

**4 UNIT MATHEMATICS FORM VI**

**Time allowed:** 3 hours (plus 5 minutes reading)

**Exam date:** 4th August, 1997

**Instructions:**

All questions may be attempted.

All questions are of equal value.

Part marks are shown in boxes in the left margin.

All necessary working must be shown.

Marks may not be awarded for careless or badly arranged work.

Approved calculators and templates may be used.

A list of standard integrals is provided at the end of the examination paper.

**Collection:**

Each question will be collected separately.

Start each question in a new answer booklet.

If you use a second booklet for a question, place it inside the first. Don't staple.

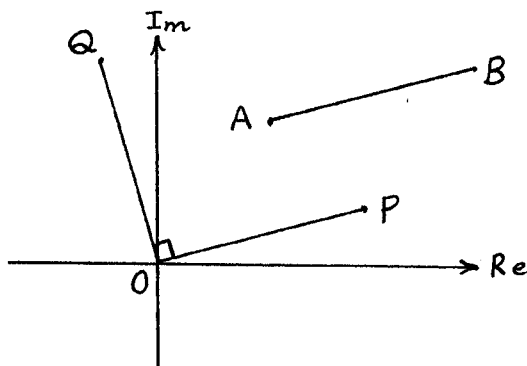
Write your candidate number on each answer booklet.

**QUESTION ONE** (Start a new answer booklet)

arks

- 2 (a) If  $w = 3 - 4i$  and  $z = 5 + 2i$ , find  $z(|w| - \bar{w})$ .
- 3 (b) Find the roots of the equation  $(2 + i)z^2 - 4z + (2 - i) = 0$ , expressing any complex roots in the form  $a + bi$ , where  $a$  and  $b$  are real.
- 3 (c) (i) Write  $-\sqrt{3} + i$  in modulus-argument form.  
 (ii) Hence find  $(-\sqrt{3} + i)^8$ , giving your answer in the form  $a + bi$ , where  $a$  and  $b$  are real.
- 4 (d) Sketch each of the following loci on separate Argand diagrams:  
 (i)  $\arg(z + 1 + i) = \frac{\pi}{4}$ ,  
 (ii)  $|z - 1| = |z + i|$ ,  
 (iii)  $z\bar{z} - i\bar{z} - iz = 0$ .

- 3 (e)



In the Argand diagram above, intervals  $AB$ ,  $OP$  and  $OQ$  are equal in length,  $OP$  is parallel to  $AB$  and angle  $POQ = \frac{\pi}{2}$ . If  $A$  and  $B$  represent the complex numbers  $3 + 5i$  and  $9 + 8i$  respectively, find the complex number which is represented by  $Q$ .

**QUESTION TWO** (Start a new answer booklet)

Marks

**4** (a) Find

(i)  $\int \frac{1}{x^2 - 6x + 8} dx,$

(ii)  $\int \frac{1}{\sqrt{x^2 - 6x + 8}} dx.$

**3** (b) By using the substitution  $t = \tan x$ , or otherwise, show that:

$$\int \operatorname{cosec} 2x \, dx = \frac{1}{2} \ln |\tan x| + c.$$

**3** (c) (i) Show that  $\int \frac{1 - x^2}{1 + x^2} dx = 2 \tan^{-1} x - x + c.$

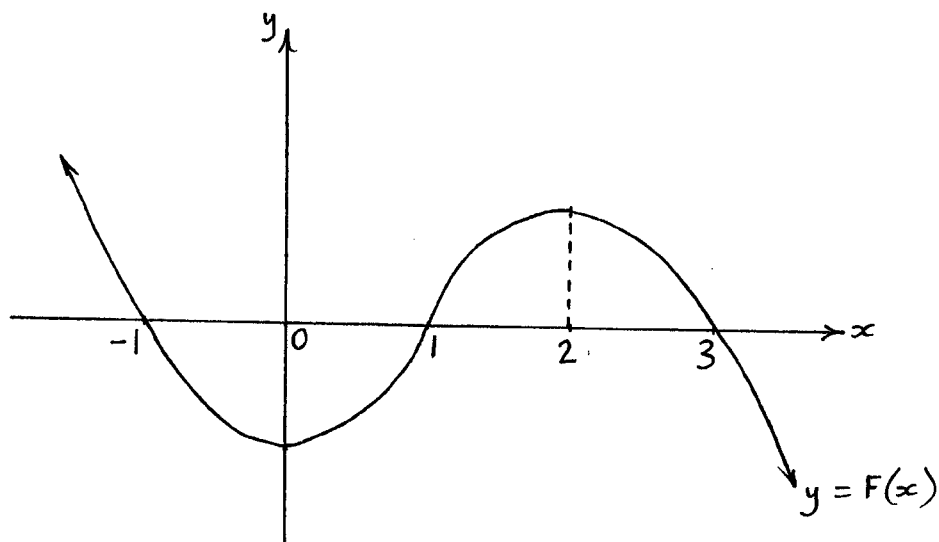
(ii) Use the substitution  $u = \cos x$  to find the exact value of  $\int_0^\pi \frac{\sin^3 x}{1 + \cos^2 x} dx.$

**3** (d) A solid has height  $H$ , and for every  $h$  between 0 and  $H$  the horizontal cross-section at height  $h$  above the base is a square, whose sides have length  $e^h$ . Find the volume of the solid.

QUESTION THREE (Start a new answer booklet)

arks

3 (a)



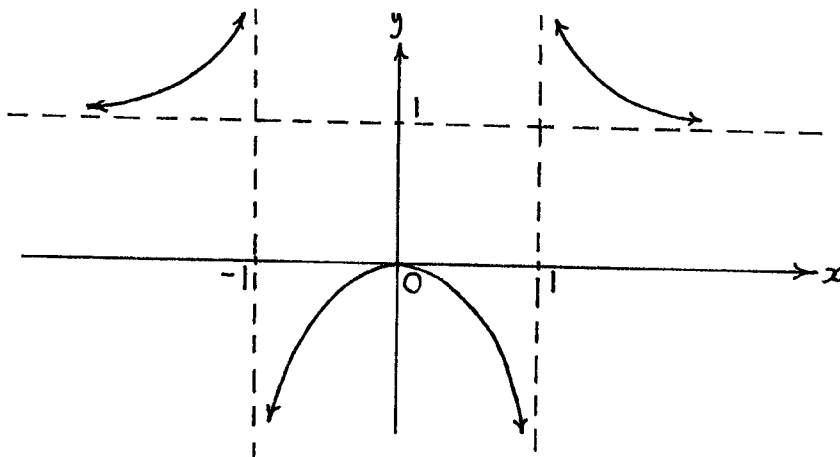
The graph of the function  $y = F(x)$  is sketched above. On separate number planes sketch the graphs of:

- (i)  $y = F(1 - x)$ ,
- (ii)  $y = F'(x)$ ,
- (iii)  $|y| = F(|x|)$ .

QUESTION THREE (Continued)

5 (b) (i) Solve the equation  $\frac{x^2}{x^2 - 1} = \frac{x^2 - 1}{x^2}$ .

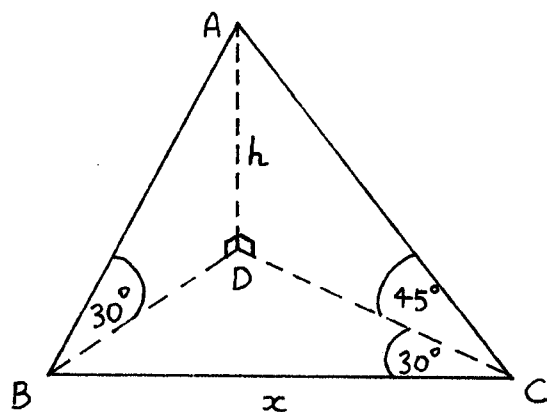
(ii)



The graph of the function  $f(x) = \frac{x^2}{x^2 - 1}$  is sketched above. Neatly copy this diagram into your answer booklet, and on it sketch the curve  $y = \frac{1}{f(x)}$ , showing the coordinates of any points of intersection of the two curves.

(iii) Hence find the values of  $x$  for which  $\frac{x^2}{x^2 - 1} < \frac{x^2 - 1}{x^2}$ .

6 (c)



$ABCD$  is a triangular pyramid with base  $BCD$  and perpendicular height  $AD$ . Use the cosine rule in triangle  $BCD$  to show that:

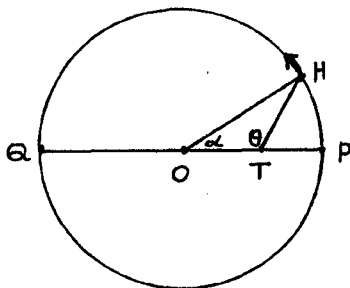
$$\frac{h}{x} = \frac{\sqrt{11} - \sqrt{3}}{4}.$$

**QUESTION FOUR** (Start a new answer booklet)

marks

- 3 (a) A particle  $P_1$  of mass  $m$  kg is dropped from point  $A$  and falls towards point  $B$ , which is directly underneath  $A$ . At the instant when  $P_1$  is dropped, a second particle  $P_2$ , also of mass  $m$  kg, is projected upwards from  $B$  towards  $A$  with an initial velocity equal to twice the terminal velocity of  $P_1$ . Each particle experiences a resistance of magnitude  $mkv$  as it moves, where  $v \text{ ms}^{-1}$  is the velocity and  $k$  is a constant.
- (i) Show that the terminal velocity of  $P_1$  is  $\frac{g}{k}$ , where  $g$  is acceleration due to gravity.
- (ii) For particle  $P_2$ , show that  $t = \frac{1}{k} \ln \left( \frac{3g}{g + kv} \right)$ , where  $v \text{ ms}^{-1}$  is the velocity after  $t$  seconds.
- (iii) Suppose the particles collide at the instant when  $P_1$  has reached 30% of its terminal velocity. Find the velocity of  $P_2$  when they collide. (Give your answer in terms of  $g$  and  $k$ .)

7 (b)



In the diagram above, a horse  $H$  is galloping with constant angular velocity  $0.2 \text{ rad/s}$  around a circular track. The radius  $OP$  is 50 metres, and there is a trainer standing at  $T$  on the diameter  $QP$ , where  $TP$  is 20 metres. Let  $\angle HOT = \alpha$  and let  $\angle HTO = \theta$ .

- (i) Show that  $30 \sin \theta = 50 \sin(\alpha + \theta)$ .
- (ii) By differentiating both sides of (i) with respect to  $t$ , show that the angular velocity of the horse about the trainer is given by  $\frac{\cos(\alpha + \theta)}{3 \cos \theta - 5 \cos(\alpha + \theta)}$ .
- Hint:  $\frac{d}{dt} \sin(\alpha + \theta) = \left( \frac{d\alpha}{dt} + \frac{d\theta}{dt} \right) \cos(\alpha + \theta)$ .
- (iii) Find the angular velocity of the horse about the trainer at the instant when the horse is at  $Q$ .

**QUESTION FIVE** (Start a new answer booklet)

- (a) The polynomial  $P(x)$  is defined by the rule  $P(x) = x^4 + Ax^2 + B$ , where  $A$  and  $B$  are positive real numbers.

(i) Explain why  $P(x)$  has no real zeros.

(ii) If the zeros of  $P(x)$  are  $bi$ ,  $-bi$ ,  $di$  and  $-di$ , where  $b$  and  $d$  are real, show that:

$$b^4 + d^4 = A^2 - 2B.$$

- (b) (i) Use integration by parts to show that:

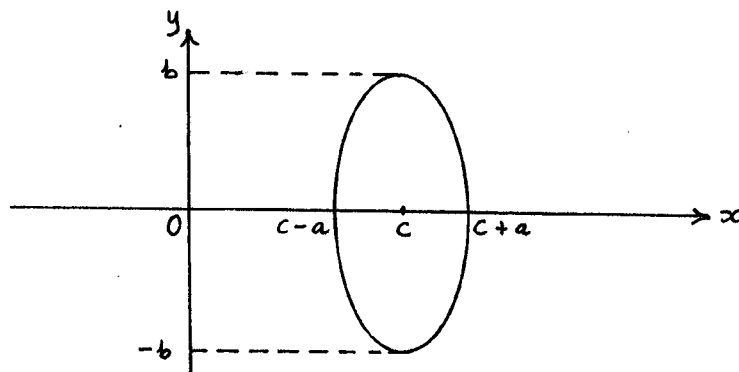
$$\int_0^1 x^{100}(1-x)^n dx = \frac{n}{101} \int_0^1 x^{101}(1-x)^{n-1} dx.$$

- (ii) If  $I_n = \int_0^1 x^{100}(1-x)^n dx$ , use part (i) to show that  $I_n = \frac{n}{101+n} I_{n-1}$ .

(iii) Hence show that  $I_n = \frac{n! 100!}{(101+n)!}$ .

- (c) (i) Sketch the curve  $x = \sqrt{b^2 - y^2}$ , and hence explain why  $\int_0^b \sqrt{b^2 - y^2} dy = \frac{\pi b^2}{4}$ .

(ii)



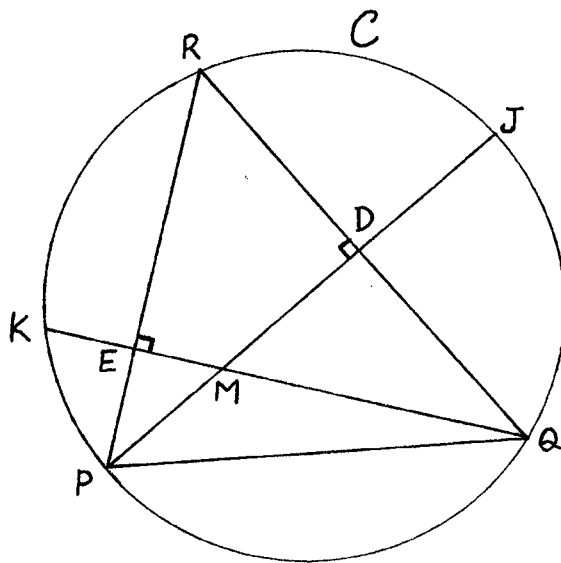
The ellipse  $\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $b > a$  and  $c > a$ , is sketched above. The region bounded by the ellipse is rotated about the  $y$  axis to form a ring. By taking slices perpendicular to the  $y$  axis, show that the ring has volume  $2abc\pi^2$ .

**QUESTION SIX** (Start a new answer booklet)

ks

- (a) (i) Use de Moivre's theorem to express  $\cos 4\theta$  and  $\sin 4\theta$  in terms of  $\cos \theta$  and  $\sin \theta$ .
- (ii) Hence show that  $\tan 4\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$ .
- (iii) Given  $\alpha = \tan^{-1} \frac{1}{4}$ , show that  $4\alpha = \tan^{-1} \frac{240}{161}$ .
- (iv) Write  $161 + 240i$  in the form  $r(\cos \theta + i \sin \theta)$ , expressing  $\theta$  in terms of  $\alpha$ .
- (v) Hence find, in the form  $a + bi$  where  $a$  and  $b$  are integers, the four fourth roots of  $161 + 240i$ .

(b)



In the diagram,  $PQR$  is a triangle inscribed in the circle  $C$ . The altitude  $PD$  is produced to meet  $C$  at  $J$ , the altitude  $QE$  is produced to meet  $C$  at  $K$  and these two altitudes intersect at  $M$ .

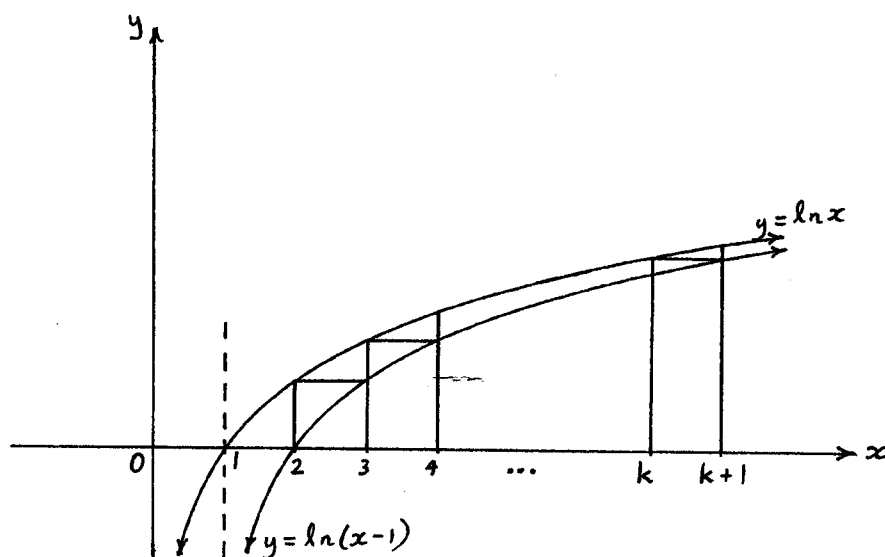
- (i) Explain why the quadrilaterals  $PQDE$  and  $REMD$  are cyclic.
- (ii) Show that  $PR$  bisects angle  $KRM$ .
- (iii) Hence, or otherwise, show that  $KR = JR$ .



**QUESTION SEVEN** (Start a new answer booklet)

Marks

**6** (a)

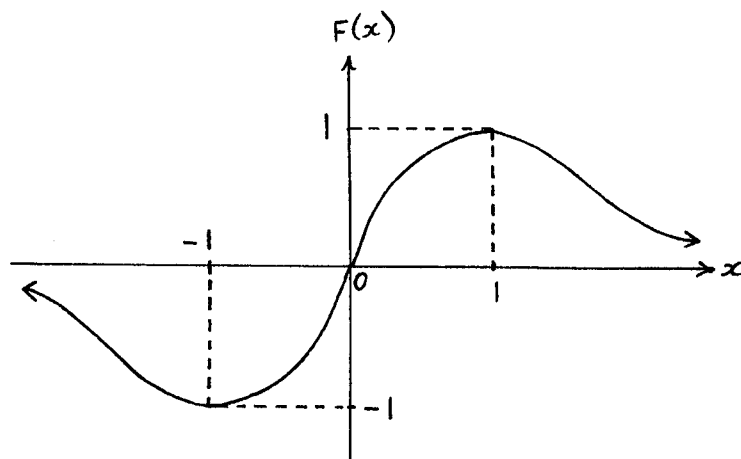


In the diagram above, the curves  $y = \ln x$  and  $y = \ln(x-1)$  are sketched and  $k-1$  rectangles are constructed between  $x=2$  and  $x=k+1$ , where  $k \geq 2$ .  
Let  $S = \ln 2 + \ln 3 + \ln 4 + \dots + \ln k$ .

- (i) Explain why  $S$  represents the sum of the areas of the  $k-1$  rectangles.
- (ii) Show that  $\int_2^{k+1} \ln(x-1) dx = k \ln k - k + 1$ .
- (iii) Hence show that  $k^k < k! e^{k-1} < \frac{1}{4}(k+1)^{k+1}$ , where  $k \geq 2$ .

QUESTION SEVEN (Continued)

(b)



The curve  $F(x) = \frac{2x}{1+x^2}$  is sketched above.

- (i) State the range of  $F(x)$ .
- (ii) Let  $x_0$  be a real number not equal to 1 or  $-1$  and consider the sequence of real numbers defined by  $x_{n+1} = F(x_n)$  for  $n = 0, 1, 2, \dots$ 
  - ( $\alpha$ ) Show that there exists a real number  $r$  such that  $x_1 = G(r)$ ,  
where  $G(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$ .
  - ( $\beta$ ) Show that  $G(2x) = \frac{2G(x)}{1 + (G(x))^2}$ .
  - ( $\gamma$ ) Hence, using parts ( $\alpha$ ) and ( $\beta$ ), show that  $x_n = G(2^{n-1}r)$ , for  $n = 1, 2, 3, \dots$

**QUESTION EIGHT** (Start a new answer booklet)

marks

- 1** (a) (i) Suppose  $f(x) = \sqrt{1+x}$ . Show that  $f'(x) < \frac{1}{6}$  for  $x > 8$ .  
 (ii) Using part (i), or otherwise, show that  $\sqrt{1+x} \leq 3 + \frac{x-8}{6}$  when  $x \geq 8$ .
- 11** (b) Suppose  $z_1, z_2, z_3$  and  $z_4$  are distinct complex numbers. The cross-ratio  $(z_1, z_2; z_3, z_4)$  of these four complex numbers is defined by:

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

(i) Show that:

$$(\alpha) (z_1, z_2; z_4, z_3) = \frac{1}{(z_1, z_2; z_3, z_4)},$$

$$(\beta) (z_1, z_3; z_2, z_4) = 1 - (z_1, z_2; z_3, z_4).$$

- (ii) Suppose  $f(z) = \frac{c^2}{z}$ , where  $c$  is a positive constant and  $z$  is a complex variable.  
 Show that:

$$(\alpha) |zf(z)| = c^2,$$

$$(\beta) \arg f(z) = \arg z.$$

- (iii) If  $f(z_1) = w_1, f(z_2) = w_2, f(z_3) = w_3$  and  $f(z_4) = w_4$ , then show that:

$$(w_1, w_2; w_3, w_4) = \overline{(z_1, z_2; z_3, z_4)}.$$

- (iv) Suppose  $z_1, z_2, z_3$  and  $z_4$  represent (in cyclic order) points in the Argand diagram which lie on a circle.

( $\alpha$ ) Show that  $(z_1, z_3; z_4, z_2)$  is a negative real number.

( $\beta$ ) Hence deduce that  $(w_1, w_2; w_3, w_4)$  is a real number greater than one.

( $\gamma$ ) Show that the points  $w_1, w_2, w_3$  and  $w_4$  are either concyclic or collinear, and find the condition on the original circle for  $w_1, w_2, w_3$  and  $w_4$  to be collinear.

DS

UKAMMAYAK PMIXX.  
SOLUTIONS TO 1997 4U TRIAL 97 (15 marks per question)

Question 1:  $2 + 3 + (1+2) + (1+1+2) + 3$

(a)  $(5+2i)(5-(3+4i))$  ✓  
 $= (5+2i)(2-4i)$   
 $= 18-16i$  ✓

(b)  $\Delta = 16 - 4(2+i)(2-i)$   
 $= 16 - 4(5)$   
 $= -4$  ✓

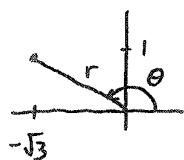
$z = \frac{4 \pm 2i}{2(2+i)}$

$= \frac{2 \pm i}{2+i}$  ✓

$= 1$  or  $\frac{2-i}{2+i} \cdot \frac{2-i}{2-i}$

$= 1$  or  $\frac{3}{5} - \frac{4}{5}i$  ✓

(c)(i)



$r = 2$   
 $\theta = \frac{5\pi}{6}$

$-\sqrt{3} + i = 2 \operatorname{cis} \frac{5\pi}{6}$  ✓

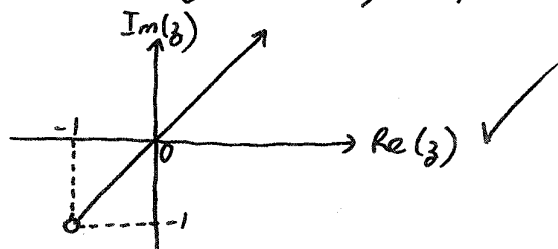
(ii)  $(-\sqrt{3} + i)^8 = 2^8 \operatorname{cis} \frac{40\pi}{6}$  ✓ (by De Moivre's thm)

$= 2^8 \operatorname{cis} \frac{2\pi}{3}$

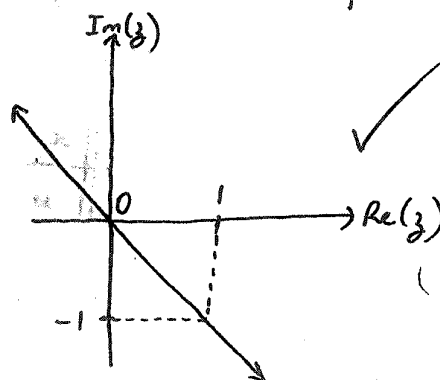
$= 256 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -128 + 128\sqrt{3}i$  ✓

(d)(i)  $\arg(z - (-1-i)) = \frac{\pi}{4}$



(ii)  $|z-1| = |z-(-i)|$

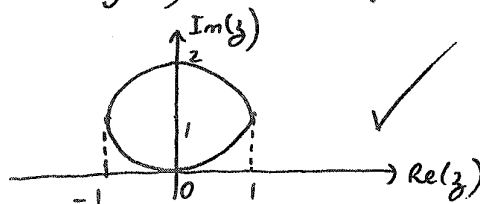


(iii) Let  $z = x+iy$

$\therefore (x+iy)(x-iy) - i(x-iy) - (-i)(x+iy) = 0$

$x^2 + y^2 - ix - y + ix - y = 0$

$x^2 + (y-1)^2 = 1$  ✓



(e)  $\vec{AB}$  represents  $(9+8i) - (3+5i)$  ✓  
 $= 6+3i$

So P represents  $6+3i$  ✓

So Q represents  $i(6+3i)$  ✓  
 $= -3+6i$

Q. 2:

$$(2+2)+3+(2+3)+3$$

(a) (i)

$$\text{Let } \frac{1}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$$

$$\therefore 1 = A(x-4) + B(x-2)$$

$$\text{Let } x=2$$

$$\therefore A = -\frac{1}{2}$$

$$\text{Let } x=4$$

$$\therefore B = \frac{1}{2}$$

So we have

$$\frac{1}{2} \int \left( \frac{-1}{x-2} + \frac{1}{x-4} \right) dx$$

$$= \frac{1}{2} \ln \left| \frac{x-4}{x-2} \right| + c$$

(no penalty for omission of "c")

$$(ii) \int \frac{1}{\sqrt{(x-3)^2 - 1}} dx$$

$$= \ln |x-3 + \sqrt{x^2 - 6x + 8}| + c$$

$$(b) \quad t = \tan x$$

$$\therefore dt = \sec^2 x \, dx$$

$$= (1 + \tan^2 x) dx$$

$$\therefore dx = \frac{dt}{1+t^2}$$

$$\text{Also } \operatorname{cosec} 2x = \frac{1+t^2}{2t}$$

$$\text{So } \int \operatorname{cosec} 2x \, dx = \int \frac{1+t^2}{2t} \cdot \frac{dt}{1+t^2}$$

$$= \frac{1}{2} \int \frac{1}{t} dt$$

$$= \frac{1}{2} \ln |t| + c$$

$$= \frac{1}{2} \ln |\tan x| + c$$

(c) (i)

$$\int \frac{1-x^2}{1+x^2} dx = \int \frac{-(1+x^2) + 2}{1+x^2} dx$$

$$= \int \left( -1 + \frac{2}{1+x^2} \right) dx$$

$$= 2 \tan^{-1} x - x + c$$

$$(ii) \quad u = \cos x$$

$$\therefore du = -\sin x \, dx$$

$$\therefore \sin x \, dx = -du$$

|     |     |       |
|-----|-----|-------|
| $x$ | $0$ | $\pi$ |
| $u$ | $1$ | $-1$  |

$$\int_0^\pi \frac{\sin^3 x}{1+\cos^2 x} dx = \int_1^{-1} \frac{1-u^2}{1+u^2} \cdot -du$$

$$= \left[ 2 \tan^{-1} u - u \right]_{-1}^1$$

(using (i))

$$= 2\left(\frac{\pi}{4}\right) - 1 - \left(2\left(-\frac{\pi}{4}\right) + 1\right)$$

$$= \frac{\pi}{2} - 1 + \frac{\pi}{2} - 1$$

$$= \pi - 2$$

(d) A typical slice has volume

$$\delta V = (e^h)^2 \delta h$$

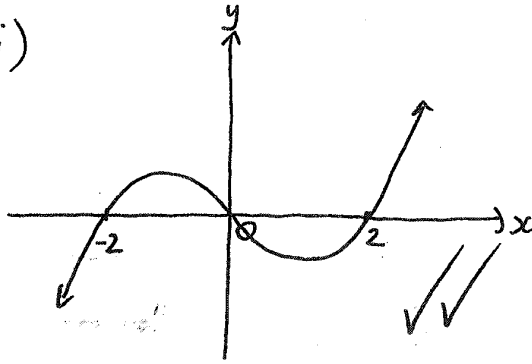
$$\text{So } V = \int_0^H e^{2h} dh$$

$$= \frac{1}{2} [e^{2h}]_0^H$$

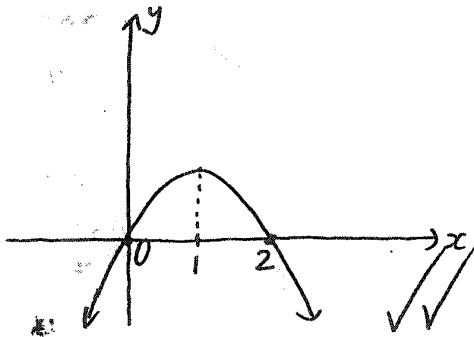
$$= \frac{1}{2} (e^{2H} - 1)$$

on 3:  $(2+2+2) + (1+2+2) + 4$

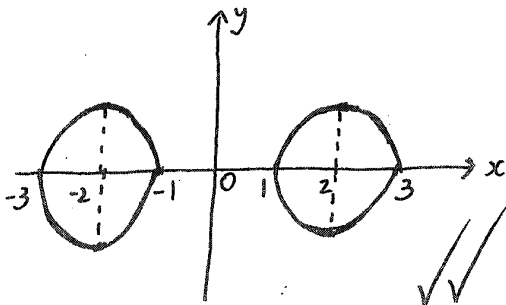
(a)(i)



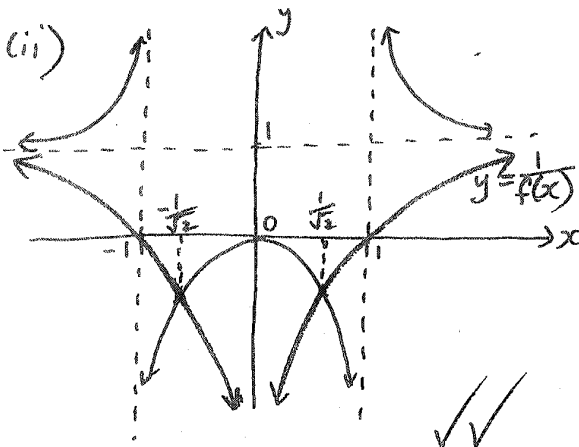
(ii)



(iii)



(b)(i)  $x^4 = x^4 - 2x^2 + 1$   
 $x = \pm \frac{1}{\sqrt{2}}$  } ✓



(iii)  $-1 < x < -\frac{1}{\sqrt{2}}$  or  $\frac{1}{\sqrt{2}} < x < 1$ .  
 (i.e.  $\frac{1}{\sqrt{2}} < |x| < 1$ ). ✓✓

(c) In  $\triangle ABD$ ,

$$\frac{h}{BD} = \tan 30^\circ$$

So  $BD = \sqrt{3}h$ .

In  $\triangle ACD$ ,

$$\frac{h}{CD} = \tan 45^\circ$$

So  $CD = h$ .

In  $\triangle BCD$ , by the cosine rule,

$$3h^2 = h^2 + x^2 - 2hx \cos 30^\circ$$

$$2h^2 = x^2 - \sqrt{3}hx$$

$$2\left(\frac{h}{x}\right)^2 + \sqrt{3}\left(\frac{h}{x}\right) - 1 = 0 \quad (x \neq 0)$$

By the quadratic formula,

$$\frac{h}{x} = \frac{-\sqrt{3} \pm \sqrt{3+8}}{4}$$

But  $\frac{h}{x} > 0$ ,

$$\text{so } \frac{h}{x} = \frac{\sqrt{11} - \sqrt{3}}{4}$$

on 4:  $(1+3+4) + (2+3+2)$

(i) For  $P_1$ ,

$$\ddot{x} = g - kv$$

Terminal velocity is achieved when  $\ddot{x} = 0$ .

i.e. when  $0 = g - kv$ .

$\therefore$  terminal velocity  $= \frac{g}{k}$

(ii) For  $P_2$ ,

$$\frac{dv}{dt} = -g - kv$$

$$t = \int \frac{-1}{g+kv} dv$$

$$= -\frac{1}{k} \ln(g+kv) + C_1$$

When  $t=0$ ,  $v = \frac{2g}{k}$

$$\therefore C_1 = \frac{1}{k} \ln(3g)$$

$$\therefore t = \frac{1}{k} \ln\left(\frac{3g}{g+kv}\right)$$

(iii) For  $P_1$ ,

$$\frac{dv}{dt} = g - kv$$

$$t = \int \frac{1}{g-kv} dv$$

$$= -\frac{1}{k} \ln(g-kv) + C_2$$

When  $t=0$ ,  $v=0$

$$\therefore C_2 = \frac{1}{k} \ln g$$

$$\therefore t = \frac{1}{k} \ln\left(\frac{g}{g-kv}\right)$$

When the particles collide,

$P_1$  has velocity  $\frac{3g}{10k}$

$$\therefore t = \frac{1}{k} \ln\left(\frac{g}{g - \frac{3g}{10}}\right)$$

$$= \frac{1}{k} \ln \frac{10}{7}$$

Using the result in (ii), we get

$$\frac{1}{k} \ln \frac{10}{7} = \frac{1}{k} \ln\left(\frac{3g}{g+kv}\right)$$

$$\therefore \frac{10}{7} = \frac{3g}{g+kv}$$

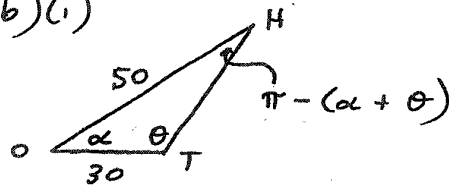
$$10g + 10kv = 21g$$

$$10kv = 11g$$

$$v = \frac{11g}{10k}$$

So at the instant the particles collide,  $P_2$  has velocity  $\frac{11g}{10k} \text{ ms}^{-1}$ .

(b)(i)



By the sine rule,

$$\frac{30}{\sin(\pi - (\alpha + \theta))} = \frac{50}{\sin \theta}$$

$$\therefore 30 \sin \theta = 50 \sin(\alpha + \theta)$$

$$(ii) 30 \cos \theta \cdot \dot{\theta} = 50 \cos(\alpha + \theta) \cdot (\dot{\alpha} + \dot{\theta})$$

We are given  $\dot{\alpha} = 0.2$ ,

$$\text{so } 30 \cos \theta \cdot \dot{\theta} = 10 \cos(\alpha + \theta)$$

$$+ 50 \cos(\alpha + \theta) \cdot \dot{\theta}$$

$$\text{so } \dot{\theta} (30 \cos \theta - 50 \cos(\alpha + \theta)) = 10 \cos(\alpha + \theta)$$

$$\text{so } \dot{\theta} = \frac{\cos(\alpha + \theta)}{3 \cos \theta - 5 \cos(\alpha + \theta)}$$

(iii) When the horse is at Q,

$$\alpha = \pi \text{ and } \theta = 0.$$

$$\therefore \dot{\theta} = \frac{\cos \pi}{3 \cos 0 - 5 \cos \pi}$$

$$= -\frac{1}{8} \text{ rad/s}$$

on 5:

$$(2+3) + (1+2+2) + (1+4)$$

(i) For real values of  $x$ , both  $x^4$  and  $x^2$  are  $\geq 0$ . Since  $A > 0$ ,  $P(x)$  has minimum value  $B$  (when  $x=0$ ). Since  $B > 0$ , it follows that  $P(x)$  is never zero.

(Other explanations are possible.)

(ii)

$A =$  sum of roots in pairs

$$= b^2 - bd + bd + bd - bd + d^2$$

$$= b^2 + d^2.$$

$B =$  product of roots

$$= b^2 d^2.$$

$$\text{So } b^4 + d^4 = (b^2 + d^2)^2 - 2b^2 d^2 = A^2 - 2B, \text{ as required.}$$

$$(b)(i) \int_0^1 x^{100} (1-x)^n dx$$

$$= \int_0^1 (1-x)^n \cdot \frac{d}{dx} \left( \frac{x^{101}}{101} \right) dx$$

$$= \left[ \frac{x^{101}}{101} (1-x)^n \right]_0^1 - \int_0^1 \frac{x^{101}}{101} \cdot n(1-x)^{n-1} dx$$

$$= \frac{n}{101} \int_0^1 x^{101} (1-x)^{n-1} dx$$

$$(ii) I_n = \frac{n}{101} \int_0^1 x^{101} (1-x)^{n-1} dx$$

$$= \frac{-n}{101} \int_0^1 \{ x^{100} (1-x)(1-x)^{n-1} - x^{100} (1-x)^{n-1} \} dx$$

$$= \frac{-n}{101} (I_n - I_{n-1})$$

$$\text{So } 101 \cdot I_n + n \cdot I_n = n \cdot I_{n-1}$$

$$\text{So } I_n = \frac{n}{101+n} I_{n-1}$$

(iii) Using (ii),

$$I_n = \frac{n}{101+n} \cdot \frac{n-1}{100+n} \cdot \frac{n-2}{99+n} \cdots \frac{1}{102} I_0$$

$$\text{where } I_0 = \int_0^1 x^{100} dx = \frac{1}{101}$$

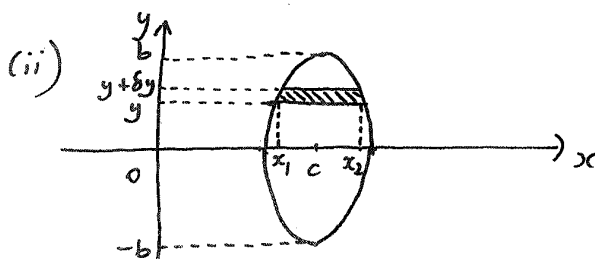
$$\text{So } I_n = \frac{n}{101+n} \cdot \frac{n-1}{100+n} \cdots \frac{1}{101} \cdot \frac{100!}{100!}$$

$$= \frac{n! 100!}{(101+n)!}, \text{ as required.}$$

(c)(i)  $\int_0^b \sqrt{b^2 - y^2} dy$  represents area of quadrant of radius  $b$ . ( $\frac{1}{4}$  circle)



$$\therefore \int_0^b \sqrt{b^2 - y^2} dy = \frac{\pi b^2}{4}$$



Make  $x$  the subject of the equation:

$$(x-c)^2 = \frac{a^2}{b^2} (b^2 - y^2)$$

$$x = c \pm \frac{a}{b} \sqrt{b^2 - y^2}$$

$$\text{So } x_2 + x_1 = 2c, x_2 - x_1 = \frac{2a}{b} \sqrt{b^2 - y^2}$$

Area of typical cross-section is

$$\pi x_2^2 - \pi x_1^2 = \pi \cdot 2c \cdot \frac{2a}{b} \sqrt{b^2 - y^2}$$

$$\text{So } V = \frac{4ac\pi}{b} \cdot 2 \int_0^b \sqrt{b^2 - y^2} dy$$

$$= \frac{8ac\pi}{b} \cdot \frac{\pi b^2}{4} \text{ (using (i))}$$



$$\sin 6 : (2+1+1+1+3) + (2+2+3)$$

$$(a)(i) \cos 4\theta + i \sin 4\theta$$

$$= (\cos \theta + i \sin \theta)^4 \quad \left( \begin{array}{l} \text{by} \\ \text{De-Moivre's} \\ \text{thm} \end{array} \right)$$

$$= \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta$$

Equating real and imaginary parts,

$$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$$

$$\text{and } \sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$$

(ii)

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$

$$= \frac{4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta}{\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta}$$

$$= \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

after dividing numerator and denominator by  $\cos^4 \theta$ .

$$(iii) \quad \alpha = \tan^{-1} \frac{1}{4}$$

$$\therefore \tan \alpha = \frac{1}{4}, \quad 0 < \alpha < \frac{\pi}{8}$$

$$\therefore \tan 4\alpha = \frac{4\left(\frac{1}{4}\right) - 4\left(\frac{1}{4}\right)^3}{1 - 6\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^4}$$

$$= \frac{256 - 16}{256 - 96 + 1}$$

(after multiplying numerator and denominator by 256)

$$= \frac{240}{161}$$

$$\therefore 4\alpha = \tan^{-1} \frac{240}{161} \quad \left( \begin{array}{l} \text{since} \\ 0 < 4\alpha < \frac{\pi}{2} \end{array} \right)$$

$$(iv) \quad |161 + 240i| = 289$$

$$\arg(161 + 240i)$$

$$= \tan^{-1} \frac{240}{161}$$

$$= 4\alpha$$

$$= 4 \tan^{-1} \frac{1}{4}$$

$$\text{So } 161 + 240i = 289 \text{cis } 4\alpha$$

$$(v) \quad \text{Let } z^4 = 161 + 240i$$

$$= 289 \text{cis}(4\alpha + 2\pi n),$$

where  $n \in \mathbb{Z}$ .

$$\therefore z = \sqrt[4]{289} \text{cis}\left(\alpha + \frac{\pi n}{2}\right), \quad \text{by De-Moivre's thm.}$$

Choosing  $n = -1, 0, 1, 2$ ,

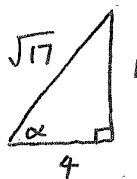
we get

$$z = \sqrt[4]{289} \text{cis}\left(\alpha - \frac{\pi}{2}\right),$$

$$\sqrt[4]{289} \text{cis } \alpha,$$

$$\sqrt[4]{289} \text{cis}\left(\alpha + \frac{\pi}{2}\right)$$

$$\text{or } \sqrt[4]{289} \text{cis}(\alpha + \pi).$$



$$\sin \alpha = \frac{1}{\sqrt{17}},$$

$$\cos \alpha = \frac{4}{\sqrt{17}}.$$

So the four fourth roots are:

$$\sqrt[4]{289} \left( \frac{1}{\sqrt{17}} - \frac{4}{\sqrt{17}}i \right), \sqrt[4]{289} \left( \frac{4}{\sqrt{17}} + \frac{1}{\sqrt{17}}i \right),$$

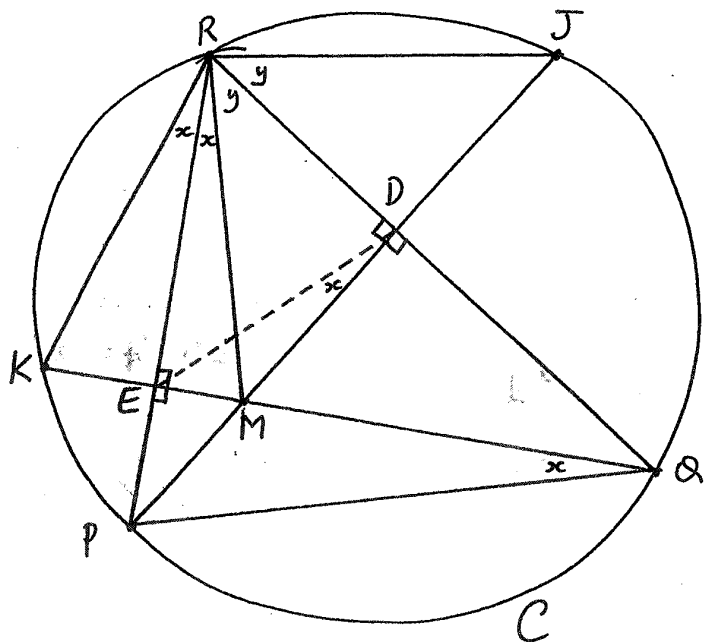
$$\sqrt[4]{289} \left( -\frac{1}{\sqrt{17}} + \frac{4}{\sqrt{17}}i \right), \sqrt[4]{289} \left( -\frac{4}{\sqrt{17}} - \frac{1}{\sqrt{17}}i \right).$$

i.e. the roots are

$$1 - 4i, 4 + i, -1 + 4i, -4 - i.$$

(b)

(b)



(i)  $\angle PEQ = \angle PDQ = \frac{\pi}{2}$  (given).

So quad PQDE is cyclic (converse of angles in semicircle)

$$\text{Also, } \angle REM + \angle RDM = \frac{\pi}{2} + \frac{\pi}{2} \text{ (given)}$$

So quad REMD is cyclic (opposite angles supplementary)

(ii) Join DE.

Let  $\angle KRP = x$ .

$$\therefore \angle KQP = x \text{ (angles at circumference standing on same arc)}$$
$$\therefore \angle EDP = x$$
$$\therefore LERM = x \quad ($$
$$\therefore L_{KRP} = L_{PRM}$$

$\therefore PR$  bisects  $\angle KRM$ .

(iii) By a similar argument to (ii),  $QR$  bisects  $\angle JRM$ .

So  $LMRD = LJRD$ .

So  $\Delta KRE \equiv \Delta MRE$  (AAS) and  $\Delta MAD \equiv \Delta JRD$  (AAS)

So  $KR = MR = JR$  (matching sides of congruent triangles)

Question 7:

$$(1+2+3) + (1+(3+2+3))$$

(a)(i) Each rectangle has width one unit, and the heights of the rectangles are  $\ln 2, \ln 3, \ln 4, \dots, \ln k$ .

So  $S$  represents the sum of the areas of the  $k-1$  rectangles.

$$(ii) \int_2^{k+1} \ln(x-1) dx$$

$$= \int_2^{k+1} \ln(x-1) \cdot \frac{d}{dx}(x-1) dx$$

$$= \left[ (x-1) \ln(x-1) \right]_2^{k+1} - \int_2^{k+1} \frac{x-1}{x-1} dx$$

$$= k \ln k - \ln 1 - [x]_2^{k+1}$$

$$= k \ln k - k + 1, \text{ as required.}$$

$$(iii) \int_2^{k+1} \ln x dx$$

$$= \int_2^{k+1} \ln x \cdot \frac{d}{dx}(x) dx$$

$$= [x \ln x]_2^{k+1} - \int_2^{k+1} \frac{x}{x} dx$$

$$= (k+1) \ln(k+1) - 2 \ln 2 - [x]_2^{k+1}$$

$$= (k+1) \ln(k+1) - \ln 4 - k + 1$$

Now,

$$\int_2^{k+1} \ln(x-1) dx < S < \int_2^{k+1} \ln x dx$$

$$So k \ln k - k + 1 < S < (k+1) \ln(k+1) - \ln 4 - k + 1$$

$$So k \ln k - k + 1 < \ln k! < (k+1) \ln(k+1) - \ln 4 - k + 1$$

$$So \ln k^k < \ln k! + k - 1 < \ln \left( \frac{(k+1)^{k+1}}{4} \right)$$

$$So k^k < e^{\ln k! + k - 1} < \frac{1}{4} (k+1)^{k+1}$$

$$So k^k < e^{\ln k!} e^{k-1} < \frac{1}{4} (k+1)^{k+1}$$

$$So k^k < k! e^{k-1} < \frac{1}{4} (k+1)^{k+1}, \text{ as required.}$$

$$(b)(i) -1 \leq F(x) \leq 1$$

$$(ii)(\alpha) x_0 \neq \pm 1$$

$$So -1 < F(x_0) < 1$$

$$\text{i.e. } -1 < x_1 < 1$$

$$\text{If } x_1 = G(r),$$

$$\text{then } x_1 = \frac{e^{2r} - 1}{e^{2r} + 1}$$

$$x_1 e^{2r} + x_1 = e^{2r} - 1$$

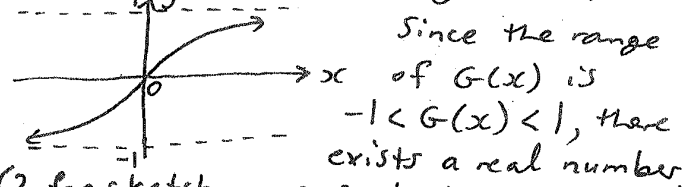
$$e^{2r} = \frac{1+x_1}{1-x_1}$$

$$r = \frac{1}{2} \ln \left( \frac{1+x_1}{1-x_1} \right)$$

Now,  $1+x_1$  and  $1-x_1$  are both positive, since  $-1 < x_1 < 1$ .

So  $r$  exists, since  $\ln \left( \frac{1+x_1}{1-x_1} \right)$  is defined when  $\frac{1+x_1}{1-x_1} > 0$ .

Alternatively, sketch  $y = G(x)$ :



(b) Sketch

Question 7: (CONTINUED)

$$(b)(ii)(\beta) \text{ RHS} = \frac{2(e^{2x}-1)}{e^{2x}+1} \div \left(1 + \left(\frac{e^{2x}-1}{e^{2x}+1}\right)^2\right)$$

$$= \frac{2(e^{2x}-1)}{e^{2x}+1} \times \frac{(e^{2x}+1)^2}{(e^{2x}+1)^2 + (e^{2x}-1)^2} \checkmark$$

$$= \frac{2(e^{2x}-1)(e^{2x}+1)}{2(e^{4x}+1)} \left. \begin{array}{l} \checkmark \\ \end{array} \right\}$$

$$= \frac{e^{4x}-1}{e^{4x}+1}$$

$$= G(2x)$$

$$= \text{LHS.}$$

(8) From part (d), the result is true when  $n=1$ .

\* Suppose  $x_k = G(2^{k-1}r)$ , where  $k$  is a positive integer.

$$x_{k+1} = F(x_k)$$

$$= \frac{2x_k}{1+x_k^2} \left. \begin{array}{l} \checkmark \\ \end{array} \right\}$$

$$= \frac{2G(2^{k-1}r)}{1+(G(2^{k-1}r))^2}$$

$$= G(2 \cdot 2^{k-1}r), \text{ by part } (\beta)$$

$$= G(2^k r).$$

So the result is true for  $n=k+1$  if it is true for  $n=k$ .

But the result is true for  $n=1$ .

So by mathematical induction, it is true for all positive integers  $n$ .

for 8:

$$2) + ((1+1) + (1+1) + 1 + (2+2+2))$$

$$(a)(i) f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$< \frac{1}{2\sqrt{1+8}} \quad (\text{since } x > 8)$$

$$= \frac{1}{6}$$

$$\text{So } f'(x) < \frac{1}{6} \text{ when } x > 8.$$

$$(ii) \text{ Let } g(x) = 3 + \frac{x-8}{6}$$

$$\text{Now, } g'(x) = \frac{1}{6} \text{ for all } x$$

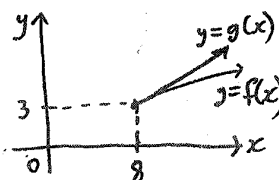
$$\text{Also, } f(8) = 3 = g(8).$$

$$\text{So we have } f(8) = g(8)$$

$$\text{and } f'(x) < g'(x) \text{ when } x > 8.$$

$$\text{It follows that } f(x) \leq g(x)$$

$$\text{for all } x > 8.$$



$$(b)(i)(\alpha)$$

$$\text{LHS} = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)}$$

$$(z_1 - z_3)(z_2 - z_4)$$

$$= \frac{1}{(z_1 - z_3)(z_2 - z_4)}$$

$$(z_1 - z_3)(z_2 - z_4)$$

$$(z_1 - z_4)(z_2 - z_3)$$

$$= \frac{1}{(z_1, z_2; z_3, z_4)}$$

$$(z_1, z_2; z_3, z_4)$$

$$= \text{RHS.}$$

$$(B)$$

$$\text{RHS} = 1 - \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

$$(z_1 - z_4)(z_2 - z_3)$$

$$= \frac{z_1 z_2 - z_1 z_3 - z_2 z_4 + z_3 z_4 - z_1 z_2 + z_1 z_3 + z_2 z_4 - z_3 z_4}{(z_1 - z_4)(z_2 - z_3)}$$

$$= \frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$$

$$(z_1 - z_4)(z_2 - z_3)$$

$$= \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

$$(z_1 - z_4)(z_3 - z_2)$$

$$= (z_1, z_3; z_2, z_4)$$

$$= \text{LHS.}$$

$$(ii)(\alpha)$$

$$|z \cdot f(z)| = |z \cdot \frac{c^2}{z}|$$

$$= |z| \cdot \left| \frac{c^2}{z} \right|$$

$$= |z| \cdot \frac{c^2}{|z|} \quad (\text{since } |\frac{c^2}{z}| = \frac{c^2}{|z|})$$

$$= c^2, \text{ as required.}$$

$$(B) \arg f(z) = \arg \left( \frac{c^2}{z} \right)$$

$$= \arg c^2 - \arg z$$

$$= 0 - (-\arg z)$$

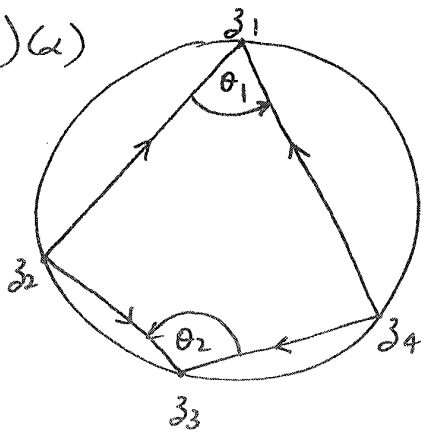
$$= \arg z, \text{ as required.}$$

Question 8 : (CONTINUED)

(b)(iii)

$$\begin{aligned}
 \text{LHS} &= (w_1, w_2; w_3, w_4) \\
 &= \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} \\
 &= \frac{\left(\frac{c^2}{\bar{z}_1} - \frac{c^2}{\bar{z}_3}\right)\left(\frac{c^2}{\bar{z}_2} - \frac{c^2}{\bar{z}_4}\right)}{\left(\frac{c^2}{\bar{z}_1} - \frac{c^2}{\bar{z}_4}\right)\left(\frac{c^2}{\bar{z}_2} - \frac{c^2}{\bar{z}_3}\right)} \\
 &= \frac{(\bar{z}_3 - \bar{z}_1)(\bar{z}_4 - \bar{z}_2)}{(\bar{z}_4 - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)} \checkmark \\
 &= \frac{(\overline{z_1 - z_3})(\overline{z_2 - z_4})}{(\overline{z_1 - z_4})(\overline{z_2 - z_3})} \\
 &= \overline{(z_1, z_2; z_3, z_4)} = \text{RHS}.
 \end{aligned}$$

(iv)(a)



$$\arg(z_1, z_3; z_4, z_2)$$

$$\begin{aligned}
 &= \left( \arg(z_1 - z_4) - \arg(z_1 - z_2) \right) \\
 &\quad + \left( \arg(z_3 - z_2) - \arg(z_3 - z_4) \right)
 \end{aligned}$$

$$= \theta_1 + \theta_2$$

$= \pi$  (since the opposite angles of a cyclic quadrilateral are supplementary.)

(Note  $\theta_1 + \theta_2 = -\pi$  if the 4 points run clockwise.)  
So  $(z_1, z_3; z_4, z_2)$  is a negative real number.  $\checkmark \checkmark$

$$(B) \quad (w_1, w_2; w_3, w_4)$$

$$= \overline{(z_1, z_2; z_3, z_4)}, \text{ by (iii).}$$

$$\text{Now, } (z_1, z_3; z_4, z_2)$$

$$= \frac{1}{(z_1, z_3; z_2, z_4)}, \text{ by (i)(a)}$$

$$= \frac{1}{1 - (z_1, z_2; z_3, z_4)}, \text{ by (i)(B).}$$

So  $(z_1, z_2; z_3, z_4)$  is a real number greater than one, since, by (iv)(a),  $(z_1, z_3; z_4, z_2)$  is a negative real number.  $\checkmark \checkmark$

So  $\overline{(z_1, z_2; z_3, z_4)}$  is a real number greater than one.

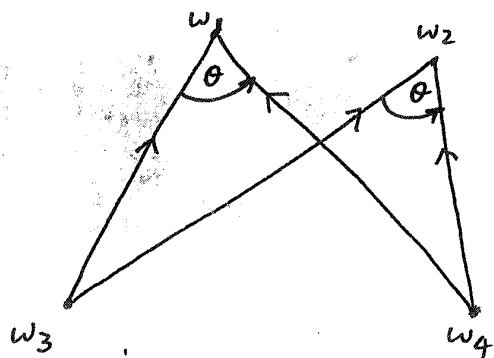
(Since the conjugate of a real number is itself.)

So  $(w_1, w_2; w_3, w_4)$  is a real number greater than one.

# on 8: (CONTINUED)

(iv)(8) Since, by (iv)(β),  $(w_1, w_2; w_3, w_4)$  is a real number greater than one,  $\arg(w_1, w_2; w_3, w_4) = 0$ .

So  $\arg(w_1 - w_4) - \arg(w_1 - w_3) = \arg(w_2 - w_4) - \arg(w_2 - w_3) = \theta$ , say.



By the converse of the angles standing on the same arc theorem,  $w_1, w_2, w_3$  and  $w_4$  are concyclic, except when  $\theta = 0$  or  $\pi$ , in which case the four points are collinear.

Finally, we note from (ii)(α) that  $|f(z)| = \frac{c^2}{|z|}$ .

So if the  $z$ -circle does not pass through  $z=0$  (i.e. the origin), the points on the  $w$ -figure have a maximum modulus of  $\frac{c^2}{m}$ , where  $m$  is the minimum distance from the origin to the  $z$ -circle, so the  $w$ -figure will be a circle. If the  $z$ -circle does pass through the origin, then the modulus of  $w$ -points is unbounded, so the  $w$ -figure cannot be a circle and will, instead, be a line.

Alternatively, let  $z = x + iy$  and  $f(z) = X + iY$ .

$$\begin{aligned} \therefore X + iY &= \frac{c^2}{x - iy} \\ &= \frac{c^2(x + iy)}{x^2 + y^2} \end{aligned}$$

If the  $w$ -points are collinear, there is a linear relationship between  $X$  and  $Y$ .