



Sydney Girls High School

2015

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- All answers should be given in simplest exact form unless otherwise specified.

Total marks – 100

Section I Pages 3 – 7

10 Marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 8 – 17

90 Marks

- Attempt Questions 11 – 16.
- Answer on the blank paper provided.
- Begin a new page for each question.
- Allow about 2 hours and 45 minutes for this section.

Name:

Teacher:

THIS IS A TRIAL PAPER ONLY

It does not necessarily reflect the format or the content of the 2015 HSC Examination Paper in this subject.

Section I

10 marks

Attempt Questions 1–10

Use the multiple-choice answer sheet for Questions 1–10.

(1) An object rotates at 40 rpm and is moving at 30 m/s. The radius of the motion is

- (A) 1.33 m
- (B) 6.37 m
- (C) 7.16 m
- (D) 20 m

(2) Let $z = 3 - i$. What is the value of $\overline{iz}$?

- (A) $-1 - 3i$
- (B) $-1 + 3i$
- (C) $1 - 3i$
- (D) $1 + 3i$

(3) Which of the following is an expression for $\int \frac{dx}{\sqrt{7-6x-x^2}}$?

(A) $\frac{1}{4} \sin^{-1} \left(\frac{x-3}{4} \right) + c$

(B) $\frac{1}{4} \sin^{-1} \left(\frac{x+3}{4} \right) + c$

(C) $\sin^{-1} \left(\frac{x-3}{4} \right) + c$

(D) $\sin^{-1} \left(\frac{x+3}{4} \right) + c$

(4) How many ways can 5 boys and 3 girls be arranged around a circular table such that no two girls sit next to each other?

(A) 144

(B) 432

(C) 720

(D) 1440

(5) What is the solution to the equation $\tan^{-1}(4x) - \tan^{-1}(3x) = \tan^{-1} \left(\frac{1}{7} \right)$?

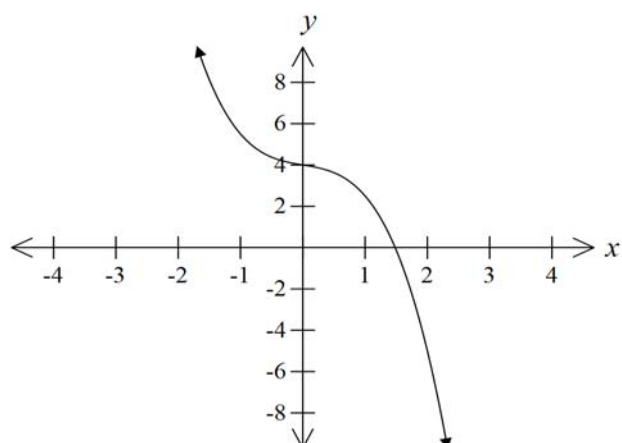
(A) $x = \frac{1}{7}$ or $x = \frac{2}{7}$

(B) $x = \frac{1}{3}$ or $x = \frac{2}{3}$

(C) $x = \frac{1}{3}$ or $x = \frac{1}{4}$

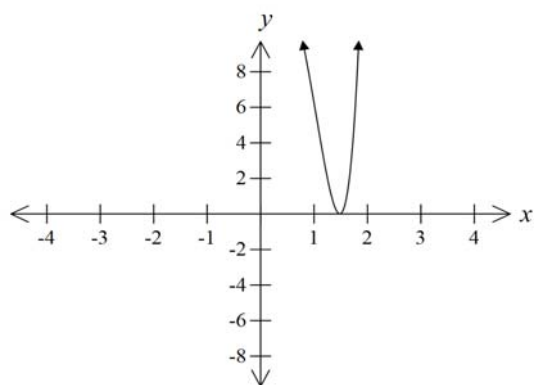
(D) $x = 3$ or $x = 4$

- (6) The diagram below shows the graph of the function $y = f(x)$.

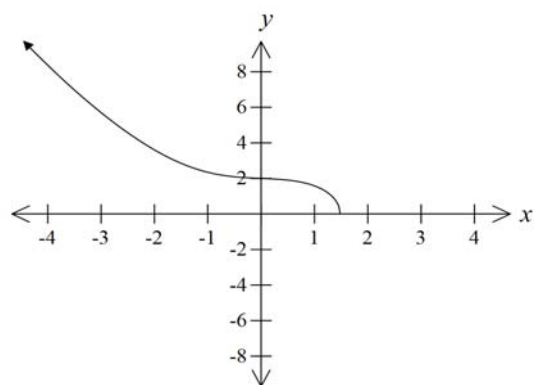


Which diagram represents the graph of $y^2 = f(x)$?

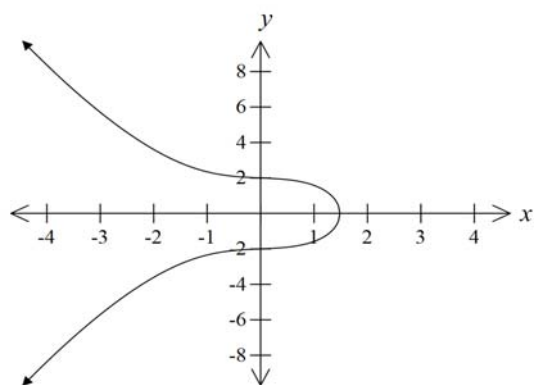
(A)



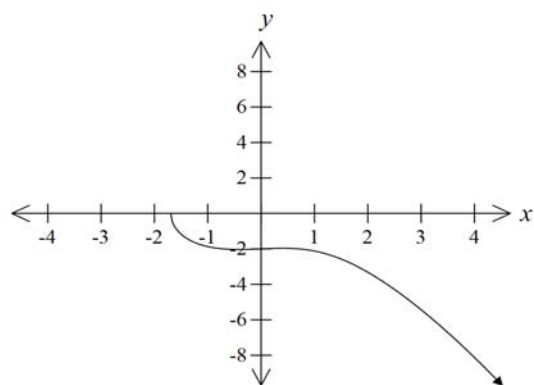
(B)



(C)



(D)



(7) Use the substitution $t = \tan \frac{x}{2}$ to find $\int -\sec x \, dx$.

(A) $\ln|(t-1)(t+1)| + c$

(B) $\ln|(1-t)(t+1)| + c$

(C) $\ln \left| \frac{1+t}{t-1} \right| + c$

(D) $\ln \left| \frac{t-1}{t+1} \right| + c$

(8) What is the eccentricity of the hyperbola $4x^2 - 25y^2 = 9$?

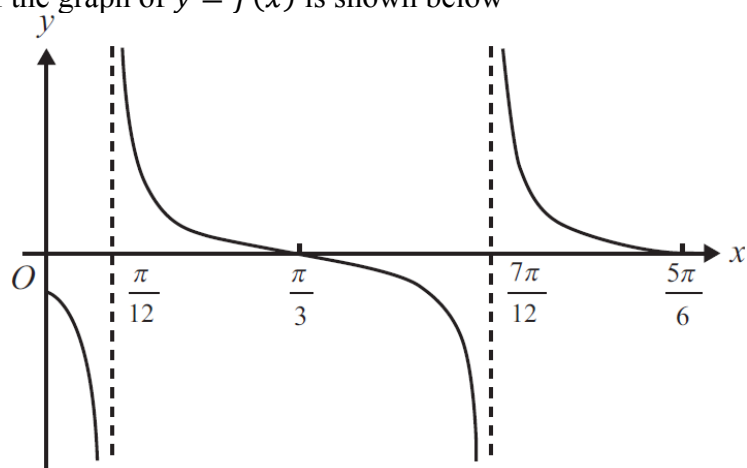
(A) $\frac{\sqrt{21}}{5}$

(B) $\frac{\sqrt{29}}{5}$

(C) $\frac{\sqrt{21}}{2}$

(D) $\frac{\sqrt{29}}{2}$

- (9) Part of the graph of $y = f(x)$ is shown below



$y = f(x)$ could be

- (A) $y = -\tan\left(2x - \frac{\pi}{6}\right)$
- (B) $y = -\tan\left(2x - \frac{\pi}{3}\right)$
- (C) $y = \cot\left(2x - \frac{\pi}{12}\right)$
- (D) $y = \cot\left(2x - \frac{\pi}{6}\right)$
- (10) The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots α , β and γ . Which one of the following polynomial equations has roots $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$ and $\alpha + \beta + 2\gamma$?

- (A) $x^3 - 6x^2 + 44x - 49 = 0$
- (B) $x^3 - 12x^2 + 44x - 49 = 0$
- (C) $x^3 + 3x^2 + 36x + 5 = 0$
- (D) $x^3 + 6x^2 + 36x + 5 = 0$

Section II

90 marks

Attempt Questions 11–16

Start each question on a NEW sheet of paper.

Question 11 (15 marks)

Use a NEW sheet of paper.

(a) If $z = (1 - i)^{-1}$

(i) Express $\bar{z}$ in modulus-argument form.

[2]

(ii) If $(\bar{z})^{13} = a + ib$ where a and b are real numbers,
find the values of a and b .

[2]

(b) Find

(i)

[1]

$$\int x^3 e^{x^4+7} dx$$

(ii)

[2]

$$\int \sec^3 x \tan x \, dx$$

(c) Find the Cartesian equation of the locus of a point P which represents
the complex number z where $|z - 2i| = |z|$

[2]

(d) Sketch the region in the complex plane where $\operatorname{Re}[(2 - 3i)z] < 12$

[2]

(e)

- (i) Express $\frac{x^2+x+2}{(x^2+1)(x+1)}$ in the form $\frac{Ax+B}{x^2+1} + \frac{C}{x+1}$,
where A, B and C are constants. [2]

- (ii) Hence find [2]

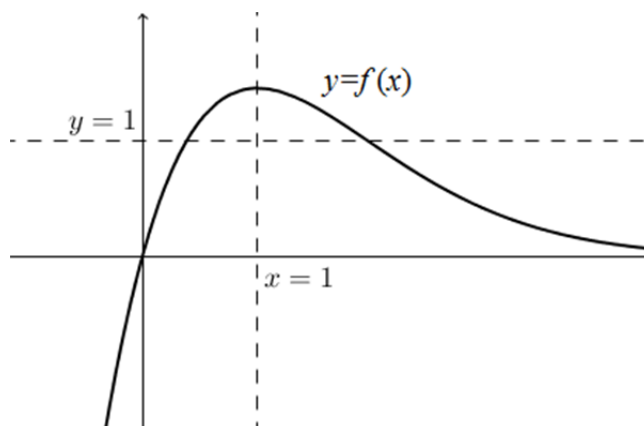
$$\int \frac{x^2 + x + 2}{(x^2 + 1)(x + 1)} dx$$

End of Question 11

Question 12 (15 marks)

Use a NEW sheet of paper.

(a)



Using four separate graphs sketch:

(i) $y = f'(x)$ [2]

(ii) $|y| = f(x)$ [2]

(iii) $y = \frac{1}{f(x)}$ [2]

(iv) $y = 3^{f(x)}$ [2]

(b) Evaluate [3]

$$\int_4^7 \frac{dx}{x^2 - 8x + 19}$$

(c) Let $f(x) = \frac{x^3+1}{x}$.

(i) Show that $\lim_{x \rightarrow \pm\infty} [f(x) - x^2] = 0$ [1]

(ii) Part (i) shows that the graph of $y = f(x)$ is asymptotic to the parabola $y = x^2$. Use this fact to help sketch the graph $y = f(x)$. [3]

End of Question 12

Question 13 (15 marks)

Use a NEW sheet of paper.

- (a) If ω is the root of $z^5 - 1 = 0$ with the smallest positive argument, find the real quadratic equation with roots $\omega + \omega^4$ and $\omega^2 + \omega^3$. [3]
- (b) Given the polynomial $P(x) = x^3 + x^2 + mx + n$ where m and n are real numbers:
- (i) If $(1 - 2i)$ is a zero of $P(x)$ factorise $P(x)$ into complex linear factors. [2]
- (ii) Find the values of m and n . [2]
- (c)
- (i) An ellipse has major and minor axes of lengths 12 and 8 respectively. Write a possible equation of this ellipse. [1]
- (ii) A solid has the elliptical base from part (i). Sections of the solid, perpendicular to its base and parallel to the minor axis, are semi-circles. Find the volume of the solid. [3]
- (d)
- (i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. [2]
- (ii) Hence find all the zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. [2]

End of Question 13

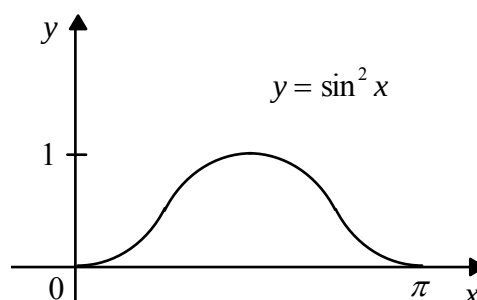
Question 14 (15 marks)

Use a NEW sheet of paper.

(a)

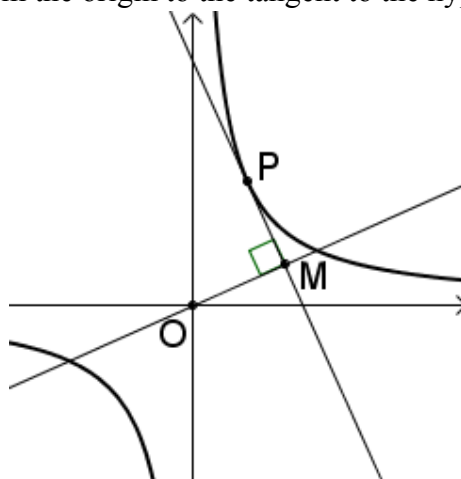
(i) Given that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, show that $\int_0^\pi x \cos 2x dx = 0$. [2]

(ii)



The area bounded by the curve $y = \sin^2 x$ and the x -axis between $x = 0$ and $x = \pi$ is rotated through one revolution about the y -axis. By taking the limiting sum of the volumes of cylindrical shells find the volume of this solid. [2]

- (b) $P\left(t, \frac{1}{t}\right)$ is a variable point on the rectangular hyperbola $xy = 1$. M is the foot of the perpendicular from the origin to the tangent to the hyperbola at P .

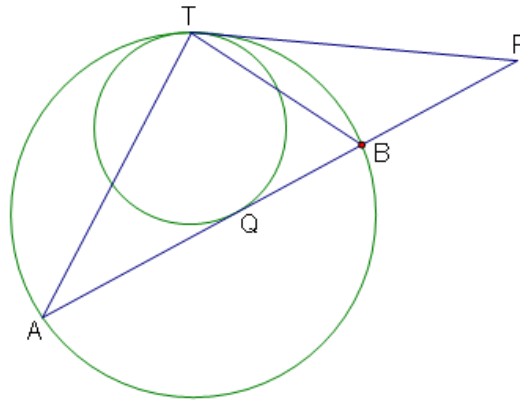


(i) Show that the tangent to the hyperbola at P has equation $x + t^2 y = 2t$. [2]

(ii) Find the equation of OM . [1]

(iii) Show that the equation of the locus of M as P varies is $x^4 + 2x^2y^2 - 4xy + y^4 = 0$ and indicate any restrictions on the values of x and y . [3]

- (c) PT is a common tangent to the circles which touch at T. PA is a tangent to the smaller circle at Q.



- (i) Prove that $\triangle BTP$ is similar to $\triangle TAP$. [2]
- (ii) Hence show that $PT^2 = PA \times PB$. [1]
- (iii) If $PT = t$, $QA = a$ and $QB = b$ prove that $t = \frac{ab}{a-b}$. [2]

End of Question 14

Question 15 (15 marks)

Use a NEW sheet of paper.

(a) Evaluate

[3]

$$\int_1^e x^7 \log_e x \, dx$$

(b)

(i) On the same diagram sketch the graphs of the ellipses

$E_1: \frac{x^2}{4} + \frac{y^2}{3} = 1$ and $E_2: \frac{x^2}{16} + \frac{y^2}{12} = 1$, showing clearly the intercepts on the axes. Find the coordinates of the foci and the equations of the directrices of the ellipse E_1 .

[2]

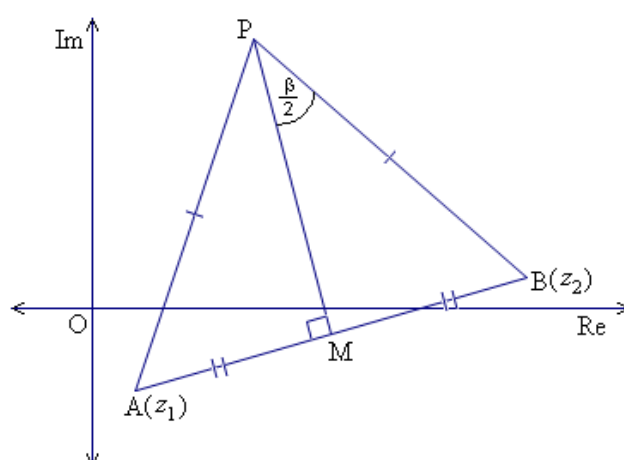
(ii) $P(2 \cos p, \sqrt{3} \sin p)$, where $0 < p < \frac{\pi}{2}$, is a point on the ellipse E_1 . Use differentiation to show that the tangent to the ellipse E_1 at P has equation $\frac{x \cos p}{2} + \frac{y \sin p}{\sqrt{3}} = 1$.

[2]

(iii) The tangent to the ellipse E_1 at P meets the ellipse E_2 at the points $Q(4 \cos q, 2\sqrt{3} \sin q)$ and $R(4 \cos r, 2\sqrt{3} \sin r)$, where $-\pi < q < \pi$ and $-\pi < r < \pi$. Show that q and r differ by $\frac{2\pi}{3}$.

[2]

- (c) The diagram shows an isosceles triangle PAB. PM is the bisector of $\angle APB$, where $\angle APB = \beta$. PM bisects AB. A and B represent the complex numbers z_1 and z_2 respectively.



- (i) Find the complex number represented by

(α) $\overrightarrow{AM}$ [1]

(β) $\overrightarrow{MP}$ [2]

- (ii) Hence show that P represents the complex number

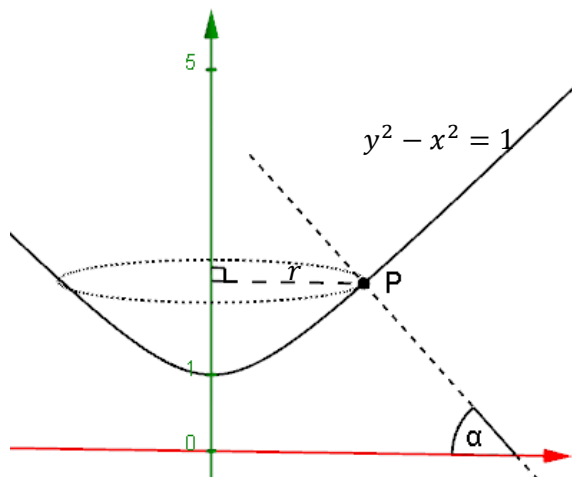
$$\frac{1}{2} \left(1 - i \cot \frac{\beta}{2} \right) z_1 + \frac{1}{2} \left(1 + i \cot \frac{\beta}{2} \right) z_2$$
 [3]

End of Question 15

Question 16 (15 marks)

Use a NEW sheet of paper.

- (a) A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \leq y \leq 5$ through 180° about the y -axis. Sometime later, a particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω .



- (i) Show that if the radius of the circle in which P moves is r , then the normal to the surface at P makes an angle α with the horizontal as shown in the diagram where $\tan \alpha = \frac{\sqrt{1+r^2}}{r}$. [2]
- (ii) Draw a diagram showing the forces on P. [1]
- (iii) Find the expressions for the radius r of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of m , g and ω . [3]
- (iv) Find the values of ω for which the described motion of P is possible. [1]

(b) Let

$$I_n = \int_1^e (1 - \ln x)^n dx \quad \text{where } n = 0, 1, 2, \dots$$

(i) Show [2]

$$I_n = -1 + nI_{n-1} \quad \text{where } n = 1, 2, 3, \dots$$

(ii) Hence evaluate [2]

$$\int_1^e (1 - \ln x)^3 dx$$

(iii) Show that [2]

$$\frac{I_n}{n!} = e - \sum_{r=0}^n \frac{1}{r!} \quad \text{where } n = 1, 2, 3, \dots$$

(iv) Show that $0 \leq I_n \leq e - 1$. [1]

(v) Deduce that [1]

$$\lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{r!} = e$$

End of Question 16

End of Exam



Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet

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Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ☐ B ☒ C ☐ D ☐

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ☒ B ☒ C ☐ D ☐

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

A ☒ B ☒ C ☐ D ☐
correct

Student Number: _____

Answers

Completely fill the response oval representing the most correct answer.

1. A ☐ B ☐ C ☒ D ☐
2. A ☐ B ☐ C ☒ D ☐
3. A ☐ B ☐ C ☐ D ☒
4. A ☐ B ☐ C ☐ D ☒
5. A ☐ B ☐ C ☒ D ☐
6. A ☐ B ☐ C ☒ D ☐
7. A ☐ B ☐ C ☐ D ☒
8. A ☐ B ☒ C ☐ D ☐
9. A ☐ B ☐ C ☐ D ☒
10. A ☐ B ☒ C ☐ D ☐

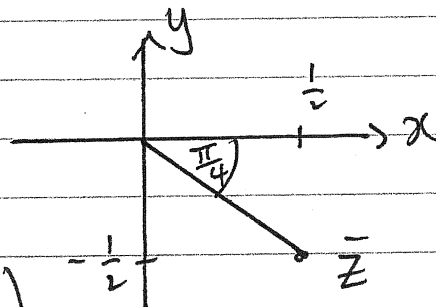
QUESTION 11

$$(a) (i) \quad z = \frac{1}{1-i} \times \frac{1+i}{1+i} = \frac{1+i}{1-i^2} = \frac{1}{2} + \frac{1}{2}i$$

$$\bar{z} = \frac{1}{2} - \frac{1}{2}i$$

$$|\bar{z}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

$$\arg(\bar{z}) = -\frac{\pi}{4} \quad \therefore \bar{z} = \frac{1}{\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$



A common error was an incorrect calculation of the argument. A diagram is usually recommended.

$$\begin{aligned} (ii) \quad (\bar{z})^{13} &= \left(\frac{1}{\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{4}\right) \right)^{13} \\ &= \left(\frac{1}{\sqrt{2}} \right)^{13} \operatorname{cis}\left(-\frac{13\pi}{4}\right) \\ &= \frac{1}{64\sqrt{2}} \operatorname{cis}\left(\frac{3\pi}{4}\right) \\ &= \frac{1}{64\sqrt{2}} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \end{aligned}$$

Some students didn't apply De Moivre's theorem correctly. Also, a reminder to answer the question being asked - some students did not state the values for a and b as asked.

$$\therefore a = -\frac{1}{128}, \quad b = \frac{1}{128}$$

$$(b) (i) \quad \int x^3 e^{x^4+7} dx = \frac{1}{4} e^{x^4+7} + C$$

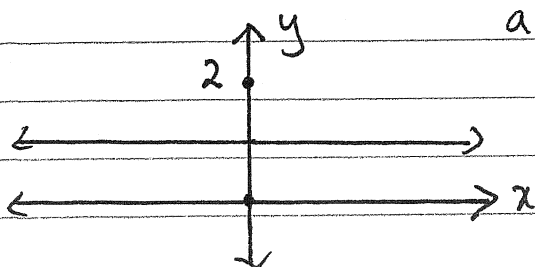
An easy question made complicated by some.

$$(ii) \quad \text{let } u = \sec x \quad du = \sec x \tan x dx$$

$$I = \int u^2 du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C$$

$$(c) \quad |z - 2i| = |z| \rightarrow \text{perpendicular bisector of } (0,0) \text{ and } (0,2).$$

Some used algebraic approach successfully.



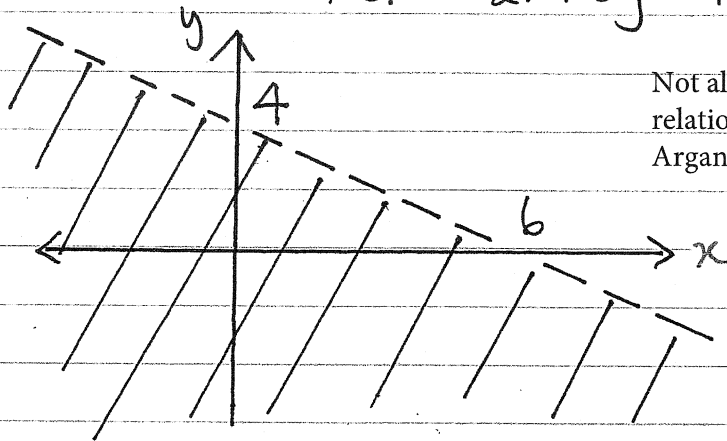
$\therefore$ The locus is the line $y=1$.

Qn 11 (continued)

(d) let $z = x + iy$ $\operatorname{Re}[(2-3i)(x+iy)] < 12$

$$\operatorname{Re}[2x + 3y + i(2y - 3x)] < 12$$

i.e. $2x + 3y < 12$



Not all students realised that the final relationship is just a region on the Argand diagram.

(e) (i) $\frac{x^2 + x + 2}{(x^2 + 1)(x + 1)} \equiv \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1}$

$$x^2 + x + 2 \equiv (Ax + B)(x + 1) + C(x^2 + 1)$$

coeff. of x^2 $1 = A + C$ (1)

coeff. of x $1 = A + B$ (2)

constant $2 = B + C$ (3)

(1) - (2) $C - B = 0$ i.e. $B = C$

subst. into (3) $B + B = 2$ $\therefore B = 1$ and $C = 1$

from (1) $A + 1 = 1$ $\therefore A = 0$

$$\therefore \frac{x^2 + x + 2}{(x^2 + 1)(x + 1)} = \frac{0x + 1}{x^2 + 1} + \frac{1}{x + 1} = \frac{1}{x^2 + 1} + \frac{1}{x + 1}$$

A reminder to ANSWER THE QUESTION. :)

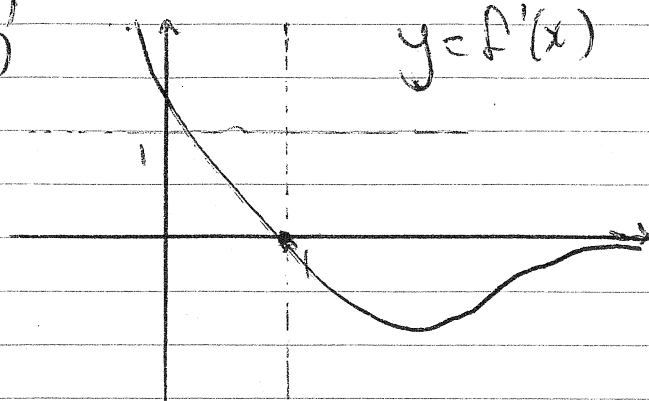
(ii) $\int \frac{x^2 + x + 2}{(x^2 + 1)(x + 1)} dx = \int \left(\frac{1}{x^2 + 1} + \frac{1}{x + 1} \right) dx$

$$= \tan^{-1} x + \ln(x + 1) + C$$

2015 THSC ext 2

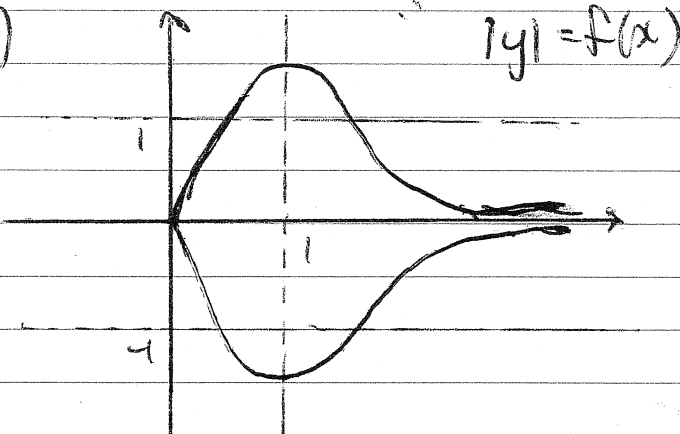
(a)

(i)

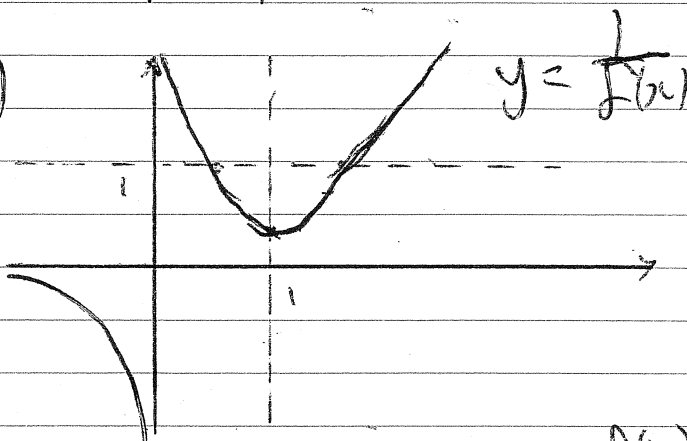


(Many student had problems with the first two graphs.)

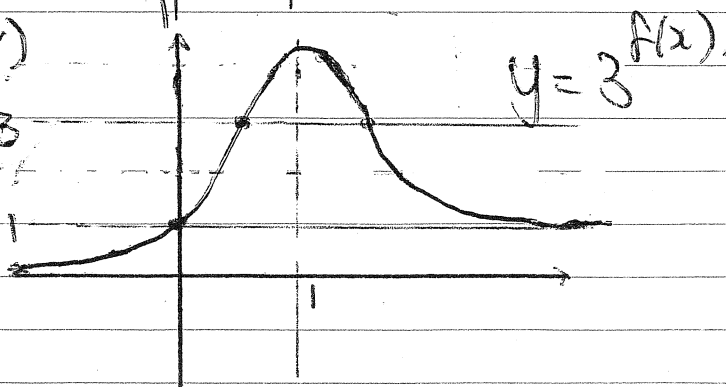
(ii)



(iii)



(iv)



$$(b) \int_4^7 \frac{dx}{x^2 - 8x + 19}$$

$$= \int_4^7 \frac{dx}{(x-4)^2 - 16 + 19}$$

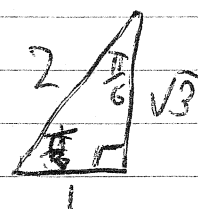
$$= \int_4^7 \frac{dx}{\sqrt{3} + (x-4)^2}$$

$$= \frac{1}{\sqrt{3}} \left[\tan^{-1} \left(\frac{x-4}{\sqrt{3}} \right) \right]_4^7$$

$$= \frac{1}{\sqrt{3}} \left(\tan^{-1} \left(\frac{3}{\sqrt{3}} \right) - \tan^{-1} 0 \right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

Most common error
was to have 3 instead
of $\sqrt{3}$. e.g.
 $\tan^{-1} \left(\frac{x-4}{3} \right)$.



$$(c) f(x) = \frac{x^3 + 1}{x}$$

$$(i) \lim_{x \rightarrow \pm\infty} \left[\frac{x^3 + 1}{x} - x^2 \right] = \lim_{x \rightarrow \pm\infty} \left[\frac{1}{x} \right]$$

$$= 0.$$

$$(ii) f'(x) = \frac{x(3x^2) - (x^3 + 1)}{x^2}$$

$$= \frac{2x^3 - 1}{x^2}$$

$$f'(x) = 0.$$

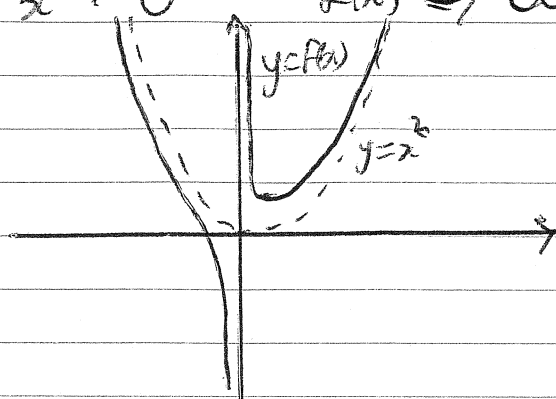
$$2x^3 - 1 = 0$$

$$x = \frac{1}{\sqrt[3]{2}}$$

x	0.7	$\frac{1}{\sqrt[3]{2}}$	0.9
$f'(x)$	-0.6	0	0.6

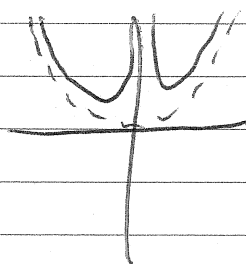
$$f\left(\frac{1}{\sqrt[3]{2}}\right) = 1.89$$

As $x \rightarrow 0^-$ $f(x) \rightarrow -\infty$
 As $x \rightarrow 0^+$ $f(x) \rightarrow \infty$.



Common for student to have
 both piece inside the parabola.

eg



Some students ignored part (i)
 completely when graphing.

$$13 (a) 1 + w + w^2 + w^3 + w^4 = 0$$

$$\alpha + \beta = w + w^4 + w^2 + w^3$$

$$= -1$$

$$\alpha \beta = w^3 + w^4 + w^6 + w^7$$

$$= w^3 + w^4 + w + w^2$$

$$= -1$$

Students
finding w
in mod-arg
form had
less success.

$$\therefore x^2 - -1x + -1 = 0$$

$$x^2 + x - 1 = 0$$

$$(b) (i) 1 - 2i + 1 + 2i + \alpha = -1$$

$$2 + \alpha = -1$$

$$\alpha = -3$$

$$\therefore P(x) = (x - 1 + 2i)(x - 1 - 2i)(x + 3)$$

$$(ii) (1 - 2i)(1 + 2i)x - 3 = -n$$

$$-15 = -n$$

$$n = 15$$

~~$$2(-3)^3 - 4x(-3)^2 + 15x - 3 + 15 = 0$$~~

$$(-3)^3 + (-3)^2 + mx - 3 + 15 = 0$$

$$-3m + 3 = 0$$

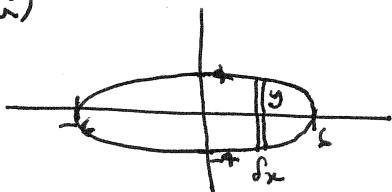
$$m = -1$$

$$(c) (i) \frac{x^2}{6^2} + \frac{y^2}{4^2} = 1$$

Many students
forgot to halve
12 and 8.

$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

(ii)



$$V_{shell} = \frac{\pi}{2} y^2 dx$$

$$V_{solid} = \frac{\pi}{2} \int_{-6}^6 y^2 dx$$

$$= \frac{\pi}{2} \int_{-6}^6 16 \left(1 - \frac{x^2}{36}\right) dx$$

$$= 16\pi \int_0^6 \left(1 - \frac{x^2}{36}\right) dx$$

$$= 16\pi \left[x - \frac{x^3}{108}\right]_0^6 = 16\pi \left(6 - \frac{6^3}{108}\right) = 64\pi u^3$$

This is not
a semi-circle

$$(d) (i) P(x) = (x-a)^3 Q(x)$$

$$P'(x) = (x-a)^3 Q'(x) + Q(x) \times 3(x-a)^2$$

$$= (x-a)^2 \{ (x-a) Q'(x) + 3Q(x) \}$$

G.E.D.

$$(ii) p'(x) = 32x^3 - 75x^2 + 54x - 1$$

$$p''(x) = 96x^2 - 150x + 54$$

$$= 6(16x^2 - 25x + 9)$$

$$= 6(16x - 9)(x - 1)$$

$$p(1) = 8 - 25 + 27 - 1 = 1$$

$$= 0$$

$\therefore 1$ is the triple root

$$1^3 \times \alpha = \frac{1}{8}$$

$$\alpha = \frac{1}{8}$$

$\therefore$ Zeros are $1, 1, 1, \frac{1}{8}$

$$\begin{aligned}
 14(a) (i) & \int_0^\pi x \cos 2x \, dx \\
 &= \int_0^\pi (\pi - x) \cos (2\pi - 2x) \, dx \\
 &= \int_0^\pi (\pi - x) \cos 2x \, dx \\
 &= \int_0^\pi \pi \cos 2x \, dx - \int_0^\pi x \cos 2x \, dx \\
 \therefore 2 \int_0^\pi x \cos 2x \, dx &= \pi \int_0^\pi \cos 2x \, dx \\
 &= \pi \left[\frac{\sin 2x}{2} \right]_0^\pi \\
 &= \pi (0 - 0) \\
 &= 0 \\
 \therefore \int_0^\pi x \cos 2x \, dx &= 0
 \end{aligned}$$

$$\begin{aligned}
 (ii) V_{shell} &= \pi \{ (x+dx)^2 - x^2 \} y \\
 &= 2\pi xy \, dx
 \end{aligned}$$

$$\begin{aligned}
 V_{solid} &= 2\pi \int_0^\pi x \sin^2 x \, dx \\
 &= 2\pi \int_0^\pi x \left(\frac{1 - \cos 2x}{2} \right) dx \\
 &= \pi \int_0^\pi (x - x \cos 2x) dx \\
 &= \pi \left[\frac{x^2}{2} \right]_0^\pi \quad \text{Using the result in (i)} \\
 &= \frac{\pi^3}{2}
 \end{aligned}$$

$$\begin{aligned}
 (b) (i) \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\
 &= -\frac{1}{t^2} \times 1 \\
 &= -\frac{1}{t^2} \\
 y - \frac{1}{t} &= -\frac{1}{t^2} (x - t) \\
 t^2 y - t &= -x + t \\
 \therefore x + t^2 y &= 2t
 \end{aligned}$$

(ii) $y = t^2 x$ $m_o m$ is perpendicular to the tangent at P

$$(iii) t^2 = \frac{y}{x}$$

$$t = \pm \sqrt{\frac{y}{x}}$$

$$x + \frac{y}{x} \times y = 2x \pm \sqrt{\frac{y}{x}}$$

$$\left(x + \frac{y^2}{x} \right)^2 = \frac{4y}{x}$$

$$x^2 + 2y^2 + \frac{y^4}{x^2} = \frac{4y}{x}$$

$$x^4 + 2x^2 y^2 + y^4 = 4xy$$

$$\text{restriction: } \frac{y}{x} > 0$$

The point M cannot lie on the hyperbola

(c) (i) In $\triangle RTP$ and $\triangle TAP$
 $\hat{P}$ is common

$$\hat{P}\hat{T}R = \hat{T}\hat{A}P \quad (\angle \text{ is external angle})$$

$\therefore \triangle RTP \parallel \triangle TAP$ (equiangular)

$$(ii) \frac{TP}{TA} = \frac{PB}{TP} \quad (\text{corresponding sides in similar } \triangle \text{ s})$$

$$\therefore TP^2 = TA \cdot PB$$

Quoting the theorem is not showing.

$$\begin{aligned}
 (iii) t^2 &= (t+a)(t-b) \\
 &= t^2 - bt + at - ab
 \end{aligned}$$

$$\begin{aligned}
 \cancel{at} &= at - bt \\
 &= t(a-b)
 \end{aligned}$$

$$\therefore A = \frac{ab}{a-b}$$

$t = PG$ (tangents from an external pt.)

Q15.
(a) $I = \int_1^e x^7 \log_e x \, dx.$

$$u = \log_e x \quad v = \frac{x^8}{8}$$

$$u' = \frac{1}{x} \quad v' = x^7$$

$$I = \frac{x^8}{8} \ln(x) \Big|_1^e - \int_1^e x^7 \, dx$$

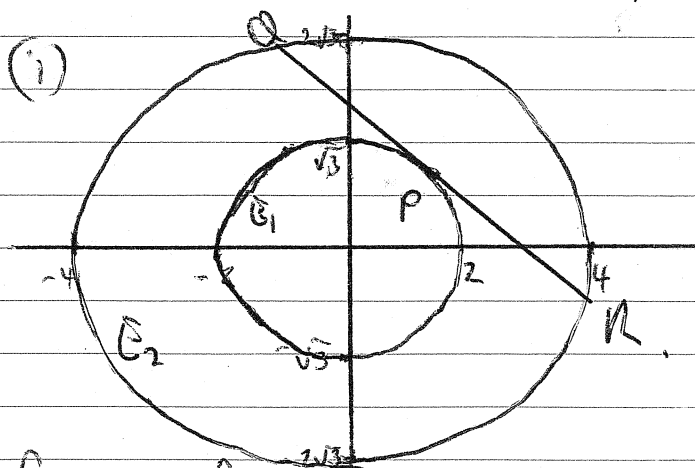
$$= \frac{e^8}{8} - \frac{1}{8} \left[\frac{x^8}{8} \right]_1^e$$

$$= \frac{e^8}{8} - \frac{x^8}{64} + \frac{1}{64}$$

$$= \frac{7e^8}{64} + \frac{1}{64}$$

Well done by most.

(b) (i)



$$E_1: \frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$E_2: \frac{x^2}{16} + \frac{y^2}{12} = 1$$

For E_1 , $3 = 4(1 - e^2)$

$$1 - \frac{3}{4} = e^2$$

$$\frac{1}{4} = e^2$$

$$e = \frac{1}{2}$$

Loci of E_1

$$S(ae, 0) \quad S'(ae, 0)$$

$$S(1, 0) \quad S'(-1, 0)$$

directrices

$$x = \pm 4$$

(b) (i) $P(2 \cos p, \sqrt{3} \sin p)$ $E_1: \frac{x^2}{4} + \frac{y^2}{3} = 1.$

$$\frac{x}{2} + \frac{2y}{3} \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{x}{2} \times \frac{3}{2y}.$$

$$\frac{dy}{dx} = -\frac{3x}{4y}.$$

At P ,

$$m_T = -\frac{\sqrt{3} \cos p}{2 \sin p}.$$

Tangent at P ,
 $y - \sqrt{3} \sin p = -\frac{\sqrt{3} \cos p}{2 \sin p} (x - 2 \cos p).$

$$2y \sin p - 2\sqrt{3} \sin^2 p = -\sqrt{3} x \cos p + 2\sqrt{3} \cos^2 p$$

$$\sqrt{3} x \cos p + 2y \sin p = 2\sqrt{3}.$$

$$\frac{x \cos p}{2} + \frac{y \sin p}{\sqrt{3}} = 1.$$

(iii) Tangent to E_1 at P meets E_2 at same point $(4 \cos t, 2\sqrt{3} \sin t).$

So

$$\frac{4 \cos t \cos p}{2} + \frac{2\sqrt{3} \sin t}{\sqrt{3}} = 1.$$

$$2(\cos t \cos p + \sin t \sin p) = 1$$

$$\cos(t-p) = \frac{1}{2}.$$

Since $0 < p < \frac{\pi}{2}$ and $-\pi < t < \pi$ $t-p = \pm \frac{\pi}{3}.$

Hence Q and R have parameters

$$q = \frac{\pi}{3} + \rho, \quad r = -\frac{\pi}{3} + \rho.$$

$$\therefore |q - r| = \frac{2\pi}{3}.$$

Some arguments for this were poor and lost a mark.

(C)(i) $\vec{AB}$ represents $(z_2 - z_1)$

$\therefore \vec{AM}$ represents $\frac{1}{2}(z_2 - z_1)$.

$$(ii) \tan \frac{\beta}{2} = \frac{|\vec{AM}|}{|\vec{PM}|}$$

$$|\vec{PM}| = |\vec{AM}| \cot \frac{\beta}{2}.$$

Also $\left| \frac{\frac{1}{2}(z_2 - z_1)}{|\vec{AM}|} \right| = 1$

Find a unit vector in the right direction then multiply by the required modulus

So $\vec{PM}$ is represented by

$$\frac{\frac{1}{2}(z_2 - z_1)}{|\vec{AM}|} \times |\vec{AM}| \cot \frac{\beta}{2}.$$

$$(iii) \rho \text{ represents } z_1 + \vec{AM} + \vec{PM} = \frac{1}{2}(z_2 - z_1) \cot \frac{\beta}{2}.$$

Must add 3 vectors to get ρ . Some student only add $\vec{AM}$ and $\vec{PM}$.

$$\begin{aligned} &= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{1}{2}(z_2 - z_1) \cot \frac{\beta}{2} \\ &= \frac{1}{2}z_1 + \frac{1}{2}z_2 + \frac{1}{2}z_2 \cot \frac{\beta}{2} - \frac{1}{2}z_1 \cot \frac{\beta}{2} \\ &= \frac{1}{2}(1 - i \cot \frac{\beta}{2})z_1 + \frac{1}{2}(1 + i \cot \frac{\beta}{2})z_2. \end{aligned}$$

Question 16

(a) (i) at P, $x=r$ and $y^2 - r^2 = 1$

$$y = \sqrt{1+r^2}$$

$$2y \frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx} = \frac{x}{y}, \text{ at P } m_T = \frac{r}{\sqrt{1+r^2}}$$

$$m_N = -\frac{\sqrt{1+r^2}}{r}$$

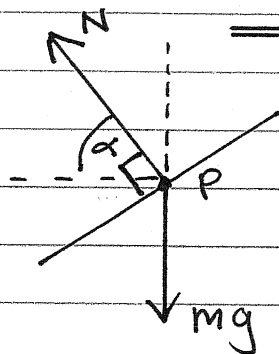
Many students did not make the connection to the gradient of the normal.

Also, the angle between the line and the x-axis was overlooked.

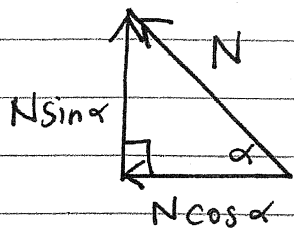
$$\tan(180 - \alpha) = -\frac{\sqrt{1+r^2}}{r}$$

$$-\tan \alpha = -\frac{\sqrt{1+r^2}}{r} \therefore \tan \alpha = \frac{\sqrt{1+r^2}}{r}$$

(ii)



(iii) Resolving forces on P



Horizontally $mrw^2 = N \cos \alpha$ (1)

Vertically $N \sin \alpha - mg = 0$

$\therefore mg = N \sin \alpha$ (2)

(2) $\div$ (1) $\tan \alpha = \frac{g}{rw^2}$

Using (i) $\frac{\sqrt{1+r^2}}{r} = \frac{g}{rw^2}$

Some students incorrectly resolved forces horizontally and vertically.

Also, the question sought an expression for r and N in terms of m , g and w only. Solutions that included r in the expression for N were incomplete.

$$1+r^2 = \left(\frac{g}{w^2}\right)^2$$

$$r^2 = \frac{g^2}{w^4} - 1$$

$$\therefore r = \sqrt{\frac{g^2}{w^4} - 1}$$

16 (a) (iii) continued

$$\begin{aligned} \textcircled{1}^2 + \textcircled{2}^2 \quad N^2(\sin^2 \alpha + \cos^2 \alpha) &= m^2 g^2 + m^2 r^2 \omega^4 \\ \therefore N^2 &= m^2 g^2 + m^2 \omega^4 \left(\frac{g^2}{\omega^4} - 1 \right) \\ &= m^2 g^2 + m^2 g^2 - m^2 \omega^4 \end{aligned}$$

$$\text{i.e. } N = \sqrt{2m^2 g^2 - m^2 \omega^4}$$

$$\text{(iv)} \quad y = \sqrt{1+r^2} \quad \text{and} \quad r^2 = \frac{g^2}{\omega^4} - 1$$

$$\therefore y = \sqrt{\frac{g^2}{\omega^4}} = \frac{g}{\omega^2}$$

For movement to occur $r > 0 \quad \therefore y = \sqrt{1+r^2} > 1$

Since $1 \leq y \leq 5$ and $y > 1$ and $y = \frac{g}{\omega^2}$

$$1 < \frac{g}{\omega^2} \leq 5$$

$$1 > \frac{\omega^2}{g} \geq \frac{1}{5} \quad \text{i.e.} \quad \sqrt{\frac{g}{5}} \leq \omega < \sqrt{g}$$

$$\therefore 1.4 \text{ rad/s} \leq \omega < 3.13 \text{ rad/s}$$

A small number of students answered this correctly. Most students failed to consider the restriction placed on ω due to the range.

Question 16 (cont.)

(b)(i) $I_n = \int_1^e (1 - \ln x)^n dx$ where $n = 0, 1, 2, \dots$

let $u = (1 - \ln x)^n$ $v' = 1$
 $u' = -\frac{n}{x} (1 - \ln x)^{n-1}$ $v = x$

Mostly well done though substitution into the definite integral would show more clearly the required result.

$$\begin{aligned} \therefore I_n &= \left[x (1 - \ln x)^n \right]_1^e - \int -n (1 - \ln x)^{n-1} dx \\ &= e (1 - \ln e)^n - 1 (1 - \ln 1)^n + n \int (1 - \ln x)^{n-1} dx \end{aligned}$$

$$\therefore I_n = -1 + n I_{n-1}$$

(ii) $I_3 = \int_1^e (1 - \ln x)^3 dx$

$$\begin{aligned} &= -1 + 3 I_2 \\ &= -1 + 3 (-1 + 2 I_1) \\ &= -4 + 6 (-1 + I_0) \\ &= -10 + 6 \int_1^e dx \\ &= -10 + 6 [x]_1^e \\ &= -10 + 6 (e - 1) \\ \therefore I_3 &= 6e - 16 \end{aligned}$$

Some students made careless calculation errors. These questions require the ability to manage multiple substitutions with accuracy.

(iii) $\frac{I_n}{n!} = \frac{-1 + n I_{n-1}}{n!} = \frac{-1}{n!} + \frac{I_{n-1}}{(n-1)!}$

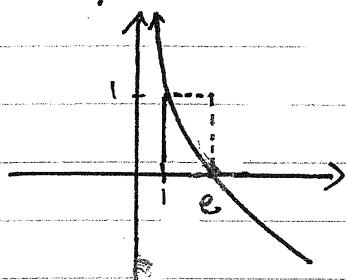
This question was not done particularly well by many. Some students glossed over certain elements of the proof and need to consider any suggestions provided on their individual paper.

$$\begin{aligned} &= \frac{-1}{n!} + \frac{-1 + (n-1) I_{n-2}}{(n-1)!} \\ &= \frac{-1}{n!} - \frac{1}{(n-1)!} + \frac{I_{n-2}}{(n-2)!} \\ &= \frac{-1}{n!} - \frac{1}{(n-1)!} - \frac{1}{(n-2)!} + \frac{I_{n-3}}{(n-3)!} \\ &= \frac{-1}{n!} - \frac{1}{(n-1)!} - \frac{1}{(n-2)!} \dots - \frac{1}{1!} + \frac{I_0}{0!} \end{aligned}$$

Qn 16 (b)(iii) Continued

$$\begin{aligned}
 \frac{I_n}{n!} &= - \left(\frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-1)!} + \frac{1}{n!} \right) + \int_1^e 1 \, dx \\
 &= - \sum_{r=1}^n \frac{1}{r!} + [x]_1^e \\
 &= - \sum_{r=1}^n \frac{1}{r!} + e - 1 \quad \text{Note: } 1 = \frac{1}{0!} \\
 &= - \left(\sum_{r=1}^n \frac{1}{r!} + \frac{1}{0!} \right) + e \\
 \text{i.e. } \frac{I_n}{n!} &= e - \sum_{r=0}^n \frac{1}{r!}
 \end{aligned}$$

(iv) Consider graph of $y = 1 - \ln x$ between $x=1$ and $x=e$.



Note that the y-values for this domain are $0 \leq y \leq 1$.

The y-values for $y = (1 - \ln x)^n$ will also be in the range $0 \leq y \leq 1$ where $n = 0, 1, 2, \dots$

Consider the area under the curve

$y = (1 - \ln x)^n$ for $1 \leq x \leq e$ where the area will

always be smaller than the rectangle shown, and always above the x-axis.

A variety of approaches could be taken. However, the question required explanation for the zero part of the inequality and not just $(e-1)$.

$$0 \leq \int_1^e (1 - \ln x)^n \, dx \leq (e-1) \times 1$$

$$\text{i.e. } 0 \leq I_n \leq e-1$$

(v) Using (iv)
$$\frac{0}{n!} \leq \frac{I_n}{n!} \leq \frac{e-1}{n!}$$

as $n \rightarrow \infty$, $\frac{e-1}{n!} \rightarrow 0 \quad \therefore 0 \leq \lim_{n \rightarrow \infty} \frac{I_n}{n!} \leq 0$

i.e. $\lim_{n \rightarrow \infty} \frac{I_n}{n!} = 0 \quad \therefore \lim_{n \rightarrow \infty} \left(e - \sum_{r=0}^n \frac{1}{r!} \right) = 0$

Many students picked up the marks for this question but could have provided a solution with greater clarity.

$$e - \lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{r!} = 0$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{r!} = e$$