOICE MARKER'S COMMENTS (Continued)

10

$$\int \frac{x^2}{(25-x^2)\sqrt{25-x^2}}\,dx$$

let $x = 5\sin\theta$

$\frac{dx}{d\theta} = 5\cos\theta$

$dx = 5\cos\theta\,d\theta$

$$= \int \frac{25\sin^2\theta \times 5\cos\theta\,d\theta}{(25-25\sin^2\theta)\sqrt{25-25\sin^2\theta}}$$

$$= \int \frac{25\sin^2\theta\, 5\cos\theta\, d\theta}{25\cos^2\theta\, 5\cos\theta}$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

$$= \int \tan^2\theta\,d\theta$$

$\therefore D$

12MXX AT4 2025 Q11 MARKER'S COMMENTS

a) i $\dfrac{3x+11}{(x-3)(x+2)} \equiv \dfrac{A}{x-3} - \dfrac{B}{x+2}$

$\therefore 3x+11 \equiv A(x+2) - B(x-3)$

when $x=-2$, $-6+11 = -B(-5)$

$5 = 5B$

$\therefore B = 1$

when $x=3$, $9+11 = A(5)$

$20 = 5A$

$\therefore A = 4$

$\therefore \dfrac{3x+11}{(x-3)(x+2)} = \dfrac{4}{x-3} - \dfrac{1}{x+2}$

2 marks Complete solution, including expressing in correct form

$1\frac{1}{2}$ marks Correctly finds A and B

1 mark Correctly finds A or B

A number of students were able to correctly evaluate A and B, but were unable to express the fraction in the requested form. In these cases $1\frac{1}{2}$ marks were awarded, regardless of whether or not the candidate was able to use the correct form in subsequent parts. Read the instructions carefully; give your answer in the requested form. Always.

12MXX AT4 2025 Q11 MARKER'S COMMENTS (continued)

a) ii $\displaystyle\int \frac{3x+11}{(x-3)(x+2)}\,dx = \int \left[\frac{4}{x-3} - \frac{1}{x+2}\right] dx$

$= 4\ln|x-3| - \ln|x+2| + C$

2 marks Complete solution

$1\frac{1}{2}$ marks Correct solution, but missing "+C"

1 mark Correctly finds $4\ln|x-3|$ or $\ln|x+2|$

b) $\begin{pmatrix}4\\2\\6\end{pmatrix} - \begin{pmatrix}-1\\2\\3\end{pmatrix} = \begin{pmatrix}5\\0\\3\end{pmatrix}$

$\therefore\ \underset{\sim}{r} = \begin{pmatrix}4\\2\\6\end{pmatrix} + \lambda\begin{pmatrix}5\\0\\3\end{pmatrix}$

2 marks Complete solution

1 mark Correctly finds direction vector

Many student misread the question and only found the direction vector, not the equation of the line.

c) i $(1-2i)^2 = (1-2i)(1-2i)$

$= 1 - 4i + 4i^2$

$= 1 - 4i - 4$

$= -3 - 4i$

12MXX AT4 2025 Q11 MARKER'S COMMENTS (continued)

c) ii $z^2 - 5z + (7+i) = 0$

$\Delta = b^2 - 4ac$

$= 25 - 4(7+i)$

$= 25 - 28 - 4i$

$= -3 - 4i$

$\therefore \pm\sqrt{\Delta} = \pm(1-2i)$

$z = \dfrac{5 \pm (1-2i)}{2}$

$= \dfrac{5+1-2i}{2}, \dfrac{5-1+2i}{2}$

$= \dfrac{6-2i}{2}, \dfrac{4+2i}{2}$

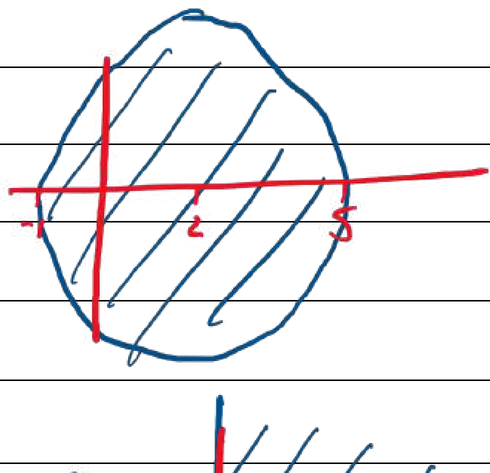
$\therefore z = 3-i, 2+i$

2 marks Complete solution

1 mark Correct substitution into correct formula

d) $|z-2| \leq 3$ ie. distance from 2 is less than or equal to 3

$\therefore$ circle, centred at 2, radius 3 (shade inside)



$z + \bar{z} \geq 0$

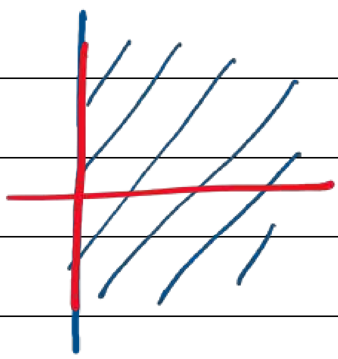
$x + iy + x - iy \geq 0$

$2x \geq 0$

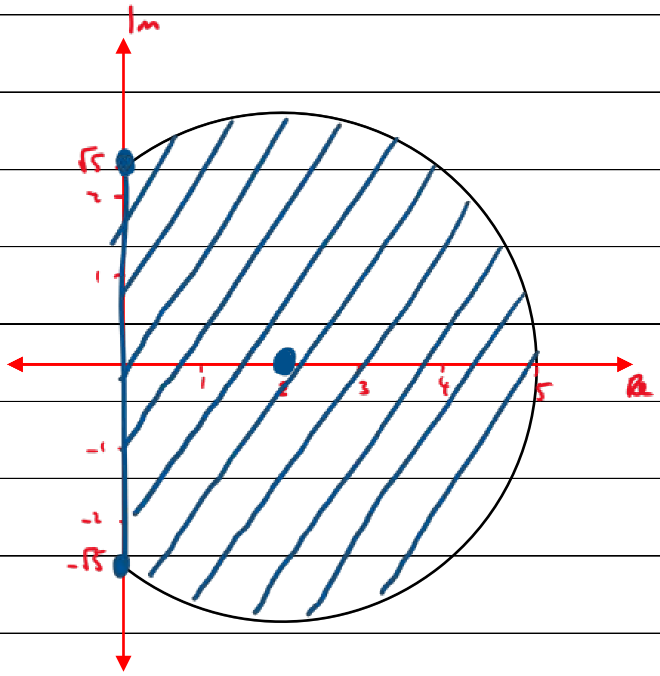
$x \geq 0$

$\therefore$ all positive x-value (and zero)

$\therefore$ non-negative real part



12MXX AT4 2025 Q11 MARKER'S COMMENTS (continued)



3 marks Complete solution

2 marks Correct cirlce, as well as a coherent and logical intersection with a straight line (only awarded for an actual intersection between a half-plane and a circle).

1 mark Correct circle or correctly shades $x \geq 0$

Not well done. Many students could not graph $x \geq 0$.
Also why is $2x \geq 0$ different to $x \geq 0$?
Better answers worked out each region individually first before combining them in a final answer.
Note also that corners should be explicitly included or excluded (using a solid or open circle).

e) $\text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} = \dfrac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}} \underset{\sim}{u}$

$= \dfrac{4-10-3}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$= \dfrac{-9}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

3 marks Complete solution

2 marks Correctly finds $v \cdot u$ and $u \cdot u$

1 mark Correctly finds $v \cdot u$ or $u \cdot u$

Generally very well done!

MARKER'S COMMENTS

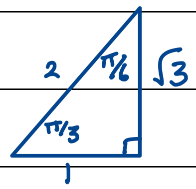
QUESTION 12

a) METHOD 1

$$|8+8\sqrt{3}i| = \sqrt{8^2+(8\sqrt{3})^2}$$
$$= \sqrt{64+64\times 3}$$
$$= \sqrt{256}$$
$$= 16$$

quad 1

$$\arg(8+8\sqrt{3}i) = \tan^{-1}\left(\frac{8\sqrt{3}}{8}\right)$$
$$= \tan^{-1}\sqrt{3}$$
$$= \frac{\pi}{3}$$



$$\therefore\ 8+8\sqrt{3}i = 16\,\text{cis}\,\frac{\pi}{3}$$

Let $z = r\,\text{cis}\,\theta$

$$z^4 = r^4\,\text{cis}\,4\theta = 8+8\sqrt{3}i$$
$$r^4\,\text{cis}\,4\theta = 8+8\sqrt{3}i$$
$$r^4\,\text{cis}\,4\theta = 16\,\text{cis}\,\pi/3$$

(1) correct equation for $r^4\text{cis}4\theta = 16\,\text{cis}\,\frac{\pi}{3}$

$r^4 = 16$ $\quad$ $4\theta = \pi/3$ $\quad$ Equally spaced

$r = 2$ $\quad$ $\theta = \pi/12$ $\quad$ $2\pi \div 4 = \frac{\pi}{2}$

(1/2) correct r value.

(1/2) correct use of equally spaced roots

$$\therefore z = 2\,\text{cis}\,\pi/12,\ 2\,\text{cis}\left(\frac{\pi}{12}+\frac{\pi}{2}\right),\ 2\,\text{cis}\left(\frac{\pi}{12}+\frac{2\pi}{2}\right),\ 2\,\text{cis}\left(\frac{\pi}{12}+\frac{3\pi}{2}\right)$$

$$= 2\,\text{cis}\,\frac{\pi}{12},\ 2\,\text{cis}\,\frac{7\pi}{12},\ 2\,\text{cis}\,\frac{13\pi}{12},\ 2\,\text{cis}\,\frac{19\pi}{12}$$

$$= 2\,\text{cis}\,\frac{\pi}{12},\ 2\,\text{cis}\,\frac{7\pi}{12},\ 2\,\text{cis}\left(\frac{-11\pi}{12}\right),\ 2\,\text{cis}\left(\frac{-5\pi}{12}\right)$$

(1) correct final solution

/3

MARKER'S COMMENTS

$\therefore \quad 8+8\sqrt{3}i = 16 \text{ cis } \frac{\pi}{3}$

Let $\quad z = r \text{ cis } \theta$

$z^4 = r^4 \text{ cis } 4\theta = 8+8\sqrt{3}i$

$r^4 \text{ cis } 4\theta = 8+8\sqrt{3}i$

$r^4 \text{ cis } 4\theta = 16 \text{ cis}\left(\pi/3 + 2k\pi\right)$

$r \text{ cis } \theta = 16^{1/4} \text{ cis}\left(\frac{\pi}{3} + 2k\pi\right)^{1/4}$

$= 2 \text{ cis}\left(\frac{\pi}{12} + \frac{2k\pi}{4}\right)$ or $2 \text{ cis}\left(\frac{\pi + 6k\pi}{12}\right)$

$k = 0, \pm 1, 2$

$\therefore z = 2\text{cis}\,\pi/12,\ 2\text{cis}\left(\frac{\pi}{12}+\frac{\pi}{2}\right),\ 2\text{cis}\left(\frac{\pi}{12}-\frac{\pi}{2}\right),\ 2\text{cis}\left(\frac{\pi}{12}+\pi\right)$

$= 2\text{cis}\frac{\pi}{12},\ 2\text{cis}\frac{7\pi}{12},\ 2\text{cis}\frac{-5\pi}{12},\ 2\text{cis}\frac{13\pi}{12}$

$= 2\text{cis}\frac{\pi}{12},\ 2\text{cis}\frac{7\pi}{12},\ 2\text{cis}\left(\frac{-5\pi}{12}\right),\ 2\text{cis}\left(\frac{-11\pi}{12}\right)$

1 correct equation for $r^4 \text{ cis } 4\theta$ in terms of cis either $\pi/3$ or $\pi/3 + 2k\pi$

½ correct r value

½ correct use of k values.

-1 correct final solution

very small minor errors incurred a ½ mark penalty

MARKER'S COMMENTS

b)

$$\int \frac{1}{\sqrt{6x-5-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2-6x+5)}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2-6x+(\frac{-6}{2})^2+5-(\frac{-6}{2})^2)}}\,dx$$

1 mark to complete the square.

$$= \int \frac{1}{\sqrt{-[(x-3)^2-4]}}$$

1 mark to obtain a correct expression to integrate

$$= \int \frac{1}{\sqrt{4-(x-3)^2}}\,dx$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}} = \sin^{-1}\frac{f(x)}{a}+c$$

$$= \sin^{-1}\left(\frac{x-3}{2}\right)+c$$

$f(x) = x-3$

$f'(x) = 1$

1 correct answer

$\boxed{/3}$

MARKER'S COMMENTS

Q12 cont...

c) STEP 1 - Prove true for $n=3$

$LHS = n^2$ $\quad$ $RHS = 2n+1$

$= 3^2$ $\quad$ $= 2(3)+1$

$= 9$ $\quad$ $= 7$

$9 \geq 7$

$\therefore LHS \geq RHS$

1 - mark to prove base case is true and assumption

STEP 2 - Assume true for $n=k$

$k^2 \geq 2k+1$

STEP 3 Prove true for $n=k+1$

Prove $(k+1)^2 \geq 2(k+1)+1$

Consider $LHS - RHS$.

$LHS - RHS = (k+1)^2 - 2(k+1) - 1$

1 mark correct use of assumption

$= k^2+2k+1-2k-3$

$= k^2 - 2$ $\quad$ $k \geq 3$ $\quad$ $\therefore k^2-2 \geq 0$

≥ 0

or use assumption again

$= k^2 - 2$

$\geq 2k+1-2$

$= 2k-1$

≥ 0

$\therefore LHS - RHS \geq 0$

1 mark to convincingly prove result.

$\therefore (k+1)^2 - 2(k+1) - 1 \geq 0$

$\therefore (k+1)^2 \geq 2(k+1)+1$

$\therefore$ True for $n=k+1$ if true for $n=k$

Some ½ mark penalties for small but crucial errors.

$\therefore$ By the principal of Mathematical Induction, true for all integer $n \geq 3$.

$\boxed{/3}$

MARKER'S COMMENTS

METHOD 2

STEP 3 Prove true for $n=k+1$

Prove $(k+1)^2 \geq 2(k+1)+1$

$LHS = (k+1)^2$

$= k^2+2k+1$

$\geq 2k+1+2k+1$ by the assumption.

$= 2k+2+2k$

$= 2(k+1)+2k$ $k \geq 3 \therefore 2k \geq 1$

$\geq 2(k+1)+1$

$\therefore (k+1)^2 \geq 2(k+1)+1$

etc ...

MARKER'S COMMENTS

Q12 continued . . .

d)
$$\int_2^3 x(1-x)^7\,dx$$

let $u = 1 - x \Rightarrow x = 1 - u$

$\frac{du}{dx} = -1$

$du = -dx$

$dx = -du$

$$= \int (1-u)u^7(-du)$$

$$= -\int (u^7 - u^8)\,du$$

$$= -\left(\frac{u^8}{8} - \frac{u^9}{9}\right) + C$$

$$= \frac{(1-x)^9}{9} - \frac{(1-x)^8}{8} + C$$

1 mark correct u-substitution and derivative relationship.

1 correct integral in terms of u.

1 Correct final answer.

$\boxed{/3}$

MARKER'S COMMENTS

Q12 cont...

e)

$$\begin{pmatrix} 2 \\ \ln a \\ \ln a \end{pmatrix} \cdot \begin{pmatrix} -3 \\ \ln a \\ 1 \end{pmatrix} = 0$$

① mark correct use of dot product of direction vectors equal to zero.

$$-6 + (\ln a)^2 + \ln a = 0$$

let $m = \ln a$

$$m^2 + m - 6 = 0$$

$$(m-2)(m+3) = 0$$

$\therefore m = 2$ or $m = -3$

① mark correct use of methods used to solve an equation reducible to a quadratic.

$\ln a = 2$ $\qquad$ $\ln a = -3$

$e^2 = a$ $\qquad$ $e^{-3} = a$

$a = e^2$ $\qquad$ $a = e^{-3}$

$\therefore a = e^2 , e^{-3}$

½ mark correct answer

½ mark correct answer.

NOTE: Many students disregarded the answer for $\log a = -3$ by incorrectly stating $\log a > 0$.

for $y = \log a$, $a > 0$ NOT $\log a > 0$

12MXX AT4 2025 Q13 MARKER'S COMMENTS

a) To meet at right angles: dot product is zero **AND** lines meet.

$$\begin{pmatrix}1\\-7\\3\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=0$$

$\therefore\ a-7b+3c=0$

try $a=1,\ b=1,\ c=2$

$$\begin{pmatrix}1\\-7\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=1-7+6$$
$$=0$$

lines will meet if they both pass through $\begin{pmatrix}0\\1\\2\end{pmatrix}$

$$\therefore\ \underset{\sim}{r}_2=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$$

2 marks	Complete solution
1 mark	Correctly finds an orthogonal direction vector

Poorly attempted. Many students gave the equation of a plane while others simply found the direction vector. Better responses realised that the question asked for a line, not the line so used that flexibility to make one up, rather than trying to set up a series of equations.

b) let $\dfrac{x}{(x-2)^2(x-1)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

$\therefore \quad x \equiv A(x-2)^2 + B(x-1)(x-2) + C(x-1)$

when $x=2$, $\quad 2=C$

when $x=1$, $\quad 1=A$

when $x=3$, $\quad 3 = A + 2B + 2C$

$\therefore \quad 3 = 1 + 2B + 4$

$2B = -2$

$B = -1$

$\therefore \quad \dfrac{x}{(x-2)^2(x-1)} = \dfrac{1}{x-1} - \dfrac{1}{x-2} + \dfrac{2}{(x-2)^2}$

$\therefore \displaystyle\int_3^4 \dfrac{x}{(x-2)^2(x-1)}\,dx = \int_3^4 \left[\dfrac{1}{x-1} - \dfrac{1}{x-2} + \dfrac{2}{(x-2)^2}\right] dx$

$= \left[\ln|x-1| - \ln|x-2| - \dfrac{2}{(x-2)}\right]_3^4$

$= \left[\ln 3 - \ln 2 - 1 - (\ln 2 - \ln 1 - 2)\right.$

$= \ln 3 - 2\ln 2 + 1$

$= \ln\frac{3}{4} + 1$

4 marks Complete solution

3 marks All correct up to and including correct integration (mistake in evaluating integral)

2 marks Correctly uses partial fraction to find $\frac{1}{x-1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}$

1 mark Correctly finds one of A, B, or C

Mostly well done. Succesful candidates most commonly used the "repeated factor" method shown above, rather than treating $(x-2)^2$ as a quadratic factor.
Care also needed to be taken when evaluating the integral – many students made mistakes with their log statements.

c) i LHS $= |w_1 + w_2|^2 + |w_1 - w_2|^2$

$= |a+bi + c+di|^2 + |a+bi-c-di|^2$

$= |(a+c) + i(b+d)|^2 + |(a-c) + i(b-d)|^2$

$= \left[\sqrt{(a+c)^2 + (b+d)^2}\right]^2 + \left[\sqrt{(a-c)^2 + (b-d)^2}\right]^2$

$= (a+c)^2 + (b+d)^2 + (a-c)^2 + (b-d)^2$

$= a^2 + 2ac + c^2 + b^2 + 2bd + d^2 + a^2 - 2ac + c^2 + b^2 - 2bd + d^2$

$= 2a^2 + 2b^2 + 2c^2 + 2d^2$

$= 2\left[a^2 + b^2 + c^2 + d^2\right]$

$= 2\left[\sqrt{a^2+b^2}^2 + \sqrt{c^2+d^2}^2\right]$

$= 2\left(|a+bi|^2 + |c+di|^2\right)$

$= 2\left(|w_1|^2 + |w_2|^2\right)$

12MXX AT4 2025 Q13 MARKER'S COMMENTS (continued)

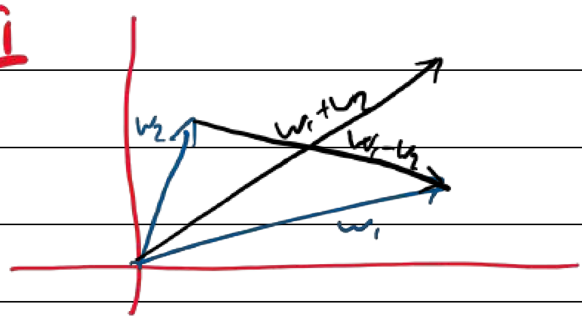
3 marks Complete solution

2 marks Correctly shows most steps

1 mark Correctly shows a useful algebraic transformation

Generally well done. Better responses showed every step. This is a "show" question - don't leave anything out! If you don't show, you won't earn the marks.

c) ii



$|w_1+w_2|$ and $|w_1-w_2|$ are the lengths of the diagonals of a parallelogram.

In a parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of the sides.

Not well done. Many students simply restated the equation, rather than interpreting it geometrically. Without reference to the parallelogram formed, no marks were awarded.

12MXX AT4 2025 Q13 MARKER'S COMMENTS (continued)

d)

let $u = \sin x \quad v = e^x$

$u' = \cos x \quad v' = e^x$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

let $u = \cos x \quad v = e^x$

$u' = -\sin x \quad v' = e^x$

$$= e^x \sin x - \left[e^x \cos x - \int e^x(-\sin x) \, dx \right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\therefore 2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\therefore \int e^x \sin x \, dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x \right] + C$$

3 marks Complete solution
$2\frac{1}{2}$ marks Complete solution, missing "+C"
2 marks Correctly applies integration by parts twice
1 mark Correctly identifies $u = \sin x$ and $v' = e^x$

Mostly very well done (as it should be).
Don't forget "+C" for integration!

e) For all odd x, there exists integers a and b such that $x = a^2 - b^2$

ie all odd numbers are the difference of two squares.

Eg $1 = 1^2 - 0^2$

$3 = 2^2 - 1^2$

$5 = 3^2 - 2^2$

$7 = 4^2 - 3^2$ (seems legit)

k	$2k+1$	a	b
0	1	1	0
1	3	2	1
2	5	3	2
3	7	4	3
4	9	5	4

Pattern seems to be $(2k+1) = (k+1)^2 - k^2$

[Odd numbers are the difference of squares of consecutive integers]

Consider two integers k and $k+1$:

$$(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2$$
$$= 2k+1$$

$\therefore$ for any integer k, $2k+1 = (k+1)^2 - k^2$

$\therefore$ for all x, there exist integers a and b such that $x = a^2 - b^2$

$\therefore$ the statement is true

e) OR

Consider all numbers of the form $2k+1$:

$$2k+1 = (1)(2k+1)$$
$$= (k+1-k)(k+1+k)$$
$$= (k+1)^2 - k^2$$

$\therefore 2k+1$ is the difference of $(k+1)^2$ and k^2, which are both integers.

$\therefore$ the statement is true

2 marks Complete solution

1 mark Clearly states a number of examples of $2k+1 = a^2 - b^2$ to try to establish the pattern.

Most of the cohort found this challenging. Note that giving one example of $x = a^2 - b^2$ isn't worth anything (even if you tried to wrap it in proof by contradiction). And not everything is about odd and even cases!

Better responses sought to understand the pattern before attempting to prove (or disprove).

QUESTION 14

MARKER'S COMMENTS

a)

$$P(z) = z^4 - 6z^3 + 18z^2 - 14z - 39$$

If $2-3i$ is a root, then $2+3i$ is also a root as they occur in conjugate pairs.

METHOD 1

$[z-(2-3i)][z-(2+3i)]$ is also a root

$$= z^2 - [(2+3i)+(2-3i)]z + (2-3i)(2+3i)$$

$$= z^2 - 4z + 13$$

① correct conjugate pairs with quadratic ready for division

$$\begin{array}{r} z^2 - 2z - 3 \\ z^2 - 4z + 13 \,\big)\, \overline{z^4 - 6z^3 + 18z^2 - 14z - 39} \\ \underline{z^4 - 4z^3 + 13z^2} \qquad\qquad \\ -2z^3 + 5z^2 - 14z - 39 \\ \underline{-2z^3 + 8z^2 - 26z} \qquad \\ -3z^2 + 12z - 39 \\ \underline{-3z^2 + 12z - 39} \\ 0 \end{array}$$

① correctly divides polynomial

$$\therefore \quad P(z) = (z^2 - 2z - 3)(z - (2-3i))(z - (2+3i))$$

$$= (z-3)(z+1)(z-(2-3i))(z-(2+3i))$$

① correct final answer.

$\boxed{/3}$

MARKER'S COMMENTS

METHOD 2

sum of roots $\alpha + \beta + 2 - 3i + 2 + 3i = 6$

$\alpha + \beta = 2$

$\beta = 2 - \alpha \quad \ldots (1)$

(1/2) mark correctly find expression for sum of roots

product of roots $\alpha\beta(2-3i)(2+3i) = -39$

$\alpha\beta(4+9) = -39$

$\alpha\beta = -3 \quad \ldots (2)$

Sub (1) into (2)

(1/2) Correctly finds expressions for product of roots

$\alpha(2-\alpha) = -3$

$2\alpha - \alpha^2 = -3$

$\alpha^2 - 2\alpha - 3 = 0$

$(\alpha - 3)(\alpha + 1) = 0$

$\alpha = 3, -1$

(1) correctly finds values for α.

when $\alpha = 3$, $\beta = -1$ when $\alpha = -1$, $\beta = 3$

$\therefore$ other roots are $3, -1$

$\therefore P(z) = (z-3)(z+1)[z-(2-3i)][z-(2+3i)]$

(1) mark correct final answer.

$\boxed{/3}$

MARKER'S COMMENTS

Q14 cont...

b) Prove true for $n=1$

$T_n = 2T_{n-1} + 1$ $\qquad$ $T_n = 2^n - 1$

$T_1 = 2T_{1-1} + 1$ $\qquad$ $T_1 = 2^1 - 1$

$= 2T_0 + 1$ $\qquad$ $= 1$

$= 2 \times 0 + 1$

$= 1$ $\qquad$ 1 mark correct base case

$\therefore$ True for $n=1$

Assume true for all integers 1 to k

ie given $T_k = 2T_{k-1} + 1$, then $T_k = 2^k - 1$

Prove true for $n = k+1$

ie Prove given $T_{k+1} = 2T_k + 1$ then $T_{k+1} = 2^{k+1} - 1$

Consider LHS of $T_{k+1} = 2^{k+1} - 1$

$LHS = T_{k+1}$ $\qquad$ 1 mark for correct use of assumption

$= 2T_k + 1$

$= 2(2^k - 1) + 1$ by the assumption.

$= 2^{k+1} - 2 + 1$ $\qquad$ 1 mark for correct setting out of proof that enabled the correct conclusion of LHS = RHS.

$= 2^{k+1} - 1$

$= RHS$

$\therefore$ True for $n = k+1$ if true for integers 1 to k

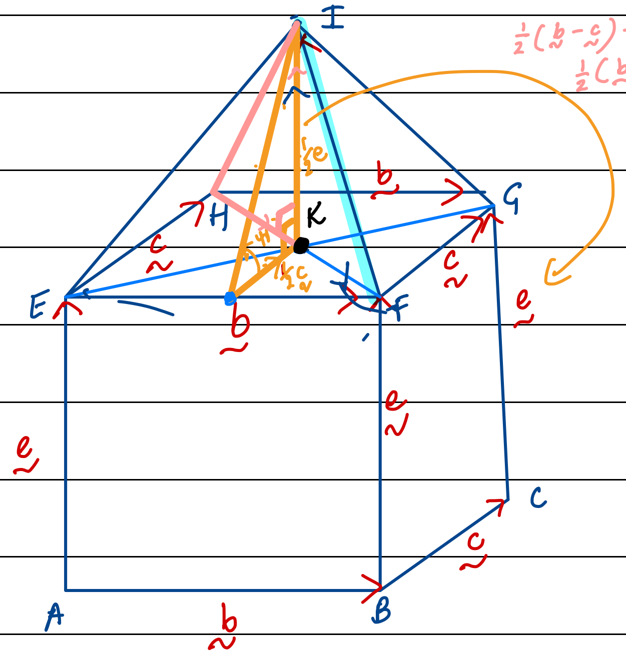
$\therefore$ By the principal of Mathematical Induction, it is true for all positive integer k.

/3

MARKER'S COMMENTS

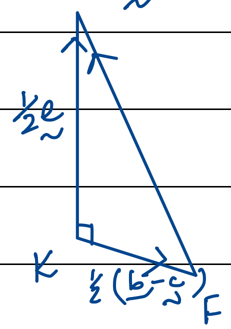
Q14 cont...

c)



going in same direction as e and length is ½ as in isos. Δ

b in i direction
c in j direction
e in k direction



(i)

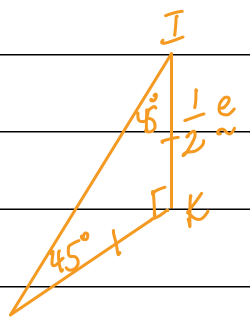
$$\overrightarrow{KF}+\overrightarrow{FI}=\overrightarrow{KI}$$

$$\overrightarrow{FI}=\overrightarrow{KI}-\overrightarrow{KF}$$

$$=\frac{1}{2}e-\frac{1}{2}(b-c)$$

$$=\frac{1}{2}e-\frac{1}{2}b+\frac{1}{2}c$$

$$=\frac{1}{2}(e-b+c)$$



(1) mark to correctly find $\overrightarrow{KF}$

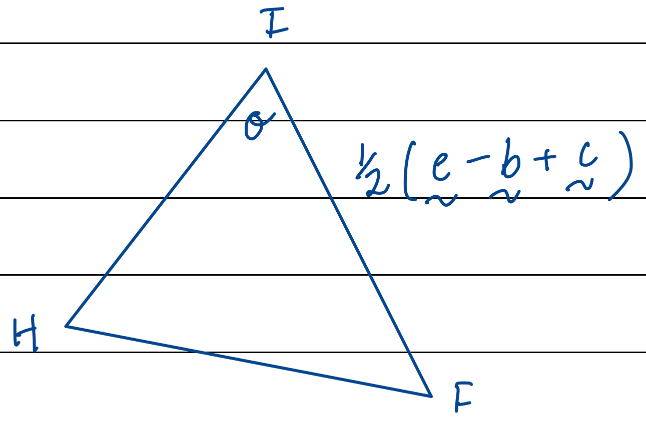
(1) correct answer.

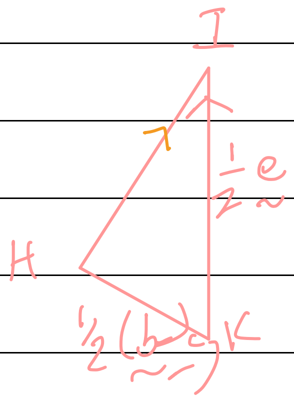
/2

NOTE: Many students found (i) and (ii) of this question very challenging, in particular finding $\overrightarrow{KI}$.

MARKER'S COMMENTS

(ii)





$\overrightarrow{HI} = \frac{1}{2}(\underset{\sim}{e} + \underset{\sim}{b} - \underset{\sim}{c})$

$\overrightarrow{IH} = \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b} - \underset{\sim}{e})$

$\overrightarrow{IF} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{e} - \underset{\sim}{c})$

$$\overrightarrow{IH} \cdot \overrightarrow{IF} = |\overrightarrow{IH}||\overrightarrow{IF}| \cos\theta$$

① mark to find $\overrightarrow{IH}$

$$\begin{pmatrix} -\frac{1}{2}\underset{\sim}{b} \\ \frac{1}{2}\underset{\sim}{c} \\ -\frac{1}{2}\underset{\sim}{e} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\underset{\sim}{b} \\ -\frac{1}{2}\underset{\sim}{c} \\ -\frac{1}{2}\underset{\sim}{e} \end{pmatrix} = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} \times \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} \cos\theta$$

$$-\frac{1}{4} + \frac{1}{4} - \frac{1}{4} = \frac{3}{4}\cos\theta$$

$$-\frac{1}{4} = \frac{3}{4}\cos\theta$$

$$\cos\theta = -\frac{1}{3}$$

$$\theta = 109.47^\circ$$

① mark final answer

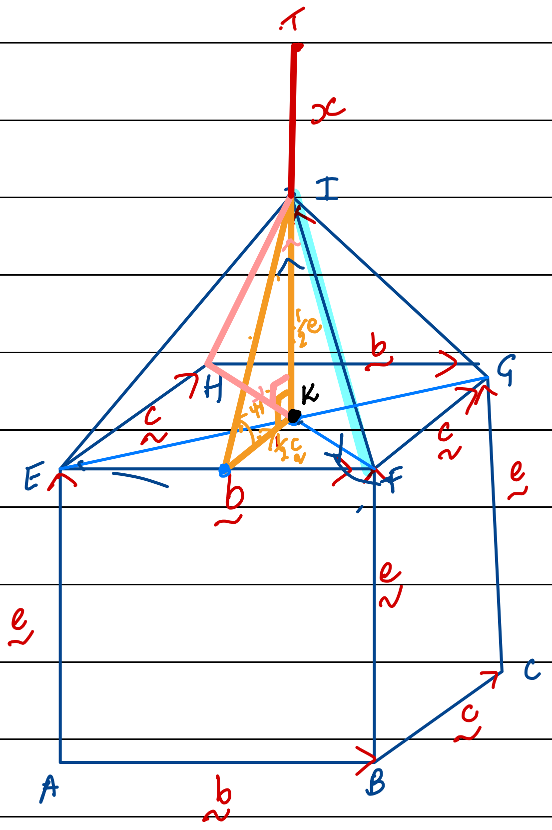
NOTE: Many students did not read the question carefully and incorrectly found $\angle IFH$ which made the question significantly easier.

$\boxed{/2}$

MARKER'S COMMENTS

Q14 c) cont...

(iii)



$$\frac{IT}{IT + \frac{3}{2}\underset{\sim}{e}} = \frac{1}{3} \quad \text{or} \quad \frac{1}{3}\left(IT + \frac{3}{2}\underset{\sim}{e}\right) = IT$$

etc...

$$3\,IT = IT + \frac{3}{2}\underset{\sim}{e}$$

$$2\,IT = \frac{3}{2}\underset{\sim}{e}$$

$$IT = \frac{3}{4}\underset{\sim}{e}$$

1 correct answer

$\boxed{/1}$

NOTE: Very poorly executed, many students did not read the question carefully and did not include the flagpole in the total height of the building.

Q14 continued . . .

MARKER'S COMMENTS

d)

$$\int \frac{\sqrt{(x^2-1)^3}}{x}\, dx$$

let $x = \sec\theta = \frac{1}{\cos\theta} = (\cos\theta)^{-1}$

$\frac{dx}{d\theta} = -(\cos\theta)^{-2} \times -\sin\theta$

$= \frac{\sin\theta}{\cos^2\theta}$

$dx = \tan\theta \sec\theta \; d\theta$

$$= \int \frac{\sqrt{(\sec^2\theta - 1)^3}}{\cancel{\sec\theta}} \times \tan\theta \cancel{\sec\theta}\, d\theta$$

$$= \int \left(\sqrt{\tan^2\theta}\right)^3 \tan\theta \; d\theta$$

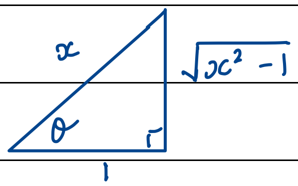
1 mark to correctly find dx.

$$= \int \tan^4\theta \; d\theta$$

1 mark to correctly form integral in terms of θ

$$= \int \tan^2\theta \tan^2\theta \; d\theta$$

$$= \int (\sec^2\theta - 1)\tan^2\theta \; d\theta$$



$\frac{x}{1} = \sec\theta$

$\therefore \cos\theta = \frac{1}{x}$

$\therefore \tan\theta = \sqrt{x^2-1}$

$$= \int \left(\sec^2\theta \tan^2\theta - \tan^2\theta\right) d\theta$$

$$= \int \left[\sec^2\theta\,[\tan\theta]^2 - (\sec^2\theta - 1)\right] d\theta$$

1 mark to simplify enough to enable integration

$$= \frac{1}{3}\tan^3\theta - \tan\theta + \theta + C$$

$$= \frac{1}{3}\left(\sqrt{x^2-1}\right)^3 - \sqrt{x^2-1} + \sec^{-1}x + C$$

1 correct answer

/4

NOTE: Small errors created big problems making integration very difficult and at times impossible.

MARKER'S COMMENTS

QUESTION 15

a) The road is wet and it did not rain.

1 mark correct negation.

$\boxed{/1}$

b) METHOD 1

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

(i) $$I_{n-2} = \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$I_n = \int_0^{\pi/4} \tan^n x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \cdot \tan^2 x \, dx$$

1 mark for correct split of $\tan^n x$.

$$= \int_0^{\pi/4} \tan^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx$$

1 mark for correct use of $\tan^2 x = \sec^2 x - 1$ and split of integral.

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \, dx$

$x = \frac{\pi}{4} \quad u = 1$

$x = 0 \quad u = 0$

$$I_n = \int_0^1 u^{n-2} \, du - I_{n-2}$$

$$I_n + I_{n-2} = \left[\frac{u^{n-1}}{n-1} \right]_0^1$$

$$= \left[\frac{1}{n-1} - 0 \right]$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

1 mark correct integration with use of either reverse chain rule or u-substitution enabling correct justification of $I_n + I_{n+2} = \frac{1}{n-1}$

$\boxed{/3}$

Q15 continued . . .

MARKER'S COMMENTS

a) <u>METHOD 2</u>

$LHS = I_n + I_{n-2}$

$= \int_0^{\pi/4} \tan^n x \, dx + \int_0^{\pi/4} \tan^{n-2} x \, dx$

(1) mark correct substitution into LHS

$= \int_0^{\pi/4} (\tan^n x + \tan^{n-2} x) \, dx$

$= \int_0^{\pi/4} \tan^{n-2} x \, (\tan^2 x + 1) \, dx$

(1) mark for correct factorisation and use of $\tan^2 + 1 = \sec^2 x$ to set up integration.

$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx$

let $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \, dx$

$x = \frac{\pi}{4} \rightarrow u = 1$

$x = 0 \rightarrow u = 0$

$= \int_0^1 u^{n-2} \, du$

$= \left[\frac{u^{n-1}}{n-1} \right]_0^1$

$= \frac{1}{n-1} - 0$

$= \frac{1}{n-1}$

(1) mark correct integration with use of either reverse chain rule or u-substitution enabling correct justification of $I_n + I_{n+2} = \frac{1}{n-1}$

$= RHS$

Q15 continued . . .

MARKER'S COMMENTS

b) (ii)

$$I_{n-2} = \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$I_7 = \frac{1}{7-1} - I_{n-2}$$

$$= \frac{1}{6} - \int_0^{\pi/4} \tan^5 x \, dx$$

① mark for correct start of the rule proved in (i)

$$= \frac{1}{6} - \left[\frac{1}{4} - \int_0^{\pi/4} \tan^3 x \, dx\right]$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \int_0^{\pi/4} \tan x \, dx$$

① mark to correctly get to the stage of integrating $\tan x$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} + \int_0^{\pi/4} \frac{-\sin x}{\cos x} \, dx$$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} + \Big[\ln|\cos x|\Big]_0^{\pi/4}$$

$$= -\frac{1}{12} + \frac{1}{2} + \left(\ln\frac{1}{\sqrt{2}} - \ln 1\right)$$

① correctly integrating $\tan x$ and correct final answer of $\frac{5}{12} + \ln\frac{1}{\sqrt{2}}$ or $\frac{5}{12} - \frac{1}{2}\ln 2$

$$= \frac{5}{12} + \ln\frac{1}{\sqrt{2}}$$

$$= \frac{5}{12} + \ln 2^{-1/2}$$

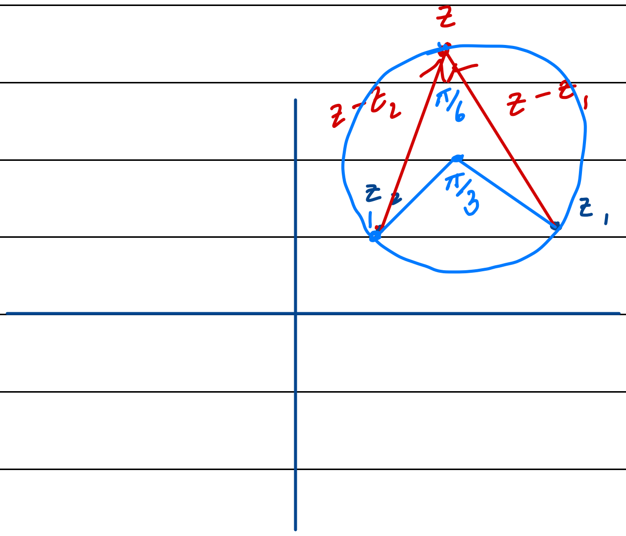
$$= \frac{5}{12} - \frac{1}{2}\ln 2$$

NOTE : Some difficulty with negatives within this question.

MARKER'S COMMENTS

Q15 continued ...

c)



$$\arg(z-(6+2i)) - \arg(z-(2+2i)) = \frac{\pi}{6}$$

(i)

$$\frac{x+iy-6-2i}{x+iy-2-2i}$$

$$= \frac{x-6+(y-2)i}{x-2+(y-2)i} \times \frac{x-2-(y-2)i}{x-2-(y-2)i}$$

① mark to multiply by the conjugate.

$$= \frac{(x-6)(x-2)-(x-6)(y-2)i+(x-2)(y-2)i+(y-2)^2}{(x-2)^2+(y-2)^2}$$

$$= \frac{(x-6)(x-2)+(y-2)^2+(y-2)[(x-2)-(x-6)]i}{(x-2)^2+(y-2)^2}$$

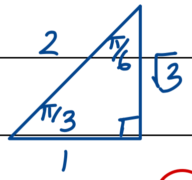
$$= \frac{(x-6)(x-2)+(y-2)^2+4(y-2)i}{(x-2)^2+(y-2)^2}$$

① mark for correct real denominator.

MARKER'S COMMENTS

Q15 c) cont ....

(ii) $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \dfrac{\pi}{6}$



$\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$ 1 mark correct tan ratio

$\therefore$ rise : run

$1 : \sqrt{3}$

$\dfrac{\text{rise}}{\text{run}} = \dfrac{Im\left(\dfrac{z-z_1}{z-z_2}\right)}{Re\left(\dfrac{z-z_1}{z-z_2}\right)} = \dfrac{1}{\sqrt{3}}$ 1 mark to relate the $\frac{\text{rise}}{\text{run}}$ with result in part (i)

$Im\left(\dfrac{z-z_1}{z-z_2}\right) = \dfrac{1}{\sqrt{3}} Re\left(\dfrac{z-z_1}{z-z_2}\right)$

$\dfrac{4(y-2)}{\cancel{(x-2)^2+(y-2)^2}} = \dfrac{1}{\sqrt{3}} \times \dfrac{(x-6)(x-2)+(y-2)^2}{\cancel{(x-2)^2+(y-2)^2}}$

$4\sqrt{3}(y-2) = (x-6)(x-2)+(y-2)^2$

$4\sqrt{3}y - 8\sqrt{3} = x^2 - 8x + 12 + {}^2 - 4y + 4$

$x^2 - 8x + \left(\frac{-8}{2}\right)^2 + y^2 - (4-4\sqrt{3})y + \left(\frac{-4-4\sqrt{3}}{2}\right)^2 = -8\sqrt{3} - 12 - 4 + \left(\frac{-8}{2}\right)^2 + \left(\frac{-4-4\sqrt{3}}{2}\right)^2$

$(x-4)^2 + \left[y-(2+2\sqrt{3})\right]^2 = -8\sqrt{3} - \cancel{16} + \cancel{16} + (2+2\sqrt{3})^2$

$(x-4)^2 + \left[y-(2+2\sqrt{3})\right]^2 = -8\sqrt{3} + 4 + 8\sqrt{3} + 12$

$(x-4)^2 + \left[y-(2+2\sqrt{3})\right] = 16$

1 correctly proves result.

$\therefore\ C\left(4,\ 2+2\sqrt{3}\right)$ $r = 4$

$\therefore\ \left|z - \{4 + (2+2\sqrt{3})i\}\right| = 4$

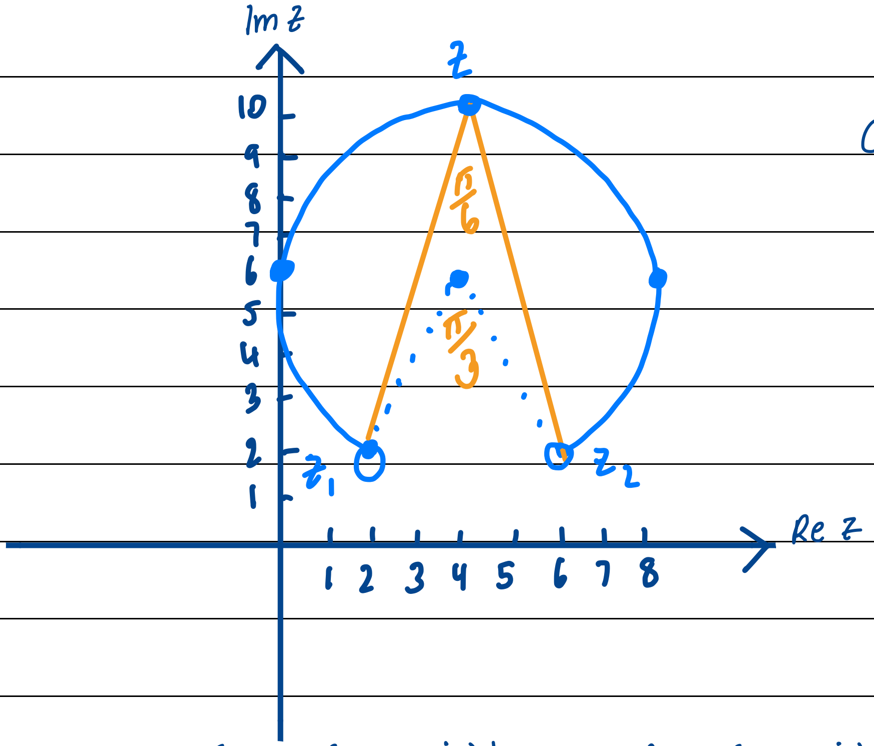
$\left|z - \{4 + 2(1+\sqrt{3})i\}\right| = 4$

NOTE: Students found this difficult to begin. Many just replaced z with $x+iy$ which led nowhere.

MARKER'S COMMENTS

Q15 c) cont....

(iii)



$C(4, 2+2\sqrt{3})$

$r = 4$

$6+2i$ and $2+2i$

$$\arg(z-(6+2i)) - \arg(z-(2+2i)) = \frac{\pi}{6}$$

1 correct centre

1 correct path of z ($\frac{1}{2}$ for open circles)

1 correct radii with correct z_1 and z_2

12MXX AT4 2025 Q16 MARKER'S COMMENTS

a) For $1 < f(x)$ consider RHS − LHS

RHS − LHS $= f(x) - 1$

$$= \frac{1+x-x^2}{1-x+x^2} - \frac{1-x+x^2}{1-x+x^2}$$

$$= \frac{1+x-x^2-(1-x+x^2)}{1-x+x^2}$$

$$= \frac{2x-2x^2}{1-x+x^2}$$

$$= \frac{2x(1-x)}{1-x+x^2}$$

$$= \frac{2x(1-x)}{(x-\frac{1}{2})+\frac{3}{4}}$$

> 0 (since $x>0$, $1-x>0$) (since $x<1$)

$\therefore$ RHS > LHS

$\therefore f(x) > 1$

$\therefore 1 < f(x)$

For $f(x) \le \frac{5}{3}$, consider RHS − LHS:

RHS − LHS $= \frac{5}{3} - \frac{1+x-x^2}{1-x+x^2}$

$$= \frac{5(1-x+x^2)-3(1+x-x^2)}{3(1-x+x^2)}$$

$$= \frac{5-5x+5x^2-3-3x+3x^2}{3(1-x+x^2}$$

$$= \frac{8x^2 - 8x + 2}{3\left[\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}\right]}$$

$$= \frac{2\left(4x^2 - 4x + 1\right)}{3\left[\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}\right]}$$

$$= \frac{2\left(2x-1\right)^2}{3\left[\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}\right]}$$

$$\geq 0$$

$\therefore RHS \geq LHS$

$\therefore \frac{5}{3} \geq f(x)$

$\therefore f(x) \leq \frac{5}{3}$

$\therefore 1 < f(x) \leq \frac{5}{3}$

3 marks	Complete solution
2 marks	Correctly shows $1 < f(x)$ or $f(x) \leq \frac{5}{3}$
1 mark	Clearly shows LHS-RHS for either $1 < f(x)$ or $f(x) \leq \frac{5}{3}$

There were other methods of completing this proof, but this was the simplest. Note that finding a collection of function values proves nothing (zero marks)

b) Prove true for $n=1$

when $n=1$, $3^{2n^2}-1 = 3^{2(1)^2}-1$

$= 3^2-1$

$= 9-1$

$= 8$ which is divisible by 8

$\therefore$ true for $n=1$

Suppose true for $n=k$

that is, $3^{2k^2}-1 = 8M$, $M \in \mathbb{Z}$

Prove true for $n=k+1$

That is, $3^{2(k+1)^2}-1$ is divisible by 8

$3^{2(k+1)^2}-1 = 3^{2k^2+4k+2}-1$

$= 3^{2k^2} \cdot 3^{4k+2}-1$

$= (8M+1)3^{4k+2}-1$ (since $3^{2k^2}-1=8M$ $\therefore 3^{2k^2}=8M+1$)

$= 8M \cdot 3^{4k+2} + 3^{4k+2}-1$

$= 8(M \cdot 3^{4k+2}) + 9 \cdot 3^{4k}-1$

$= 8(M \cdot 3^{4k+2}) + 8 \cdot 3^{4k} + 3^{4k}-1$

$= 8[M \cdot 3^{4k+2} + 3^{4k}] + 3^{4k}-1$

Consider $3^{4k}-1$

Prove $3^{4k}-1$ is divisible by 8 by mathematical induction.

Prove true for $n=1$

$3^{4k}-1 = 3^4-1$

$=81-1$

$=80$

$=8(10)$ which is divisible by 8

$\therefore$ true for $n=1$

Suppose true for $n=k$

That is, $3^{4k}-1 = 8P$, $P \in \mathbb{Z}$

Prove true for $n=k+1$

That is, $3^{4(k+1)}-1$ is divisible by 8

Given $3^{4k}-1 = 8P$

$3^4 . 3^{4k} - 3^4 = 3^4 \times 8P$

$3^{4k+4} - 81 = 8P \times 3^4$

$3^{4(k+1)} - 1 = 8\left[P \times 3^4\right] + 80$

$= 8\left[P \times 3^4 + 10\right]$

which is divisible by 8

$\therefore$ $3^{4k}-1$ is also divisible by 8

$\therefore 8\left[M . 3^{4k+7} + 3^{4k}\right] + 3^{4k}-1$ is divisble by 80

$\therefore$ true for $n=k+1$ when it is true for $n=k$

$\therefore$ true for all n by the principle of mathematical induction

12MXX AT4 2025 Q16 MARKER'S COMMENTS (continued)

4 marks	Complete solution
3 marks	Correct solution up to and including $8\left[M\cdot 3^{4k+2}+3^{4k}\right]+3^{4k}-1$
2 marks	Correctly proves base case and productively uses inductive hypothesis
1 mark	Correctly proves base case or productively uses inductive hypothesis

This was a challenging question that tested your mathematical stamina and initiative. Better responses used lots of space and paper – only one equals sign per line, please! Readability matters in proof – write cleanly. Many students let themselves down with errors in index laws – your algebra needs to be excellent at this level.

c)

$$(r+n)(s+n)(t+n)=n^6$$

$$(rs+rn+ns+n^2)(t+n)=n^6$$

$$rst+rnt+nst+n^2t+rsn+rn^2+n^2s+n^3=n^6$$

$$rst+n(rt+st+rs)+n^2(r+s+t)+n^3=n^6 \quad (1)$$

Also $(r+s+t)^3 \geq 27rst$

$$\therefore r+s+t \geq 3\sqrt[3]{rst} \quad (2)$$

And $(rt+st+rs)^3 \geq 27(rt \times st \times rs)$

$\geq 27r^2s^2t^2$

$\therefore rt+st+rs \geq 3\sqrt[3]{r^2s^2t^2}$ ③

sub ② and ③ into ①

$$rst + n\left(3\sqrt[3]{r^2s^2t^2}\right) + n^2\left(3\sqrt[3]{rst}\right) + n^3 \leq n^6$$

$$rst + 3n\left[(rst)^{\frac{1}{3}}\right]^2 + 3n^2\left[(rst)^{\frac{1}{3}}\right] + n^3 \leq n^6$$

$$\left[(rst)^{\frac{1}{3}}\right]^3 + 3n\left[(rst)^{\frac{1}{3}}\right]^2 + 3n^2(rst)^{\frac{1}{3}} + n^3 \leq n^6$$

$$\therefore \left[(rst)^{\frac{1}{3}} + n\right]^3 \leq n^6$$

$$(rst)^{\frac{1}{3}} + n \leq n^2$$

$$(rst)^{\frac{1}{3}} \leq n^2 - n$$

$$\therefore rst \leq (n^2-n)^3$$

3 marks Complete solution

2 marks Correctly substitutes ② and ③ into ①

1 mark Correctly finds equation ①, ② or ③ (exactly, not a transformed version).

Very challenging for most students.

Common error was $(a>b \wedge c>b \Rightarrow a>c)$

d) $S_n = \underbrace{1 + z + z^2 + z^3 + \ldots + z^n}$

G.P with $a = 1$
$r = z$
$n = n+1$

$\frac{z}{1} = z$

$\frac{z^2}{z} = z \quad \therefore$ G.P.

$$S_n = \frac{a(r^n - 1)}{r-1}$$
$$= \frac{1(z^{n+1} - 1)}{z - 1}$$
$$= \frac{(e^{i\theta})^{n+1} - 1}{e^{i\theta} - 1} \quad (\text{since } z = e^{i\theta})$$
$$= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

Many students did not show enough to earn the mark. This is a "show" question; don't leave anything out!

Most common error was $\frac{e^{in\theta} - 1}{e^{i\theta} - 1}$

ii $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$ is the imaginary part of $S_n = 1 + z + z^2 + z^3 + \ldots + z^n$

$\therefore$ find imaginary part of $S_n = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$

$$S_n = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

$$= \frac{e^{\frac{i(n+1)\theta}{2}}\left[e^{\frac{i(n+1)\theta}{2}} - e^{-\frac{i(n+1)\theta}{2}}\right]}{e^{\frac{i\theta}{2}}\left[e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}\right]}$$

$$= \frac{e^{\frac{i(n+1)\theta}{2}}\left(2i\sin\left[\frac{(n+1)\theta}{2}\right]\right)}{e^{\frac{i\theta}{2}}\left(2i\sin\frac{\theta}{2}\right)}$$

$$\left(\begin{aligned} z - z^{-1} &= \text{cis}\,\theta - \text{cis}(-\theta) \\ &= \cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta)) \\ &= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) \\ &= 2i\sin\theta \end{aligned}\right)$$

$$= e^{\left(\frac{i(n+1)\theta}{2} - \frac{i\theta}{2}\right)} \times \frac{\sin\left[\frac{(n+1)}{2}\theta\right]}{\sin\frac{\theta}{2}}$$

$$= e^{\frac{in\theta}{2}} \times \frac{\sin\left(\frac{n+1}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

Take imaginary part:

$$= \frac{\sin\frac{n\theta}{2} \times \sin\left(\frac{n+1}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

4 marks	Complete solution
3 marks	Correct solution up to and including using $z + z^{-1} = 2i\sin\theta$
2 marks	Recognises correct factorisation using $e^{\frac{i(n+1)\theta}{2}}$
1 mark	Clearly states need to find imaginary part.

Very challenging question. Better responses recognised the denominator of 2 as a hint to split the powers.