

Sydney Girls High School

2017

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 3 – 6

10 Marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 8 – 19

90 Marks

- Attempt Questions 11 – 16
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 2 hours and 45 minutes for this section

Name:

Teacher:

THIS IS A TRIAL PAPER ONLY

It does not necessarily reflect the format or the content of the 2017 HSC Examination Paper in this subject.

Section I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

(1) What is the double root of the equation $x^3 - 5x^2 + 8x - 4 = 0$?

(A) $x = -2$

(B) $x = -1$

(C) $x = 1$

(D) $x = 2$

(2) A small car of mass 1200 kg is rounding a curve of radius 500 metres on a level road at 84km/h.

What force of friction is necessary between the wheels and the ground?

(A) $3 \cdot 36 \text{ N}$

(B) $52 \cdot 27 \text{ N}$

(C) $1306 \cdot 67 \text{ N}$

(D) $16\,934.4 \text{ N}$

(3) Which of the following parametric equations represent the hyperbola $x^2 - y^2 = 4$?

(A) $x = 2 \tan \theta$ and $y = 2 \sec \theta$

(B) $x = 4 \tan \theta$ and $y = 4 \sec \theta$

(C) $x = 2 \sec \theta$ and $y = 2 \tan \theta$

(D) $x = 4 \sec \theta$ and $y = 4 \tan \theta$

(4) Which of the following is the modulus-argument form of $2-2i$?

(A) $2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$

(B) $2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

(C) $2\text{cis}\left(\frac{7\pi}{4}\right)$

(D) $2\text{cis}\left(-\frac{7\pi}{4}\right)$

(5) The graph of $y = \frac{x^2}{x^2 - 4}$ has:

(A) a single vertical asymptote, two horizontal asymptotes and no turning points

(B) a single horizontal asymptote, two vertical asymptotes and no turning points

(C) a single vertical asymptote, two horizontal asymptotes and one turning point

(D) a single horizontal asymptote, two vertical asymptotes and one turning point

(6) The point P on the Argand diagram represents the complex number z .

The point P moves such that $|z|^2 + |z + 2i|^2 = 10$.

Which of the following best describes the path traced out by P ?

(A) An ellipse

(B) A hyperbola

(C) A circle

(D) A straight line

(7) A committee of 5 people is to be chosen from a group of 6 girls and 4 boys.

How many different committees could be formed that have at least one boy.

(A) ${}^{10}C_5 - 1$

(B) ${}^4C_1 + {}^6C_4$

(C) ${}^4C_1 \times {}^6C_4$

(D) ${}^{10}C_5 - 6$

(8) The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x .

Which of the following is the expression for $\frac{dy}{dx}$?

(A) $\frac{y^2 - x}{x^2 + y}$

(B) $\frac{y^2 + x}{x^2 - y}$

(C) $\frac{x^2 + y}{y^2 - x}$

(D) $\frac{x^2 - y}{y^2 + x}$

(9) At time t seconds, $t \geq 0$, the velocity v m/s of a particle moving in a straight line is given by

$v = \sqrt{3}\cos(t) + \sin(t) - 2$. For what value of t does the particle first attain its maximum speed of 4 m/s ?

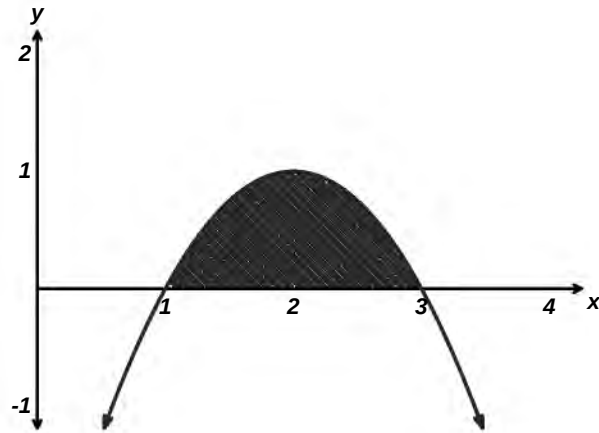
(A) $t = \frac{\pi}{6}$

(B) $t = \frac{7\pi}{6}$

(C) $x = \frac{4\pi}{3}$

(D) The particle never attains a speed of 4 m/s .

(10)



The diagram above shows the graph $y = 4x - x^2 - 3$.

The shaded region bounded by the graph and the x -axis is rotated around the y -axis to form a solid.

Which of the integrals below gives the volume of the solid?

(A) $8\pi \int_0^1 \sqrt{1-y} \, dy$

(B) $8\pi \int_0^1 2 + \sqrt{1-y} \, dy$

(C) $\pi \int_0^1 1 + \sqrt{1-y} \, dy$

(D) $\pi \int_0^1 \sqrt{1-y} \, dy$

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer on the blank lined paper provided. Begin a new page for each question
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11

(15 Marks)

Use a NEW sheet of paper.

(a) Find:

i) $\int e^x (1 + e^x)^5 dx.$ [1]

ii) $\int \frac{dt}{\sqrt{7 + 6t - t^2}}.$ [2]

(b) Let $\alpha = -\sqrt{3} + i$ and $\beta = 1 - i$.

i) Express $\bar{\alpha}$ and β in modulus-argument form. [2]

ii) Find $\bar{\alpha}\beta$ in modulus-argument form. [1]

iii) Hence, or otherwise, find the exact value of $\tan \frac{11\pi}{12}$. [2]

Express your answer in its simplest form.

Question 11 continues on the next page

Question 11 (Continued)

(c) For the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

- i) Find the eccentricity. [1]
- ii) Find the coordinates of the foci S and S' . [1]
- iii) Find the equations of the directrices. [1]
- iv) Show that the coordinates of any point P on the ellipse can be represented by $(5\cos\theta, 4\sin\theta)$. [2]
- v) Show that $PS + PS'$ is a constant. [2]

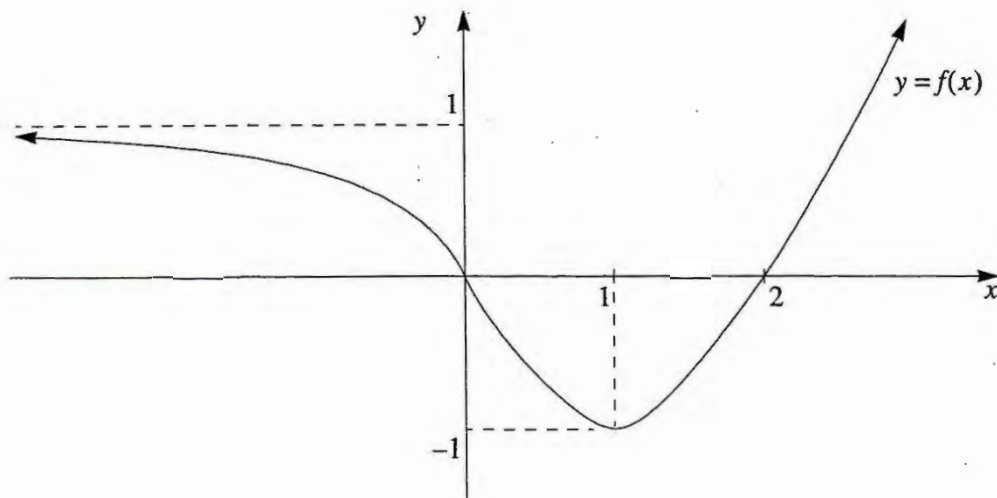
End of Question 11

Question 12

(15 Marks)

Use a NEW sheet of paper.

(a)



Given the function $y = f(x)$ in the diagram above, sketch on separate diagrams, showing all intercepts, turning points and asymptotes:

i) $y = f(|x|)$ [1]

ii) $|y| = f(x)$ [2]

iii) $y = f(2x)$ [2]

iv) $y = \frac{1}{f(x)}$ [2]

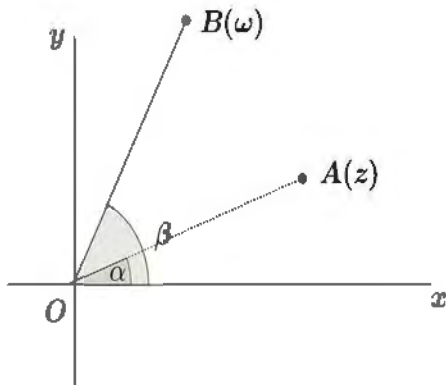
v) $y = e^{f(x)}$ [2]

Question 12 continues on the next page

Question 12 (Continued)

(b) Using the substitution $t = \tan \frac{\theta}{2}$, find $\int \frac{2}{4 + 3 \sin \theta} d\theta$. [3]

(c) [3]



The points A and B on the Argand diagram above represent the complex numbers z and w respectively and $|z| = |w| = 2$.

If $\arg z = \alpha$ and $\arg w = \beta$ show that $|z + w| = 4 \cos \left(\frac{\beta - \alpha}{2} \right)$.

End of Question 12

Question 13

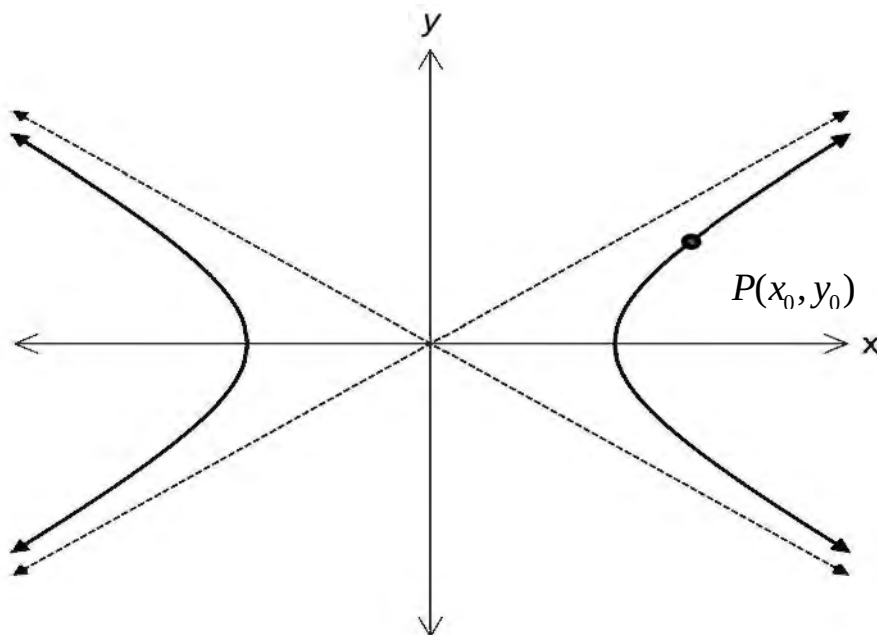
(15 Marks)

Use a NEW sheet of paper.

- (a) The roots of the equation $x^3 - 9x^2 + 31x + m = 0$ are in an arithmetic sequence. [3]

Find the roots of the equation and hence the value of m .

- (b) The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$.



- i) Write down the equations of the two asymptotes of the hyperbola. [1]

- ii) Show that the acute angle α between the two asymptotes satisfies [2]

$$\tan \alpha = \frac{2ab}{a^2 - b^2}.$$

- iii) If M and N are the feet of the perpendiculars drawn from P to the [3]

asymptotes, show that $MP \times NP = \frac{a^2 b^2}{a^2 + b^2}$.

- iv) Hence find the area of $\triangle PMN$ in terms of a and b . [2]

Question 13 continues on the next page

Question 13 (Continued)

(c)

i) Find the rational values of A , B and C given:

[2]

$$\frac{y^2 + 8}{(y - 2)(y^2 + 2y + 4)} \equiv \frac{A}{y - 2} + \frac{By + C}{y^2 + 2y + 4}.$$

ii) Hence find $\int \frac{y^5 - 7y^2 + 8}{y^3 - 8} dy$.

[2]

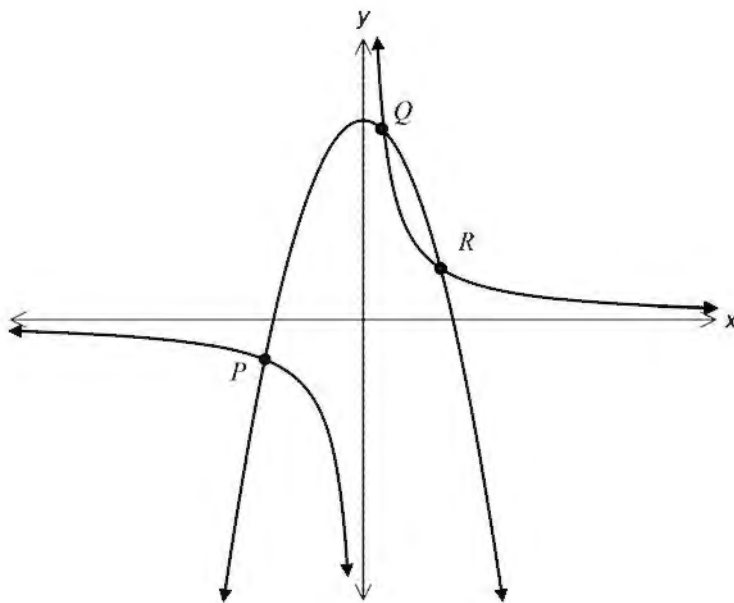
End of Question 13

Question 14

(15 Marks)

Use a NEW sheet of paper.

(a)



The curves $y = \frac{1}{x}$ and $y = k - x^2$, for some real number k , intersect at the points P , Q and R where the x -coordinates are $x = \alpha$, $x = \beta$ and $x = \gamma$ respectively.

i) Show that the monic cubic equation with coefficients in terms of k [3]

whose roots are α^2, β^2 and γ^2 is given by $x^3 - 2kx^2 + k^2x - 1 = 0$.

ii) Find the monic cubic equation with coefficients in terms of k whose [1]

roots are: $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$.

iii) Hence find in simplest form $OP^2 + OQ^2 + OR^2$ in terms of k , [2]
where O is the origin.

Question 14 continues on the next page

Question 14 (Continued)

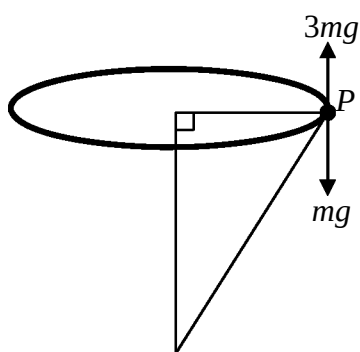
(b)

i) Show that a reduction formula for $I_n = \int (\ln x)^{\frac{n}{2}} dx$, where n is a positive [2]

integer, is $I_n = x(\ln x)^{\frac{n}{2}} - \frac{n}{2} I_{n-2}$.

ii) Hence, or otherwise, evaluate $\int_1^e (\ln x)^4 dx$. [2]

(c)



A model aircraft P , of mass $m = 8 \text{ kg}$ is attached to the end of a 10 m long inelastic wire, with the other end fixed to the ground.

The model flies in a horizontal circle so that the wire makes an angle of 30° with the ground. The uplift created by the wings of the aircraft is a vertical force $3mg$. (take $g = 10 \text{ ms}^{-1}$)

i) By resolving the forces at P , calculate the tension in the wire. [3]

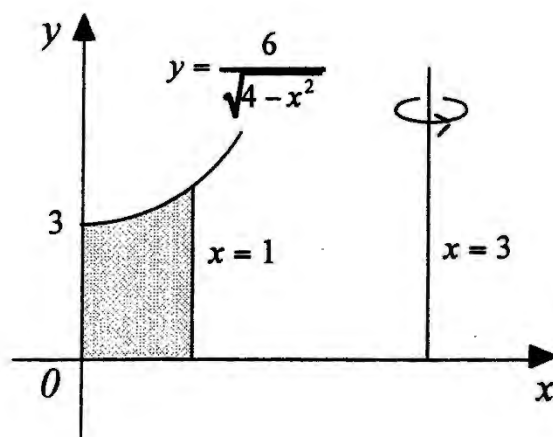
ii) Calculate the angular velocity about the centre of the horizontal circle. [2]

End of Question 14

Question 15**(15 Marks)**

Use a NEW sheet of paper.

(a)



A mould for a section of concrete piping is made by rotating the region bounded by the curve $y = \frac{6}{\sqrt{4-x^2}}$ and the x axis between the lines $x = 0$ and $x = 1$ through one complete revolution about the line $x = 3$. All measurements are in metres.

- i) By considering strips of width δx parallel to the axis of rotation, show that the volume $V \text{ m}^3$ of the concrete used in the piping is given by [2]

$$V = 12\pi \int_0^1 \frac{3-x}{\sqrt{4-x^2}} dx.$$

- ii) Hence find the volume of the concrete used in the piping, giving your answer correct to the nearest cubic metre. [2]

Question 15 continues on the next page

Question 15 (Continued)

b)

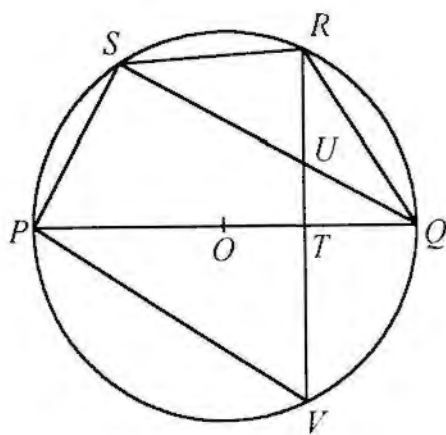
i) Use De Moivre's Theorem to show that $\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$. [2]

ii) Show that the equation $16x^4 - 16x^2 + 1 = 0$ has roots [2]

$$x = \cos \frac{\pi}{12}, x = -\cos \frac{\pi}{12}, x = \cos \frac{5\pi}{12} \text{ and } x = -\cos \frac{5\pi}{12}.$$

iii) By considering this equation as a quadratic equation in x^2 , find the exact value of $\cos \frac{5\pi}{12}$. [3]

c) In the diagram, PQ is the diameter of the circle with centre O .



RV intersects SQ and PQ at U and T respectively.

If $\angle QRT = \angle RSQ$, prove that:

i) $\angle TPV = \angle RSQ$. [1]

ii) $\angle RTQ$ is a right angle. [2]

iii) PU is a diameter of the circle passing through P, T, U and S . [1]

End of Question 15

Question 16

Use a NEW sheet of paper.

(15 Marks)

(a) Use the letters of the word *STRETCH* to answer the following.

- i) How many two-letter arrangements can be made? [1]
- ii) If the letters are selected at random to create a two-letter arrangement, what is the probability that the two-letter arrangement will be “*TT*” ? [1]
- iii) The creation of two-letter arrangements from the word *STRETCH* is repeated. [2]

How many two-letter arrangements need to be created to ensure that the probability of obtaining the arrangement “*TT*” at least once, exceeds 90%?

(b)

- i) Show that $\sin(2r+1)\theta - \sin(2r-1)\theta = 2\sin\theta\cos 2r\theta$, where r is a positive integer. [1]
- ii) Hence show that for $n \geq 1$ [2]

$$\sin\theta \sum_{r=1}^n \cos 2r\theta = \frac{1}{2} \{ \sin(2n+1)\theta - \sin\theta \}.$$

- iii) Hence evaluate $\sum_{r=1}^{100} \cos^2\left(\frac{r\pi}{100}\right)$. [3]

Question 16 continues on the next page

Question 16 (Continued)

(c)

i) Use the principle of mathematical induction to prove that $3^n > n^3$ for all integers $n \geq 4$. [4]

ii) Hence or otherwise show that $\sqrt[3]{3} > \sqrt[n]{n}$ for all integers $n \geq 4$. [1]

End of paper



Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet

Trial HSC Mathematics Extension 2

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ☐ B ☒ C ☐ D ☐

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ☒ B ☒ C ☐ D ☐

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

A ☒ B ☒ C ☐ D ☐
correct

Student Number: _____

ANSWERS

Completely fill the response oval representing the most correct answer.

1. A ☐ B ☐ C ☐ D ☒

2. A ☐ B ☐ C ☒ D ☐

3. A ☐ B ☐ C ☒ D ☐

4. A ☐ B ☒ C ☐ D ☐

5. A ☐ B ☐ C ☐ D ☒

6. A ☐ B ☐ C ☒ D ☐

7. A ☐ B ☐ C ☐ D ☒

8. A ☐ B ☐ C ☒ D ☐

9. A ☐ B ☒ C ☐ D ☐

10. A ☒ B ☐ C ☐ D ☐

QUESTION 11

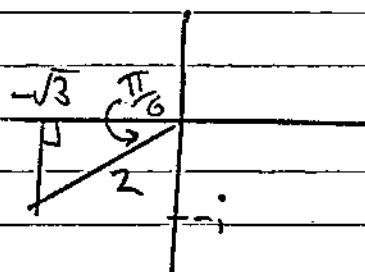
(a) (i) $\int e^x (1+e^x)^5 dx$ let $u = 1+e^x$
 $= \int u^5 du$ $du = e^x dx$
 $= \frac{u^6}{6} + C$
 $= \frac{(1+e^x)^6}{6} + C$

(ii) $-(t^2 - 6t - 7) = -(t-3)^2 + 9 + 7$
 $= 16 - (t-3)^2$

$\int \frac{dt}{\sqrt{7+6t-t^2}} = \int \frac{dt}{\sqrt{16-(t-3)^2}}$
 $= \sin^{-1}\left(\frac{t-3}{4}\right) + C$

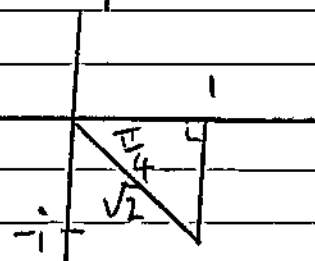
(b) $\alpha = -\sqrt{3} + i$ $\beta = 1 - i$

(i) $\bar{\alpha} = -\sqrt{3} - i$
 $\bar{\alpha} = 2 \operatorname{cis}\left(-\frac{5\pi}{6}\right)$



$\beta = 1 - i$

$\beta = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$



(ii) $\bar{\alpha}\beta = 2\sqrt{2} \operatorname{cis}\left(-\frac{5\pi}{6} - \frac{\pi}{4}\right)$

$\bar{\alpha}\beta = 2\sqrt{2} \operatorname{cis}\left(-\frac{13\pi}{12}\right)$

$\bar{\alpha}\beta = 2\sqrt{2} \operatorname{cis}\left(\frac{11\pi}{12}\right)$

$$(ii) \bar{z}_\beta = 2\sqrt{2} \cos\left(\frac{11\pi}{12}\right) + i 2\sqrt{2} \sin\left(\frac{11\pi}{12}\right)$$

$$\bar{z}_\beta = (-\sqrt{3}-i)(1-i)$$

$$= -\sqrt{3} + i\sqrt{3} - i - 1$$

$$= (-\sqrt{3}-1) + (\sqrt{3}-1)i$$

$$\text{So } 2\sqrt{2} \sin\left(\frac{11\pi}{12}\right) = \sqrt{3}-1$$

$$\text{and } 2\sqrt{2} \cos\left(\frac{11\pi}{12}\right) = -\sqrt{3}-1$$

$$\text{Thus } \tan\left(\frac{11\pi}{12}\right) = \frac{\sqrt{3}-1}{-\sqrt{3}-1}$$

$$\text{that is } \tan\left(\frac{11\pi}{12}\right) = \sqrt{3}-2$$

$$(c) (i) b^2 = a^2(1-e^2)$$

$$16 = 25(1-e^2)$$

$$\frac{16}{25} = 1-e^2$$

$$e^2 = \frac{25}{25} - \frac{16}{25}$$

$$e^2 = \frac{9}{25}$$

$$e = \frac{3}{5}$$

$$(ii) \quad S(ae, 0) \quad S'(-ae, 0)$$

$$S(5 \times \frac{3}{5}, 0)$$

$$S(3, 0) \quad S'(-3, 0).$$

$$(iii) \quad x = \pm \frac{a}{e}.$$

$$x = \pm \frac{5}{\frac{3}{5}}.$$

$$x = \pm \frac{25}{3}.$$

$$(iv) \quad LHS = \frac{x^2}{25} + \frac{y^2}{16}$$

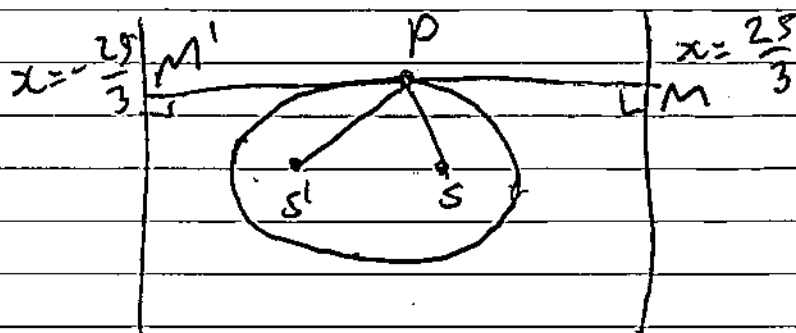
$$= \frac{5^2 \cos^2 \theta}{25} + \frac{4^2 \sin^2 \theta}{16}$$

$$= \cos^2 \theta + \sin^2 \theta$$

$$= 1$$

$$= RHS.$$

(v) Let M and M' be the feet of the perpendiculars to the corresponding directrices from the point P .



Thus $M(\frac{25}{3}, 4\sin\theta)$

$M'(-\frac{25}{3}, 4\sin\theta)$.

By the locus definition of an ellipse $\frac{PS}{PM} = e$ $\frac{PS'}{PM'} = e$.

$$\text{So } PS + PS' = ePM + ePM'$$

$$= \frac{3}{5} \left(\frac{25}{3} - 4\sin\theta \right)$$

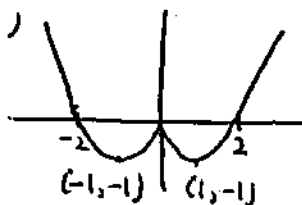
$$+ \frac{3}{5} \left(4\sin\theta + \frac{25}{3} \right).$$

$$= 5 - \frac{12}{5}\sin\theta + \frac{12}{5}\sin\theta + 5$$

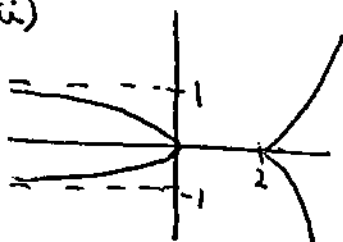
$$= 10.$$

Which is a constant.

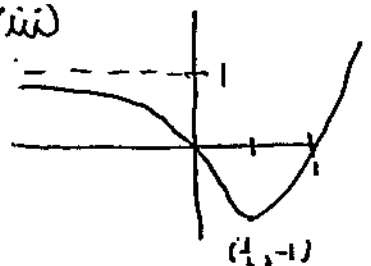
12 (a) (i)



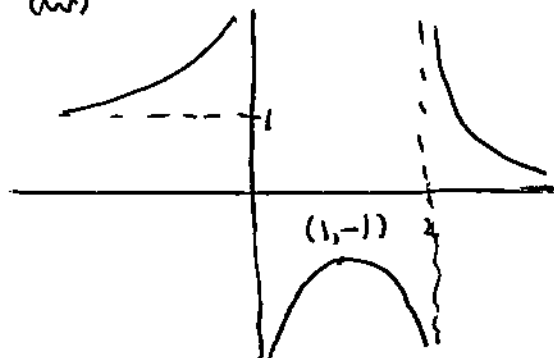
(ii)



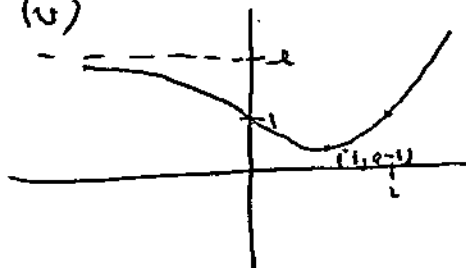
(iii)



(iv)



(v)



$$(b) \quad \frac{dt}{d\theta} = \frac{\sec^2 \frac{\theta}{2}}{2}$$

$$= \frac{1+t^2}{2}$$

$$\therefore \frac{2dt}{1+t^2} = d\theta$$

$$\int \frac{2}{4+3\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{4dt}{4+4t^2+6t}$$

$$= \int \frac{dt}{t^2+\frac{3}{2}t+1}$$

$$= \int \frac{dt}{(t+\frac{3}{4})^2 + \frac{7}{16}}$$

$$= \frac{4}{\sqrt{7}} \tan^{-1} \left\{ \frac{4(t+\frac{3}{4})}{\sqrt{7}} \right\}$$

Many made errors in completing the square

(c) let C represent $z+w$

OABC is a rhombus ($|z|=|w|$)

$\cos A = \frac{B^2+C^2-A^2}{2BC}$ (diagonals of a rhombus bisect the $\angle$ through which they pass)

$$|w|^2 = |z|^2 + |z+w|^2 - 2|z||z+w|\cos\left(\frac{B-A}{2}\right)$$

$$2^2 = 2^2 + |z+w|^2 - 2 \times 2 \times |z+w|\cos\left(\frac{B-A}{2}\right)$$

$$4 = 4 + |z+w|^2 - 4|z+w|\cos\left(\frac{B-A}{2}\right)$$

$$|z+w| (|z+w| - 4\cos\left(\frac{B-A}{2}\right))$$

$$\therefore |z+w| = 4\cos\left(\frac{B-A}{2}\right)$$

Many students lost marks for insufficient reasons.

Question 13

(15 Marks)

Use a NEW sheet of paper.

- (a) The roots of the equation $x^3 - 9x^2 + 31x + m = 0$ are in an arithmetic sequence. [3]

Find the roots of the equation and hence the value of m .

Let the roots be : $\alpha - d, \alpha, \alpha + d$.

$$\text{Sum (1 at a time)} : \alpha - d + \alpha + \alpha + d = -\frac{b}{a}$$

$$3\alpha = -\frac{(-9)}{1}$$

$$\therefore \alpha = 3. \quad \checkmark$$

$$\text{Sum (2 at a time)} : \alpha^2 - \alpha d + \alpha^2 - d^2 + \alpha^2 + \alpha d = \frac{c}{a}$$

$$3\alpha^2 - d^2 = \frac{31}{1}$$

$$3(3)^2 - d^2 = 31$$

$$-d^2 = 4$$

$$d = \pm 2i \quad \checkmark$$

$\therefore$ Roots are : $3 - 2i, 3, 3 + 2i$

$$\text{Product} : (\alpha^2 - d^2)\alpha = -\frac{d}{a}$$

$$[3^2 - (-4)] \times 3 = -\frac{m}{1}$$

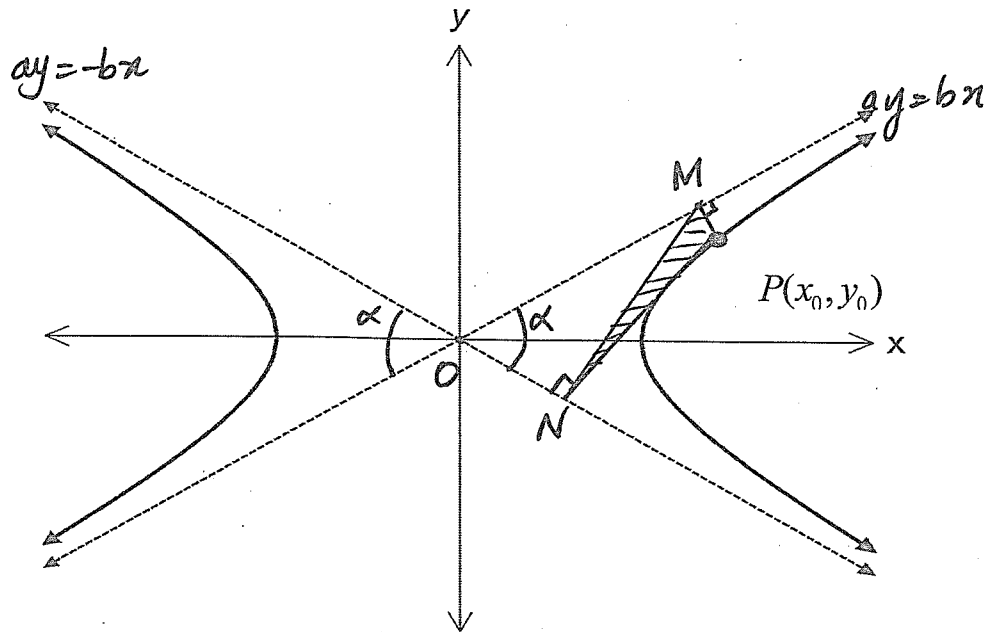
$$(9 + 4) \times 3 = -m$$

$$-39 = m$$

$$\therefore m = -39. \quad \checkmark$$

★ Students did well in this part.

- (b) The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$.



- i) Write down the equations of the two asymptotes of the hyperbola. [1]
 ii) Show that the acute angle α between the two asymptotes satisfies [2]

$$\tan \alpha = \frac{2ab}{a^2 - b^2}.$$

i) asymptotes: $y = \frac{bx}{a}$, $y = -\frac{bx}{a}$ ✓

$$\begin{aligned} \text{ii) } \tan \alpha &= \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \\ &= \left| \frac{(b/a) - (-b/a)}{1 + (b/a)(-b/a)} \right| \end{aligned}$$

$$= \left| \frac{\frac{2b}{a}}{1 - \frac{b^2}{a^2}} \right|$$

$$= \left| \frac{\frac{2b}{a} \times a^2}{a^2 - b^2} \right|$$

✓ * Absolute value signs should be included and then a reference to $a > b > 0$

✓ $a > b > 0$.

$$\therefore \tan \alpha = \frac{2ab}{a^2 - b^2}, \text{ as required.}$$

iii) If M and N are the feet of the perpendiculars drawn from P to the

[3]

asymptotes, show that $MP \times NP = \frac{a^2 b^2}{a^2 + b^2}$.

iv) Hence find the area of $\triangle PMN$ in terms of a and b .

[2]

iii) $MP = \text{distance from } P(x_0, y_0) \text{ to } bx - ay = 0$
 $NP = \text{distance from } P(x_0, y_0) \text{ to } bx + ay = 0$

$$\therefore MP = \frac{|bx_0 - ay_0|}{\sqrt{a^2 + b^2}} \quad NP = \frac{|bx_0 + ay_0|}{\sqrt{a^2 + b^2}} \quad \checkmark$$

$$\therefore MP \times NP = \frac{|bx_0 - ay_0|}{\sqrt{a^2 + b^2}} \times \frac{|bx_0 + ay_0|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|b^2 x_0^2 - a^2 y_0^2|}{a^2 + b^2} \quad \checkmark$$

* The simplest approach was to use the perpendicular distance formula

Now $P(x_0, y_0)$ lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

* Reference had to be made at this step

$$\therefore \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1 \quad \checkmark$$

$$\therefore b^2 x_0^2 - a^2 y_0^2 = a^2 b^2$$

$$\therefore MP \times NP = \frac{a^2 b^2}{a^2 + b^2} = \frac{a^2 b^2}{a^2 + b^2}, \text{ as required}$$

iv) Area $\triangle PMN$

$$= \frac{1}{2} MP \cdot NP \sin \angle MPN$$

$$= \frac{1}{2} \cdot MP \cdot NP \sin (180^\circ - \alpha) \leftarrow (\text{MPNO is a cyclic quad})$$

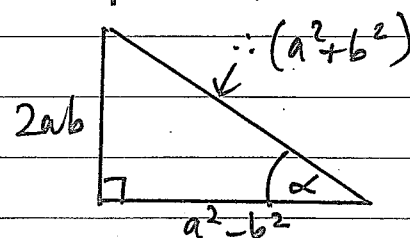
$$= \frac{1}{2} \cdot \frac{a^2 b^2}{a^2 + b^2} \cdot \sin \alpha \leftarrow$$

$$= \frac{1}{2} \cdot \frac{a^2 b^2}{a^2 + b^2} \cdot \frac{2ab}{a^2 + b^2}$$

$$= \frac{a^3 b^3}{(a^2 + b^2)^2} \quad \checkmark \checkmark$$

* Students who assumed that $\angle MPN = 90^\circ$ did not receive any marks

(From part ii)



Question 13 (Continued)

(c)

i) Find the rational values of A, B and C given:

[2]

$$\frac{y^2+8}{(y-2)(y^2+2y+4)} \equiv \frac{A}{y-2} + \frac{By+C}{y^2+2y+4}$$

ii) Hence find $\int \frac{y^5-7y^2+8}{y^3-8} dy$.

[2]

$$\begin{aligned} \text{i) } y^2+8 &\equiv A(y^2+2y+4) + (By+C)(y-2) \\ &\equiv Ay^2+2Ay+4A+By^2-2By+Cy-2C \\ &\equiv (A+B)y^2 + (2A-2B+C)y + (4A-2C) \end{aligned}$$

$$\begin{aligned} \therefore A+B &= 1 \\ 2A-2B+C &= 0 \\ 4A-2C &= 8 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} A=1 \\ B=0 \\ C=-2 \end{array} \quad \checkmark \checkmark$$

$$\begin{aligned} \text{ii) } \frac{y^5-7y^2+8}{y^3-8} &= \frac{y^5-8y^2+y^2+8}{(y-2)(y^2+2y+4)} \\ &= \frac{y^2(y^3-8) + (y^2+8)}{(y-2)(y^2+2y+4)} \\ &= y^2 + \frac{y^2+8}{(y-2)(y^2+2y+4)} \quad \checkmark \end{aligned}$$

$$\therefore \int \frac{y^5-7y^2+8}{y^3-8} dy = \int \left[y^2 + \frac{1}{y-2} - \frac{2}{y^2+2y+4} \right] dy$$

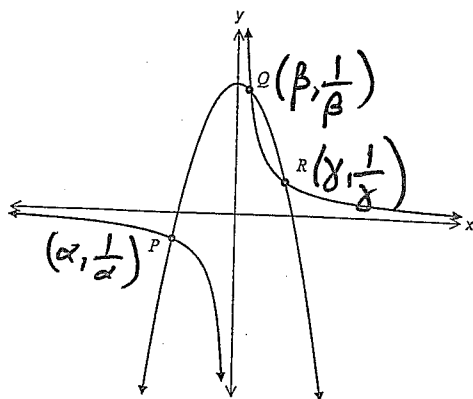
★ Some students did not read the question carefully, and integrated the answer directly from part i)

$$\begin{aligned} &= \int \left[y^2 + \frac{1}{y-2} - \frac{2}{(\sqrt{3})^2 + (y+1)^2} \right] dy \\ &= \frac{y^3}{3} + \ln|y-2| - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{y+1}{\sqrt{3}} \right) + C \quad \checkmark \end{aligned}$$

Question 14

Use a NEW sheet of paper.

(a)



The curves $y = \frac{1}{x}$ and $y = k - x^2$, for some real number k , intersect at the points P , Q and R where the x -coordinates are $x = \alpha$, $x = \beta$ and $x = \gamma$ respectively.

- [3] i) Show that the monic cubic equation with coefficients in terms of k whose roots are α^2 , β^2 and γ^2 is given by $x^3 - 2kx^2 + k^2x - 1 = 0$.
- [1] ii) Find the monic cubic equation with coefficients in terms of k whose roots are: $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$.
- [2] iii) Hence find in simplest form $OP^2 + OQ^2 + OR^2$ in terms of k , where O is the origin.

i) α, β, γ are the roots of $k - x^2 = \frac{1}{x}$

$$\Rightarrow x^3 - kx + 1 = 0.$$

$\alpha^2, \beta^2, \gamma^2$ are the roots of the equation:

$$(\sqrt{x})^3 - k(\sqrt{x}) + 1 = 0. \quad (\text{replace } x \text{ by } \sqrt{x})$$

$$x\sqrt{x} - k\sqrt{x} + 1 = 0.$$

$$(x - k)\sqrt{x} = -1$$

$$(x - k)^2 (\sqrt{x})^2 = (-1)^2$$

$$(x^2 - 2kx + k^2)x = 1$$

$$\therefore x^3 - 2kx^2 + k^2x - 1 = 0, \text{ as required.}$$

ii) Replace x by $\frac{1}{x}$:

$$\left(\frac{1}{x}\right)^3 - 2k\left(\frac{1}{x}\right)^2 + k^2\left(\frac{1}{x}\right) - 1 = 0$$

$$\frac{1}{x^3} - \frac{2k}{x^2} + \frac{k^2}{x} - 1 = 0.$$

$$\Rightarrow 1 - 2kx + k^2x^2 - x^3 = 0$$

$$\therefore x^3 - k^2x^2 + 2kx - 1 = 0. \checkmark$$

This step had to be shown

$$\text{iii) } OP^2 + OQ^2 + OR^2$$

$$= \left(\alpha^2 + \frac{1}{\alpha^2} \right) + \left(\beta^2 + \frac{1}{\beta^2} \right) + \left(\gamma^2 + \frac{1}{\gamma^2} \right)$$

$$= (\alpha^2 + \beta^2 + \gamma^2) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} \right). \checkmark$$

$$\text{From i) } \alpha^2 + \beta^2 + \gamma^2 = 2k$$

$$\text{From ii) } \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = k^2 \quad \checkmark$$

$$\therefore OP^2 + OQ^2 + OR^2 = k^2 + 2k$$

(b)

i) Show that a reduction formula for $I_n = \int (\ln x)^{\frac{n}{2}} dx$, where n is a positive

[2]

integer, is $I_n = x(\ln x)^{\frac{n}{2}} - \frac{n}{2} I_{n-2}$.

$$\begin{aligned} I_n &= \int (\ln x)^{\frac{n}{2}} dx \\ &= x(\ln x)^{\frac{n}{2}} - \int x \cdot \frac{n}{2} (\ln x)^{\frac{n}{2}-1} \cdot \frac{1}{x} dx \\ &= x(\ln x)^{\frac{n}{2}} - \frac{n}{2} \int (\ln x)^{\frac{n}{2}-1} dx \end{aligned} \quad \left| \begin{array}{l} u = (\ln x)^{\frac{n}{2}} \\ du = \frac{n}{2} (\ln x)^{\frac{n}{2}-1} \cdot \frac{1}{x} dx \\ dv = dx \\ v = x \quad \checkmark \end{array} \right.$$

$$\therefore I_n = x(\ln x)^{\frac{n}{2}} - \frac{n}{2} \int (\ln x)^{\frac{n-2}{2}} dx.$$

$$= x(\ln x)^{\frac{n}{2}} - \frac{n}{2} I_{n-2}. \quad \checkmark$$

★ {This was achieved by most students.}

ii) Hence, or otherwise, evaluate $\int_1^e (\ln x)^4 dx$.

[2]

$$\frac{n}{2} = 4$$

$$n = 8$$

$$\therefore I_8 = \left[x(\ln x)^4 \right]_1^e - 4I_6$$

$$= e - 4 \left(\left[x(\ln x)^3 \right]_1^e - 3I_4 \right)$$

$$= e - 4e + 12I_4$$

$$= -3e + 12 \left(\left[x(\ln x)^2 \right]_1^e - 2I_2 \right)$$

$$= -3e + 12e - 24I_2$$

$$= 9e - 24 \left(\left[x(\ln x) \right]_1^e - I_0 \right) \checkmark$$

$$= 9e - 24e + 24I_0$$

$$= 9e - 24e + 24 \left(\left[x(\ln x)^0 \right]_1^e \right)$$

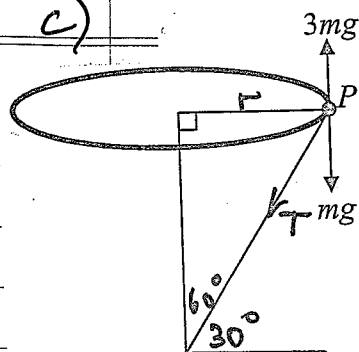
$$= 9e - \cancel{24e} + \cancel{24e} - 24$$

$$= 9e - 24 \checkmark$$

★ Most common error was finding $I_4 = e - 2$ instead of I_8 . Loss of one mark.



c)

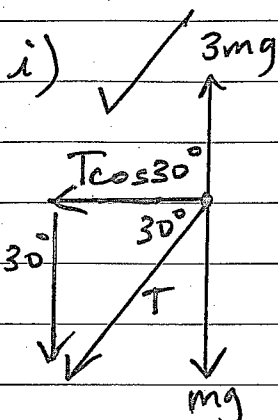


A model aircraft P , of mass $m = 8 \text{ kg}$ is attached to the end of a 10 m long inelastic wire, with the other end fixed to the ground.

The model flies in a horizontal circle so that the wire makes an angle of 30° with the ground. The uplift created by the wings of the aircraft is a vertical force $3mg$. (take $g = 10 \text{ ms}^{-1}$)

[3] i) By resolving the forces at P , calculate the tension in the wire.

[2] ii) Calculate the angular velocity about the centre of the horizontal circle.

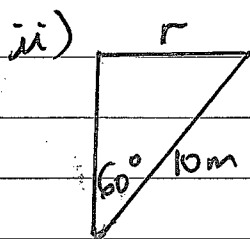


vertically: $T \sin 30^\circ + mg = 3mg$ ✓
 $T \sin 30^\circ = 2mg$ ①

horizontally: $T \cos 30^\circ = mr\omega^2$ ②

Tension: $T \sin 30^\circ = 2 \times 8 \times 10$

Most common error was labelling the angle made with the ground
 $\frac{1}{2}T = 160$ ✓
 $\therefore T = 320 \text{ Newtons}$



From ②:

$$T \cos 30^\circ = mr\omega^2$$

$$320 \times \frac{\sqrt{3}}{2} = 8 \times 5\sqrt{3} \times \omega^2$$

$$160 = 40\omega^2$$

$$\omega^2 = 4$$

$$\therefore \omega = 2 \text{ rad/sec } (\omega \neq 0)$$

$$r = 10 \sin 60^\circ$$

$$r = 10 \sin 60^\circ$$

$$r = \frac{10\sqrt{3}}{2}$$

$$r = 5\sqrt{3} \text{ cm.}$$

$$\begin{aligned}
 15(a)(i) \quad V_{\text{shell}} &= \pi (R^2 - r^2) L \\
 &= \pi \{ (3-x)^2 - (3-x-\delta x)^2 \} y \\
 &= 2\pi (3-x) y \delta x
 \end{aligned}$$

$$\begin{aligned}
 V_{\text{solid}} &= \lim_{\delta x \rightarrow 0} \sum_{x=0}^1 2\pi (3-x) y \delta x \\
 &= 2\pi \int_0^1 (3-x) y \, dx \\
 &= 2\pi \int_0^1 (3-x) \times \frac{6}{\sqrt{4-x^2}} \, dx \\
 &= 12\pi \int_0^1 \frac{3-x}{\sqrt{4-x^2}} \, dx \quad \text{Q.E.D.}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad V &= 12\pi \left(\int_0^1 \left(\frac{3}{\sqrt{4-x^2}} - \frac{x}{\sqrt{4-x^2}} \right) dx \right) \\
 &= 12\pi \left[3 \sin^{-1} \frac{x}{2} \right]_0^1 - 12\pi \int_0^1 \frac{x}{\sqrt{4-x^2}} \times \frac{du}{2x} \quad \begin{array}{l} u = x^2 \\ \frac{du}{dx} = 2x \\ \frac{du}{2x} = dx \end{array} \\
 &= 12\pi \times 3 \times \frac{\pi}{6} - 12\pi \left[-(4-u)^{\frac{1}{2}} \right]_0^1 \\
 &= 6\pi^2 + 12\pi \times \sqrt{3} - 24\pi \\
 &= 6\pi^2 + 12\sqrt{3}\pi - 24\pi \quad \text{A calculator answer}
 \end{aligned}$$

$$\begin{aligned}
 (1) (\cos \theta + i \sin \theta)^4 &= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta \\
 \cos^4 \theta + i \sin^4 \theta &= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2 \\
 \cos^4 \theta &= \cos^4 \theta - 6 \cos^2 \theta + 6 \cos^4 \theta + 1 - 2 \cos^2 \theta + \cos^4 \theta \\
 &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1 \quad \text{Q.E.D.}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \text{let } x &= \cos \theta \\
 16 \cos^4 \theta - 16 \cos^2 \theta + 1 &= 0 \\
 2(8 \cos^4 \theta - 8 \cos^2 \theta + 1) - 1 &= 0 \\
 2 \cos^4 \theta - 1 &= 0 \\
 \cos^4 \theta &= \frac{1}{2} \\
 \pm \theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\
 \theta &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12} \\
 x &= \cos \frac{\pi}{12}, \cos \frac{5\pi}{12}, \cos \frac{7\pi}{12}, \cos \frac{11\pi}{12} \\
 x &= \cos \frac{\pi}{12}, \cos \frac{5\pi}{12}, -\cos \frac{5\pi}{12}, -\cos \frac{\pi}{12}
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad x &= \frac{16 \pm \sqrt{16^2 - 4 \times 16 \times 1}}{2 \times 16} \\
 &= \frac{16 \pm \sqrt{192}}{32} \\
 &= \frac{16 \pm 8\sqrt{3}}{32} \quad \because \cos \frac{5\pi}{12} = \frac{2-\sqrt{3}}{4} \\
 &= \frac{2 \pm \sqrt{3}}{4} \quad \left[\text{because } \cos \frac{5\pi}{12} < \cos \frac{\pi}{12} \right]
 \end{aligned}$$

(i) $\angle TPV = \angle RPT$ ($\angle$ in same segment)

$\angle RPT = \angle RST$ (given)

$\therefore \angle TPV = \angle RST$ G.E.D.

Many students forgot to mention this.

(ii) In $\triangle PGV$ and $\triangle VGT$

$\angle PGV$ is common

$\angle TVG = \angle RST$ ($\angle$ in same segment)
 $= \angle TPV$ (from i))

$\therefore \triangle PGV \parallel \triangle VGT$ (equiangular)

$\therefore \angle TVG = \angle PVG$ (corresponding $\angle$ in similar $\triangle$)

$\angle PVG = 90^\circ$ ($\angle$ in a semi circle)

$\therefore \angle TVG = 90^\circ$

$\angle RPT + \angle TVG = 180^\circ$ (straight $\angle$)

$\therefore \angle RPT = 90^\circ$ G.E.D.

(iii) $\angle RST = 90^\circ$ ($\angle$ in a semi circle)

$\therefore PU$ is a diameter of a circle through P, S and V

$\therefore \angle RST = \angle PVS$

$\therefore PTUS$ is a cyclic quadrilateral (interior $\angle$ equals exterior opposite $\angle$)

$\therefore PU$ is a diameter of the circle

passing through P, T, U and S.
G.E.D.

Both elements are required.

Question 16

(a)(i) Method 1

Case 0, No Ts: $5 \times 4 = 20$ Case 1, One T: $5 \times 2! = 10$

Case 2, TT: 1

Total = 31

Method 2

Distinct letters: ${}^6C_2 \times 2! = {}^6P_2 = 30$

Same letter (TT): 1

Total = 31

(ii) $P(TT) = \frac{1}{31}$

Most students had $\frac{{}^7P_2}{2!} + 1 = 21$ but you only need to divide by $2!$ if you have the two TTs.

(iii) $P(TT) = 1 - P(\text{not } TT)$

$$1 - \underbrace{\frac{30}{31} \times \frac{30}{31} \times \cdots \times \frac{30}{31}}_n > 0.9$$

$$0.1 > \left(\frac{30}{31}\right)^n$$

This part was answered poorly by most students.

$$\log\left(\frac{30}{31}\right)^n < \log(0.1)$$

$$n > \frac{\log 0.1}{\log\left(\frac{30}{31}\right)} \quad \text{Since } \log\left(\frac{30}{31}\right) < 0$$

$$n > 70.22$$

That is $n = 71$

You would need to create 71 arrangements to have over 90% chance of getting at least one TT.

$$(b)(i) \quad LHS = \sin(2r\theta + \theta) - \sin(2r\theta - \theta)$$

$$= \cancel{\sin(2r\theta)} \cos \theta + \sin \theta \cos(2r\theta) - \cancel{\sin(2r\theta)} \cos \theta + \sin \theta \cos(2r\theta)$$

$$= 2 \sin \theta \cos(2r\theta)$$

$$= RHS$$

$$(ii) \quad 2 \sin \theta \sum_{r=1}^n \cos 2r\theta$$

$$= 2 \sin \theta \cos 2\theta + 2 \sin \theta \cos 4\theta + 2 \sin \theta \cos 6\theta + \dots$$

$$+ 2 \sin \theta \cos 2(n-1)\theta + 2 \sin \theta \cos 2n\theta$$

$$= \cancel{\sin 3\theta} - \sin \theta + \cancel{\sin 5\theta} - \cancel{\sin 3\theta} + \sin 7\theta - \cancel{\sin 5\theta} + \dots$$

$$+ \sin(2n-1)\theta - \cancel{\sin(2n-3)\theta} + \sin(2n+1)\theta - \cancel{\sin(2n-1)\theta}$$

$$= -\sin \theta + \sin(2n+1)\theta$$

$$\text{So } 2 \sin \theta \sum_{r=1}^n \cos 2r\theta = \sin(2n+1)\theta - \sin \theta$$

$$\text{That is } \sin \theta \sum_{r=1}^n \cos 2r\theta = \frac{1}{2} \{ \sin(2n+1)\theta - \sin \theta \}$$

$$(iii) \quad \sum_{r=1}^{100} \cos^2 \left(\frac{r\pi}{100} \right)$$

Some students could not see the connection between parts (ii) and (iii).

$$= \sum_{r=1}^{100} \left(\frac{1}{2} + \frac{1}{2} \cos \left(\frac{2r\pi}{100} \right) \right)$$

$$= \frac{1}{2} \sum_{r=1}^{100} 1 + \frac{1}{2} \sum_{r=1}^{100} \cos \left(2r \frac{\pi}{100} \right)$$

$$= \frac{1}{2} \times 100 + \frac{1}{2} \left\{ \frac{1}{2 \sin \left(\frac{\pi}{100} \right)} \left[\sin \left(201 \frac{\pi}{100} \right) - \sin \left(\frac{\pi}{100} \right) \right] \right\} \quad \text{from part (ii)}$$

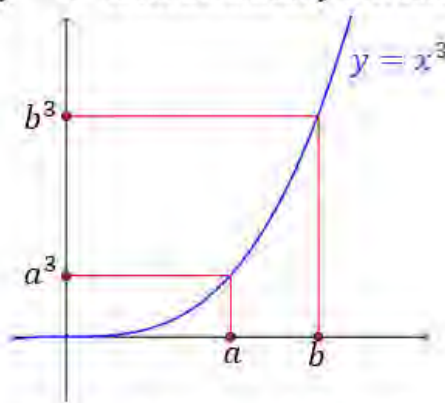
$$= 50 + \frac{1}{4 \sin \left(\frac{\pi}{100} \right)} \left[\sin \left(2\pi + \frac{\pi}{100} \right) - \sin \left(\frac{\pi}{100} \right) \right]$$

$$= 50 + \frac{1}{4 \sin \left(\frac{\pi}{100} \right)} \left[\sin \left(\frac{\pi}{100} \right) - \sin \left(\frac{\pi}{100} \right) \right]$$

$$= 50$$

(c)(i) Method 1

Cubing positive numbers preserves order, since $y = x^3$ is monotonically increasing for all $x > 0$.



That is, if $0 < a < b$ then $a^3 < b^3$.

Call this result (*)

Given that $k \geq 4$

$$\Rightarrow k > 2.27$$

$$\Rightarrow k > \frac{1}{\sqrt[3]{3} - 1}$$

$$\Rightarrow \sqrt[3]{3}k - k > 1$$

$$\Rightarrow \sqrt[3]{3}k > k + 1 \quad \text{use result (*)}$$

$$\Rightarrow 3k^3 > (k + 1)^3 \quad \text{Call this result (**)}$$

Students tended to do these side calculations as part of the induction structure and got themselves very confused.

Prove $3^n > n^3$ for $n = 4$.

$$\begin{array}{ll} RHS = 4^3 & LHS = 3^4 \\ = 64 & = 81 \\ & > RHS \end{array}$$

Assume for $n = k$.

$$3^k > k^3$$

Prove for $n = k + 1$.

$$3^{k+1} = 3 \times 3^k$$

$$> 3k^3 \quad \text{by assumption}$$

$$> (k + 1)^3 \quad \text{by result (**)}$$

Required to prove
 $3^{k+1} > (k + 1)^3$

Therefore by the principle of mathematical induction $3^n > n^3$ for all $n \geq 4$.

(c)(i) Method 2

Most students that gained full marks for this question used this method.

$$\text{Let } f(k) = 2k^3 - 3k^2 - 3k - 1$$

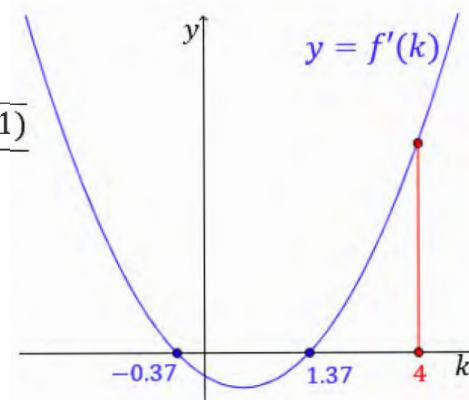
$$\text{then } f'(k) = 6k^2 - 6k - 3$$

$$y = 3(2k^2 - 2k - 1)$$

$$k = \frac{2 \pm \sqrt{4 - 4 \times 2 \times (-1)}}{4}$$

$$k = \frac{1 \pm \sqrt{3}}{2}$$

$$k \approx -0.37, 1.37$$



So for $k \geq 4$, $f'(k) > 0$.

That is $f(k)$ is an increasing function for $k \geq 4$.

$$f(4) = 2(4)^3 - 3(4)^2 - 3(4) - 1$$

$$f(4) = 67$$

At $k = 4$, $f(k)$ is positive and $f(k)$ is an increasing function for $k \geq 4$.

So $f(k) > 0$ for $k \geq 4$. Call this result (+)

Prove $3^n > n^3$ for $n = 4$.

$$\begin{array}{ll} RHS = 4^3 & LHS = 3^4 \\ = 64 & = 81 \\ & > RHS \end{array}$$

Assume for $n = k$.

$$3^k > k^3$$

Prove for $n = k + 1$.

Required to prove
 $3^{k+1} > (k+1)^3$

$$\begin{aligned} 3^{k+1} - (k+1)^3 &= 3 \times 3^k - (k^3 + 3k^2 + 3k + 1) \\ &> 3k^3 - k^3 - 3k^2 - 3k - 1 \quad \text{by assumption} \\ &= 2k^3 - 3k^2 - 3k - 1 \\ &= f(k) \\ &> 0 \quad \text{for } k \geq 4 \text{ by result (+)} \end{aligned}$$

$$\text{So } 3^{k+1} - (k+1)^3 > 0$$

$$\text{That is } 3^{k+1} > (k+1)^3$$

Therefore by the principle of mathematical induction $3^n > n^3$ for all $n \geq 4$.

(c)(i) Method 3

Prove $3^n > n^3$ for $n = 4$.

$$\begin{array}{ll}
 RHS = 4^3 & LHS = 3^4 \\
 = 64 & = 81 \\
 & > RHS
 \end{array}$$

Assume for $n = k$.

$$3^k > k^3$$

That is $3^k - k^3 > 0$

Prove for $n = k + 1$.

Nobody used this method correctly.
you must break the algebra down to
obviously true statements.

Required to prove
 $3^{k+1} > (k + 1)^3$

$$\begin{aligned}
 3^{k+1} - (k + 1)^3 &= 3 \times 3^k - (k^3 + 3k^2 + 3k + 1) \\
 &= 3(3^k - k^3) + 2k^3 - 3k^2 - 3k - 1 \\
 &= 3(3^k - k^3) + (k^3 - 3k^2 + 3k - 1) + (k^3 - 6k) \\
 &= 3(3^k - k^3) + (k - 1)^3 + k(k^2 - 6)
 \end{aligned}$$

Now $3(3^k - k^3)$ is positive by assumption.

and $(k - 1)^3$ is positive since $k \geq 4$.

and $k(k^2 - 6)$ is positive since $k \geq 4$.

So $3^{k+1} - (k + 1)^3 > 0$

That is $3^{k+1} > (k + 1)^3$

Therefore by the principle of mathematical induction $3^n > n^3$ for all $n \geq 4$.

(ii) Method 1

$$3^n > n^3$$

$$(3^n)^{\frac{1}{3n}} > (n^3)^{\frac{1}{3n}}$$

$$\sqrt[3]{3} > \sqrt[n]{n}$$

$$\sqrt[3]{3} > \sqrt[n]{n}$$

Method 2

$$3^n > n^3$$

$$\log(3^n) > \log(n^3)$$

$$n \log(3) > 3 \log(n)$$

$$\frac{1}{3} \log(3) > \frac{1}{n} \log(n)$$

$$\log(\sqrt[3]{3}) > \log(\sqrt[n]{n})$$

$$\sqrt[3]{3} > \sqrt[n]{n}$$