Sydney Girls High School
2025
Trial Higher School Certificate
Examination

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using a black pen • Calculators approved by NESA may be used • A reference sheet is provided • In Questions 11-16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks** • Attempt Questions 1-10 • Allow about 15 minutes for this section **Section II – 90 marks** • Attempt Questions 11-16 • Allow about 2 hours and 45 minutes for this section

Name: …………………………………………………… Teacher: ……………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2025 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

Questions **Marks**

1 What is the converse of the statement "if you can't see my mirrors, then I can't see you"? **1**

A. If I can't see your mirrors, then you can't see me.

B. If I can't see you, then you can't see my mirrors.

C. If you can see me, then I can see your mirrors.

D. If I can see you, then you can see my mirrors.

2 What is the contrapositive of the statement "if you can't see my mirrors, then I can't see you"? **1**

A. If I can't see your mirrors, then you can't see me.

B. If I can't see you, then you can't see my mirrors.

C. If you can see me, then I can see your mirrors.

D. If I can see you, then you can see my mirrors.

3 If $\underset{\sim}{u} = 5\underset{\sim}{i} + (t-3)\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + (t+9)\underset{\sim}{j} + 2\underset{\sim}{k}$ are perpendicular, what are the possible values of t? **1**

A. $t = 8,\ 2$

B. $t = -8,\ -2$

C. $t = 8,\ -2$

D. $t = -8,\ 2$

4 Suppose $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are three vectors such that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$, $|\underset{\sim}{a}| = 3$, $|\underset{\sim}{b}| = 5$, and $|\underset{\sim}{c}| = 7$. **1**

What is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$?

A. $\dfrac{\pi}{6}$

B. $\dfrac{2\pi}{3}$

C. $\dfrac{5\pi}{3}$

D. $\dfrac{\pi}{3}$

5 A particle of mass m moves so that its velocity v is given by $v = f(x)$. **1**

The resultant force that causes this motion is given by:

A. $mf'(x)$

B. $mxf'(x)$

C. $mf'\left(\dfrac{1}{2}x^2\right)$

D. $mf(x)f'(x)$

6 Let ω be a non-real cube root of unity. **1**

What is the value of $(1 + 2\omega + 3\omega^2)(1 + 3\omega + 2\omega^2)$?

A. 3

B. 6

C. 9

D. 12

7 Suppose that z_1 and z_2 are two distinct non-zero complex numbers satisfying $|z_1 + z_2| = |z_1 - z_2|$. **1**

Which of the following is always true?

A. $z_1 z_2$ is purely imaginary.

B. $\dfrac{z_1}{z_2}$ is purely imaginary.

C. $\dfrac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

D. None of the above.

8 A particle moves from the initial conditions $t = t_0$, $x = x_0$, $v = v_0$ to the final conditions $t = t_F$, $x = x_F$, $v = v_F$ under a force that produces an acceleration of $\ddot{x}$. **1**

Which of the following is incorrect?

A. If $\ddot{x} = f(t)$, then $v_F = \displaystyle\int_{t_0}^{t_F} f(t)\, dt + v_0$.

B. If $\ddot{x} = f(v)$, then $x_F = \displaystyle\int_{v_0}^{v_F} \frac{f(v)}{v}\, dv + x_0$.

C. If $\ddot{x} = f(v)$, then $t_F = \displaystyle\int_{v_0}^{v_F} \frac{1}{f(v)}\, dv + t_0$.

D. If $\ddot{x} = f(x)$, then $(v_F)^2 = 2\displaystyle\int_{x_0}^{x_F} f(x)\, dx + (v_0)^2$.

9 Suppose that a, b and c are non-zero real numbers satisfying **1**

$$\frac{a^2}{b^2+c^2} < \frac{b^2}{c^2+a^2} < \frac{c^2}{a^2+b^2}.$$

Which of the following statements can be deduced from the preceding inequality?

A. $a < b < c$

B. $|a| < |b| < |c|$

C. $c < b < a$

D. $|c| < |b| < |a|$

10 Three non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ on a number plane are such that $\underset{\sim}{b} \cdot \underset{\sim}{c} = 2|\underset{\sim}{a}|$, $\underset{\sim}{a} \cdot \underset{\sim}{c} = 3|\underset{\sim}{b}|$, and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$. **1**

What is the minimum value of $|\underset{\sim}{c}|$?

A. $2\sqrt{2}$

B. $2\sqrt{3}$

C. $2\sqrt{5}$

D. None of the above.

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question on the writing paper supplied. Start each question on a NEW page. Extra writing paper are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start on a NEW page **Marks**

(a) Give a counterexample to disprove the following statement: **1**

For all $x, y \in \mathbb{R}$, if $x^2 \geq y^2$, then $x \geq y$.

(b) Write down the negation of the following statement: **1**

There exists $k \in \mathbb{R}$ such that $k^2 < 0$.

(c) Consider the vectors $\underset{\sim}{u} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = -5\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{w} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$.

(i) Prove that $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are perpendicular. **1**

(ii) Verify that $\left|\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\right| \leq \left|\underset{\sim}{u}\right| + \left|\underset{\sim}{v}\right| + \left|\underset{\sim}{w}\right|$. **2**

(d) Given the points $A(1, 3, 5)$, $B(2, -4, 11)$ and $C(-3, -1, 0)$, find $\angle ABC$. Give your answer correct to the nearest minute. **3**

(e) By using integration by parts, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x \cos x \, dx$. **3**

(f) Evaluate $\operatorname{Arg}\left(i^{2025}\right)$, where $\operatorname{Arg}(z)$ denotes the principal argument of z. **2**

(g) Prove that $\sin\theta = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. **2**

End of Question 11

Question 12 (15 marks) Start on a NEW page **Marks**

(a) Find $\displaystyle\int \cos^3 x \sin^2 x \, dx$. **2**

(b) By using an appropriate t-substitution, or otherwise, find $\displaystyle\int \frac{d\theta}{3 + 2\cos 4\theta}$. **3**

(c) (i) By expanding $(\cos\theta + i\sin\theta)^3$ and using de Moivre's theorem, show that **2**

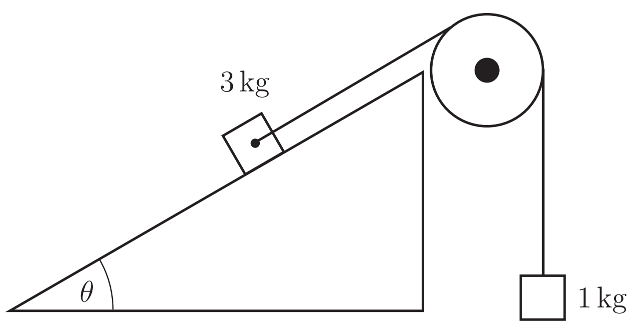
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

(ii) Hence, or otherwise, solve $8x^3 - 6x + 1 = 0$ where $x = \cos\theta$. **3**

(iii) Deduce that **2**

$$\cos\frac{\pi}{9} - \cos\frac{2\pi}{9} - \cos\frac{4\pi}{9} = 0.$$

(d) The diagram below shows particles of mass 1 kg and 3 kg connected by a light inextensible string passing over a smooth pulley. The 3 kg mass lies on a smooth plane inclined at an angle of θ to the horizontal, and T is the tension in the string. **3**



Find θ so that the system will be in equilibrium. Give your answer in degrees, correct to one decimal place.

End of Question 12

Question 13 (15 marks) Start on a NEW page **Marks**

(a) The velocity v of a particle moving along the x-axis is given by

$$v^2 = 12 + 8x - 4x^2,$$

where x is the displacement of the particle from the origin O.

(i) Prove that the particle is undergoing simple harmonic motion. **2**

(ii) Find the period, amplitude and centre of motion. **2**

(b) Prove by contradiction that $\sqrt[3]{5}$ is irrational. **2**

(c) Sketch the region of the complex plane defined by $\mathrm{Re}(z^2) \leq (\mathrm{Re}\, z)^2$ and $-\dfrac{2\pi}{3} \leq \mathrm{Arg}(z+i) \leq 0$. **3**

(d) By using the substitution $u^6 = x$, or otherwise, evaluate $\displaystyle\int_1^{64} \frac{1}{\sqrt{x}+\sqrt[3]{x}}\, dx$. **3**

(e) Find $\displaystyle\int \frac{x^2+1}{x^4+8x^2+12}\, dx$. **3**

End of Question 13

Question 14 (15 marks) Start on a NEW page **Marks**

(a) Two spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ have Cartesian equations $x^2 + y^2 + z^2 = 9$ and $(x-3)^2 + \left(y+\sqrt{6}\right)^2 + (z-7)^2 = 25$ respectively.

(i) Prove that $\mathcal{S}_1$ and $\mathcal{S}_2$ are externally tangent. **3**

(ii) Find the vector equation of ℓ_1, where ℓ_1 is the line through the centres of $\mathcal{S}_1$ and $\mathcal{S}_2$. Give your answer in the form $\underset{\sim}{r} = \lambda \underset{\sim}{d}$. **2**

(iii) Consider the line ℓ_2 with vector equation **2**

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -\sqrt{6} \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Determine any points of intersection between lines ℓ_1 and ℓ_2.

(iv) Deduce, with mathematical justification, whether the lines ℓ_1 and ℓ_2 are skew. **2**

(b) The rise and fall of the tide at a certain port may be considered as simple harmonic, with the time difference between successive high tides being 10 hours. The harbour entrance has a depth of 20 m at high tide and 8 m at low tide. **3**

If low tide occurs at 10 a.m. on a certain day, find the earliest time that a cargo ship requiring a minimum depth of 15 m of water can pass through the entrance.

(c) Let a and b be consecutive positive integers with $a < b$. **3**

By considering the expansion of $(a^2 + a + 1)^2$, prove that if $c = ab$, then $\sqrt{a^2 + b^2 + c^2}$ is an odd integer.

End of Question 14

Question 15 (15 marks) Start on a NEW page **Marks**

(a) Let z and w be two distinct non-zero complex numbers. **3**

Show that if $|z|^2 = |w|^2 = 2\,\mathrm{Re}(z\overline{w})$, then the triangle with vertices at 0, z and w is equilateral.

(b) Prove that if $x > 0$, $y > 0$ and $x + y = 1$, then **2**

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9$$

(c) Let $I_n = \displaystyle\int \frac{\sin nx}{\cos x}\, dx$ for integers $n \geq 0$. **3**

Show that

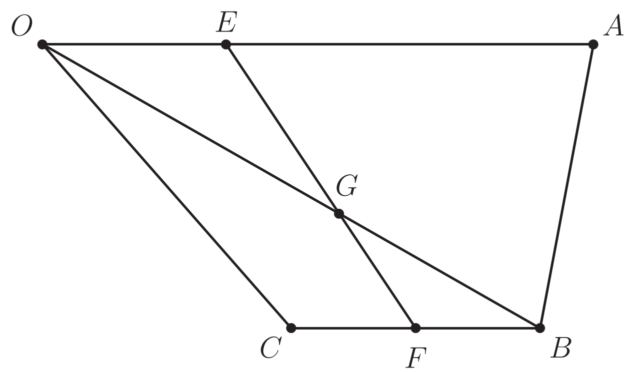
$$I_n = \frac{2}{1-n}\cos\Big((n-1)x\Big) - I_{n-2}, \text{ for } n \geq 2.$$

(d) Prove by mathematical induction that for all integers $n \geq 0$, **3**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \leq 1 + n.$$

Examination continues overleaf...

(e) The sides OA and CB of a trapezium $OABC$ are parallel, with $OA > CB$. The point E on OA is such that $OE : EA = 1 : 2$, and F is the midpoint of CB. The point G is the intersection of OB and EF.



Let $\underset{\sim}{a}$ and $\underset{\sim}{c}$ be the position vectors of points A and C respectively.

(i) Show that B has position vector $\lambda\underset{\sim}{a} + \underset{\sim}{c}$ for some scalar λ. **1**

(ii) Hence, or otherwise, show that the position vector of G is **3**

$$\left(\frac{2\lambda}{2+3\lambda}\right)\underset{\sim}{a} + \left(\frac{2}{2+3\lambda}\right)\underset{\sim}{c}.$$

End of Question 15

Question 16 (15 marks) Start on a NEW page **Marks**

(a) Suppose that a, b and c are the lengths of a right-angled triangle, where c is the hypotenuse. **3**

Show that if $n \leq 2$, then $c^n \leq a^n + b^n$.

(b) A particle is projected vertically upwards from the Earth's surface with initial speed U ms^{-1}. **4**

Experiments show that the acceleration due to gravity of an object at a distance x from the centre of the Earth is proportional to the inverse square of the distance x, towards the centre of the Earth; i.e.

$$a = -\frac{k}{x^2},$$

for some positive constant k. (Do NOT prove this.)

Show that the distance travelled by the particle before it has lost three-quarters of its initial speed is

$$\frac{15RU^2}{32gR - 15U^2} \text{ m},$$

where R is the radius of the Earth and g is the acceleration at the Earth's surface.

(c) Let $I_n = \displaystyle\int_0^{\alpha} (\sec x + \tan x)^n \, dx$, where n is a non-negative integer and $0 < \alpha < \frac{\pi}{2}$.

(i) For $n \geq 1$, show that **3**

$$\frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}\left((\sec\alpha + \tan\alpha)^n - 1\right).$$

(ii) Deduce that **2**

$$I_n < \frac{1}{n}\left((\sec\alpha + \tan\alpha)^n - 1\right), \text{ for } n \geq 1.$$

Examination continues overleaf...

(d) Suppose that z_1, z_2 and $z_3 \in \mathbb{C}$ satisfy the following conditions simultaneously. **3**

$$\begin{cases} |z_1| = |z_2| = |z_3| = 1 \\[2ex] \dfrac{z_1}{z_2} + \dfrac{z_2}{z_3} + \dfrac{z_3}{z_1} = 1 \\[2ex] z_2 \neq z_3 \text{ and } z_3 \neq z_1. \end{cases}$$

Prove that if a, b and $c \in \mathbb{R}$, then $|az_1 + bz_2 + cz_3| = \sqrt{a^2 + b^2 + c^2 + 2ab}$.

End of paper

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

Questions **Marks**

1 What is the converse of the statement "if you can't see my mirrors, then I can't see you"? **1**

A. If I can't see your mirrors, then you can't see me.

B. If I can't see you, then you can't see my mirrors.

C. If you can see me, then I can see your mirrors.

D. If I can see you, then you can see my mirrors.

2 What is the contrapositive of the statement "if you can't see my mirrors, then I can't see you"? **1**

A. If I can't see your mirrors, then you can't see me.

B. If I can't see you, then you can't see my mirrors.

C. If you can see me, then I can see your mirrors.

D. If I can see you, then you can see my mirrors.

3 If $\underset{\sim}{u} = 5\underset{\sim}{i} + (t-3)\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + (t+9)\underset{\sim}{j} + 2\underset{\sim}{k}$ are perpendicular, what are the possible values of t? **1**

A. $t = 8,\ 2$

B. $t = -8,\ -2$

C. $t = 8,\ -2$

D. $t = -8,\ 2$

$5 + (t-3)(t+9) + 6 = 0$

$t^2 + 6t - 16 = 0$

$(t+8)(t-2) = 0$

→ forms a Δ

4 Suppose $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are three vectors such that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$, $|\underset{\sim}{a}| = 3$, $|\underset{\sim}{b}| = 5$, and $|\underset{\sim}{c}| = 7$. **1**

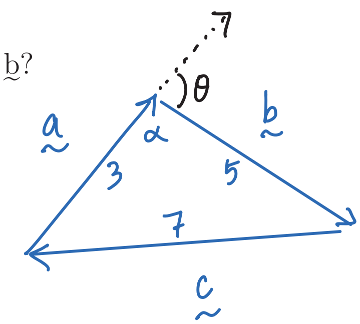
What is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$?

A. $\dfrac{\pi}{6}$

B. $\dfrac{2\pi}{3}$

C. $\dfrac{5\pi}{3}$

D. $\dfrac{\pi}{3}$



$$\cos\alpha = \frac{3^2 + 5^2 - 7^2}{2(3)(5)}$$

$\alpha = \dfrac{2\pi}{3}$ $\quad\therefore \theta = \dfrac{\pi}{3}$

5 A particle of mass m moves so that its velocity v is given by $v = f(x)$. **1**

The resultant force that causes this motion is given by:

A. $mf'(x)$

B. $mxf'(x)$

C. $mf'\left(\frac{1}{2}x^2\right)$

D. $mf(x)f'(x)$

$F = ma$

$= m\,v\,\dfrac{dv}{dx}$

$= m\,f(x) \cdot f'(x)$

6 Let ω be a non-real cube root of unity. **1**

$$\begin{cases} 1 + w + w^2 = 0 \\ w^3 = 1 \end{cases}$$

What is the value of $(1 + 2\omega + 3\omega^2)(1 + 3\omega + 2\omega^2)$?

A. 3

B. 6

C. 9

D. 12

$$(1 + w + w^2 + w + 2w^2)(1 + w + w^2 + 2w + w^2)$$
$$= (w + 2w^2)(2w + w^2)$$
$$= 2w^2 + w^3 + 4w^3 + 2w^4$$
$$= 2w^2 + 5 + 2w$$
$$= 2(w^2 + w) + 5 = 2(-1) + 5 = 3$$

7 Suppose that z_1 and z_2 are two distinct non-zero complex numbers satisfying $|z_1 + z_2| = |z_1 - z_2|$. **1**

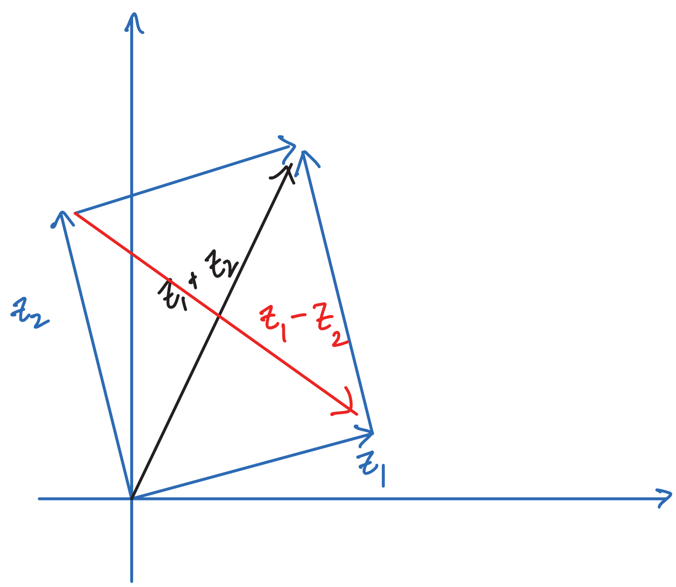
Which of the following is always true?

A. $z_1 z_2$ is purely imaginary.

B. $\dfrac{z_1}{z_2}$ is purely imaginary.

C. $\dfrac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

D. None of the above.



$|z_1 + z_2| = |z_1 - z_2|$

$\therefore$ equal diagonals

$\therefore$ parallelogram must be a rectangle

$\therefore \operatorname{Arg}(z_2) - \operatorname{Arg}(z_1) = \dfrac{\pi}{2}$

$\therefore \operatorname{Arg}\left(\dfrac{z_2}{z_1}\right) = \dfrac{\pi}{2}$, ie $\dfrac{z_2}{z_1}$ is purely imaginary

$\therefore \dfrac{z_1}{z_2}$ is purely imaginary.

8 A particle moves from the initial conditions $t = t_0$, $x = x_0$, $v = v_0$ to the final conditions $t = t_F$, $x = x_F$, $v = v_F$ under a force that produces an acceleration of $\ddot{x}$. **1**

Which of the following is incorrect?

A. If $\ddot{x} = f(t)$, then $v_F = \displaystyle\int_{t_0}^{t_F} f(t)\,dt + v_0$.

B. If $\ddot{x} = f(v)$, then $x_F = \displaystyle\int_{v_0}^{v_F} \frac{f(v)}{v}\,dv + x_0$.

C. If $\ddot{x} = f(v)$, then $t_F = \displaystyle\int_{v_0}^{v_F} \frac{1}{f(v)}\,dv + t_0$.

D. If $\ddot{x} = f(x)$, then $(v_F)^2 = 2\displaystyle\int_{x_0}^{x_F} f(x)\,dx + (v_0)^2$.

$$\frac{v\,dv}{dx} = f(v)$$

$$\int_{v_0}^{v_F} \frac{v\,dv}{f(v)} = \int_{x_0}^{x_F} dx$$

9 Suppose that a, b and c are non-zero real numbers satisfying **1**

$$\frac{a^2}{b^2+c^2} < \frac{b^2}{c^2+a^2} < \frac{c^2}{a^2+b^2}.$$

Which of the following statements can be deduced from the preceding inequality?

A. $a < b < c$

B. $|a| < |b| < |c|$

C. $c < b < a$

D. $|c| < |b| < |a|$

$$\frac{b^2+c^2}{a^2} > \frac{c^2+a^2}{b^2} > \frac{a^2+b^2}{c^2}$$

Add 1 to each side

$$\frac{a^2+b^2+c^2}{a^2} > \frac{a^2+b^2+c^2}{b^2} > \frac{a^2+b^2+c^2}{c^2}$$

$$\therefore \frac{1}{a^2} > \frac{1}{b^2} > \frac{1}{c^2}$$

$$\therefore a^2 < b^2 < c^2 \qquad \therefore |a| < |b| < |c|$$

10 Three non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ on a number plane are such that $\underset{\sim}{b} \cdot \underset{\sim}{c} = 2|\underset{\sim}{a}|$, $\underset{\sim}{a} \cdot \underset{\sim}{c} = 3|\underset{\sim}{b}|$, and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$. **1**

What is the minimum value of $|\underset{\sim}{c}|$?

A. $2\sqrt{2}$

B. $2\sqrt{3}$

C. $2\sqrt{5}$

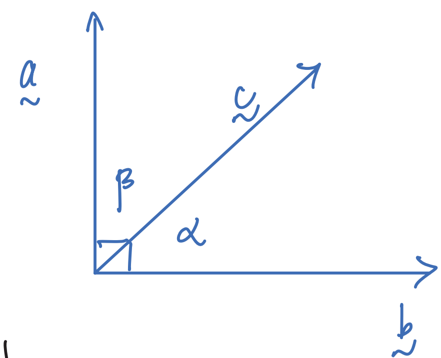
D. None of the above.

$|\underset{\sim}{b}||\underset{\sim}{c}| \cos\alpha = 2|\underset{\sim}{a}|$ (note: $0 \le \alpha \le \pi$)

$\therefore \cos\alpha > 0 \quad \therefore \alpha$ is acute

Similarly,

$|\underset{\sim}{a}||\underset{\sim}{c}| \cos\beta = 3|\underset{\sim}{b}| \quad \therefore \beta$ is acute



$\therefore \alpha + \beta = \frac{\pi}{2}$

$\therefore |\underset{\sim}{a}||\underset{\sim}{c}| \sin\alpha = 3|\underset{\sim}{b}|$

Now $|\underset{\sim}{c}| \sin\alpha = \dfrac{3|\underset{\sim}{b}|}{|\underset{\sim}{a}|}$

Examination continues overleaf…

and $|\underset{\sim}{c}| \cos\alpha = \dfrac{2|\underset{\sim}{a}|}{|\underset{\sim}{b}|}$

$$\therefore |\underset{\sim}{c}|^2 = \frac{9|\underset{\sim}{b}|^2}{|\underset{\sim}{a}|^2} + \frac{4|\underset{\sim}{a}|^2}{|\underset{\sim}{b}|^2}$$

$$= \frac{9|\underset{\sim}{b}|^4 + 4|\underset{\sim}{a}|^4}{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2}$$

$$\ge \frac{2(3|\underset{\sim}{b}|^2)(2|\underset{\sim}{a}|^2)}{|\underset{\sim}{a}|^2|\underset{\sim}{b}|^2}$$

$$= 12$$

$\therefore |\underset{\sim}{c}| \ge 2\sqrt{3}$

Question 11 (15 marks) Start on a NEW page

(a) Give a counterexample to disprove the following statement:

For all $x, y \in \mathbb{R}$, if $x^2 \geq y^2$, then $x \geq y$.

Suppose $x = -1$, $y = 0$

$(-1)^2 \geq 0$ ✓ but $-1 \not\geq 0$ ✓

$\therefore$ This disapproves the statement.

Comments:

Students have to clearly disprove the statement through a conclusion.

(b) Write down the negation of the following statement:

There exists $k \in \mathbb{R}$ such that $k^2 < 0$.

$\sim (\exists k \in \mathbb{R}, k^2 < 0) \equiv \forall k \in \mathbb{R}, k^2 \geq 0$ ✓

Comments:

Some students left their statement as an existential statement and failed to include zero in their inequality.

(c) Consider the vectors $\underset{\sim}{u} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = -5\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{w} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + 5\underset{\sim}{k}$.

(i) Prove that $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are perpendicular.

RTP: $\underset{\sim}{v} \cdot \underset{\sim}{w} = 0$

$$LHS = \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} = 2 \times 0 - 5 \times 2 + 2 \times 5 = 0$$

Since $\underset{\sim}{v} \cdot \underset{\sim}{w} = 0$, $\underset{\sim}{v} \perp \underset{\sim}{w}$ ✓

Comments:

Done well.

(ii) Verify that $|\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}| \leq |\underset{\sim}{u}| + |\underset{\sim}{v}| + |\underset{\sim}{w}|$.

$$LHS = \left| \begin{pmatrix} 3+0+2 \\ 2-5+2 \\ 1+2+5 \end{pmatrix} \right| = \left| \begin{pmatrix} 5 \\ -1 \\ 8 \end{pmatrix} \right| = \sqrt{25+1+64} = \sqrt{90} \doteqdot 9.49$$ ✓

$$RHS = \sqrt{9+4+1} + \sqrt{25+4} + \sqrt{4+4+25} = \sqrt{14} + \sqrt{29} + \sqrt{33} \doteqdot 14.87$$

$\therefore LHS \leq RHS$ ✓

OR $\therefore \underbrace{|\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}|}_{\doteqdot 14.87} \leq \underbrace{|\underset{\sim}{u}| + |\underset{\sim}{v}| + |\underset{\sim}{w}|}_{\doteqdot 9.49}$

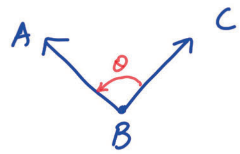
Mark	Criteria
2	Correct solution.
1	Finds $\lvert\underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}\rvert$ and $\lvert\underset{\sim}{u}\rvert + \lvert\underset{\sim}{v}\rvert + \lvert w\rvert$, but fails to clearly show that one is greater than the other through correct decimal rounding.

Comments:

Students had to **clearly** show this inequality was true by decimal rounding. Students that proved the result using the triangle identity were awarded with full marks.

(d) Given the points $A(1,3,5)$, $B(2,-4,11)$ and $C(-3,-1,0)$, find $\angle ABC$. Give your answer correct to the nearest minute.

$$\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$



$$\vec{BA} = \vec{OA} - \vec{OB} = \begin{pmatrix}1\\3\\5\end{pmatrix} - \begin{pmatrix}2\\-4\\11\end{pmatrix}$$

$$= \begin{pmatrix}-1\\7\\-6\end{pmatrix}$$

Similarly, $\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix}-3\\-1\\0\end{pmatrix} - \begin{pmatrix}2\\-4\\11\end{pmatrix}$

$$= \begin{pmatrix}-5\\3\\-11\end{pmatrix}$$

$$\therefore \cos\theta = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}|\,|\vec{BC}|} = \frac{5+21+66}{\sqrt{1+49+36}\,\sqrt{25+9+121}} = \frac{92}{\sqrt{86}\sqrt{155}}$$

$$\therefore \theta = 37^\circ 10'$$

Mark	Criteria
3	Correct solution.
2	Makes error in their final calculation or in obtaining their vectors $\overrightarrow{BA}$, $\overrightarrow{BC}$.
1	Makes 2 subsequent errors in finding their vectors and in calculating their angle.

Comments:

Students were not deducted for rounding. Many students made mistakes with finding $\overrightarrow{CB}$ instead of $\overrightarrow{BC}$. Students made errors in calculating their angle by accidentally omitting the square roots or mistyping their expression in their calculator.

(e) By using integration by parts, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\, dx$.

let $u = x$ $\quad u' = 1$

$v' = \cos x$ $\quad v = \sin x$

$$I = \left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} 1.\sin x\, dx$$

$$= \frac{\pi}{2}(1) - 0 + \left[\cos x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$$

Mark	Criteria
3	Correct solution.
2	Makes error in their integration by parts.
1	Makes 2 subsequent errors in integration by parts.

Comments:

Most students obtained at least 2 marks. Some errors that occurred included mistakes with the negative and forgetting to substitute the zero into the antiderivative.

(f) Evaluate $\text{Arg}(i^{2025})$, where $\text{Arg}(z)$ denotes the principal argument of z.

$$\arg(i^{2025}) = 2025 \times \frac{\pi}{2}$$

Note: $\boxed{\arg z^n = n \arg z}$

$$= \frac{2025}{2}\pi$$

$$= \pi(1012 + 0.5)$$

$$= \underbrace{1012\pi}_{2k\pi} + \frac{\pi}{2}$$

$$\Rightarrow \text{Arg}(i^{2025}) = \frac{\pi}{2}$$

Mark	Criteria
2	Correct solution.
1	Makes substantial progress or makes an error.

Comments:

Done well.

(g) Prove that $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

let $z = e^{i\theta} = \cos\theta + i\sin\theta$ (1)

$z^{-1} = e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$ (2)

(1) - (2) $\Rightarrow e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ [$\div 2i$]

$\therefore \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

Comments:

Done well.

Q12

a. $\int \cos^3 x \sin^2 x \, dx$

$= \int \sin^2 x \cos^2 x \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$= \int (\sin^2 x \cos x - \sin^4 x \cos x) \, dx$

$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + c$

Alternate Method: let $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$I = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$= \int (u^2 - u^4) \, du$ ✓

$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + c$

$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 + c$ ✓

correct working and answer : AW 2

correct integrand in terms of u : AW 1

Q12

b. $\displaystyle\int \frac{d\theta}{3+2\cos 4\theta}$

Let $t = \tan 2\theta$

$$\frac{dt}{d\theta} = 2\sec^2\theta \, d\theta$$

$$d\theta = \frac{dt}{2(t^2+1)}$$

$$I = \int \frac{\frac{dt}{2(t^2+1)}}{3 + \frac{2(1-t^2)}{1+t^2}}$$

$$I = \frac{1}{2}\int \frac{dt}{t^2+5}$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{t}{\sqrt{5}}\right)$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{\tan 2\theta}{\sqrt{5}}\right) + c$$

Express $d\theta$ in terms of t
AW: 1

Correct integrand in terms of t AW: 2

Correct working and Ans AW: 3

$\boxed{C/i}$

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\sin\theta\, i + 3\cos i^2 \sin^2\theta + i^3\sin^3\theta$$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) + \left(\sin(1-\sin^2\theta) - \sin^3\theta\right) i$$

$$= 4\cos^3\theta - 3\cos\theta + (\sin\theta - 2\sin^3\theta)\, i \quad 1$$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \quad 2 \qquad \text{De Moivre's}$$

Equating real part from 1 and 2

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Use binomial expansion correctly AW: 1

C/ii Let $x = \cos\theta$: $8x^3 - 6x + 1 = 0$

$$2(4x^3 - 3x) = -1$$

$$4x^3 - 3x = -\frac{1}{2}$$

$$4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$$ ✓

$$\cos 3\theta = -\frac{1}{2}$$ from (i)

$$3\theta = \pi - \frac{\pi}{3},\ \pi + \frac{\pi}{3},\ 3\pi - \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{9},\ \frac{4\pi}{9},\ \frac{8\pi}{9}$$ ✓

The solutions: $x = \cos\frac{2\pi}{9},\ \cos\frac{4\pi}{9},\ \cos\frac{8\pi}{9}$ ✓

Express $\cos 3\theta = -\frac{1}{2}$ AW: 1

Having $\cos 3\theta = -\frac{1}{2}$ and $\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$ AW: 2

Provide correct working and solutions AW: 3

$\boxed{\text{C/iii}}$ $8x^3 - 6x + 1 = 0$

Sum of roots: $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = \frac{-b}{a} = 0$ ✓

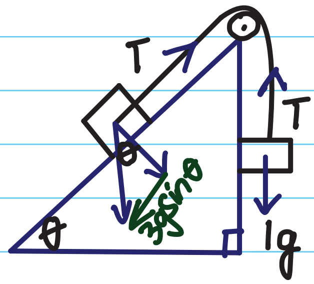
$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\left(\pi - \frac{\pi}{9}\right) = 0$$

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9} = 0$$

$$\cos\frac{\pi}{9} - \cos\frac{2\pi}{9} - \cos\frac{4\pi}{9} = 0$$ ✓

- Use sum of the roots $= \frac{-b}{a} = 0$ AW: 1
- Express $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$: AW 1

12/d



The System is in equilibrium $\underset{\sim}{a}=0$

Forces act on 3 kg object

$$T = 3g\sin\theta$$

Forces act on 1 kg object

$$T = 1g$$

$$T = T \quad : \quad 3g\sin\theta = 1g$$

$$\sin\theta = \frac{1}{3}$$

$$\theta = 19.5^\circ$$

provide $T = 3g\sin\theta$ OR $T = 1g$ AW 1

solve equation and provide $\sin\theta = \frac{1}{3}$ AW 2

provide correct working and solution AW 3

Question 13

(a)

(i) $\frac{1}{2}v^2 = 6 + 4x - 2x^2$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= 4 - 4x$

$= -4(x-1)$

$\therefore$ particle is undergoing SHM.

Comment

- Generally well done
- Students that have shown that

$$v^2 = n^2\left(a^2 - (x-c)^2\right)$$

were not awarded any marks (as this makes (a)(ii) very easy and the marks are awarded there).

⟨ To prove a particle is undergoing SHM, $\ddot{x} = -n^2(x-c)$ must be shown. ⟩

(ii) $n = 2$, $\therefore$ period $= \dfrac{2\pi}{2} = \pi$

C.O.M, $x = 1$

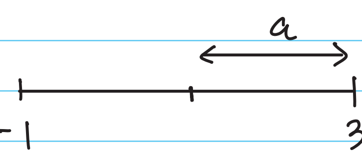
Let $v = 0$

$$-4x^2 + 8x + 12 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3, -1$$



$$\therefore a = \frac{3-(-1)}{2}$$

$$= 2$$

Comment

- Generally well done

(b) Assume that $\sqrt[3]{5} = \frac{p}{q}$, where p and q are coprime positive integers.

$$5 = \frac{p^3}{q^3}$$

$$5q^3 = p^3$$

$\therefore$ 5 is a factor of p^3 and hence a factor of p

(since 5 is prime)

better to include, however it was not penalised this time if students did not write it.

$\therefore p = 5k, \ k \in \mathbb{Z}$

$\therefore 5q^3 = 125k^3$

$$q^3 = 25k^3 = 5(5k^3)$$

$\therefore$ 5 is a factor of q^3, and hence a factor of q

This contradicts the assumption that p and q are coprime

$\therefore \sqrt[3]{5}$ must be irrational.

(c) $\quad Re(z^2) \le (Re\, z)^2$

Let $z = x + iy$

$Re(x^2 + 2ixy - y^2) \le x^2$

$x^2 - y^2 \le x^2$

$-y^2 \le 0$

$\therefore y^2 \ge 0$

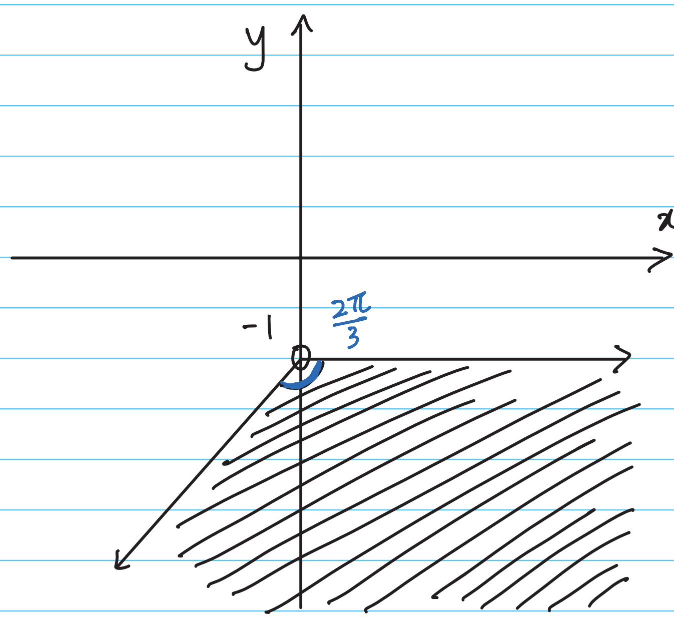
$\therefore$ all real y

Must show

Must write

AND

$-\frac{2\pi}{3} \le Arg(z - (-i)) \le 0$



(d) $\displaystyle\int_{1}^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx$

$u^6 = x$

$6u^5\,du = dx$

$x = 1,\ u = 1$

$x = 64,\ u = 2$

$$= \int_{1}^{2} \frac{6u^5\,du}{u^3 + u^2}$$

$$= 6\int_{1}^{2} \frac{u^3}{u+1}\,du$$

$$= 6\int_{1}^{2} \frac{u^3+1}{u+1} - \frac{1}{u+1}\,du$$

Some students had difficulty in obtaining this integrand

$$= 6\int_{1}^{2} u^2 - u + 1 - \frac{1}{u+1}\,du$$

$\left[u^3 + 1 = (u+1)(u^2 - u + 1)\right]$

$$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1)\right]_1^2$$

$$= 6\left[\left(\frac{8}{3} - 2 + 2 - \ln 3\right) - \left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2\right)\right]$$

$$= 11 + 6\ln\frac{2}{3}$$

(e) Note that $\dfrac{x^2+1}{x^4+8x^2+12}$

$$= \frac{y+1}{y^2+8y+12} \quad \text{where } y = x^2 \quad (\star)$$

$$= \frac{y+1}{(y+6)(y+2)}$$

$$= \frac{\frac{5}{4}}{y+6} + \frac{-\frac{1}{4}}{y+2}$$

can be obtained easily with the cover up rule

$$\therefore I = \int \frac{\frac{5}{4}}{x^2+6} - \frac{\frac{1}{4}}{x^2+2}\, dx$$

$$= \frac{5}{4\sqrt{6}} \tan^{-1}\left(\frac{x}{\sqrt{6}}\right) - \frac{1}{4\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Comment

- Only a handful of students recognised that the integrand can be simplified immensely as shown in $(\star)$ above.
- Majority of the students' partial fractions were $\dfrac{Ax+B}{x^2+6} + \dfrac{Cx+D}{x^2+2}$.

This was time costly and susceptible to errors.

Q14.

a/i. $S_1 : x^2 + y^2 + z^2 = 9$
$S_2 : (x-3)^2 + (y+\sqrt{6})^2 + (z-7)^2 = 25$

Let d be the distance between the centres of the spheres.

$d = \sqrt{(3-0)^2 + (-\sqrt{6}-0)^2 + (7-0)^2}$ ✓
$d = 8$
The sum of two radii $= 3 + 5 = 8$ ✓

Since the sum of two radii is the same as the distance between the centres of the two spheres thus the spheres must touch each other at a single point or S_1 and S_2 are externally tangent. ✓

- Use the formula to provide the correct answer for the distance between two spheres AW: 1
- Having both the sum of the two radii and the distance between the centres of two spheres AW: 2
- correct working and answer AW: 3

a/ii
$C_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} 3 \\ -\sqrt{6} \\ 7 \end{pmatrix}$

$\overrightarrow{C_1C_2} = \begin{pmatrix} 3 \\ -\sqrt{6} \\ 7 \end{pmatrix}$ ✓

$l_1 : r = \lambda \underset{\sim}{d} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -\sqrt{6} \\ 7 \end{pmatrix}$

$l_1 : \underset{\sim}{r} = \lambda \begin{pmatrix} 3 \\ -\sqrt{6} \\ 7 \end{pmatrix}$ ✓

provide vector direction of l_1 : AW 1

Q14

a/iii $l_2: \underset{\sim}{r_2} = \begin{pmatrix}1\\-\sqrt{6}\\2\end{pmatrix} + \mu\begin{pmatrix}-1\\1\\2\end{pmatrix}$

$l_2: \underset{\sim}{r_1} = \lambda\begin{pmatrix}3\\-\sqrt{6}\\7\end{pmatrix}$

$$3\lambda = 1 - \mu \quad ①$$
$$-\sqrt{6}\lambda = -\sqrt{6} + \mu \quad ② \checkmark$$
$$7\lambda = 2 + 2\mu \quad ③$$

① + ②

$$\lambda(3-\sqrt{6}) = 1-\sqrt{6}$$
$$\lambda = \frac{1-\sqrt{6}}{3-\sqrt{6}} = \frac{-3-2\sqrt{6}}{3} \text{ subs into ①}$$

$$3\left(\frac{-3-2\sqrt{6}}{3}\right) = 1-\mu$$
$$\mu = 4+2\sqrt{6}$$

Note: the values of λ and μ can vary $\left(\lambda = \frac{4}{13}, \mu = \frac{1}{13}\right)$

Subs λ and μ into ③

$$7\left(\frac{-3-2\sqrt{6}}{3}\right) = 2 + 2(4+2\sqrt{6})$$
$$\frac{-21-14\sqrt{6}}{3} \neq 10 + 4\sqrt{6} \checkmark$$

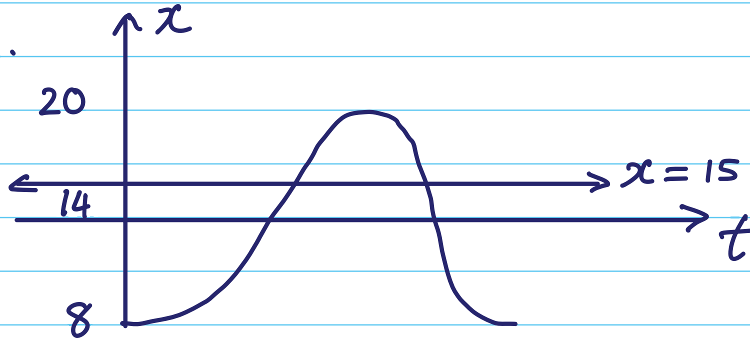
l_1 and l_2 do Not intersect. No points of intersection.

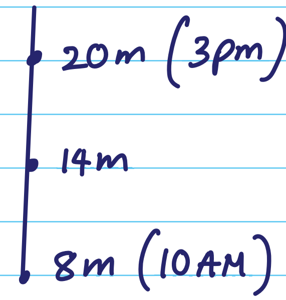
a/iv. Since l_1 and l_2 do Not intersect $\checkmark$ from (iii) and direction vector of l_1 and l_2 are Not multiple Scalar of each other

$$\begin{pmatrix}3\\-\sqrt{6}\\7\end{pmatrix} \neq a\begin{pmatrix}-1\\1\\2\end{pmatrix}, \checkmark a \in R \quad \therefore l_1 \not\parallel l_2$$

1M for stating l_1 and l_2 Not intersecting and another mark for stating l_1 is Not $\parallel$ l_2.

Q14/b.





Centre of motion $c = \dfrac{20+8}{2} = 14$

amplitude: $a = 20 - 14 = 14 - 8 = 6$ ✓

Period: $T = \dfrac{2\pi}{n} = \dfrac{2\pi}{n} = 10 \rightarrow n = \dfrac{\pi}{5}$

$x = a\cos(nt + \alpha) + c$

At $t = 0$ (8AM), $x = 8$ m

$8 = 6\cos\alpha + 14$

$\cos\alpha = -1 \longrightarrow \alpha = \pi$

$x = 6\cos(\pi + \frac{\pi}{5}t) + 15$

$x = -6\cos(\frac{\pi}{5}t) + 14$ ✓

Ship requires minimum of 15m depth.

$15 = -6\cos(\frac{\pi}{5}t) + 14$

$\cos\frac{\pi}{5}t = -\frac{1}{6}$

$t = \dfrac{\pi - \cos^{-1}(\frac{1}{6})}{\pi/5} = 2.7665$ hrs

$= 2$ hrs and 46 mins ✓

$\therefore$ Earliest time 10AM + 2hrs, 46 mins = 12:46 PM

provide either amplitude, centre of motion or $n = \frac{\pi}{5}$ AW: 1

Correct equation: $x = -6\cos(\frac{\pi}{5}t) + 14$ AW: 2

Correct working and Solution AW: 3

Q14/C.

Let the two consecutive numbers be :

$a, \quad b = a + 1$

$c = ab = a(a+1)$

$$(a^2 + a + 1)^2 = a^4 + 2a^2(a+1) + (a+1)^2$$

$$= a^4 + 2a^3 + 2a^2 + a^2 + 2a + 1$$

$$= \underline{a^4 + 2a^3 + 3a^2 + 2a + 1} \quad (*) \checkmark$$

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{a^2 + (a+1)^2 + a^2(a+1)^2}$$

$$= \sqrt{a^2 + a^2 + 2a + 1 + a^2(a^2 + 2a + 1)}$$

$$= \sqrt{a^4 + 2a^3 + 3a^2 + 2a + 1}$$

$$= \sqrt{(a^2 + a + 1)^2} \quad \text{from } (*)$$

$$= a^2 + a + 1 \quad \checkmark$$

$$= \underbrace{a(a+1)}_{\text{Even}} \underbrace{+}_{+} \underbrace{1}_{1 \; = \; \text{odd}} \quad \text{This is odd.}$$

$a(a+1)$ = the product of two consecutive numbers is always even plus 1, gives

$(a^2 + a + 1)$ odd.

$\therefore \sqrt{a^2 + b^2 + c^2} = a^2 + a + 1$ is an odd integer. ✓

Expand $(a^2 + a + 1)^2$ correctly : AW 1

Express $\sqrt{a^2+b^2+c^2} = a^2 + a + 1$: AW 2M

Correct proof : AW 3

Question Q15(a)	Marks
· Provides correct solution.	3 marks
· Proves $\lvert z-w\rvert^2 = \lvert z\rvert^2 - (z\overline{w} + w\overline{z}) + \lvert w\rvert^2$.	2 marks
· Proves that $\lvert z\rvert = \lvert w\rvert$ and attempts to show $\lvert z-w\rvert = \lvert z\rvert$.	1 mark

Solution (Method 1 of 4, Sum of two squares identity)
The triangle with vertices at 0, z and w has sides of length $|z|$, $|w|$ and $|z-w|$.

Since $|z|^2 = |w|^2$ we have $|z| = |w|$.

Hence, it suffices to prove that $|z-w| = |z|$, because then $|z-w| = |z| = |w|$ and the triangle is equilateral as required. ✓

Consider $|z-w|^2$, and use the fact that for any complex number u, $|u|^2 = u\overline{u}$:

$$\begin{aligned}|z-w|^2 &= (z-w)\overline{(z-w)}\\ &= (z-w)(\overline{z}-\overline{w})\\ &= z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w}\\ &= |z|^2 - (z\overline{w} + w\overline{z}) + |w|^2 \checkmark\\ &= |z|^2 + |w|^2 - (z\overline{w} + \overline{z\overline{w}})\\ &= |z|^2 + |w|^2 - 2\mathrm{Re}(z\overline{w})\end{aligned}$$

Now use the given information that $|w|^2 = |z|^2$ and $2\mathrm{Re}(z\overline{w}) = |z|^2$:

$$\begin{aligned}|z-w|^2 &= |z|^2 + |z|^2 - |z|^2\\ &= |z|^2\end{aligned}$$

which implies $|z-w| = |z|$, which is what we were aiming to show. ✓

Question 15(a) (continued)

Solution (Method 2 of 4, Cartesian form)
Let $z = a + ib$ and let $w = x + iy$. Then

$$\begin{aligned}\mathrm{Re}(z\overline{w}) &= \mathrm{Re}\big((a+ib)(x-iy)\big)\\ &= \mathrm{Re}\,(ax+by+ibx-iay)\\ &= ax+by\end{aligned}$$

So the given information $|z|^2 = |w|^2 = 2\mathrm{Re}(z\overline{w})$ becomes $a^2+b^2 = x^2+y^2 = 2ax+2by$ $(*)$.

$$\begin{aligned}|z-w|^2 &= |(a-x)+i(b-y)|^2\\ &= (a-x)^2+(b-y)^2\\ &= a^2-2ax+x^2+b^2-2by+y^2\\ &= a^2+b^2+x^2+y^2-2(ax+by)\\ &= a^2+b^2+a^2+b^2-(a^2+b^2) \quad \text{using } (*)\\ &= a^2+b^2\\ &= |z|^2\end{aligned}$$

Hence $|z-w|^2 = |z|^2 = |w|^2$ and so $|z-w| = |z| = |w|$, which implies all sides are equal and the triangle is equilateral.

Question 15(a) (continued)

Solution (Method 3 of 4, Mod-arg form)

Since $|z|^2 = |w|^2$, we have $|z| = |w|$.
Hence, let $z = r \operatorname{cis} \theta$ and $w = r \operatorname{cis} \phi$.
Since $2\text{Re}(z\overline{w}) = |z|^2$, we have

$$\begin{aligned}
2\text{Re}(z\overline{w}) &= r^2 \\
2\text{Re}(r^2 \operatorname{cis} \theta \operatorname{cis}(-\phi)) &= r^2 \\
2\text{Re}(r^2 \operatorname{cis}(\theta - \phi)) &= r^2 \\
\text{Re}\left(\operatorname{cis}(\theta - \phi)\right) &= \frac{1}{2} \\
\cos(\theta - \phi) &= \frac{1}{2} \\
\theta - \phi &= \pm\frac{\pi}{3}
\end{aligned}$$

Therefore the angle between z and w is $\frac{\pi}{3}$.
Since $|z| = |w|$, the triangle is isosceles, but by angle sum of a triangle, all angles are $\frac{\pi}{3}$, which implies the triangle is equilateral.

Question 15(a) (continued)

Solution (Method 4 of 4, Vectors)

As in Method 2, let $z = a + ib$ and $w = x + iy$ so that $\mathrm{Re}(z\overline{w}) = ax + by$. ✓

If θ is the angle between the vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\begin{aligned}
\cos\theta &= \frac{\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}{\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| \left|\begin{pmatrix} x \\ y \end{pmatrix}\right|} \\
&= \frac{ax + by}{\sqrt{a^2 + b^2}\sqrt{x^2 + y^2}} \quad ✓ \\
&= \frac{ax + by}{a^2 + b^2} \quad \text{since } a^2 + b^2 = x^2 + y^2 \\
&= \frac{ax + by}{2(ax + by)} \quad \text{since } |z|^2 = 2\mathrm{Re}(z\overline{w}) \\
&= \frac{1}{2}
\end{aligned}$$

Hence $\theta = \frac{\pi}{3}$, which implies the angle between z and w is $\frac{\pi}{3}$.

As in Method 3, this implies the triangle is equilateral. ✓

Comments

· This was a rich question with many successful approaches, as shown above.
· There were significant notation mistakes that were not penalised.
· For example, $\overrightarrow{zw}$ instead of $\overrightarrow{ZW}$.
· Another example: $|zw|$ instead of $|ZW|$. This was a very confusing error of notation.
· The correct expression was $|z - w|$ which is very different from $|zw| = |z \times w|$.

Question Q15(b)	Marks
· Provides correct solution.	2 marks
· Obtains LHS $= 1 + \frac{2}{xy}$ or manages to prove LHS $\geqslant 8$	1 mark

Solution

Expand the brackets:

$$\begin{aligned}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) &= 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{xy}\\ &= 1+\frac{x+y}{xy}+\frac{1}{xy}\\ &= 1+\frac{1}{xy}+\frac{1}{xy}\\ &= 1+\frac{2}{xy}, \checkmark\end{aligned}$$

where we used $x+y=1$. We will now show $\dfrac{2}{xy} \geqslant 8$, which will imply the result.

$$\begin{aligned}(x-y)^2 &\geqslant 0\\ x^2+y^2 &\geqslant 2xy\\ x^2+2xy+y^2 &\geqslant 4xy\\ (x+y)^2 &\geqslant 4xy\\ 1^2 &\geqslant 4xy \quad \text{since } x+y=1\\ \frac{1}{xy} &\geqslant 4\\ \frac{2}{xy} &\geqslant 8\end{aligned}$$

Therefore we have

$$\begin{aligned}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) &= 1+\frac{2}{xy}\\ &\geqslant 1+8\\ &= 9 \checkmark\end{aligned}$$

Comments:

· Expanding the brackets was a simple but efficient approach.
· Many students launched into AM-GM, but this was not well suited to this problem.
· Students are encouraged to try simple approaches first before complicating matters.

Question Q15(c)	Marks
· Provides correct solution.	3 marks
· Applies products to sums with $\sin B \cos A$ where $A = (n-1)x$ and $B = x$.	2 marks
· Applies angle expansion formula for sin with $A = (n-1)x$ and $B = x$.	1 mark

Solution (Method 1 of 2, Slow but easier to find)

$$
\begin{aligned}
I_n &= \int \frac{\sin nx}{\cos x}\, dx \\
&= \int \frac{\sin\big((n-1)x + x\big)}{\cos x}\, dx \quad \text{(now use angle expansion formula)} \\
&= \int \frac{\sin\big((n-1)x\big)\cos x + \sin x \cos\big((n-1)x\big)}{\cos x}\, dx \checkmark \\
&= \int \frac{\sin\big((n-1)x\big)\cos x}{\cos x}\, dx + \int \frac{\sin x \cos\big((n-1)x\big)}{\cos x}\, dx \\
&= \int \sin\big((n-1)x\big)\, dx + \int \frac{\sin x \cos\big((n-1)x\big)}{\cos x}\, dx \\
&= -\frac{1}{n-1}\cos\big((n-1)x\big) + \int \frac{\sin x \cos\big((n-1)x\big)}{\cos x}\, dx \quad \text{(now use products to sums)} \\
&= -\frac{1}{n-1}\cos\big((n-1)x\big) + \frac{1}{2}\int \frac{\sin\big((n-1)x + x\big) - \sin\big((n-1)x - x\big)}{\cos x}\, dx \checkmark \\
&= -\frac{1}{n-1}\cos\big((n-1)x\big) + \frac{1}{2}\int \frac{\sin(nx) - \sin\big((n-2)x\big)}{\cos x}\, dx \\
&= -\frac{1}{n-1}\cos\big((n-1)x\big) + \frac{1}{2}\int \frac{\sin(nx)}{\cos x}\, dx - \frac{1}{2}\int \frac{\sin\big((n-2)x\big)}{\cos x}\, dx \\
I_n &= -\frac{1}{n-1}\cos\big((n-1)x\big) + \frac{1}{2}I_n - \frac{1}{2}I_{n-2} \\
2I_n &= -\frac{2}{n-1}\cos\big((n-1)x\big) + I_n - I_{n-2} \\
I_n &= \frac{2}{1-n}\cos\big((n-1)x\big) - I_{n-2} \checkmark
\end{aligned}
$$

Question 15(c) (continued)

Solution (Method 2 of 2, Fast but harder to find)

$$\begin{aligned}
I_n + I_{n-2} &= \int \frac{\sin(nx) + \sin\big((n-2)x\big)}{\cos x}\, dx \quad \text{(now use sums to products)} \\
&= \int \frac{2\sin\big((n-1)x\big)\cos x}{\cos x} \\
&= 2\int \sin\big((n-1)x\big) \\
&= -\frac{2}{n-1}\cos\big((n-1)x\big) \\
&= \frac{2}{1-n}\cos\big((n-1)x\big)
\end{aligned}$$

Therefore $I_n = \dfrac{2}{1-n}\cos\big((n-1)x\big) - I_{n-2}$ as required.

Comments:

· It is amazing how much easier it is to consider $I_n + I_{n-2}$ than I_n on its own.

· Students should take note of this trick because it might save time in the HSC exam.

Question Q15(d)	Marks
· Provides correct solution.	3 marks
· Substitutes assumption $(*)$ and correctly lists all terms.	2 marks
· Establishes base case $n = 0$.	1 mark

Solution

Base case: For $n = 0$,

$$\begin{aligned} \text{LHS} &= \frac{1}{2^0} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 1 + 0 \\ &= 1 \end{aligned}$$

Hence LHS $\leqslant$ RHS, since both equal 1. ✓

Assume true for $n = k$:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} \leqslant 1 + k \quad (*)$$

Prove true for $n = k + 1$

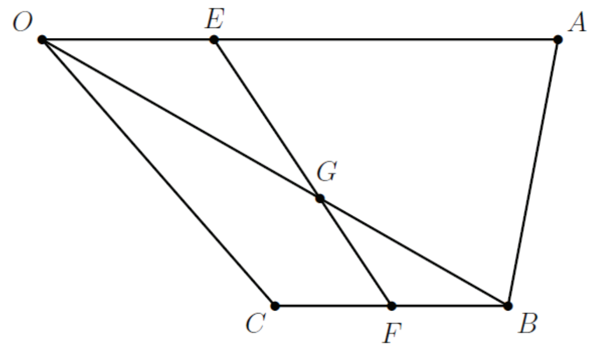
$$\text{R.T.P:} \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \leqslant 1 + (k + 1)$$

$$\begin{aligned} \text{LHS} &= \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}}_{\text{use } (*)} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \\ &\leqslant 1 + k + \underbrace{\frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}}}_{2^{k+1} - (2^k + 1) + 1 \text{ terms}} \quad ✓ \quad \text{(note that } 2^{k+1} - (2^k + 1) + 1 = 2^k\text{)} \\ &\leqslant 1 + k + \underbrace{\frac{1}{2^k} + \cdots + \frac{1}{2^k}}_{2^k \text{ terms}} \quad \text{(A smaller denominator makes the number larger)} \\ &\leqslant 1 + k + 2^k \times \frac{1}{2^k} \\ &= 1 + k + 1 \\ &= \text{RHS} \quad ✓ \end{aligned}$$

Comments:

· Unfortunately, almost all students misunderstood how the series changed for $n = k + 1$.
· They wrote LHS $= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$.
· However, as above, the correct expression is LHS $= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}}$
· Consequently, the median mark for this question was 1 out of 3.

Question 15(e)(i)	Marks
· Provides correct solution.	1 mark



Solution

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OC} + \overrightarrow{CB}\\ &= \underset{\sim}{c} + \lambda\,\overrightarrow{OA}, \quad \text{for some scalar } \lambda, \text{ since } OA \,||\, BC\\ &= \underset{\sim}{c} + \lambda\,\underset{\sim}{a}\end{aligned}$$

Question 15(e)(ii)	Marks
· Provides correct solution.	3 marks
· Finds $\overrightarrow{OG}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ using both $\overrightarrow{OE}+\overrightarrow{EG}$ and $\mu\overrightarrow{OB}$	2 marks
· Finds $\overrightarrow{OG}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ using $\mu\overrightarrow{OB}$.	1 mark

Solution

The plan is to find $\overrightarrow{OG}$ in two different ways, then equate coefficients.
In order to achieve this, we will need to introduce two new scalars, μ and ν.

$$\begin{aligned}\overrightarrow{OG} &= \mu\,\overrightarrow{OB} \quad \text{for some scalar } \mu, \text{ since } OG \,||\, OB\\ &= \mu(\underset{\sim}{c} + \lambda\underset{\sim}{a})\\ &= \lambda\mu\underset{\sim}{a} + \mu\underset{\sim}{c}\end{aligned}$$

We also have

$$\begin{aligned}\overrightarrow{OG} &= \overrightarrow{OE} + \overrightarrow{EG}\\ &= \overrightarrow{OE} + \nu\,\overrightarrow{EF} \quad \text{for some scalar } \nu, \text{ since } EG \,||\, EF\\ &= \frac{1}{3}\underset{\sim}{a} + \nu\left(\overrightarrow{OF} - \overrightarrow{OE}\right) \quad \text{since } OE : EA = 1 : 2\\ &= \frac{1}{3}\underset{\sim}{a} + \nu\left(\underset{\sim}{c} + \frac{1}{2}\overrightarrow{CB} - \frac{1}{3}\underset{\sim}{a}\right) \quad \text{since } F \text{ is the midpoint of } CB\\ &= \frac{1}{3}\underset{\sim}{a} + \nu\left(\underset{\sim}{c} + \frac{\lambda}{2}\underset{\sim}{a} - \frac{1}{3}\underset{\sim}{a}\right) \quad \text{since } \overrightarrow{CB} = \lambda\,\overrightarrow{OA} \text{ from part (i)}\\ &= \left(\frac{1}{3} + \frac{\lambda\nu}{2} - \frac{\nu}{3}\right)\underset{\sim}{a} + \nu\underset{\sim}{c}\end{aligned}$$

Question 15(e) (continued)

Equate the two expressions for $\overrightarrow{OG}$:

$$\lambda\mu\underset{\sim}{a} + \mu\underset{\sim}{c} = \left(\frac{1}{3} + \frac{\lambda\nu}{2} - \frac{\nu}{3}\right)\underset{\sim}{a} + \nu\underset{\sim}{c}$$

Equate coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ (valid, since $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are not parallel).

$$\lambda\mu = \frac{1}{3} + \frac{\lambda\nu}{2} - \frac{\nu}{3} \quad (1)$$

$$\mu = \nu \quad (2)$$

Substitute (2) into (1):

$$\begin{aligned}
\lambda\mu &= \frac{1}{3} + \frac{\lambda\mu}{2} - \frac{\mu}{3} \\
\frac{\lambda\mu}{2} &= \frac{1-\mu}{3} \\
3\lambda\mu &= 2 - 2\mu \\
\mu\left(3\lambda + 2\right) &= 2 \\
\mu &= \frac{2}{2+3\lambda}
\end{aligned}$$

Therefore, since $\overrightarrow{OG} = \lambda\mu\underset{\sim}{a} + \mu\underset{\sim}{c}$ from earlier, we have

$$\begin{aligned}
\overrightarrow{OG} &= \lambda\left(\frac{2}{2+3\lambda}\right)\underset{\sim}{a} + \left(\frac{2}{2+3\lambda}\right)\underset{\sim}{c} \\
&= \left(\frac{2\lambda}{2+3\lambda}\right)\underset{\sim}{a} + \left(\frac{2}{2+3\lambda}\right)\underset{\sim}{c} \checkmark
\end{aligned}$$

Comments:
- Part (i) was very well done.
- Part (ii) was challenging.
- There was a common misconception that $\overrightarrow{OG}$ could be found by equating $\overrightarrow{EF}$ and $\overrightarrow{OB}$.
- The source of this misconception is that G is the intersection of the lines EF and OB.
- To make progress, students had to introduce two new scalars and solve simultaneously.

Question 16 (15 marks) Start on a NEW page

(a) Suppose that a, b and c are the lengths of a right-angled triangle, where c is the hypotenuse.

Show that if $n \leq 2$, then $c^n \leq a^n + b^n$.

since $a^2+b^2=c^2$

$c \geq a$ (Δ identity)

$\therefore c^{n-2} \geq a^{n-2}$ as $n-2 \leq 0$

$a^2 c^{n-2} \geq a^n$ ①

Similarly, $b^2 c^{n-2} \leq b^n$ ②

① + ② $\Rightarrow a^2c^{n-2} + b^2c^{n-2} \leq a^n + b^n$

$c^{n-2}(a^2+b^2) \leq a^n + b^n$

$c^{n-2} \cdot c^2 \leq a^n + b^n$

$\therefore c^n \leq a^n + b^n$

Mark	Criteria
3	Correct solution.
2	Makes substantial progress in their proof by contradiction or setting up of inequality using the triangle inequality.
1	Correctly quotes the triangle inequality. OR Uses induction up to the inductive step but fails to go further.

Comments:

Poorly done. Students were awarded a mark if they successfully quoted the triangle inequality. Many students attempted induction, but this ultimately will not prove for all **real** numbers less than 2, i.e $\{n \in \mathbb{R} | n \leq 2\}$. Additionally, students that proved by contradiction had to clearly show why the contradiction exists for all cases for n.

(b) Show that the distance travelled by the particle before it has lost three-quarters of its initial speed is

$$\frac{15RU^2}{32gR - 15U^2} \text{ m},$$

where R is the radius of the Earth and g is the acceleration at the Earth's surface.

$\ddot{x} = \frac{-k}{x^2}$

$v\frac{dv}{dx} = \frac{-k}{x^2}$

$\int_u^{\frac{1}{4}u} v\,dv = \int_R^x \frac{-k}{x^2}\,dx$

$\frac{1}{2}v^2 \Big|_u^{\frac{1}{4}u} = k\left[\frac{1}{x}\right]_R^x$

$\frac{1}{2}u^2\left(\frac{1}{16}-1\right) = k\left(\frac{1}{x} - \frac{1}{R}\right)$

$\frac{-15}{32}u^2 = k\left(\frac{1}{x} - \frac{1}{R}\right)$

$x = \left(\frac{-15u^2}{32k} + \frac{1}{R}\right)^{-1}$

$x=R, \ddot{x} = -g \Rightarrow g = \frac{k}{R^2}$ sub in!

$\boxed{k = R^2 g}$

$x = \left(\frac{-15u^2 + 32Rg}{32R^2g}\right)^{-1}$

$x = \frac{32R^2g}{32Rg - 15u^2}$

But x = distance from O

i.e distance (d) travelled $= x - R$

$\boxed{d = x - R}$

$d = \frac{\cancel{32g}R^{\cancel{2}} - R(\cancel{32Rg} - 15u^2)}{32Rg - 15u^2}$

$\therefore d = \frac{15u^2R}{32Rg - 15u^2}$ as required.

Mark	Criteria
4	Correct solution.
3	Obtains $x = \frac{32R^2g}{32Rg - 15U^2}$ without error.
2	Obtains $x = \frac{32R^2g}{32Rg+15U^2}$ by using $k = -R^2g$.
1	Makes some correct progress in obtaining x. OR Incorrectly obtains $x = \frac{32R^2g}{32Rg-15U^2}$ from using $k = -R^2g$ OR Obtains $k = R^2g$

Comments:

Students made many errors from their **signs**. Students were commonly penalised for incorrectly obtaining $k = -R^2g$ and then using this expression to obtain $x = \frac{32R^2g}{32Rg-15U^2}$ which cannot be done. **By forcing this expression, they were further penalised.** Additionally, at the end, many students did not realise they needed to subtract R from x as x is the distance from the centre of earth.

(c) Let $I_n = \int_0^{\alpha} (\sec x + \tan x)^n \, dx$, where n is a non-negative integer and $0 < \alpha < \frac{\pi}{2}$.

(i) For $n \geq 1$, show that

$$\frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}\left((\sec\alpha + \tan\alpha)^n - 1\right).$$

Mark	Criteria
3	Correct solution.
2	Makes substantial progress.
1	Uses trig identities to simplify the trig expression to $2\sec^2 x + 2\sec x \tan x - 1$

Comments:

Poorly done. Many students did not know what to differentiate and integrate when it came to setting up the integration by parts.

$$I_n = \int_0^{\alpha} (\sec x + \tan x)^{n-2}(\sec^2 x + \tan^2 x + 2\sec x \tan x)\, dx$$

$$\boxed{\begin{aligned} \text{Note: } \frac{d}{dx}(\sec x + \tan x) &= \sec x \tan x + \sec^2 x \\ &= \sec x(\tan x + \sec x) \\ \sec^2 x &= \tan^2 x + 1 \\ \tan^2 x &= \sec^2 x - 1 \end{aligned}}$$

$$I_n = \int_0^{\alpha} (\sec x + \tan x)^{n-2}(\underbrace{2\sec^2 x + 2\sec x \tan x}_{2\sec x(\sec x + \tan x)} - 1)$$

$$= \int_0^{\alpha} \underbrace{2\sec x(\sec x + \tan x)(\sec x + \tan x)^{n-2}}_{f'(x)[f(x)]^{n-2} - \text{Rev. chain}} - \underbrace{(\sec x + \tan x)^{n-2}}_{I_{n-2}}\, dx$$

$$I_n = 2\left[\frac{(\sec x + \tan x)^{n-1}}{n-1}\right]_0^{\alpha} - I_{n-2}$$

$$= 2\left(\frac{(\sec\alpha + \tan\alpha)^{n-1}}{n-1} - \left(\frac{1-0}{n-1}\right)\right) - I_{n-2}$$

$$I_n + I_{n-2} = \frac{2}{n-1}\left((\sec\alpha + \tan\alpha)^{n-1} - 1\right) \quad [\div 2]$$

$$\frac{1}{2}(I_n + I_{n-2}) = \frac{1}{n-1}\left((\sec\alpha + \tan\alpha)^{n-1} - 1\right)$$

Replace n by $n+1$,

$$\therefore \frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}\left((\sec\alpha + \tan\alpha)^n - 1\right)$$

(ii) Deduce that

$$I_n < \frac{1}{n}\left((\sec\alpha + \tan\alpha)^n - 1\right), \text{ for } n \geq 1.$$

RTP: $I_n < \frac{1}{2}\left(I_{n+1} + I_{n-1}\right)$

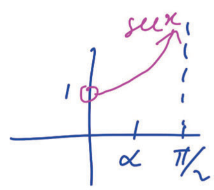
$$\begin{aligned}
\frac{1}{2}\left(I_{n+1} + I_{n-1}\right) &= \frac{1}{2}\int_0^\alpha (\sec x + \tan x)^{n+1} + (\sec x + \tan x)^{n-1}\,dx \\
&= \frac{1}{2}\int_0^\alpha (\sec x + \tan x)^{n-1}\left(\sec^2 x + \tan^2 x + 2\sec x\tan x + 1\right) \\
&= \frac{1}{2}\int_0^\alpha (\sec x + \tan x)^{n-1}\left(2\sec^2 x + 2\sec x\tan x\right)dx \\
&= \frac{1}{2}\int_0^\alpha (\sec x + \tan x)^{n-1}\, 2\underbrace{\sec x(\sec x + \tan x)}\,dx
\end{aligned}$$

$\sec x(\sec x + \tan x) = \frac{d}{dx}(\sec x + \tan x)$

$$= \int_0^\alpha (\sec x + \tan x)^n \sec x\,dx \quad ✓$$

Since $\sec x > 1$ for $0 < x < \alpha$ where $0 < \alpha < \frac{\pi}{2}$

$$> \underbrace{\int_0^\alpha (\sec x + \tan x)^n}_{I_n} \times 1\,dx$$



✓

$\therefore\ I_n < \frac{1}{2}\left(I_{n+1} + I_{n-1}\right)$ as required.

Mark	Criteria
2	Correct solution.
1	Makes substantial progress by proving that the function is concave up in the domain $\left(0, \frac{\pi}{2}\right)$.

Comments:

Poorly done. To obtain full marks students had to **clearly** prove that the $\sec x > 1$ in $x \in \left(0, \frac{\pi}{2}\right)$.

(d) Suppose that z_1, z_2 and $z_3 \in \mathbb{C}$ satisfy the following conditions simultaneously.

$$\begin{cases} |z_1| = |z_2| = |z_3| = 1 \\ \dfrac{z_1}{z_2} + \dfrac{z_2}{z_3} + \dfrac{z_3}{z_1} = 1 \quad ☆ \\ z_2 \neq z_3 \text{ and } z_3 \neq z_1. \end{cases}$$

Prove that if a, b and $c \in \mathbb{R}$, then $|az_1 + bz_2 + cz_3| = \sqrt{a^2 + b^2 + c^2 + 2ab}$.

Mark	Criteria
3	Correct solution.
2	Makes substantial progress or proves that $z_3 = \pm i z_1$.
1	Obtains $z_1 = z_2$

Comments:

Poorly done. Many students incorrectly used the vector formula $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$ to prove this result. However, this is invalid as z_1, z_2 and z_3 are complex numbers and have $i = \sqrt{-1}$ in their expression.

let $z_1 = \text{cis}\,\alpha$, $z_2 = \text{cis}\,\beta$, $z_3 = \text{cis}\,\gamma$

as $|z_1| = |z_2| = |z_3| = 1$

$\therefore \dfrac{\text{cis}\,\alpha}{\text{cis}\,\beta} + \dfrac{\text{cis}\,\beta}{\text{cis}\,\gamma} + \dfrac{\text{cis}\,\gamma}{\text{cis}\,\alpha} = 1$ from ☆

$\text{cis}(\alpha-\beta) + \text{cis}(\beta-\gamma) + \text{cis}(\gamma-\alpha) = 1$

Equate the im. parts,

$\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha) = 0$

$2\sin\left(\dfrac{\alpha-\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) + 2\sin\left(\dfrac{\beta-\alpha}{2}\right)\cos\left(\dfrac{\beta-2\gamma+\alpha}{2}\right) = 0$

$\sin(-\theta) = -\sin\theta$

$2\sin\left(\dfrac{\alpha-\beta}{2}\right)\left[\cos\left(\dfrac{\alpha-\beta}{2}\right) - \cos\left(\dfrac{\beta-2\gamma+\alpha}{2}\right)\right] = 0$

$\therefore \sin\left(\dfrac{\alpha-\beta}{2}\right) = 0$ OR $\cos\left(\dfrac{\alpha-\beta}{2}\right) = \cos\left(\dfrac{\beta-2\gamma+\alpha}{2}\right)$

$\therefore \boxed{z_1 = z_2}$ ✓ as $\alpha = \beta$

$\alpha - \cancel{\beta} = \cancel{\beta} - 2\gamma + \alpha$

$\alpha = \gamma$

But $\alpha \neq \gamma$ as $\boxed{z_1 \neq z_3}$

$z_1 = z_2 = \text{cis}\,\alpha \Rightarrow 1 + \dfrac{z_1}{z_3} + \dfrac{z_3}{z_1} = 1$

$\dfrac{z_1}{z_3} + \dfrac{z_3}{z_1} = 0$

$z_1^2 + z_3^2 = 0$

$z_3^2 = -z_1^2 \Rightarrow z_3 = \pm i z_1$ ✓

☆ $|z+\omega|^2 \neq (z+\omega)\cdot(z+\omega)$

$$\therefore |a z_1 + b z_2 + c z_3| = |a z_1 + b z_1 \pm i c z_1|$$

$$= |z_1|\,|a + b \pm i|$$

$$= |z_1| \sqrt{(a+b)^2 + c^2}$$

$$= \sqrt{a^2 + b^2 + c^2 + 2ab}$$

as required.

Alternate Sol^n

Let $\alpha = \frac{z_1}{z_2}$, $\beta = \frac{z_2}{z_3}$, $\gamma = \frac{z_3}{z_1}$

$\Sigma \alpha = 1$ (given)

$\alpha\beta\gamma = \frac{z_1}{z_2} \cdot \frac{z_2}{z_3} \cdot \frac{z_3}{z_1} = 1$

$\alpha\beta + \beta\gamma + \beta\gamma = \frac{z_1}{z_3} + \frac{z_3}{z_2} + \frac{z_2}{z_1}$

Now $|z_k| = 1$ $\therefore |z_k|^2 = 1$

$\Rightarrow z_k \bar{z}_k = 1$

$\therefore \bar{z}_k = \frac{1}{z_k}$

sub back in!

$$\therefore \Sigma\alpha\beta = \frac{\bar{z}_3}{\bar{z}_1} + \frac{\bar{z}_2}{\bar{z}_3} + \frac{\bar{z}_1}{\bar{z}_2} = \left(\overline{\frac{z_3}{z_1}}\right) + \left(\overline{\frac{z_2}{z_3}}\right) + \left(\overline{\frac{z_1}{z_2}}\right)$$

$$= \overline{\frac{z_3}{z_1} + \frac{z_2}{z_3} + \frac{z_1}{z_2}}$$

$$= \bar{1} = 1$$

$\therefore \alpha, \beta, \gamma$ - roots of

$x^3 - x^2 + x - 1 = 0$

$x^2(x-1) + (x-1) = 0$

$(x^2+1)(x-1) = 0 \Rightarrow x = \pm i, 1$

Now since $z_2 \neq z_3$ and $z_3 \neq z_1$

$\therefore \frac{z_2}{z_3} \neq 1$, $\frac{z_3}{z_1} \neq 1$ i.e. $\alpha = 1 \Rightarrow \boxed{z_1 = z_2}$

Now the rest follows!