SYDNEY BOYS HIGH SCHOOL

NESA Number:

Name:

Tick next to your teacher's name

AF		**PSP**	
JM		**JT**	
JJ			

2025

YEAR 12
TASK 3
TRIAL HSC

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Only NESA approved calculators may be used
- A reference sheet is provided with this paper
- All answers, unless otherwise stated, should be left in simplified, exact form
- Marks may **NOT** be awarded for messy or badly arranged work
- For questions in Section II, show ALL relevant mathematical reasoning and/or calculations as answers that rely totally on calculator technology may not necessarily receive full marks.

Total Marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 16)

- Attempt all Questions in Section II
- Allow about 2 hours and 45 minutes for this section

Examiner:
External Examiner

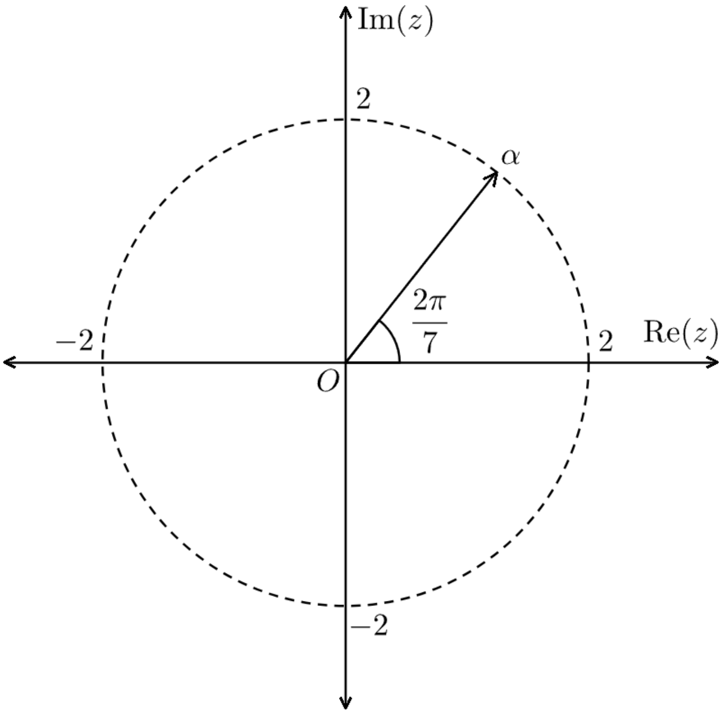
Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

Question 1

Let α be the complex number plotted on the Argand diagram below.



Which expression is equal to $(\overline{\alpha})^{-1}$?

(A) $\dfrac{1}{2}\left(\cos\dfrac{2\pi}{7} - i\sin\dfrac{2\pi}{7}\right)$

(B) $2\left(\cos\dfrac{2\pi}{7} - i\sin\dfrac{2\pi}{7}\right)$

(C) $\dfrac{1}{2}\left(\cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}\right)$

(D) $2\left(\cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}\right)$

Question 2

A polynomial with positive integer coefficients has roots 2 and $4-5i$. Which of the following must also be a root?

(A) $5-4i$

(B) $4+5i$

(C) $5+4i$

(D) $-4+5i$

Question 3

Using a t-substitution, which of the following is equivalent to $\displaystyle\int \frac{dx}{1-\sin x-\cos x}$?

(A) $\displaystyle\int \frac{dt}{t^2-t}$

(B) $\displaystyle\int \frac{2dt}{t^2-t}$

(C) $\displaystyle-\int \frac{dt}{t}$

(D) $\displaystyle-\int \frac{2dt}{t}$

Question 4

If $|z|=1$ and $w=\dfrac{z-1}{z+1}$ (where $z\neq -1$), then which of the following is equal to $\mathrm{Re}(w)$?

(A) 0

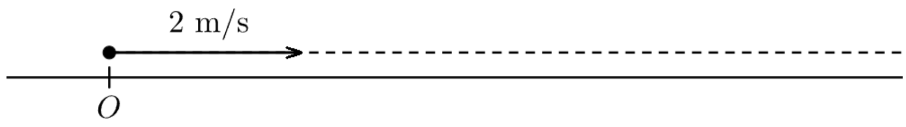
(B) $\dfrac{1}{|z+1|^2}$

(C) $\left|\dfrac{1}{z+1}\right|\cdot\dfrac{1}{|z+1|^2}$

(D) $\dfrac{\sqrt{2}}{|z+1|^2}$

Question 5

A particle moves in a straight line such that $a = \dfrac{1}{\sqrt{x+1}}$ m/s^2.



It is initially at the origin, moving to the right at 2 m/s.

At what position will it first reach a speed of 4 m/s?

(A) $x = 8$

(B) $x = 9$

(C) $x = 15$

(D) $x = 16$

Question 6

Let $z = x + iy$. Which of the following options is equal to $|e^{iz}|$?

(A) 1

(B) e^{y}

(C) e^{-y}

(D) e^{x}

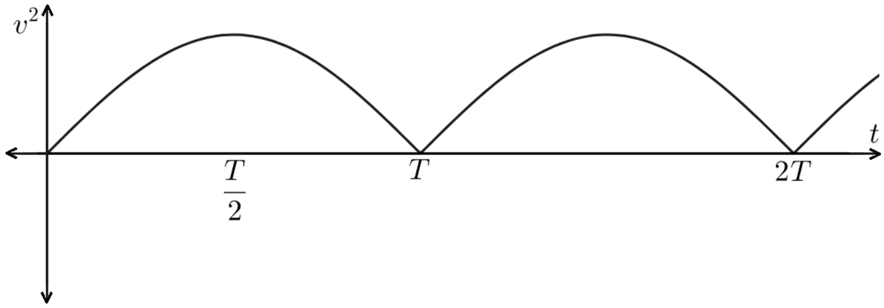
Question 7

A particle is executing simple harmonic motion with a period of T seconds.

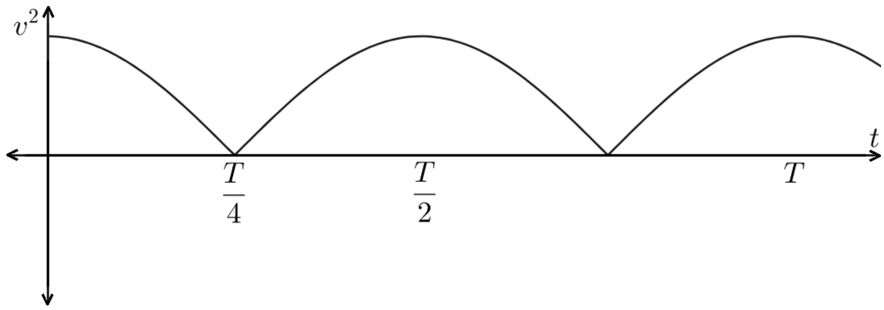
At time $t = 0$, it is at its centre of motion.

Which of the following would be the best graph of v^2 against time?

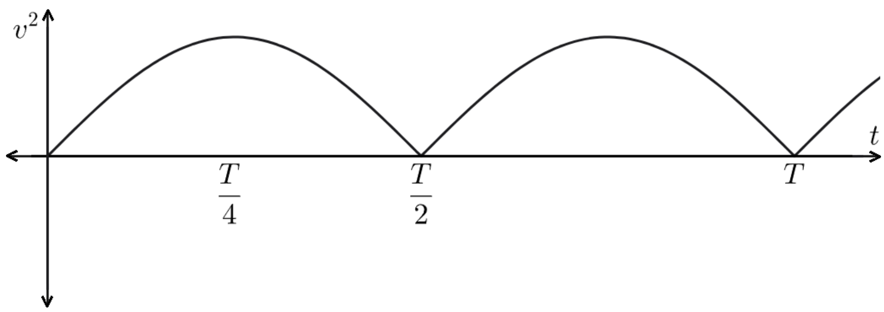
(A)



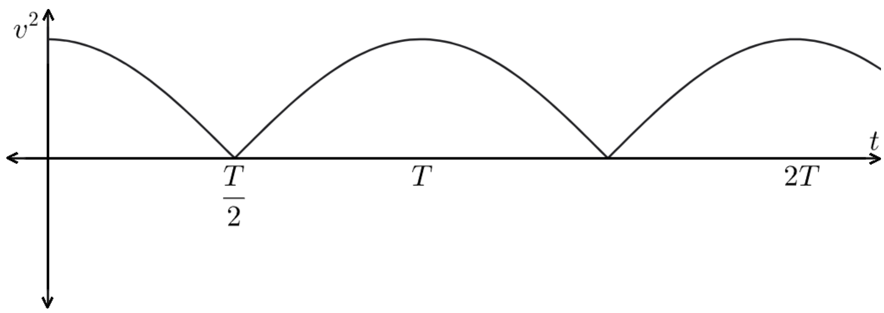
(B)



(C)



(D)



Question 8

Find the magnitude of the vector $\underset{\sim}{u} = \underset{\sim}{i} + \tan\theta \underset{\sim}{j}$ for $\theta \in \left(\frac{\pi}{2}, \pi\right)$.

(A) 1

(B) $\sec\theta$

(C) $\tan\theta$

(D) $-\sec\theta$

Question 9

ω is a non-real cube root of unity. Evaluate

$$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots(1+\omega^{2^n})$$

where n is some even positive integer.

(A) 1

(B) ω

(C) $\frac{1}{\omega}$

(D) $-\frac{1}{\omega}$

Question 10

Let $\alpha, \beta \in \mathbb{C}$ so that $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$ and $\mathrm{Re}(\alpha) > \mathrm{Re}(\beta)$.

Determine which of the following equations can be used to find the set of values z where

$$\mathrm{Arg}(z-\alpha) = 2\,\mathrm{Arg}(z-\beta).$$

Note that $\mathrm{Arg}(z)$ denotes the principal argument of z.

(A) $|z-\alpha| = |\beta-\alpha|$

(B) $|z-\beta| = |\beta-\alpha|$

(C) $|z-\alpha| = |z-\beta|^2$

(D) $|z-\beta| = |z-\alpha|^2$

Section II

Attempt Questions 11 – 16
Allow 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet.
Extra writing booklets are available.
In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Define $z = 5 - 12i$ and $w = 3 + 4i$. Find the value of:

(i) v such that $3\overline{v} = z - 2\overline{w}$; **2**

(ii) $\dfrac{2}{w} + z$; **1**

(iii) u such that u is a square root of zw. **3**

Leave your answer in the form $a + ib$, where a, b are real numbers with $a > 0$.

(b) Find the values of a if the vectors $\underset{\sim}{p} = \begin{pmatrix} 9a - 4 \\ 5 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} a \\ -1 \end{pmatrix}$ are perpendicular. **2**

(c) (i) Find the values of constants A, B and C such that **2**

$$\frac{5x^2 - 9x + 4}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx + C}{x^2+2}.$$

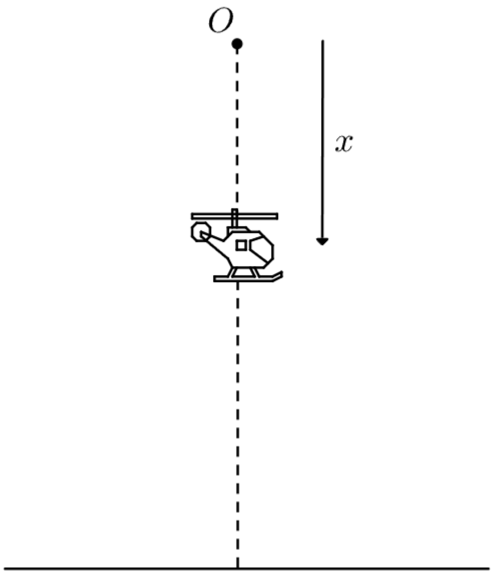
(ii) Hence, or otherwise, find $\displaystyle\int \frac{5x^2 - 9x + 4}{(x-3)(x^2+2)}\, dx$. **2**

(d) Find $\displaystyle\int \cos^{-1}(2x)\, dx$. **3**

Question 12 (14 marks) Use a SEPARATE writing booklet.

(a) Calculate the value of $k \in \mathbb{R}$ so that there are only two points of intersection between $\left|\operatorname{Arg}(z-k-i)-\frac{\pi}{6}\right| \geq \frac{\pi}{6}$ and $|z-2-2i|=1$. **2**

(b) A 2 kg toy helicopter is low on battery and provides a thrust of 10 N upwards. It starts falling from rest, and is subject to gravity and air resistance numerically equal to kv^2 where the constant of proportionality is $k = 0.096$. Take $g = 9.8$ m/s^2.



Let x be the downward displacement of the helicopter from its initial position in metres, along with acceleration a m/s^2 and velocity v m/s.

(i) Explain why the motion of the helicopter follows the acceleration equation **1**

$$a = 4.8 - 0.048v^2.$$

(ii) Prove that the velocity of the helicopter is given by **3**

$$v = \frac{10(e^{0.96t}-1)}{e^{0.96t}+1}.$$

(iii) Given that the helicopter hits the ground with speed 6.5 m/s, calculate the height the helicopter was initially at, correct to 2 decimal places. **2**

Question 12 continues on page 11

Question 12 (continued)

(c) (i) Using De Moivre's Theorem, prove that **3**

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

(ii) Hence, by making the substitution $x = A\cos\theta$ for a suitable A, or otherwise, find all the roots of the polynomial **3**

$$P(x) = x^6 - 6x^4 + 9x^2 - 2.$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A ball moves in a straight line according to

$$v^2 = 9(12 + 4x - x^2)$$

where v is the velocity of the ball in cm/s and x is the displacement of the ball from the origin in cm.

(i) Prove that the ball is moving in simple harmonic motion. **1**

(ii) By considering $v^2 \geq 0$, or otherwise, find the range of x values that the ball can move in. **1**

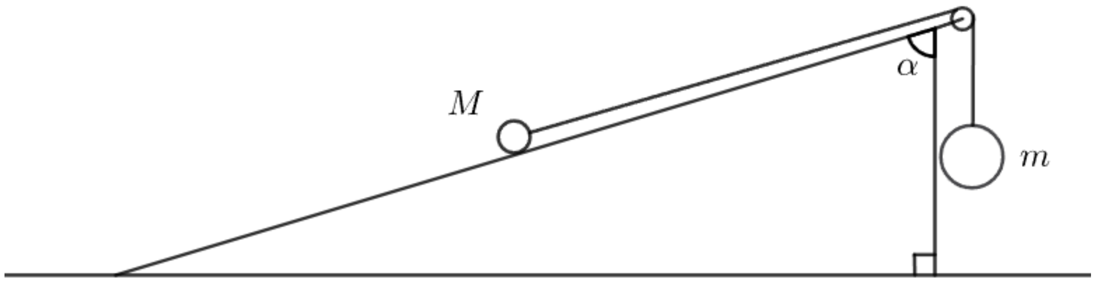
(iii) The ball is considered to be in the extreme zone when it deviates from its central position for more than 2 cm. **3**
It completes one whole oscillation and returns to its starting position.
For what percentage of time, correct to the nearest whole number, is the ball in the extreme zone?

(b) By considering the expansion of $(z+1)^4$, or otherwise, solve **3**

$$z^4 + 4z^3 + 8z^2 + 8z + 3 = 0.$$

Question 13 continues on page 13

(c) The two weights in the diagram are of mass m kg and M kg as shown.



For the weight on the slope, M kg, there is a frictional force, F acting on it.

It is given that $F = \mu N$, where N is the normal reaction force of the slope on the weight and μ is a real constant.

Assume that the mass of the string is negligible.

(i) The m kg mass is moving downwards. **2**

By resolving parallel and perpendicular to the slope, show that the acceleration, a, of the masses is given by

$$a = \frac{g(m - \mu M \sin\alpha - M\cos\alpha)}{m + M},$$

where g is the acceleration due to gravity.

(ii) Find the tension in the string. **2**

(iii) Prove that if the vertically hung mass is moving downwards, then **3**

$$\alpha < \sin^{-1}\left(\frac{m}{M\sqrt{1+\mu^2}}\right) - \tan^{-1}\frac{1}{\mu}.$$

You may assume that $\alpha + \tan^{-1}\dfrac{1}{\mu} < \dfrac{\pi}{2}$.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A complex number z satisfies $z^5 + i = 0$, where $z \neq -i$.

(i) Show that $z^4 - iz^3 - z^2 + iz + 1 = 0$. **1**

(ii) Prove that $(z+i)\left(z^2 + 2iz\sin\frac{\pi}{10} - 1\right)\left(z^2 - 2iz\sin\frac{3\pi}{10} - 1\right) = 0$. **3**

(iii) Hence, prove that $\sin\frac{\pi}{10} - \sin\frac{3\pi}{10} = -\frac{1}{2}$. **2**

(iv) Hence, find the exact value of $\sin\frac{\pi}{10}$ and $\sin\frac{3\pi}{10}$. **3**

(b) Let $I_n = \int_0^{\frac{1}{2}} x^2(1-x^3)^n \, dx$ for $n \geq 0$.

(i) Prove that $(k+1)I_k - kI_{k-1} = \frac{1}{24}\left(\frac{7}{8}\right)^k$, for $k \geq 1$. **3**

(ii) Hence, by taking an appropriate sum on both sides, prove for all $n \geq 0$, **3**

$$I_n = \frac{1}{3(n+1)}\left[1 - \left(\frac{7}{8}\right)^{n+1}\right].$$

Question 15 (15 marks) Use a SEPARATE writing booklet.

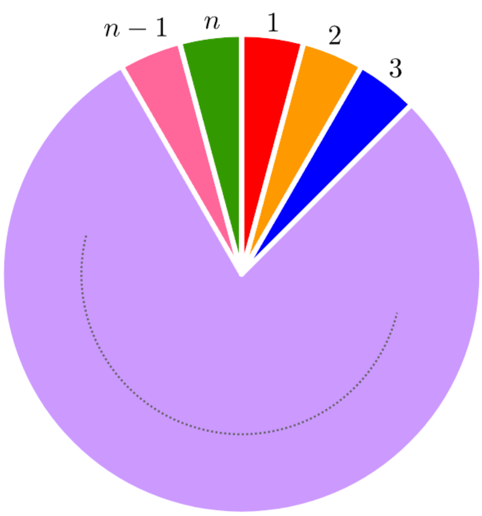
(a) (i) Prove that $\displaystyle\int_0^a f(x)\,dx = \int_0^{\frac{a}{2}} [f(x) + f(a-x)]\,dx$. **1**

(ii) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx$. **1**

By using (i), or otherwise, prove that $I = \displaystyle\int_0^{\frac{\pi}{4}} \ln\left(\frac{\sin 2x}{2}\right) dx$.

(iii) Hence, evaluate I. **1**

(b) Seven distinct colours are used to colour the numbered sectors of a circle shown below.



Let a_n denote the number of ways to colour a circle with n sectors such that each sector is coloured by one colour and any two adjacent sectors must be coloured by different colours.

(i) Explain why $a_n + a_{n-1} = 7 \times 6^{n-1}$ for positive integers $n \geq 3$. **2**

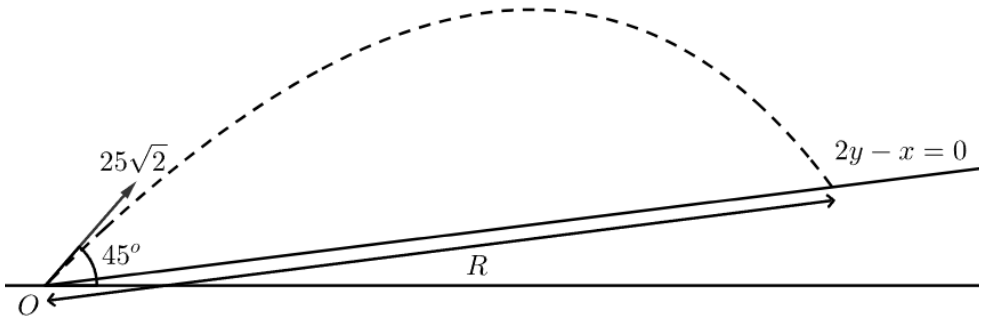
(ii) Use mathematical induction to show for positive integers $n \geq 3$, **3**

$$a_n = 6 \times (-1)^n + 6^n.$$

Question 15 continues on page 16

Question 15 (continued)

(c) A 100 g football is kicked up a slope with equation $2y - x = 0$ as shown.



It is subject to gravity 10 m/s^2 and air resistance $-0.02\underset{\sim}{v}$ N where $\underset{\sim}{v}$ is the velocity vector of the football.

The football is launched with speed $25\sqrt{2}$ m/s at angle 45^o to the horizontal.

The acceleration of the football is given by

$$\underset{\sim}{a} = -\underset{\sim}{g} - 0.2\underset{\sim}{v}$$

where $\underset{\sim}{g} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$.

(i) Show that the vertical displacement of the football is **2**

$$y = -50t + 375(1 - e^{-0.2t}).$$

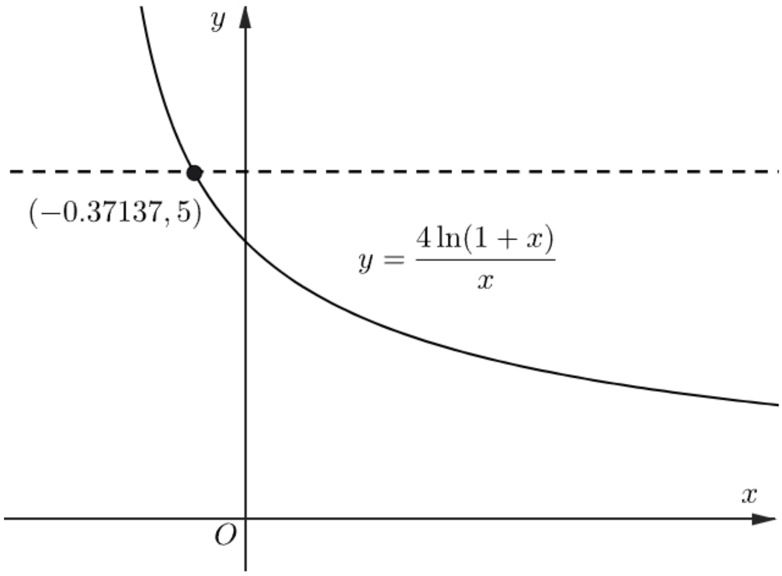
(ii) You are given that the horizontal displacement, in metres, of the football is $x = 125(1 - e^{-0.2t})$. (Do NOT prove this.) **2**

Prove that the Cartesian equation of the trajectory is given by

$$y = 3x + 250\ln\left(1 - \frac{x}{125}\right).$$

Question 15 continues on page 17

(c) (iii) The graph shows the point where $y = 5$ on the curve $y = \dfrac{4\ln(1+x)}{x}$. **3**



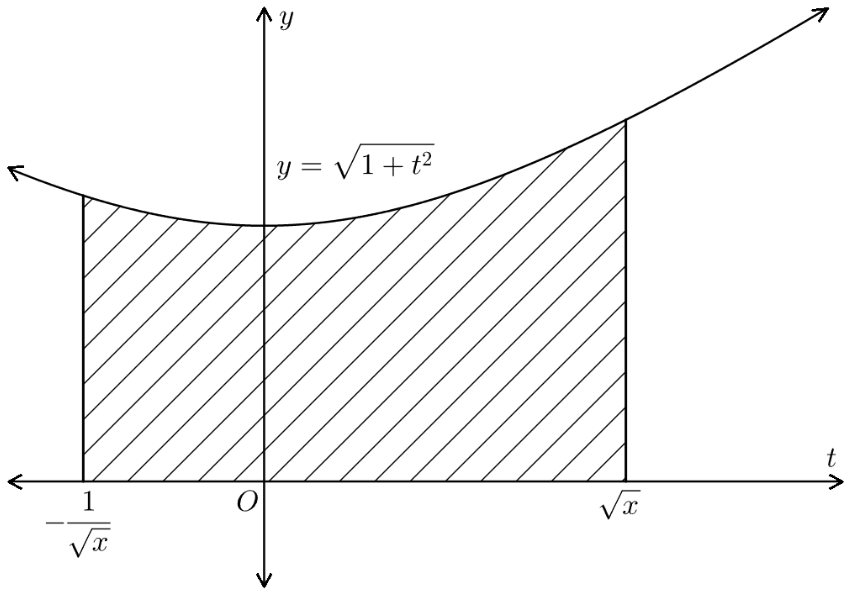
Hence, or otherwise, find the range R up the slope, correct to 4 decimal places.

End of Question 15

Question 16 (16 marks) Use a SEPARATE writing booklet.

(a) (i) Explain why $\dfrac{d}{dx}\left[\displaystyle\int_0^{g(x)} h(t)\ dt\right] = g'(x)h(g(x))$. **1**

(ii) Consider the area under the graph $y = \sqrt{1+t^2}$ between $t = -\dfrac{1}{\sqrt{x}}$ and $t = \sqrt{x}$. **3**



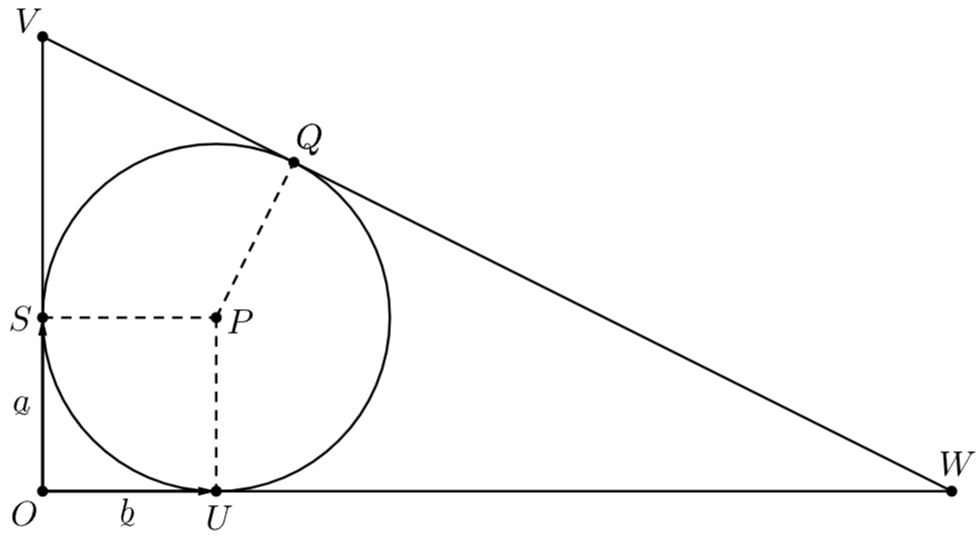
By using the result from (i), find the value of x for which the area is minimised.

Question 16 continues on page 19

(b) A circle centred at P is inscribed inside a right-triangle $\triangle OVW$, as shown.

It is tangent to the triangle at the points S, U and Q.

Q lies on the hypotenuse, such that $\overrightarrow{VQ} = k\overrightarrow{VW}$ where $k > 0$. Let $\overrightarrow{OS} = \underset{\sim}{a}$ and $\overrightarrow{OU} = \underset{\sim}{b}$.



(i) Let $\overrightarrow{OV} = \mu\underset{\sim}{a}$ and $\overrightarrow{OW} = \lambda\underset{\sim}{b}$ for real numbers $\lambda, \mu > 1$. **1**

Explain why $\overrightarrow{VQ} = k(\lambda\underset{\sim}{b} - \mu\underset{\sim}{a})$.

(ii) Hence, by considering VS, prove that $k = \dfrac{\mu - 1}{\sqrt{\lambda^2 + \mu^2}}$. **3**

(iii) Prove that $\overrightarrow{PQ} = [(1-k)\mu - 1]\underset{\sim}{a} + (k\lambda - 1)\underset{\sim}{b}$. **1**

(iv) Hence, or otherwise, prove that $k = \dfrac{\lambda + \mu^2 - \mu}{\lambda^2 + \mu^2}$. **2**

(v) Hence, prove that $\lambda = \dfrac{2(\mu - 1)}{\mu - 2}$. **2**

(vi) Using the results of the previous parts, or otherwise, show that the area of the inscribed circle is **3**

$$\frac{\pi}{4}(OV + OW - VW)^2.$$

End of Paper

SYDNEY BOYS HIGH SCHOOL

2025

YEAR 12

HSC TASK 3

Mathematics Extension 2

Sample Solutions

NOTE:

This process of checking your mark is about reading the solutions and the comments.

Before putting in an appeal re marking, first consider that the mark is not linked to the amount of writing you have done.

Just because you have shown some 'working' does not justify that your solution is worth any marks.

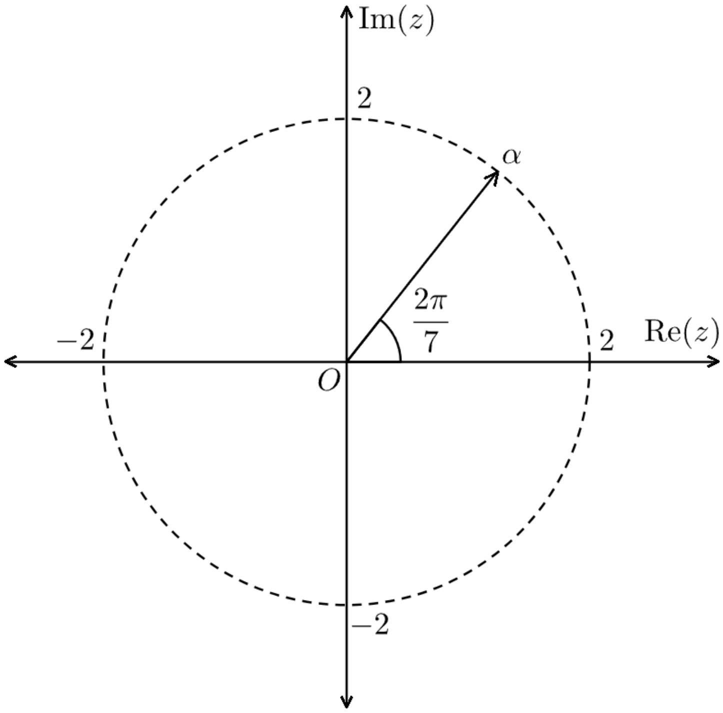
Students who used pencil, an erasable pen and/or whiteout, may NOT be able to appeal.

MC Answers

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
C	B	A	A	C	C	B	D	D	A

Question 1

Let α be the complex number plotted on the Argand diagram below.



Which expression is equal to $(\overline{\alpha})^{-1}$?

(A) $\frac{1}{2}\left(\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}\right)$

(B) $2\left(\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}\right)$

(C) $\frac{1}{2}\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)$

(D) $2\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)$

From the diagram, $\alpha = 2\operatorname{cis}\frac{2\pi}{7}$, so

$$\therefore (\overline{\alpha})^{-1} = \left[2\operatorname{cis}\left(-\frac{2\pi}{7}\right)\right]^{-1}$$
$$= \frac{1}{2}\operatorname{cis}\frac{2\pi}{7} \text{ (De Moivre's Theorem)}$$

A	13
B	0
C	98
D	7

Question 2

A polynomial with positive integer coefficients has roots 2 and $4-5i$. Which of the following must also be a root?

(A) $5-4i$

(B) $4+5i$

(C) $5+4i$

(D) $-4+5i$

$\because$ All integer coefficients, which are also real coefficients,

$\therefore$ By conjugate root theorem, $z = \overline{4-5i} = 4+5i$ is also a root.

A	0
B	118
C	0
D	0

Everyone got this right!

Question 3

Using a t-substitution, which of the following is equivalent to $\displaystyle\int \frac{dx}{1-\sin x-\cos x}$?

(A) $\displaystyle\int \frac{dt}{t^2-t}$

(B) $\displaystyle\int \frac{2dt}{t^2-t}$

(C) $\displaystyle-\int \frac{dt}{t}$

(D) $\displaystyle-\int \frac{2dt}{t}$

The only viable options are A & B

$$t = \tan\frac{x}{2} \Rightarrow x = 2\tan^{-1} t$$

$$\therefore dx = \frac{2}{1+t^2}\,dt$$

$$\begin{aligned}\int \frac{dx}{1-\sin x-\cos x} &= \int \frac{\frac{2\,dt}{1+t^2}}{1-\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \\ &= \int \frac{2\,dt}{1+t^2-2t-1+t^2} \\ &= \int \frac{2\,dt}{2t^2-2t} \\ &= \int \frac{dt}{t^2-t}.\end{aligned}$$

A	105
B	8
C	2
D	2

Question 4

If $|z| = 1$ and $w = \dfrac{z-1}{z+1}$ (where $z \neq -1$), then which of the following is equal to $\mathrm{Re}(w)$?

(A) 0

(B) $\dfrac{1}{|z+1|^2}$

(C) $\left|\dfrac{1}{z+1}\right| \cdot \dfrac{1}{|z+1|^2}$

(D) $\dfrac{\sqrt{2}}{|z+1|^2}$

A	75
B	8
C	12
D	23

Let $z = e^{i\theta}$ since $|z| = 1$.

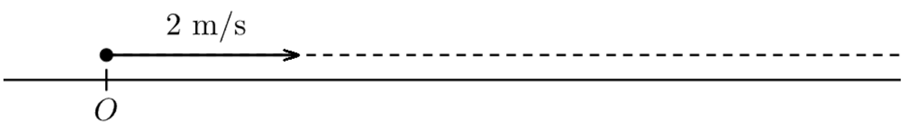
$$\begin{aligned}
\therefore w &= \frac{z-1}{z+1} \\
&= \frac{e^{i\theta}-1}{e^{i\theta}+1} \\
&= \frac{e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})}{e^{i\theta/2}(e^{i\theta/2} + e^{-i\theta/2})} \\
&= \frac{2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} \quad \left((\because z + \overline{z} = 2\mathrm{Re}(z), z - \overline{z} = 2i\mathrm{Im}(z)\right) \\
&= i\tan\frac{\theta}{2}.
\end{aligned}$$

Alternatively: w is related to the angle in a semi-circle

i.e. $\arg w = 90°$

Question 5

A particle moves in a straight line such that $a = \dfrac{1}{\sqrt{x+1}}$ m/s^2.



It is initially at the origin, moving to the right at 2 m/s.

At what position will it first reach a speed of 4 m/s?

(A) $x = 8$

(B) $x = 9$

(C) $x = 15$

(D) $x = 16$

A	9
B	6
C	102
D	1

$$a = v\frac{dv}{dx} = \frac{1}{\sqrt{x+1}}$$

$$\int_2^4 v\, dv = \int_0^x \frac{1}{\sqrt{x+1}}\, dx$$

$$\left[\frac{1}{2}v^2\right]_2^4 = 2\left[\sqrt{x+1}\right]_0^x$$

$$\frac{1}{2}(4^2 - 2^2) = 2(\sqrt{x+1} - \sqrt{1})$$

$$3 = \sqrt{x+1} - 1$$

$$\sqrt{x+1} = 4$$

Suggestion: At least do definite integrals in MC!

$$x + 1 = 16$$

$$x = 15$$

Question 6

Let $z = x + iy$. Which of the following options is equal to $|e^{iz}|$?

(A) 1

(B) e^y

(C) e^{-y}

(D) e^x

$$
\begin{aligned}
|e^{iz}| &= |e^{i(x+iy)}| \\
&= |e^{ix-y}| \\
&= |e^{-y}e^{ix}| \\
&= |e^{-y}(\cos x + i \sin x)| \\
&= e^{-y} > 0
\end{aligned}
$$

A	25
B	7
C	81
D	5

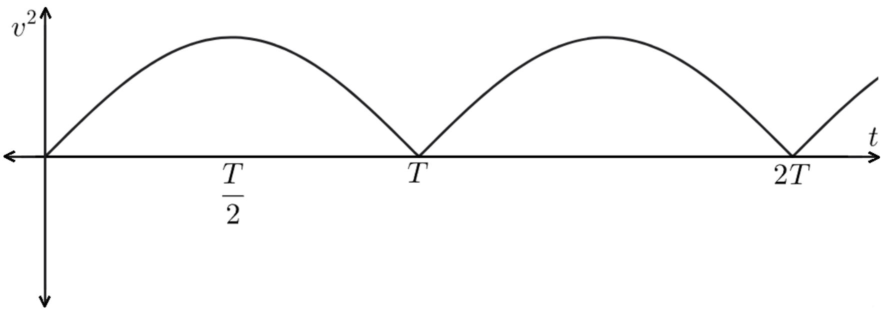
Question 7

A particle is executing simple harmonic motion with a period of T seconds.

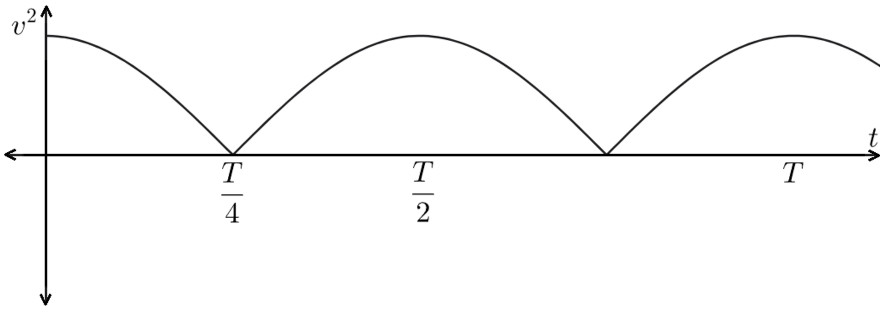
At time $t = 0$, it is at its centre of motion.

Which of the following would be the best graph of v^2 against time?

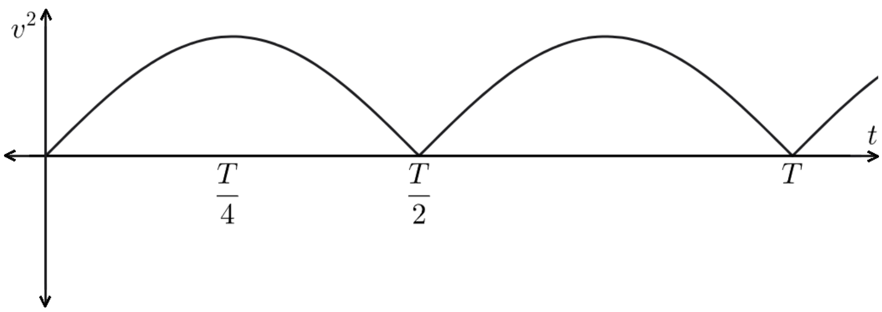
(A)



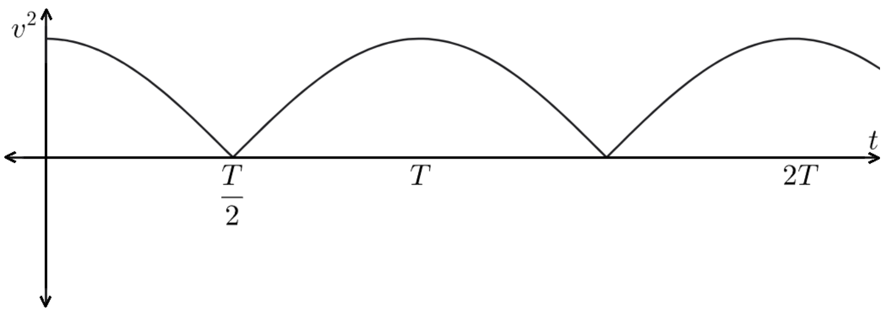
(B)



(C)



(D)



A	12
B	66
C	18
D	22

$\because$ Starts at centre

$\therefore v^2$ is at a maximum at $t = 0$, cannot be (A) or (C)

$\because$ It will return to centre and reach a maximum speed again at half the period

$\therefore$ Cannot be (D). Hence, (B).

Question 8

Find the magnitude of the vector $\underset{\sim}{u} = \underset{\sim}{i} + \tan\theta \underset{\sim}{j}$ for $\theta \in \left(\frac{\pi}{2}, \pi\right)$.

(A) 1

(B) $\sec\theta$

(C) $\tan\theta$

(D) $-\sec\theta$

A	0
B	43
C	0
D	75

$$\begin{aligned}
|\underset{\sim}{u}| &= |\underset{\sim}{i} + \tan\theta \underset{\sim}{j}| \\
&= \sqrt{1+\tan^2\theta} \\
&= \sqrt{\sec^2\theta} \\
&= |\sec\theta| \\
&= -\sec\theta > 0 \ \left(\because \theta \in \left(\frac{\pi}{2}, \pi\right)\right)
\end{aligned}$$

Question 9

ω is a non-real cube root of unity. Evaluate

$$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\dots(1+\omega^{2^n})$$

where n is some even positive integer.

(A) 1

(B) ω

(C) $\frac{1}{\omega}$

(D) $-\frac{1}{\omega}$

A	57
B	7
C	30
D	24

$\because \omega$ is a cube root of unity, $\therefore \omega^3 = 1$ (1)

$\because \omega^3 - 1 = 0$, $\therefore (\omega-1)(\omega^2+\omega+1) = 0 \Rightarrow \omega^2+\omega+1 = 0$ (2) $(\because \omega \neq 1 \in \mathbb{R})$

$\therefore (1+\omega)(1+\omega^2) = 1+\omega+\omega^2+\omega^3 = 0+1 = 1$ (From (1) and (2))

Also, $\omega^4 = \omega^3 \times \omega = \omega,\ \omega^8 = \omega^6 \times \omega^2 = \omega^2,\ \omega^{16} = \omega^{15} \times \omega = \omega \dots$

$$\therefore \omega^{2^k} = \begin{cases} \omega, & \text{if } k \text{ is even,} \\ \omega^2, & \text{if } k \text{ is odd} \end{cases} \quad (3)$$

$$\begin{aligned}
&\therefore (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\dots(1+\omega^{2^n}) \\
&= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\dots(1+\omega) \ \text{(From (3))} \\
&= 1 \times 1 \times \dots \times 1 \times (1+\omega) \\
&= -\omega^2 \ \text{(From (2))} \\
&= -\frac{1}{\omega} \ \left(\text{From (1), } \omega^2 = \frac{1}{\omega}\right)
\end{aligned}$$

Question 10

Let $\alpha, \beta \in \mathbb{C}$ so that $\text{Im}(\alpha) = \text{Im}(\beta)$ and $\text{Re}(\alpha) > \text{Re}(\beta)$.

Determine which of the following equations can be used to find the set of values z where

$$\text{Arg}(z - \alpha) = 2\,\text{Arg}(z - \beta).$$

Note that $\text{Arg}(z)$ denotes the principal argument of z.

(A) $|z - \alpha| = |\beta - \alpha|$

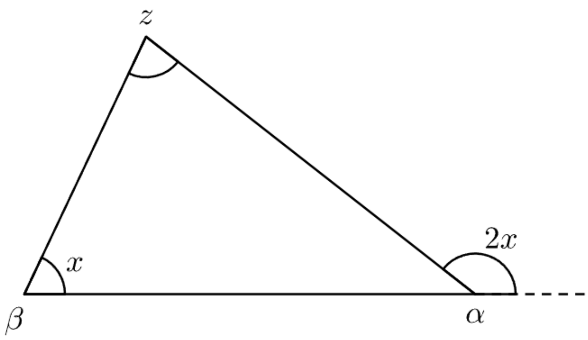
(B) $|z - \beta| = |\beta - \alpha|$

(C) $|z - \alpha| = |z - \beta|^2$

(D) $|z - \beta| = |z - \alpha|^2$

A	17
B	15
C	60
D	24

Consider z, α, β on the Argand diagram. Let $x = \text{Arg}\,(z - \beta)$.



Using exterior angles of a triangle, the triangle is isosceles.

$\therefore |z - \alpha| = |\beta - \alpha| = |\alpha - \beta|$.

Alternatively:

$\because \text{Arg}(z - \alpha) = 2\,\text{Arg}(z - \beta) \therefore$ Angle at the centre is twice the angle at the circumference.

$\therefore$ The centre is at $z = \alpha$, with radius $|\beta - \alpha| \Rightarrow |z - \alpha| = |\beta - \alpha|$.

(a) Define $z = 5 - 12i$ and $w = 3 + 4i$. Find the value of:

(i) v such that $3\overline{v} = z - 2\overline{w}$; **2**

$3\overline{v} = 5 - 12i - 2(3 - 4i)$

$= 5 - 12i - 6 + 8i$

$= -1 - 4i$

$\therefore 3v = -1 + 4i$

$\therefore v = \frac{1}{3}(-1 + 4i)$ OR $-\frac{1}{3} + \frac{4}{3}i$

(ii) $\frac{2}{w} + z$; **1**

$$\frac{2}{w} + z = \frac{2}{3 + 4i} + 5 - 12i$$

$$= \frac{2(3 - 4i)}{25} + 5 - 12i$$

$$= 5\frac{6}{25} - \left(12\frac{8}{25}\right)i$$

Comments: These were done well.

Some students forgot the conjugate or didn't know how to expand brackets properly.

(a) (continued)

(iii) u such that u is a square root of zw. **3**

Leave your answer in the form $a + ib$, where a, b are real numbers with $a > 0$.

$zw = (5 - 12i)(3 + 4i) = 63 - 16i$

Method 1: Standard

Let $u^2 = (a + ib)^2 = 63 - 16i$

$$\therefore a^2 - b^2 = 63 \qquad -(1)$$
$$2ab = -16 \qquad -(2)$$
$$|u^2| = a^2 + b^2 = 65 \qquad -(3)$$

(3) + (1): $2a^2 = 128 \Rightarrow a^2 = 64$

$\therefore a = 8 \ (a > 0)$

$\therefore$ From (2): $b = -1$

$\therefore u = 8 - i$

Method 2: "Inspection"

$$\begin{aligned} 63 - 16i &= 63 - 2 \times 8\, i \\ &= 63 - 2 \times (8 \times 1)\, i \\ &= (64 - 1) - 2 \times (8 \times 1)\, i \\ &= (8 - i)^2 \end{aligned}$$

$\therefore u = 8 - i$

Method 3: "Quartic Polynomial"

Avoid!

Comment: It was surprising the number of students who opted for the quartic polynomial. The goal is to reduce time in Q11, as you will need more time later on.

A lot of students didn't read about a being positive, and so could not get full marks.

(b) Find the values of a if the vectors $\underset{\sim}{p} = \begin{pmatrix} 9a-4 \\ 5 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} a \\ -1 \end{pmatrix}$ are perpendicular. **2**

$$\begin{pmatrix} 9a-4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} a \\ -1 \end{pmatrix} = 0$$

$$\therefore 9a^2 - 4a - 5 = 0$$

$$\therefore (9a + 5)(a - 1) = 0$$

$$\therefore a = -\frac{5}{9}, 1$$

Comments: Almost everyone got at least 1 out of 2.

Getting full marks depended on one's ability to solve a quadratic equation.

(c) (i) Find the values of constants A, B and C such that **2**

$$\frac{5x^2 - 9x + 4}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}.$$

Method 1: Standard

$$5x^2 - 9x + 4 = A\left(x^2 + 2\right) + (Bx + C)(x - 3)$$

Let $x = 3$: $5(9) - 9(3) + 4 = A(9 + 2)$
$\therefore A = 2$

$A + B = 5$ (coefficient of x^2) $\Rightarrow B = 3$

Let $x = 0$: $4 = 2(2) + C(-3)$
$\therefore C = 0$

Method 2: Non-Standard

$$\frac{5x^2 - 9x + 4}{x^2 + 2} = A + \frac{(Bx + C)(x - 3)}{x^2 + 2}$$

Let $x = 3$: $A = \dfrac{5(3)^2 - 9(3) + 4}{(3)^2 + 2} = 2$

$A + B = 5$ (coefficient of x^2) $\Rightarrow B = 3$

Let $x = 0$: $\dfrac{4}{2} = 2 + \dfrac{(C)(-3)}{2} \Rightarrow C = 0$

Comment: This was done well.

Again for doing questions quickly (or not), many students opted for the method where they expanded the brackets in Method 1 and then did simultaneous equations.

Students who used the “cover up” rule need to be careful if they get a “Show that” question in the HSC as the “cover up” rule doesn’t “show” anything.

(c) (continued)

(ii) Hence, or otherwise, find $\displaystyle\int \frac{5x^2 - 9x + 4}{(x-3)(x^2+2)}\,dx$. **2**

$$\int \frac{5x^2-9x+4}{(x-3)(x^2+2)}\,dx = \int \frac{2}{x-3} + \frac{3x}{x^2+2}\,dx$$

$$= \int \frac{2}{x-3} + \frac{3}{2} \times \frac{2x}{x^2+2}\,dx$$

$$= 2\ln|x-3| + \frac{3}{2}\ln|x^2+2| + C$$

Comment: Done well, though surprising those who thought they had an integral involving arctan x.

Students who forgot the "absolute value" were penalised here. Of course, the absolute value is not needed in the second term.

(d) Find $\int \cos^{-1}(2x)\,dx$. **3**

$$\int \cos^{-1} 2x\,dx = \int 1 \times \cos^{-1} 2x\,dx$$

$$= \int \frac{d}{dx}(x) \times \cos^{-1} 2x\,dx$$

$$= x\cos^{-1} 2x - \int x \times \frac{d}{dx}\left(\cos^{-1} 2x\right) dx$$

$$= x\cos^{-1} 2x - \int x \times \left(-\frac{2}{\sqrt{1-4x^2}}\right) dx$$

$$= x\cos^{-1} 2x - \frac{1}{4}\int \frac{-8x}{\sqrt{1-4x^2}}\,dx$$

Use Reference Sheet here

$$= x\cos^{-1} 2x - \frac{1}{4} \times 2\left(1-4x^2\right)^{\frac{1}{2}} + C$$

$$= x\cos^{-1} 2x - \frac{1}{2}\left(1-4x^2\right)^{\frac{1}{2}} + C$$

Comment: Many students had errors on "–", and also the constant (here $\frac{1}{4}$).

Students who couldn't differentiate cos $^{-1}(2x)$ properly (apart from the "–") could only get a maximum of 2 marks as they made their answer even simpler than it already was.

Students weren't penalised this time if they left their answer in terms of cos $^{-1}(2x)$ after doing a substitution. Again, I don't know why as they just put off the steps to doing Integration by Parts.
If you are going to do a substitution then make sure you know how it works!

Some students didn't manage Integration by Parts successfully as they were left with an tan $^{-1}(\ )$ in their answer.

The goal is to minimise time: doing integration by substitution is not helping this goal.

Students who forgot the "+ C" were penalised here.

From the Reference sheet as mentioned above.

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

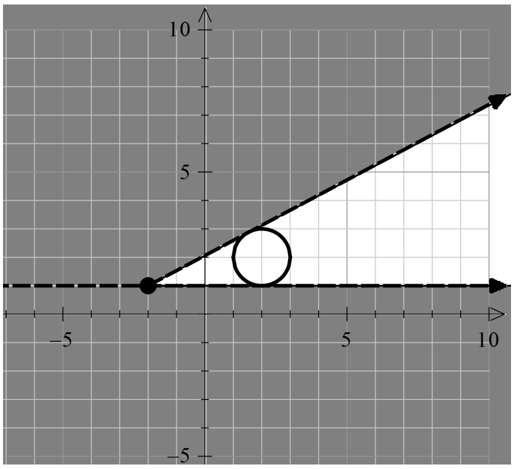
Question 12 Solutions

Question 12 (14 marks) Use a SEPARATE writing booklet.

(a) Calculate the value of $k \in \mathbb{R}$ so that there are only two points of intersection between $\left|\operatorname{Arg}(z-k-i)-\frac{\pi}{6}\right| \geq \frac{\pi}{6}$ and $|z-2-2i|=1$. **2**

$$\operatorname{Arg}(z-(k+i))-\frac{\pi}{6} \geq \frac{\pi}{6} \Rightarrow \operatorname{Arg}(z-(k+i)) \geq \frac{\pi}{3}$$
$$-\operatorname{Arg}(z-(k+i))+\frac{\pi}{6} \geq \frac{\pi}{6} \Rightarrow \operatorname{Arg}(z-(k+i)) \leq 0$$

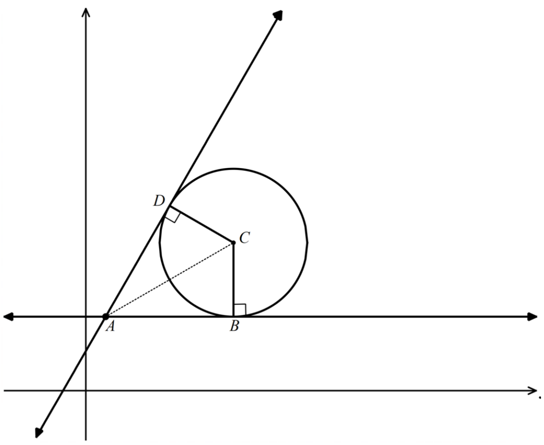
$\Rightarrow$ The only region this doesn't cover is $0 < \operatorname{Arg}(z-(k+i)) < \frac{\pi}{3}$



Let A be the point $k+i$. Let B be the point where Im (z) = 1 touches the circle. Let D be the point where

$$Arg(z-(k+i))=\frac{\pi}{3}$$

touches the circle and is a tangent. Let C be the centre of the circle.



Since $CB = 1 = CD$, AB and AD are tangents and AC is common, then triangles ACD and ABD are congruent such that

$$\angle BAC = \frac{\pi}{6}.$$

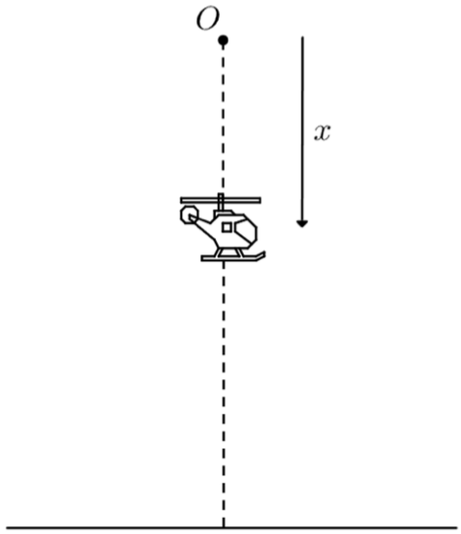
$\therefore$ AB must equal $\sqrt{3}$.

Hence $k = 2 - \sqrt{3}$.

Note: Most pupils tried to use coordinate geometry to solve this and I don't think any of them were successful.

(b) A 2 kg toy helicopter is low on battery and provides a thrust of 10 N upwards.

It starts falling from rest, and is subject to gravity and air resistance proportional to v^2 where the constant of proportionality is $k = 0.096$. Take $g = 9.8$ m/s^2.



Let x be the downward displacement of the helicopter from its initial position in metres, along with acceleration a m/s^2 and velocity v m/s.

(i) Explain why the motion of the helicopter follows the acceleration equation **1**

$$a = 4.8 - 0.048v^2.$$

$$\begin{aligned} 2a &= -10 + 2g - kv^2 \\ a &= -5 + 9.8 - \frac{k}{2}v^2 \\ &= 4 \cdot 8 - 0.048v^2 \end{aligned}$$

Note: If a pupil did not consider the 10 N upthrust or treated g as a force, then they could NOT get full marks. The expression for resistance was given as a force - most pupils figured that out based on the equation they were trying to "show".

(ii) Prove that the velocity of the helicopter is given by **3**

$$v = \frac{10(e^{0.96t} - 1)}{e^{0.96t} + 1}.$$

$$\begin{aligned}\frac{dv}{dt} &= \frac{4800}{1000} - \frac{48}{1000}v^2 \\ &= \frac{48}{1000}\left(100 - v^2\right)\end{aligned}$$

$$20 \times \frac{1}{20}\left(\frac{1}{10-v} + \frac{1}{10+v}\right)\frac{dv}{dt} = \frac{48}{1000} \times 20$$

$$\int_0^v \left(\frac{1}{10-V} + \frac{1}{10+V}\right) dV = \frac{48}{50}\int_0^t dT$$

$$\left[\ln\left|\frac{10+V}{10-V}\right|\right]_0^v = \left[\frac{48}{50}T\right]_0^t \qquad \#$$

$$\ln\left|\frac{10+v}{10-v}\right| - 0 = \frac{48}{50}t - 0$$

$$\frac{10+v}{10-v} = e^{\frac{24}{25}t}\,, v < 10.$$

$$10 + v = (10 - v)\,e^{\frac{24}{25}t}$$

$$v\left(1 + e^{\frac{24}{25}t}\right) = 10\left(e^{\frac{24}{25}t} - 1\right)$$

$$\therefore v = 10\frac{e^{\frac{24}{25}t} - 1}{e^{\frac{24}{25}t} + 1}$$

Note: Pupils that used a different function to the one given had to obtain $v(t)$ for the one they gave. Those obstinate enough to do so, were not able to find the appropriate $v(t)$ for the equation they used.

(iii) Given that the helicopter hits the ground with speed 6.5 m/s, calculate the height the helicopter was initially at, correct to 2 decimal places. **2**

$$a = \frac{12}{125}\left(100 - v^2\right)$$

$$v\frac{dv}{dx} = \frac{12}{125}\left(100 - v^2\right)$$

$$\int_0^v \frac{-2}{100 - V^2}\,dV = -\frac{24}{125}\int_0^x dX$$

$$\left[\ln\left(100 - V^2\right)\right]_0^v = \left[-\frac{24}{125}X\right]_0^x$$

$$\ln\left(100 - v^2\right) - 2\ln 10 = -\frac{24}{125}x + 0$$

$$\Rightarrow x = \frac{125}{12}\ln\frac{10}{\sqrt{100 - v^2}}$$

$$x\left(\frac{13}{2}\right) = \frac{125}{12}\ln\frac{10}{\sqrt{\frac{400 - 169}{4}}}$$

$$= \frac{125}{12}\ln\frac{20}{\sqrt{231}}$$

$$\approx 5.72$$

Note: This was reasonably well done. A solution has been provided (a few pages on) for students who used part (ii) to solve this.

(c) (i) Using De Moivre's Theorem, prove that **3**

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

Let $z = \cos\theta + i\sin\theta$

1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$z^6 = \cos^6\theta + i6\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - i20\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + i6\cos^5\theta\sin\theta - \sin^6\theta \text{ ①}$$

$$z^6 = \cos 6\theta + i\sin 6\theta \text{ ②}$$

Re (*RHS* ②) = *Re* (*RHS* ①)

$$\cos 6\theta = \cos^6\theta - 15\cos^4\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$\sin^2 0 = 1 - \cos^2\theta \Rightarrow \sin^4\theta = 1 - 2\cos^2\theta + \cos^4\theta$$

$$\begin{array}{rcl}
15\cos^2\theta\sin^4\theta & = & 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta \\
-15\cos^4\theta\sin^2\theta & = & -15\cos^4\theta + 15\cos^6\theta \\
-\sin^6\theta & = & -1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta \\
\cos^6\theta & = & \cos^6\theta \\
\hline
\cos 6\theta & = & -1 + 18\cos^2\theta - 45\cos^4\theta + 32\cos^6\theta
\end{array}$$

Note: Some pupils failed to expand the terms in terms of cosines only and could not "show" how the result comes about and hence couldn't get full marks.

(ii) Hence, by making the substitution $x = A\cos\theta$ for a suitable A, or otherwise, find all the roots of the polynomial **3**

$$P(x) = x^6 - 6x^4 + 9x^2 - 2.$$

$$P(A\cos\theta) = A^6\cos^6\theta - 6A^4\cos^4\theta + 9A^2\cos^2\theta - 2$$
$$\frac{0}{2} = \frac{A^c}{2}\cos^6\theta - 3A^4\cos^4\theta + \frac{9}{2}A^2\cos^2\theta - 1$$

When $A = 2 \Rightarrow 0 = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

$$0 = \cos 6\theta, -\pi < \theta \le \pi \Rightarrow -6\pi < 6\theta \le 6\pi$$

$$6\theta = -\frac{11\pi}{2}, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

$$\theta = -\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$$

$$x = \pm 2\cos\frac{\pi}{12}, \pm\frac{2}{\sqrt{2}}, \pm 2\cos\frac{5\pi}{12}$$

End of Question 12

Note: Some pupils obtained an incorrect value for A. Some solved for one instead of zero. Finally, many pupils did not provide six unique solutions. They had forgotten that the cosine of x is a even function.

b (ii) Prove that the velocity of the helicopter is given by **3**

$$v = \frac{10(e^{0.96t} - 1)}{e^{0.96t} + 1}.$$

$$\frac{dv}{dt} = \frac{4800}{1000} - \frac{48}{1000}v^2$$

$$= \frac{48}{1000}\left(100 - v^2\right)$$

$$20 \times \frac{1}{20}\left(\frac{1}{10-v} + \frac{1}{10+v}\right)\frac{dv}{dt} = \frac{48}{1000} \times 20$$

$$\int \left(\frac{1}{10-v} + \frac{1}{10+v}\right)\frac{dv}{dt}dt = \frac{48}{50}\int dt$$

$$\ln\left|\frac{10+v}{10-v}\right| = \frac{48}{50}t + C \qquad \#$$

$$\frac{10+v}{10-v} = Ae^{\frac{24}{25}t} \qquad *$$

$$10 + v = (10 - v)\,Ae^{\frac{24}{25}t}$$

$$v\left(1 + Ae^{\frac{24}{25}t}\right) = 10\left(Ae^{\frac{24}{25}t} - 1\right)$$

$$v = 10\frac{Ae^{\frac{24}{25}t} - 1}{Ae^{\frac{24}{25}t} + 1}$$

$$v(0) = 0 \Rightarrow A = 1 \qquad (\text{from } *)$$

$$\therefore v = 10\frac{e^{\frac{24}{25}t} - 1}{e^{\frac{24}{25}t} + 1}$$

b (iii) Given that the helicopter hits the ground with speed 6.5 m/s, calculate the height the helicopter was initially at, correct to 2 decimal places. **2**

$$\begin{aligned}
a &= \frac{12}{125}\left(100 - v^2\right) \\
v\frac{dv}{dx} &= \frac{12}{125}\left(100 - v^2\right) \\
\int \frac{-2}{100 - v^2} v' dx &= -\frac{24}{125}\int dx \\
\ln\left(100 - v^2\right) &= -\frac{24}{125}x + C
\end{aligned}$$

$$\Rightarrow x = -\frac{12}{125}\ln\sqrt{100 - v^2} + D$$

$$\begin{aligned}
x\left(0\right) = 0 \Rightarrow D &= \frac{125}{12}\ln 10 \\
x &= \frac{125}{12}\ln\frac{10}{\sqrt{100 - v^2}}
\end{aligned}$$

$$\begin{aligned}
x\left(\frac{13}{2}\right) &= \frac{125}{12}\ln\frac{10}{\sqrt{\dfrac{400 - 169}{4}}} \\
&= \frac{125}{12}\ln\frac{20}{\sqrt{231}} \\
&\approx 5.72
\end{aligned}$$

b (iii) Given that the helicopter hits the ground with speed 6.5 m/s, calculate the height the helicopter was initially at, correct to 2 decimal places. **2**

$$v\left(t_1\right) = \frac{13}{2}\text{m/s}$$

$$\ln\left(\frac{10+\frac{13}{2}}{10-\frac{13}{2}}\right) = \frac{24}{25}t_1, \text{ from \# where } C = 0, \text{ since } A = 1.$$

$$\frac{25}{24}\ln\left(\frac{\frac{33}{2}}{\frac{7}{2}}\right) = t_1$$

$$t_1 = \frac{25}{24}\ln\left(\frac{33}{7}\right)$$

$$\int \frac{dx}{dt}dt = 10\int \frac{e^{\frac{24}{25}t}-1}{e^{\frac{24}{25}t}+1}dt$$

Let $u = e^{\frac{24}{25}t}+1 \qquad \Rightarrow \frac{du}{dt} = \frac{24}{25}e^{\frac{24}{25}t}$

$e^{\frac{24}{25}t} = u-1 \qquad \Rightarrow \frac{25}{24}\frac{du}{(u-1)} = dt$

$$\begin{aligned} x &= \frac{250}{24}\int \frac{u-2}{u(u-1)}du \\ &= \frac{125}{12}\int\left(\frac{2}{u}-\frac{1}{u-1}\right)du \\ &= \frac{125}{6}\ln\left(e^{\frac{24}{25}t}+1\right) - \frac{125}{12}\ln\left(e^{\frac{24}{25}t}\right) + C \\ &= -10t + \frac{125}{6}\ln\left(e^{\frac{24}{25}t}+1\right) + C \end{aligned}$$

$$x(0) = 0 \Rightarrow 0 = -0 + \frac{125}{12}\ln 2 + C$$

$$x = -10t + \frac{125}{6}\left(\ln\left(e^{\frac{24}{25}t} + 1\right) - \ln 2\right)$$

$$\begin{aligned}
x\left(\frac{25}{24}\ln\left(\frac{33}{7}\right)\right) &= -10 \times \frac{25}{24}\ln\left(\frac{33}{7}\right) + \frac{125}{6}\left(\ln\left(e^{\frac{24}{25}\times\frac{25}{24}\ln\left(\frac{33}{7}\right)} + 1\right) - \ln 2\right) \\
&= -\frac{125}{6} \times \frac{1}{2}\ln\frac{33}{7} + \frac{125}{6}\left(\ln\frac{40}{7} - \ln 2\right) \\
&= \frac{125}{6}\left(\ln\frac{20}{7} - \frac{1}{2}\ln\frac{33}{7}\right) \\
&\approx 5.72
\end{aligned}$$

Question 13 Solutions (15 marks)

(a) A ball moves in a straight line according to

$$v^2 = 9(12 + 4x - x^2)$$

where v is the velocity of the ball in cm/s and x is the displacement of the ball from the origin in cm.

(i) Prove that the ball is moving in simple harmonic motion. **1**

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left[\frac{9}{2}(12 + 4x - x^2)\right]$$

$$= \frac{9}{2}(4 - 2x)$$

$$= -3^2(x - 2)$$ which is in the form of $a = -n^2(x - x_0)$

$\therefore$ The ball is moving in simple harmonic motion.

- Generally well done.
- $v^2 = n^2\left[A^2 - (x - x_0)^2\right]$ is not quotable.

(ii) By considering $v^2 \geq 0$, or otherwise, find the range of x values that the ball can move in. **1**

$$v^2 = -9(x^2 - 4x - 12)$$

$$= -9(x - 6)(x + 2) \geq 0$$

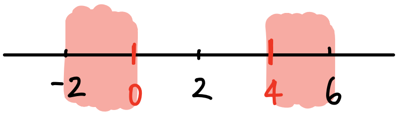
$$\therefore -2 \leq x \leq 6$$

- Generally well done.

(iii) The ball is considered to be in the extreme zone when it deviates from its central position for more than 2 cm. **3**

It completes one whole oscillation and returns to its starting position.

For what percentage of time, correct to the nearest whole number, is the ball in the extreme zone?



$\text{Amplitude} = \dfrac{6-(-2)}{2} = 4$

$n = 3$

$x_0 = 2$

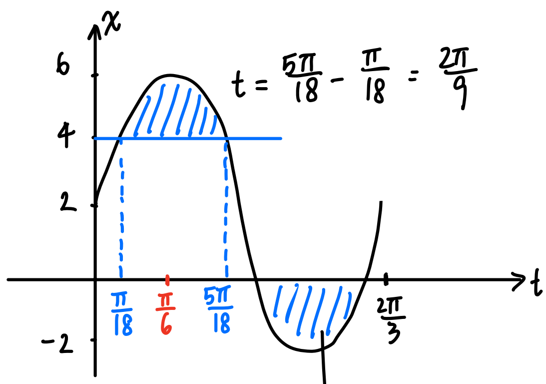
Let $x = 4\sin 3t + 2$

$4\sin 3t + 2 = 4$

$\sin 3t = \dfrac{1}{2}$

$3t = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$t = \dfrac{\pi}{18}, \dfrac{5\pi}{18}$



$t = \dfrac{5\pi}{18} - \dfrac{\pi}{18} = \dfrac{2\pi}{9}$

$t = \dfrac{2\pi}{9}$ (By symmetry)

$\therefore \dfrac{\frac{2\pi}{9} \times 2}{\frac{2\pi}{3}} = \dfrac{2}{3} = 67\%$ (nearest whole number)

- Since the motion is SHM, you can assume any of these 2 equations.

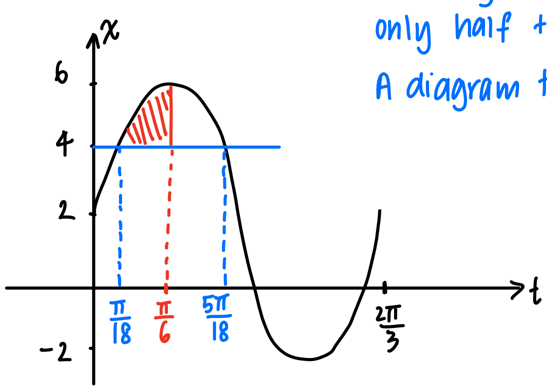
  $x = a\cos(nt+\alpha) + c$ (formula sheet)

  $x = a\sin(nt+\alpha) + c$

  Don't waste your time trying to derive it.

- $n = 3$ is given to us, so students should not assume $n = 1$ or any other number – despite still giving the same answer.

- A common error was thinking the time between $x = 4 \sim 6$ is the red area, when it is only half that time. As a result 33% was a popular answer. A diagram to visualise the motion would have helped.



(b) By considering the expansion of $(z+1)^4$, or otherwise, solve **3**

$$z^4 + 4z^3 + 8z^2 + 8z + 3 = 0.$$

$(z+1)^4 = 1 + 4z + 6z^2 + 4z^3 + z^4$

$$z^4 + 4z^3 + 8z^2 + 8z + 3 = z^4 + 4z^3 + 6z^2 + 4z + 1 + 2z^2 + 4z + 2$$

$$= (z+1)^4 + 2(z+1)^2 = 0$$

$$(z+1)^2\left[(z+1)^2 + 2\right] = 0$$

$z = -1, \quad (z+1)^2 = -2$

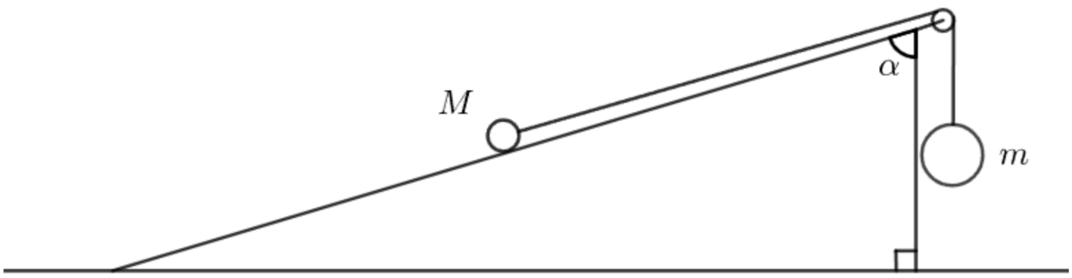
$$z + 1 = \pm\sqrt{2}\,i$$

$$z = -1 \pm \sqrt{2}\,i$$

$\therefore \quad z = -1, -1 \pm \sqrt{2}\,i$

- Generally well done.
- It is implied you need to solve over $\mathbb{C}$.
- Many students expanded and used quadratic formula to solve $(z+1)^2 + 2 = 0$.
- Common error: $\pm\sqrt{-2} = \pm 2i$

(c) The two weights in the diagram are of mass m kg and M kg as shown.



For the weight on the slope, M kg, there is a frictional force, F acting on it.

It is given that $F = \mu N$, where N is the normal reaction force of the slope on the weight and μ is a real constant.

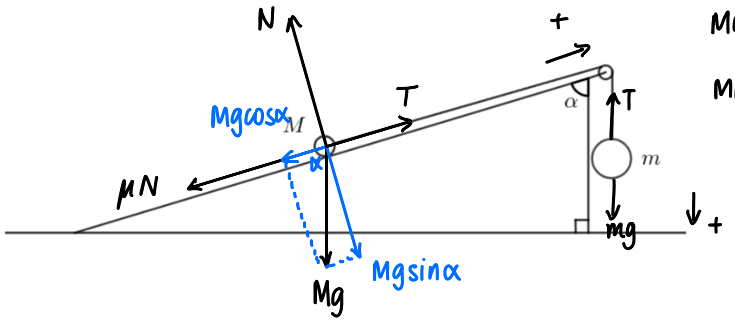
Assume that the mass of the string is negligible.

(i) The m kg mass is moving downwards. **2**

By resolving parallel and perpendicular to the slope, show that the acceleration, a, of the masses is given by

$$a = \frac{g(m - \mu M \sin\alpha - M\cos\alpha)}{m + M},$$

where g is the acceleration due to gravity.



Mass m: $ma = mg - T$ ①

Mass M

Vertically: $N = Mg\sin\alpha$ ②

Horizontally: $Ma = T - Mg\cos\alpha - \mu N$ ③

② → ①+③: $ma + Ma = mg - Mg\cos\alpha - \mu Mg\sin\alpha$

$$a(m+M) = g(m - \mu M\sin\alpha - M\cos\alpha)$$

$$\therefore\ a = \frac{g(m - \mu M\sin\alpha - M\cos\alpha)}{m+M} \quad ④$$

- Poorly done, seeing as this is a show question, it was very difficult for students to receive many marks if their working only looked like this.

$$a(m+M) = g(m - \mu M\sin\alpha - M\cos\alpha)$$

$$\therefore\ a = \frac{g(m - \mu M\sin\alpha - M\cos\alpha)}{m+M}$$

This is basically reverse engineering, you haven't shown anything.

(ii) Find the tension in the string. **2**

Method 1

④ → ① : $T = mg - \dfrac{mg(m - \mu M\sin\alpha - M\cos\alpha)}{m+M}$

$= \dfrac{mg}{m+M}\left[\cancel{m} + M - \cancel{m} + \mu M\sin\alpha + M\cos\alpha\right]$

$= \dfrac{mMg}{m+M}\left(1 + \mu\sin\alpha + \cos\alpha\right)$

Method 2

②, ④ → ③ : $\dfrac{Mg(m - \mu M\sin\alpha - M\cos\alpha)}{m+M} = T - Mg\cos\alpha - \mu Mg\sin\alpha$

$T = \dfrac{Mg(m - \mu M\sin\alpha - M\cos\alpha)}{m+M} + Mg\cos\alpha + \mu Mg\sin\alpha$

$= \dfrac{Mg}{m+M}\left[m - \mu M\sin\alpha - M\cos\alpha + (m+M)(\cos\alpha + \mu\sin\alpha)\right]$

$= \dfrac{Mg}{m+M}\left[m - \cancel{\mu M\sin\alpha} - \cancel{M\cos\alpha} + m\cos\alpha + m\mu\sin\alpha + \cancel{M\cos\alpha} + \cancel{\mu M\sin\alpha}\right]$

$= \dfrac{Mgm}{m+M}\left(1 + \cos\alpha + \mu\sin\alpha\right)$

Method 3

② → ③ − ①: $Ma - ma = 2T - Mg\cos\alpha - \mu Mg\sin\alpha - mg$ ⑤

④ → ⑤: $\dfrac{(M-m)g\,(m - \mu M\sin\alpha - M\cos\alpha)}{m+M} = 2T - Mg\cos\alpha - \mu Mg\sin\alpha - mg$

$$2T = \frac{(M-m)g\,(m - \mu M\sin\alpha - M\cos\alpha)}{m+M} + Mg\cos\alpha + \mu Mg\sin\alpha + mg$$

$$T = \frac{g}{2(M+m)}\Big[(M-m)(m - \mu M\sin\alpha - M\cos\alpha) + (m+M)(M\cos\alpha + \mu M\sin\alpha + m)\Big]$$

$$= \frac{g}{2(M+m)}\Big[Mm - \cancel{\mu M^2\sin\alpha} - \cancel{M^2\cos\alpha} - \cancel{m^2} + Mm\mu\sin\alpha + Mm\cos\alpha +$$
$$Mm\cos\alpha + \mu Mm\sin\alpha + \cancel{m^2} + \cancel{M^2\cos\alpha} + \cancel{\mu M^2\sin\alpha} + Mm\Big]$$

$$= \frac{g}{\cancel{2}(M+m)}\Big[\cancel{2}Mm + \cancel{2}Mm\mu\sin\alpha + \cancel{2}Mm\cos\alpha\Big]$$

$$= \frac{Mmg}{M+m}\Big[1 + \mu\sin\alpha + \cos\alpha\Big]$$

- Since many reverse engineered (i), finding the tension proved difficult.
- Method 1 was by far the easiest approach.

(iii) Prove that if the vertically hung mass is moving downwards, then **3**

$$\alpha < \sin^{-1}\left(\frac{m}{M\sqrt{1+\mu^2}}\right) - \tan^{-1}\frac{1}{\mu}.$$

You may assume that $\alpha + \tan^{-1}\frac{1}{\mu} < \frac{\pi}{2}$.

Moving downwards when $a = \dfrac{g(m - \mu M\sin\alpha - M\cos\alpha)}{m + M} > 0$

$m - \mu M\sin\alpha - M\cos\alpha > 0$

$m > M(\mu\sin\alpha + \cos\alpha)$

$\dfrac{m}{M} > \mu\sin\alpha + \cos\alpha$

Let $\mu\sin\alpha + \cos\alpha = R\sin(\alpha + \theta)$

$= R\sin\alpha\cos\theta + R\cos\alpha\sin\theta$

$\mu = R\cos\theta$ ①

$1 = R\sin\theta$ ②

①^2 + ②^2: $\mu^2 + 1 = R^2(\cos^2\theta + \sin^2\theta)$

$R = \sqrt{1+\mu^2}$ $(R>0)$

$\dfrac{②}{①}$: $\tan\theta = \dfrac{1}{\mu}$

$\theta = \tan^{-1}\dfrac{1}{\mu}$

$\therefore \dfrac{m}{M} > \sqrt{1+\mu^2}\sin\left(\alpha + \tan^{-1}\dfrac{1}{\mu}\right)$

$\sin\left(\alpha + \tan^{-1}\dfrac{1}{\mu}\right) < \dfrac{m}{M\sqrt{1+\mu^2}}$

$\alpha + \tan^{-1}\dfrac{1}{\mu} < \sin^{-1}\left(\dfrac{m}{M\sqrt{1+\mu^2}}\right)$ $\left(\alpha + \tan^{-1}\dfrac{1}{\mu} < \dfrac{\pi}{2}\right)$

students had to explicitly mention this in their response.

$\therefore \alpha < \sin^{-1}\left(\dfrac{m}{M\sqrt{1+\mu^2}}\right) - \tan^{-1}\dfrac{1}{\mu}$

- Many students recognised $a>0$, but didn't get much further. Since there is $\frac{m}{M}$ in the result, students should try to get that somehow.
- Reverse engineering was very popular again, but didn't gain many marks.

Question 14 – Solutions

(a) (i) $\because z^5 + i = 0,\ z^5 + i^5 = 0$

$\therefore (z+i)(z^4 - z^3 i + z^2 i^2 - z i^3 + i^4) = 0$

$\therefore z^4 - z^3 i + z^2 i^2 - z i^3 + i^4 = 0\ (\because z \neq -i, z + i \neq 0)$

$\therefore z^4 - iz^3 - z^2 + iz + 1 = 0$

OR

i) $z^5 + i = z^5 + i^5$

$= (z+i)(z^4 - z^3 i + z^2 i^2 - z i^3 + i^4)$

$= (z+i)(z^4 - z^3 i - z^2 + iz + 1)$

since $z^5 + i = 0$ and $z \neq -i$

$z + i \neq 0$

$\therefore (z+i)(z^4 - z^3 i - z^2 + iz + 1) = 0$

$\therefore z^4 - z^3 i - z^2 + iz + 1 = 0$ ...... [1]

Markers Comments:
Very well done. Many students also showed this by taking the sum of a Geometric Progression.

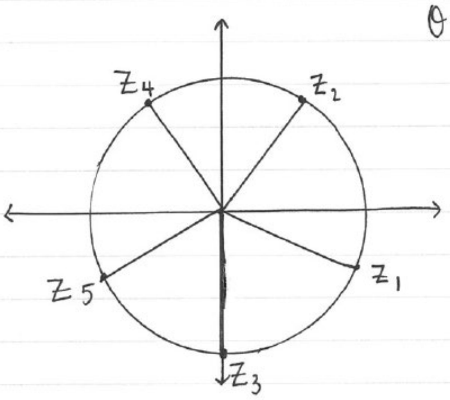
ii) $z^5 = -i$

$(re^{i\theta})^5 = 1 . e^{-\frac{\pi}{2} i}$

$r^5 e^{i5\theta} = 1\ e^{-\frac{\pi}{2} i}$

$\therefore r = 1$ and $5\theta = -\frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

$\theta = \frac{-\pi + 4k\pi}{10}$



$z_1 = e^{-\frac{\pi}{10} i}$

$z_2 = e^{\frac{3\pi}{10} i}$

$z_3 = e^{-\frac{\pi}{2} i}\ (= -i)$

$z_4 = e^{\frac{7\pi}{10} i}$

$z_5 = e^{-\frac{9\pi}{10} i}$ .... [1]

$z^5 + i = (z - -i)\left(z - e^{-\pi i/10}\right)\left(z - e^{\frac{3\pi}{10} i}\right)\left(z - e^{\frac{7\pi}{10} i}\right)\left(z - e^{-\frac{9\pi i}{10}}\right)$

$= (z+i)\left(z^2 - \left(e^{-\frac{\pi}{10} i} + e^{-\frac{9\pi}{10} i}\right) z + e^{-\pi i}\right)$
$\left(z^2 - \left(e^{\frac{3\pi i}{10}} + e^{\frac{7\pi}{10} i}\right) z + e^{\pi i}\right)$

$= (z+i)\left(z^2 - \left(\cancel{\cos\frac{\pi}{10}} - i\sin\frac{\pi}{10} - \cancel{\cos\frac{\pi}{10}} - i\sin\frac{\pi}{10}\right) z - 1\right)$

$\times\left(z^2 - \left(\cancel{\cos\frac{3\pi}{10}} + i\sin\frac{3\pi}{10} - \cancel{\cos\frac{3\pi}{10}} + i\sin\frac{3\pi}{10}\right) - 1\right)$ ...... [1]

$= (z+i)\left(z^2 + 2i\sin\frac{\pi}{10}\, z - 1\right)\left(z^2 - 2i\sin\frac{3\pi}{10}\, z - 1\right)$ - [1]

OR

(ii) $\because z^5 = -i = \text{cis}\left(-\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z}$

$$\therefore z = \text{cis}\left(-\frac{\pi}{10} + \frac{2k\pi}{5}\right),\ \ k = 0, \pm1, \pm2 \text{ (De Moivre's Theorem)}$$

$$\therefore z = \text{cis}\left(-\frac{\pi}{10}\right), \text{cis}\left(\frac{3\pi}{10}\right), \text{cis}\left(\frac{7\pi}{10}\right), \text{cis}\left(-\frac{5\pi}{10}\right) = -i, \text{cis}\left(-\frac{9\pi}{10}\right)$$

$$\therefore z^5 + i = (z+i)(z - e^{-i\frac{\pi}{10}})(z - e^{-i\frac{9\pi}{10}})(z - e^{i\frac{3\pi}{10}})(z - e^{i\frac{7\pi}{10}})$$

$$= (z+i)(z - e^{-i\frac{\pi}{10}})(z + e^{i\frac{\pi}{10}})(z - e^{i\frac{3\pi}{10}})(z + e^{-i\frac{3\pi}{10}})\ (\because e^{i\pi} = e^{-i\pi} = -1)$$

$$= (z+i)\left(z^2 - 2i\sin\left(-\frac{\pi}{10}\right)z - 1\right)\left(z^2 - 2i\sin\left(\frac{3\pi}{10}\right)z - 1\right)$$

$$(\because (z-w)(z+\overline{w}) = z^2 - (w - \overline{w})z - w\overline{w} = z^2 - 2i\ \text{Im}(w)z - |w|^2)$$

$$= (z+i)\left(z^2 + 2iz\sin\frac{\pi}{10} - 1\right)\left(z^2 - 2iz\sin\left(\frac{3\pi}{10}\right) - 1\right)$$

$$\therefore (z+i)\left(z^2 + 2iz\sin\frac{\pi}{10} - 1\right)\left(z^2 - 2iz\sin\left(\frac{3\pi}{10}\right) - 1\right) = 0$$

<u>Markers Comments:</u>
Very poorly done.
Students didn't have the correct roots.

[1]... for the correct roots
[1]... for recognising the conjugate pairs
[1]... for the correct solution

(iii) Using (ii), $\because z \neq -i,\ \therefore \left(z^2 + 2iz\sin\frac{\pi}{10} - 1\right)\left(z^2 - 2iz\sin\left(\frac{3\pi}{10}\right) - 1\right) = 0$

Also, from (i), $z^4 - iz^3 - z^2 + iz + 1 = 0$

$$\therefore z^4 - iz^3 - z^2 + iz + 1 = \left(z^2 + 2iz\sin\frac{\pi}{10} - 1\right)\left(z^2 - 2iz\sin\left(\frac{3\pi}{10}\right) - 1\right)\ (1)$$

Equate coefficients of z^3,

$$2i\sin\frac{\pi}{10} - 2i\sin\frac{3\pi}{10} = -i \quad \ldots\ldots [1]$$

$$\therefore \sin\frac{\pi}{10} - \sin\frac{3\pi}{10} = -\frac{i}{2i} = -\frac{1}{2} \quad \ldots\ldots [1]$$

<u>Markers Comments:</u>
Reasonably well done, for those students who attempted this question.
Some students didn't equate coefficients correctly.
[1]... for the equating coefficients
[1]... for the correct solution

(iv) Using 1 from (iii), equate coefficients of z^2,

$$-1+4\sin\frac{\pi}{10}\sin\frac{3\pi}{10}-1=-1$$
$$\therefore \sin\frac{\pi}{10}\sin\frac{3\pi}{10}=\frac{1}{4}$$
$$\therefore \sin\frac{3\pi}{10}=\frac{1}{4\sin\frac{\pi}{10}} \quad (2)$$

Let $x=\sin\frac{\pi}{10}$. Sub 2 into equation from (iii):

$$x-\frac{1}{4x}=-\frac{1}{2}$$
$$4x^2+2x-1=0$$
$$\therefore x=\frac{-2\pm\sqrt{2^2-4\times(4)\times(-1)}}{2\times 4} \qquad \text{......[1]}$$
$$=\frac{-2\pm\sqrt{20}}{8}$$
$$=\frac{-1\pm\sqrt{5}}{4}.$$

$\because x=\sin\frac{\pi}{10}>0$ (acute angle)

$$\therefore x=\sin\frac{\pi}{10}=\frac{-1+\sqrt{5}}{4} \qquad \text{......[1]}$$

$\therefore$ From 2,

$$\sin\frac{3\pi}{10}=\frac{1}{4\times\frac{-1+\sqrt{5}}{4}}$$
$$=\frac{1}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}+1}$$
$$=\frac{1+\sqrt{5}}{4} \qquad \text{......[1]}$$

Markers Comments:
Very poorly done, many students didn't attempt this question.
Some students didn't equate coefficients correctly and some didn't see the quadratic equation.
[1]... for the quadratic equation
[1]... for each of the correct exact values of sine.

(b) (i) Using integration by parts,

$$
\begin{aligned}
I_k &= \int_0^{\frac{1}{2}} x^2(1-x^3)^k \, dx \\
&= \int_0^{\frac{1}{2}} (1-x^3)^k \frac{d(\frac{1}{3}x^3)}{dx} dx \\
&= \left[\frac{1}{3}x^3(1-x^3)^k\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{1}{3}x^3 \frac{d((1-x^3)^k)}{dx} dx \qquad \text{......[1]} \\
&= \frac{1}{3} \times \frac{1}{8} \times \left(1 - \frac{1}{8}\right)^k - \int_0^{\frac{1}{2}} \frac{1}{3}x^3 \cdot k \cdot (1-x^3)^{k-1} \cdot (-3x^2) dx \\
&= \frac{1}{24}\left(\frac{7}{8}\right)^k + k\int_0^{\frac{1}{2}} x^5(1-x^3)^{k-1} dx \\
&= \frac{1}{24}\left(\frac{7}{8}\right)^k + k\int_0^{\frac{1}{2}} x^2[1-(1-x^3)](1-x^3)^{k-1} dx \\
&= \frac{1}{24}\left(\frac{7}{8}\right)^k + k\int_0^{\frac{1}{2}} \left[x^2(1-x^3)^{k-1} - x^2(1-x^3)^k\right] dx \qquad \text{......[1]} \\
&= \frac{1}{24}\left(\frac{7}{8}\right)^k + k(I_{k-1} - I_k)
\end{aligned}
$$

$$\therefore (k+1)I_k - kI_{k-1} = \frac{1}{24}\left(\frac{7}{8}\right)^k \qquad \text{......[1]}$$

Markers Comments:

Very poorly done.

Some students didn't use integration by parts correctly.

[1]... for use of integration by parts

[1]... for correctly simplifying like terms

[1]... for the correct solution

(ii) Using the result from (i), $\sum_{k=1}^{n}\left[(k+1)I_k - kI_{k-1}\right] = \sum_{k=1}^{n}\frac{1}{24}\left(\frac{7}{8}\right)^k$. (1)

The left hand side can be written as

$$
\begin{aligned}
&\sum_{k=1}^{n}\left[(k+1)I_k - kI_{k-1}\right] \\
&= (2I_1 - I_0) \\
&\quad + (3I_2 - 2I_1) \\
&\quad + (4I_3 - 3I_2) \\
&\quad + \cdots + \\
&\quad + nI_{n-1} - (n-1)I_{n-2} \\
&\quad + (n+1)I_n - nI_{n-1} \\
&= (n+1)I_n - I_0 \text{ (2)} \qquad \text{......[1]}
\end{aligned}
$$

Hence, from (1) and (2),

$$\therefore (n+1)I_n - I_0 = \frac{1}{24} \times \frac{\frac{7}{8}\left[1-\left(\frac{7}{8}\right)^n\right]}{1-\frac{7}{8}} \quad \left(\text{G.P. with } a=\frac{7}{8}, r=\frac{7}{8}, n \text{ terms}\right) \quad \ldots\ldots[1]$$

$$(n+1)I_n - I_0 = \frac{7}{24} \times \left[1-\left(\frac{7}{8}\right)^n\right] \text{ (3)}$$

Note that $I_0 = \int_0^{\frac{1}{2}} x^2 \, dx$

$$= \left[\frac{1}{3}x^3\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{24} \text{ (4)}$$

$$\therefore \text{From (3) and (4)}: (n+1)I_n - \frac{1}{24} = \frac{7}{24} \times \left[1-\left(\frac{7}{8}\right)^n\right]$$

$$\therefore (n+1)I_n = \frac{1}{24} + \frac{7}{24} \times \left[1-\left(\frac{7}{8}\right)^n\right]$$

$$= \frac{1}{3} - \frac{7}{24}\left(\frac{7}{8}\right)^n$$

$$= \frac{1}{3} - \frac{1}{3} \times \frac{7}{8}\left(\frac{7}{8}\right)^n$$

$$= \frac{1}{3}\left[1-\frac{7}{8}\left(\frac{7}{8}\right)^n\right]$$

$$\therefore I_n = \frac{1}{3(n+1)}\left[1-\left(\frac{7}{8}\right)^{n+1}\right] \quad \ldots\ldots[1]$$

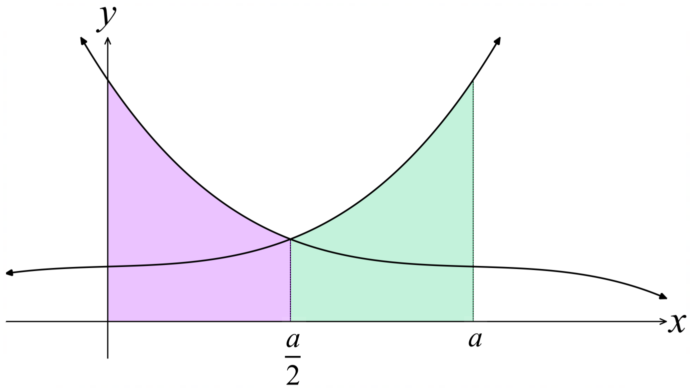
Markers Comments:

Very poorly done, many students didn't attempt this question.
Some students didn't use an appropriate sum of both sides correctly.
Some students also completed this question using mathematical induction.
[1]... for correctly simplifying the telescoping series.
[1]... for identity and summing of the geometric series.
[1]... for the correct solution

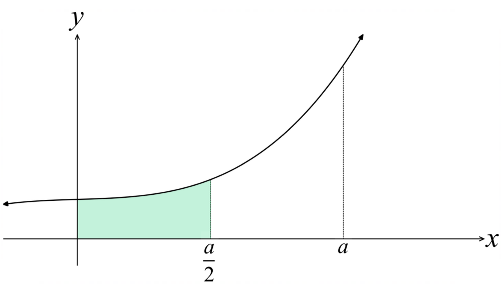
(a) (i) Prove that $\int_0^a f(x)\,dx = \int_0^{\frac{a}{2}} [f(x) + f(a-x)]\,dx$. **1**

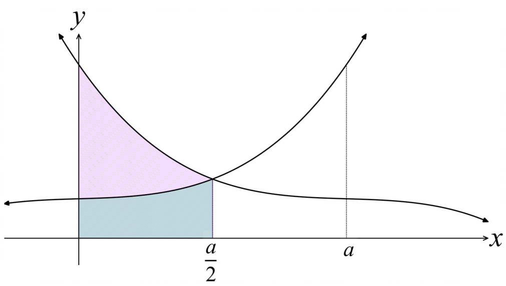
Method 1: Graphical solution

$y = f(a-x)$ is a reflection of $y = f(x)$ in the line $x = \frac{a}{2}$, as can be seen by the first diagram.



The next two diagrams effectively "prove" the result.





(a) (i) (continued)

Method 2: Properties of Definite Integrals

$$\int_0^a f(x)\,dx = \int_0^{\frac{a}{2}} f(x)\,dx + \int_{\frac{a}{2}}^a f(x)\,dx$$

Need to prove that $\displaystyle\int_0^{\frac{a}{2}} f(a-x)\,dx = \int_{\frac{a}{2}}^a f(x)\,dx$.

Considering the RHS: Let $u = a - x \Rightarrow du = -dx$

x: $0 \sim \dfrac{a}{2} \Rightarrow u = a \sim \dfrac{a}{2}$

$$\text{RHS} = \int_0^{\frac{a}{2}} f(a-x)\,dx$$

$$= \int_a^{\frac{a}{2}} f(u)\,(-du)$$

$$= \int_{\frac{a}{2}}^a f(u)\,du$$

$$= \int_{\frac{a}{2}}^a f(x)\,dx \qquad \text{[Dummy variable]}$$

$$= \text{LHS}$$

Comment: This was a "Prove that", and so just like a "Show that" students had to show some detail.

Some students assumed that $f(x) = f(a-x)$ i.e. $f(x)$ is symmetric in $x = \frac{a}{2}$.

Students who did Method 1 or Method 2 were successful.

Those who only proved $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ were not as successful.

(a) (continued)

(ii) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \ln(\sin x)\ dx$. **1**

By using (i), or otherwise, prove that $I = \displaystyle\int_0^{\frac{\pi}{4}} \ln\left(\frac{\sin 2x}{2}\right) dx$.

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \ln(\sin x) + \ln\left(\sin\left(\frac{\pi}{2} - x\right)\right)\ dx$$

$$= \int_0^{\frac{\pi}{4}} \ln(\sin x) + \ln(\cos x)\ \ dx$$

$$= \int_0^{\frac{\pi}{4}} \ln(\sin x \cos x)\ \ \ dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{\sin 2x}{2}\right)\ \ dx$$

Comment: Done very well.

Unfortunately this integral is undefined in the strictest sense, but is defined in that the value exists.

Students who didn't substitute $(\frac{\pi}{2} - x)$ properly could not get full marks.

(a) (continued)

(iii) Hence, evaluate I. **1**

$$I = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\sin 2x}{2}\right) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \ln(\sin 2x) - \ln 2 \, dx$$

[Use $u = 2x$]

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin u)\, du - \int_0^{\frac{\pi}{4}} \ln 2 \, dx$$

$$\therefore \frac{1}{2}I = -\frac{\pi}{4} \times \ln 2$$

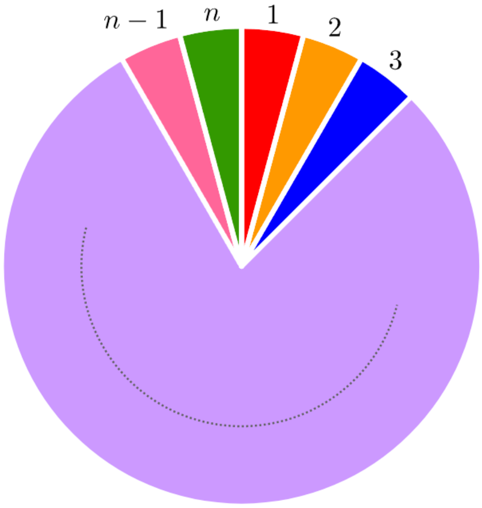
$$\therefore I = -\frac{\pi}{2}\ln 2$$

Comment: This question was not done well at all. Half marks were not awarded.

Most students opted for an "Integration by Parts" approach that they either could not do, or fudged.

As it was only worth 1 mark, students didn't need to show all the details of the substitution. That said, those who took the approach above, the common mistake was at the substitution.

(b) Seven distinct colours are used to colour the numbered sectors of a circle shown below.



Let a_n denote the number of ways to colour a circle with n sectors such that each sector is coloured by one colour and any two adjacent sectors must be coloured by different colours.

(i) Explain why $a_n + a_{n-1} = 7 \times 6^{n-1}$ for positive integers $n \geq 3$. **2**

If we colour the first sector then there are 7 ways to do this, and if we just have the next adjacent sector different, there are 6 ways to do each of the $(n-1)$ remaining sectors i.e. $7 \times 6^{n-1}$ ways. But then the case arises where the 1st sector could be the same as the nth.

The $n = 5$ case is illustrated on the next page, and explains about below.

$a_n = 7 \times 6^{n-1} -$ (#ways of 1st and nth sectors being the same colour)

[#ways of 1st and nth sectors being the same colour = a_{n-1}]

$\therefore\ a_n = 7 \times 6^{n-1} - a_{n-1},\ n \geq 3$

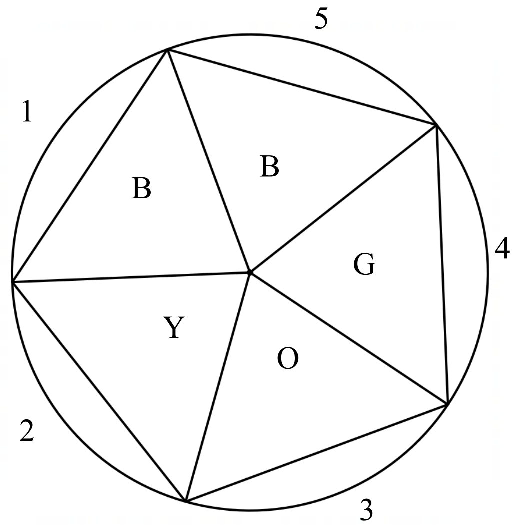
$\therefore\ a_n + a_{n-1} = 7 \times 6^{n-1},\ n \geq 3$

We need $n \geq 3$, since $a_n = 7 \times 6^{n-1},\ n = 1, 2$ i.e. the "contradictory" colourings don't arise.

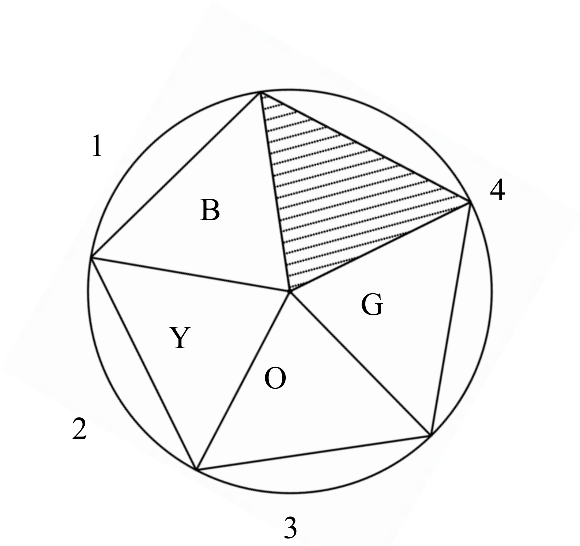
(b) (i) (continued)

Example $n = 5$:

Starting at sector 1, by only ensuring the next sector is different we could get the following:



This is contradictory to the fact that both adjacent sectors are different.



The number of "contradictory" colourings corresponds to "ignoring" the 5^{th} sector i.e. counting the number of ways with 4 sectors i.e. a_4 as the 4^{th} sector must be different from the hidden (5^{th}) sector it cannot be the same as the 1^{st} one.

So $a_5 = 7 \times 6^4 - a_4$

Comment: This question was not done well.

A lot of students ignored the "Explain why" and so could not get many marks.
Writing down a series of calculations (usually products) without explanation is not "Explaining why".
In some cases the calculations were fudged to get the result.

(b) (ii) Use mathematical induction to show for positive integers $n \geq 3$, **3**

$$a_n = 6 \times (-1)^n + 6^n.$$

Test $n = 3$: $a_3 + a_2 = 210 + 42 = 7 \times 6^2$

$a_2 = 7 \times 6 = 42$

LHS = $a^3 = 210$ [Or $a_3 = 7 \times 6 \times 5 = 210$]

RHS = $6 \times (-1)^3 + 6^3 = -6 + 216 = 210$

$\therefore$ LHS = RHS

$\therefore$ True for $n = 3$.

Comment: This step i.e. "showing $n = 3$" was worth 1 mark and no half marks were awarded.

This is the place where students faltered.
Some students just wrote something like:

$$\text{LHS} = 6 \times (-1)^3 + 6^3 = -6 + 216 = 210 = \text{RHS}$$

They did not calculate a_3.

Some students did not know their LHS from their RHS, but as long as they "showed" two different things they were not penalised

Some students tried to use (a) (i), but then they got stuck as they needed $a_1 = 0$, whereas $a_1 = 7$.
There was a reason why that result was only true for $n = 3, 4, \ldots$

(b) (ii) (continued)

Assume true for $n = k$ i.e. $a_k = 6\times(-1)^k + 6^k$

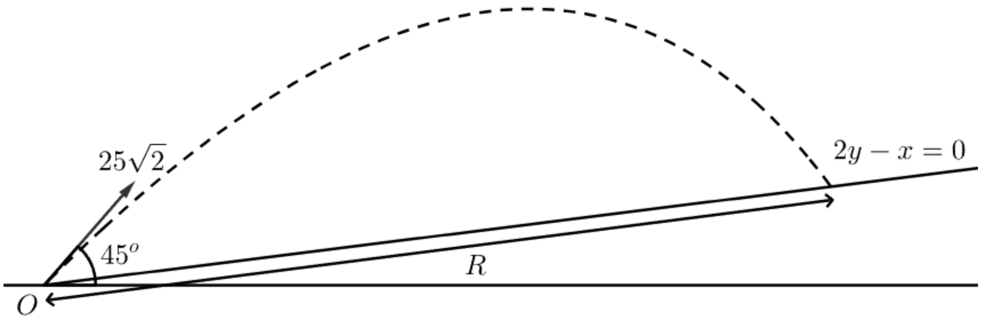
Need to prove true for $n = k + 1$ i.e. $a_{k+1} = 6\times(-1)^{k+1} + 6^{k+1}$

$$
\begin{aligned}
\text{LHS} &= a_{k+1} \\
&= 7\times 6^k - a_k \\
&= 7\times 6^k - 6\times(-1)^k - 6^k \\
&= \left(7\times 6^k - 6^k\right) + 6\times(-1)\times(-1)^k \\
&= 6\times(-1)^{k+1} + 6^{k+1} \\
&= \text{RHS}
\end{aligned}
$$

Comment: Almost all students who got this far got 2 marks as long as they used “induction”.

Some students did not use induction, so they could only get the mark from the previous page.

(c) A 100 g football is kicked up a slope with equation $2y - x = 0$ as shown.



It is subject to gravity 10 m/s^2 and air resistance $-0.02\underset{\sim}{v}$ N where $\underset{\sim}{v}$ is the velocity vector of the football.

The football is launched with speed $25\sqrt{2}$ m/s at angle 45^o to the horizontal.

The acceleration of the football is given by

$$\underset{\sim}{a} = -\underset{\sim}{g} - 0.2\underset{\sim}{v}$$

where $\underset{\sim}{g} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$.

(i) Show that the vertical displacement of the football is **2**

$$y = -50t + 375(1 - e^{-0.2t}).$$

$$\underset{\sim}{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -\begin{pmatrix} 0.2\,\dot{x} \\ 10 + 0.2\,\dot{y} \end{pmatrix}$$

Let $\ddot{y} = \dfrac{dv}{dt}$ and $\dot{y} = v$

$$\therefore \frac{dv}{dt} = -\left(10 + \frac{1}{5}v\right) = -\frac{1}{5}(50 + v)$$ **NOTE: This is simplified**

Separating variables:

$$\therefore \frac{1}{50 + v}\,dv = -\frac{1}{5}\,dt$$

Continued on page 10

(c) (i) (continued)

At $t = 0$, $v = 25\sqrt{2}\sin 45° = 25$ **NOTE: This is simplified**

$$\therefore \int_{25}^{V} \frac{1}{50+V}\,dV = -0.2\int_{0}^{t} dT$$

$$\therefore \left[\ln(50+V)\right]_{25}^{v} = -0.2(t-0)$$ [**Note**: $50 + v > 0$]

$$\therefore \ln\left(\frac{50+v}{75}\right) = -0.2t$$

$$\therefore \frac{50+v}{75} = e^{-0.2t}$$

$$\therefore v = 75e^{-0.2t} - 50$$ **NOTE: This is simplified**

At $t = 0$, $y = 0$, $v = \dfrac{dy}{dt}$

$$\therefore \frac{dy}{dt} = 75e^{-0.2t} - 50$$

$$\therefore \int_{0}^{y} dY = \int_{0}^{t} 75e^{-0.2T} - 50\,dT$$

$$\therefore y - 0 = \left[\frac{75}{-0.2}e^{-0.2T} - 50T\right]_{0}^{t}$$

$$\therefore y = -375e^{-0.2t} - 50t - (-375 - 0)$$
$$= 375\left(1 - e^{-0.2t}\right) - 50t$$

Comment: It was good that "absolute value" was only penalised in Question 11, otherwise many students would have lost marks here.

Embrace simplifying and substitution at the beginning, and also embrace the definite integral approach. You don't need to use "other" pronumerals e.g. T, V, and Y.

Students should show their substitutions!

(ii) You are given that the horizontal displacement, in metres, of the football is $x = 125(1 - e^{-0.2t})$. (Do NOT prove this.) **2**

Prove that the Cartesian equation of the trajectory is given by

$$y = 3x + 250\ln\left(1 - \frac{x}{125}\right).$$

$$\begin{aligned} y &= -50t + 3\times 125\left(1 - e^{-0.2t}\right) \\ &= -50t + 3x \end{aligned}$$

$$\frac{x}{125} = 1 - e^{-0.2t} \Rightarrow e^{-0.2t} = 1 - \frac{x}{125}$$

$$\therefore -0.2t = \ln\left(1 - \frac{x}{125}\right)$$

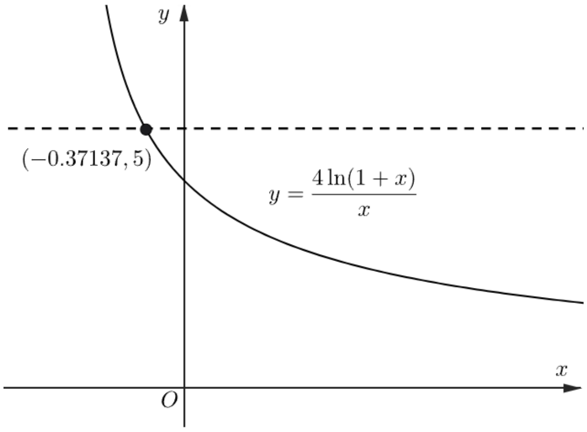
$$\therefore -50t = \frac{50}{0.2}\ln\left(1 - \frac{x}{125}\right) = 250\ln\left(1 - \frac{x}{125}\right)$$

$$\therefore y = 3x + 250\ln\left(1 - \frac{x}{125}\right)$$

Comment: This was done very well.

Those who tried it were successful.

(c) (iii) The graph shows the point where $y = 5$ on the curve $y = \dfrac{4\ln(1+x)}{x}$. **3**



Hence, or otherwise, find the range R up the slope, correct to 4 decimal places.

Intersecting $y = 3x + 250\ln\left(1 - \dfrac{x}{125}\right)$ with $x = 2y$

$$\therefore\ 3x + 250\ln\left(1 - \frac{x}{125}\right) = \frac{x}{2} \Rightarrow 5x + 500\ln\left(1 - \frac{x}{125}\right) = 0$$

$$\therefore\ 5x = -4 \times 125\ln\left(1 - \frac{x}{125}\right)$$

$$\therefore\ \frac{5}{-125}x = 4 \times \ln\left(1 - \frac{x}{125}\right)$$

(*)

$$\therefore\ 5\left(-\frac{x}{125}\right) = 4\ln\left(1 - \frac{x}{125}\right) \qquad \left[z = -\frac{x}{125}\right]$$

$$\therefore\ 5z = 4\ln(1+z) \Rightarrow 5 = \frac{4\ln(1+z)}{z}$$

(**)

$$\therefore\ z = -0.37137 \Rightarrow -\frac{x}{125} = -0.37137$$

$$\therefore\ x = 125 \times 0.37137 = 46.42125 \Rightarrow y = \frac{1}{2}x$$

$$\therefore\ R = \sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{1}{4}x^2} = \frac{\sqrt{5}}{2}x \approx 51.9005$$

Comment: Students who could get to (*) or equivalent got ½ – 1 marks.
Most students who got to (**) were able to continue, though most only found x not R.

See **HSC 2023 ME 2** for a similarly styled question.

16) a) i) $LHS = \frac{d}{dx}\left[\int_0^{g(x)} h(t)\,dt\right]$

$$= \frac{d}{dx}\left[H(t)\right]_0^{g(x)} \quad \text{where } H(t) \text{ is a primitive of } h(t)$$

$$= \frac{d}{dx}\left[H(g(x)) - H(0)\right]$$

$$= H'(g(x)).g'(x) - 0 \quad \text{since } H(0) \text{ is a constant}$$

$$= g'(x)h(g(x))$$

$$= RHS$$

OR

$$\frac{d}{dx}\left[\int_0^{g(x)} h(t)\,dt\right]$$

let $u = g(x)$

$\frac{du}{dx} = g'(x)$

$dx = \frac{du}{g'(x)}$

$$= \frac{d}{\left(\frac{du}{g'(x)}\right)}\left[\int_0^{u} h(t)\,dt\right]$$

$$= g'(x)\underbrace{\frac{d}{du}\left[\int_0^{u} h(t)\,dt\right]}_{\text{differential form of the fundamental theorem of calculus.}}$$

$$= g'(x)\,h(u)$$

$$= g'(x)h(g(x))$$

COMMENT:

Students could not just write down fundamental theorem of calculus and chain rule. They need to demonstrate how they are applied.

Some students introduced notation to talk about primitives even though we already have a notation for this.

Some students omitted the lower limit.

16)a)ii) $A(x) = \int_{-\frac{1}{\sqrt{x}}}^{\sqrt{x}} \sqrt{1+t^2}\,dt$

$$= \int_{-\frac{1}{\sqrt{x}}}^{0} \sqrt{1+t^2}\,dt + \int_{0}^{\sqrt{x}} \sqrt{1+t^2}\,dt$$

$$= \int_{0}^{\sqrt{x}} \sqrt{1+t^2}\,dt - \int_{0}^{-\frac{1}{\sqrt{x}}} \sqrt{1+t^2}\,dt$$

$$A'(x) = \frac{d}{dx}\left[\int_{0}^{\sqrt{x}} \sqrt{1+t^2}\,dt - \int_{0}^{-\frac{1}{\sqrt{x}}} \sqrt{1+t^2}\,dt\right]$$

$$= \frac{d}{dx}\left[\int_{0}^{\sqrt{x}} \sqrt{1+t^2}\,dt\right] - \frac{d}{dx}\left[\int_{0}^{-\frac{1}{\sqrt{x}}} \sqrt{1+t^2}\,dt\right]$$

Note: $g(x) = \sqrt{x}$

$g'(x) = \frac{1}{2\sqrt{x}}$

Note: $g(x) = -\frac{1}{\sqrt{x}} = -x^{-\frac{1}{2}}$

$g'(x) = \frac{1}{2}x^{-\frac{3}{2}}$

$= -\frac{1}{2x\sqrt{x}}$

$$A'(x) = \frac{1}{2\sqrt{x}}\sqrt{1+(\sqrt{x})^2} - \frac{1}{2x\sqrt{x}}\sqrt{1+\left(-\frac{1}{\sqrt{x}}\right)^2}$$

$$= \frac{\sqrt{1+x}}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\sqrt{1+\frac{1}{x}}$$

$$A'(x) = \frac{\sqrt{1+x}}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\sqrt{\frac{x+1}{x}}$$

$$= \frac{\sqrt{1+x}}{2\sqrt{x}} - \frac{\sqrt{1+x}}{2x^2}$$

$$= \frac{\sqrt{1+x}}{2x^2}\left(x^{3/2} - 1\right)$$

For stat. points let $A'(x) = 0$

$1 + x = 0$ or $x^{3/2} - 1 = 0$

$x = -1$ $\qquad$ $x = 1$

However, $x > 0$ so that $-\frac{1}{\sqrt{x}}$ is defined.

x	0.9	1	1.1
$A'(x)$	−0.124	0	0.092
	$\searrow$	—	$\nearrow$

$\therefore$ Minimum Area when $x = 1$.

Note: $$\int_{-\frac{1}{\sqrt{x}}}^{0} \sqrt{1+t^2}\,dt = \int_{0}^{\frac{1}{\sqrt{x}}} \sqrt{1+t^2}\,dt$$

since $f(t) = \sqrt{1+t^2}$ is even.

Note: $$\int_{-\frac{1}{\sqrt{x}}}^{\sqrt{x}} \sqrt{1+t^2}\,dt = \int_{0}^{\sqrt{x}+\frac{1}{\sqrt{x}}} \sqrt{1+\left(t-\frac{1}{\sqrt{x}}\right)^2}\,dt$$

"shift the curve $\frac{1}{\sqrt{x}}$ units to the right and move the limits also."

COMMENT:

Very few students gained full marks for this question..

Students needed to use the result from (i). This means that the lower limit needs to be zero.

Students could use $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ to help or they could use even symmetry or translate the area to allow this.

Many students failed to differentiate correctly in particular $\frac{d}{dx}\left(-\frac{1}{\sqrt{x}}\right)$.

Very few students justified why the value of x gave a minimum value.

This could be done with values either side in the first derivative, finding the value of $A''(1)$ or by noting that as $x \to 0^+$, $-\frac{1}{\sqrt{x}} \to -\infty$ and so $A(x) \to \infty$
and as $x \to \infty$, $\sqrt{x} \to \infty$ and so $A(x) \to \infty$

If this was a hence, or otherwise, it could be done differently.

Due to the even symmetry of the function we are looking to minimise the length of the interval from $-\frac{1}{\sqrt{x}}$ to $\sqrt{x}$.

ie minimise the length of $\sqrt{x} + \frac{1}{\sqrt{x}}$.

$$\left(\sqrt[4]{x} - \frac{1}{\sqrt[4]{x}}\right)^2 \geq 0$$

$$\sqrt{x} - 2 + \frac{1}{\sqrt{x}} \geq 0$$

$$\sqrt{x} + \frac{1}{\sqrt{x}} \geq 2$$

this occurs when $x = 1$.

16)b)i) $\vec{VW} = \vec{VO} + \vec{OW}$

$= -\vec{OV} + \vec{OW}$

$= -\mu \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$\vec{VQ} = k\vec{VW}$

$= k(-\mu \underset{\sim}{a} + \lambda \underset{\sim}{b})$

$= k(\lambda \underset{\sim}{b} - \mu \underset{\sim}{a})$

COMMENT:

This was well answered by students.

16)b)ii) $\vec{OV} = \vec{OS} + \vec{SV}$

$\vec{SV} = \vec{OV} - \vec{OS}$

$= \mu \underset{\sim}{a} - \underset{\sim}{a}$

$= (\mu - 1)\underset{\sim}{a}$

$|\vec{SV}| = |(\mu-1)\underset{\sim}{a}|$

$= |\mu - 1||\underset{\sim}{a}|$

$= (\mu - 1)|\underset{\sim}{a}|$ since $\mu > 1$.

$|\vec{VQ}| = |k(\lambda \underset{\sim}{b} - \mu \underset{\sim}{a})|$

$= |k||\lambda \underset{\sim}{b} - \mu \underset{\sim}{a}|$

$= k|\lambda \underset{\sim}{b} - \mu \underset{\sim}{a}|$ since $k > 0$

$= k\sqrt{|\lambda \underset{\sim}{b} - \mu \underset{\sim}{a}|^2}$

$= k\sqrt{(\lambda \underset{\sim}{b} - \mu \underset{\sim}{a}) \cdot (\lambda \underset{\sim}{b} - \mu \underset{\sim}{a})}$

$= k\sqrt{\lambda^2 \underset{\sim}{b} \cdot \underset{\sim}{b} - 2\lambda\mu \underset{\sim}{a} \cdot \underset{\sim}{b} + \mu^2 \underset{\sim}{a} \cdot \underset{\sim}{a}}$

$= k\sqrt{\lambda^2|\underset{\sim}{b}|^2 - \cancel{2\lambda\mu(0)} + \mu^2|\underset{\sim}{a}|^2}$ since $a \cdot b = 0$ as $\underset{\sim}{a} \perp \underset{\sim}{b}$

$= k\sqrt{\lambda^2|\underset{\sim}{a}|^2 + \mu^2|\underset{\sim}{a}|^2}$ since $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$= k|\underset{\sim}{a}|\sqrt{\lambda^2 + \mu^2}$

$|\overrightarrow{SV}| = |\overrightarrow{VQ}|$ since tangents from an external point are equal.

$(\mu - 1)\cancel{|\underset{\sim}{a}|} = k\cancel{|\underset{\sim}{a}|}\sqrt{\lambda^2 + \mu^2}$

$$k = \frac{\mu - 1}{\sqrt{\lambda^2 + \mu^2}}$$

OR

$|\overrightarrow{VW}|^2 = |\overrightarrow{OV}|^2 + |\overrightarrow{OW}|^2$ as ΔOVW is right angled.

$= |\mu\underset{\sim}{a}|^2 + |\lambda\underset{\sim}{b}|^2$

$= (|\mu||\underset{\sim}{a}|)^2 + (|\lambda||\underset{\sim}{b}|)^2$

$= \mu^2|\underset{\sim}{a}|^2 + \lambda^2|\underset{\sim}{b}|^2$

$= \mu^2|\underset{\sim}{a}|^2 + \lambda^2|\underset{\sim}{a}|^2$ since $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$= |\underset{\sim}{a}|^2(\mu^2 + \lambda^2)$

$\therefore |\overrightarrow{VW}| = |\underset{\sim}{a}|\sqrt{\mu^2 + \lambda^2}$

$|\overrightarrow{VQ}| = |k\overrightarrow{VW}|$

$= |k||\overrightarrow{VW}|$

$= k|\underset{\sim}{a}|\sqrt{\lambda^2 + \mu^2}$ since $k > 0$.

etc.

COMMENT:
This is a prove that question which means we know what we are working towards. The majority of students didn't justify certain steps or omitted certain terms that should have been considered. Only a few students gained full marks for this question.

16) b) iii) $\vec{PQ} = \vec{PS} + \vec{SV} + \vec{VQ}$

$= -\vec{OU} + \vec{SV} + \vec{VQ}$

$= -\underset{\sim}{b} + (\mu - 1)\underset{\sim}{a} + k(\lambda \underset{\sim}{b} - \mu \underset{\sim}{a})$

$= (\mu - 1 - k\mu)\underset{\sim}{a} + (-1 + k\lambda)\underset{\sim}{b}$

$= ((1-k)\mu - 1)\underset{\sim}{a} + (k\lambda - 1)\underset{\sim}{b}$

COMMENT:
This was answered well by students

16) b) iv) $\vec{PQ} \cdot \vec{VW} = 0$ the radius is perpendicular to tangent.

$\left(((1-k)\mu - 1)\underset{\sim}{a} + (k\lambda - 1)\underset{\sim}{b}\right) \cdot (\lambda \underset{\sim}{b} - \mu \underset{\sim}{a}) = 0$

$\cancel{((1-k)\mu - 1)\underset{\sim}{a} \cdot \underset{\sim}{b}} - \mu((1-k)\mu - 1)\underset{\sim}{a} \cdot \underset{\sim}{a} + \lambda(k\lambda - 1)\underset{\sim}{b} \cdot \underset{\sim}{b} - \cancel{\mu(k\lambda - 1)\underset{\sim}{a} \cdot \underset{\sim}{b}} = 0$ again since $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\mu((1-k)\mu - 1)\cancel{|\underset{\sim}{a}|^2} = \lambda(k\lambda - 1)\cancel{|\underset{\sim}{b}|^2}$ since $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$\mu^2 - k\mu^2 - \mu = k\lambda^2 - \lambda$

$k\lambda^2 + k\mu^2 = \mu^2 - \mu + \lambda$

$k(\lambda^2 + \mu^2) = \lambda + \mu^2 - \mu$

$$k = \frac{\lambda + \mu^2 - \mu}{\lambda^2 + \mu^2}$$

COMMENT:

Few students were able to correctly reach the result for k.

It is not possible to reach the expression for k by using the fact that $|\vec{PQ}| = |\underset{\sim}{a}| = |\underset{\sim}{b}|$

It does lead to the equation:

$$k^2(\mu^2+\lambda^2) - 2k(\mu^2-\mu+\lambda) + (\mu-1)^2 = 0$$

and this leads to

$$k = \frac{\mu^2-\mu+\lambda \pm \sqrt{\lambda\mu(\lambda(2-\mu)+2(\mu-1))}}{\lambda^2+\mu^2}$$

Note: $\Delta = 0$ if you use the result from (v).

1.6) b) v)
$$\frac{\mu-1}{\sqrt{\lambda^2+\mu^2}} = \frac{\lambda+\mu^2-\mu}{\lambda^2+\mu^2}$$

$$(\mu-1)\sqrt{\lambda^2+\mu^2} = \lambda + \mu(\mu-1)$$

$$\sqrt{\lambda^2+\mu^2} = \frac{\lambda}{\mu-1} + \mu$$

$$\lambda^2+\cancel{\mu^2} = \frac{\lambda^2}{(\mu-1)^2} + \frac{2\lambda\mu}{\mu-1} + \cancel{\mu^2}$$

$$\lambda^2(\mu-1)^2 = \lambda^2 + 2\lambda\mu(\mu-1)$$

since $\lambda \neq 0$

$$\lambda(\mu^2-2\mu+1) = \lambda + 2\mu(\mu-1)$$

$$\lambda(\mu^2-2\mu+1-1) = 2\mu(\mu-1)$$

$$\lambda(\mu^2-2\mu) = 2\mu(\mu-1)$$

$$\lambda = \frac{2\mu(\mu-1)}{\mu^2-2\mu}$$

$$\lambda = \frac{2\cancel{\mu}(\mu-1)}{\cancel{\mu}(\mu-2)}$$

$$\lambda = \frac{2(\mu-1)}{\mu-2}$$

COMMENT:

It should have been obvious how to start this question.
Parts (ii) and (iv) had expressions for k in terms of μ & λ.
Combining gives an equation just in terms of μ & λ.
To get rid of the square root both sides need to be squared.
A lot of students struggled with the algebra.

16)b)vi)

$$\frac{1}{2} \times OV \times OW = \frac{1}{2} \times OW \times PU + \frac{1}{2} \times OV \times PS + \frac{1}{2} \times VW \times PQ$$

$$OV \times OW = (OW + OV + VW)r \qquad \text{since } PU = PS = PQ = r$$

$$r = \frac{OV \times OW}{OW + OV + VW}$$

But $VW^2 = OV^2 + OW^2$

$$0 = OV^2 + 2\times OV\times OW + OW^2 - VW^2 - 2\times OV\times OW$$

$$2\times OV\times OW = (OV + OW)^2 - VW^2$$

$$2\times OV\times OW = (OV+OW-VW)(OV+OW+VW)$$

$$\frac{OV\times OW}{OW+OV+VW} = \frac{OV+OW-VW}{2}$$

$$\therefore\ r = \frac{OV+OW-VW}{2}$$

$$A = \pi r^2$$

$$= \pi\left(\frac{OV+OW-VW}{2}\right)^2$$

$$= \frac{\pi}{4}(OV+OW-VW)^2$$

OR

since $|\underset{\sim}{a}| = |\underset{\sim}{b}| = r$

$OV = \mu r$, $OW = \lambda r$
$= \frac{2(\mu-1)}{\mu-2} r$, $VW = \sqrt{\mu^2+\lambda^2}\,.\,r$
$= \sqrt{\mu^2 + \left(\frac{2(\mu-1)}{\mu-2}\right)^2}\ r$

$$VW = \sqrt{\mu^2 + \frac{4(\mu-1)^2}{(\mu-2)^2}} \cdot r$$

$$= \sqrt{\frac{\mu^2(\mu-2)^2 + 4(\mu-1)^2}{(\mu-2)^2}} \cdot r$$

$$= \sqrt{\frac{\mu^2(\mu^2-4\mu+4) + 4(\mu^2-2\mu+1)}{(\mu-2)^2}} \cdot r$$

$$= \sqrt{\frac{\mu^4 - 4\mu^3 + 4\mu^2 + 4\mu^2 - 8\mu + 4}{(\mu-2)^2}} \cdot r$$

$$= \sqrt{\frac{\mu^4 - 4\mu^3 + 8\mu^2 - 8\mu + 4}{(\mu-2)^2}} \cdot r$$

consider $(\mu^2 - 2\mu + 2)^2 = (\mu^2 - 2\mu + 2)(\mu^2 - 2\mu + 2)$

$$= \mu^4 - 2\mu^3 + 2\mu^2 - 2\mu^3 + 4\mu^2 - 4\mu + 2\mu^2 - 4\mu + 4$$

$$= \mu^4 - 4\mu^3 + 8\mu^2 - 8\mu + 4$$

$$\therefore VW = \sqrt{\frac{(\mu^2-2\mu+2)^2}{(\mu-2)}} \cdot r$$

$$= \frac{\mu^2 - 2\mu + 2}{\mu - 2} \cdot r$$

Note: $x^2 - 2x + 2$ is positive definite

from (v) $\mu - 2 > 0$.

$$OV + OW - VW = \mu r + \frac{2(\mu-1)}{\mu-2} r - \frac{\mu^2-2\mu+2}{\mu-2} \cdot r$$

$$= \frac{\mu(\mu-2) + 2(\mu-1) - \mu^2 + 2\mu - 2}{\mu-2} \cdot r$$

$$= \frac{\cancel{\mu^2} \cancel{-2\mu} + \cancel{2\mu} - 2 - \cancel{\mu^2} + 2\mu - 2}{\mu-2} \cdot r$$

$$= \frac{2\mu - 4}{\mu-2} \cdot r$$

$$= \frac{2\cancel{(\mu-2)}}{\cancel{\mu-2}} r$$

$$= 2r$$

$$\therefore \frac{\pi}{4}(OV + OW - VW)^2 = \frac{\pi}{4}(2r)^2$$

$$= \pi r^2$$

the area of the inscribed circle.

OR

$OV = \mu r$

$\mu = \frac{OV}{r}$ —— ①

$OW = \lambda r$

$OW = \frac{2(\mu-1)}{\mu-2} r$ —— ②

sub ① into ②

$$OW = \frac{2\left(\frac{OV}{r} - 1\right)}{\frac{OV}{r} - 2} r$$

$$\frac{OW \times OV}{r} - 2(OW) = 2(OV) - 2r$$

$$2r = 2(OV) + 2(OW) - \frac{OW \times OV}{r}$$

$$2r^2 = 2r(OV + OW) - OW \times OV$$

$$2r^2 - 2(OV+OW)r + OW \times OV = 0$$

$$r = \frac{--2(OV+OW) \pm \sqrt{(-2(OV+OW))^2 - 4(2)(OW \times OV)}}{2(2)}$$

$$r = \frac{2(OV+OW) \pm \sqrt{4(OV+OW)^2 - 8(OW \times OV)}}{4}$$

$$r = \frac{2(OV+OW) \pm \sqrt{4((OV+OW)^2 - 2(OW \times OV))}}{4}$$

$$r = \frac{2(OV+OW) \pm 2\sqrt{(OV)^2 + \cancel{2(OV \times OW)} + (OW)^2 - \cancel{2(OW \times OV)}}}{4}$$

$$r = \frac{OV + OW \pm \sqrt{OV^2 + OW^2}}{2}$$

$$r = \frac{OV + OW \pm VW}{2}$$

~~assssudea~~ $r = \frac{OV + OW + VW}{2}$

$$OV + OW + VW = 2r$$
$$= d$$

clearly $OV > d$, $OW > d$ & $VW > d$

$$OV + OW + VW > 3d$$

A contradiction.

Assumption is false

$$r = \frac{OV + OW - VW}{2}$$

$$A = \pi r^2$$

$$= \pi \left(\frac{OV + OW - VW}{2}\right)^2$$

$$= \frac{\pi}{4}\left(OV + OW - VW\right)^2$$

COMMENT:

No students were able to get this question out. Very few gained marks at all.
Of the solutions provided the first solution links the radius of the circle with the length of the sides of the triangle. There is some algebra that needs to be done with pythagoras' theorem to get the required form.
The second solution is probably the most obvious use of previous parts. However, it requires the square root of a quartic. Without a lead-in it is very difficult.
The third solution which is the suggested solution from the examiner also has issues. It is not obvious initially that linking two of the sides of the triangle with the radius will lead to an expression also involving the third side.