4 UNIT MATHEMATICS FORM VI

Time allowed: 3 hours (plus five minutes reading) **Exam date:** 11th August, 1998

Instructions:

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

Tear out Page 11 and place it inside the answer booklet for Question Seven.

QUESTION ONE (Start a new answer booklet)

(a) Evaluate $\displaystyle\int_0^1 \frac{4}{\sqrt{3x+1}}\,dx$.

(b) Evaluate $\displaystyle\int_1^e \log_e x\,dx$.

(c) Evaluate $\displaystyle\int_0^\pi \sin^3 x\,dx$.

(d) Find $\displaystyle\int \frac{dx}{2+\sin x}$ using the substitution $t = \tan\frac{1}{2}x$. Leave your answer in terms of t.

(e) (i) Find constants A, B and C so that $\displaystyle\frac{x^2+x+1}{x^3+3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$.

(ii) Find $\displaystyle\int \frac{x^2+x+1}{x^3+3x^2}\,dx$.

QUESTION TWO (Start a new answer booklet)

(a) Simplify $(2-5i)^3$.

(b) Solve $z^2 = 5 - 12i$, giving your answer in the form $a + ib$.

(c) (i) Express $-1 + i$ in modulus argument form.

(ii) Hence evaluate $(-1+i)^{10}$.

(d) (i) Sketch the locus of $\arg z = \frac{\pi}{4}$.

(ii) Sketch the locus of $\arg \overline{z} = \frac{\pi}{4}$.

(iii) Sketch the locus of $\arg(-z) = \frac{\pi}{4}$.

(e) The points O, I, Z and P on the Argand diagram represent the complex numbers 0, 1, z and $z+1$ respectively, where:

$$z = \cos\theta + i\sin\theta$$

is any complex number of modulus 1, with $0 < \theta < \pi$.

(i) Explain why $OIZP$ is a rhombus.

(ii) Show that $\dfrac{z-1}{z+1}$ is purely imaginary.

(iii) Find the modulus and argument of $z+1$.

Exam continues next page ...

QUESTION THREE (Start a new answer booklet)

Marks

5 (a) Let ω be a non-real root of $z^7 - 1 = 0$.

(i) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$.

(ii) Show that $(1 + \omega)(1 + \omega^2)(1 + \omega^4) = 1$.

(iii) Simplify $(\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3)$.

(iv) Sketch on the Argand diagram all seven roots of $z^7 - 1 = 0$.

4 (b) It is known that $2 + i$ is a root of the equation:

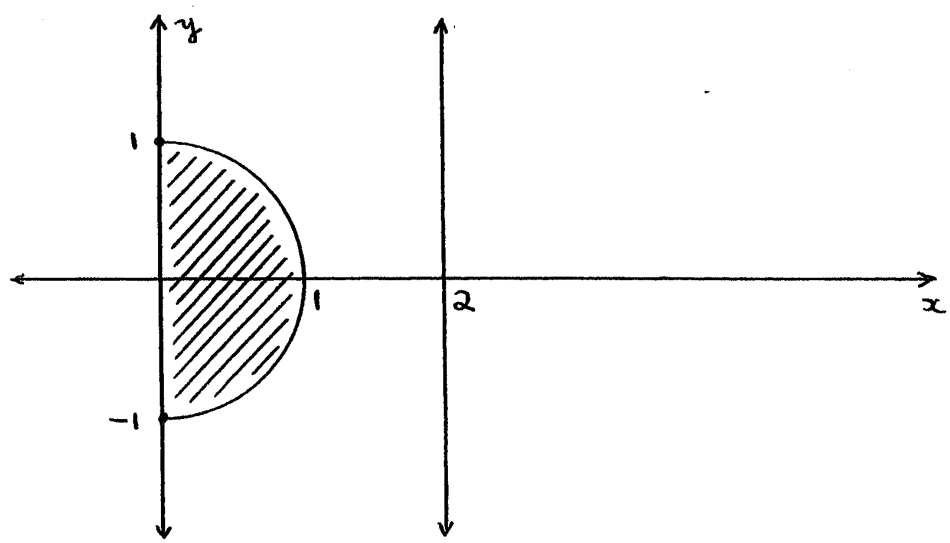
$$x^6 - 7x^4 + 31x^2 - 25 = 0.$$

(i) Give a reason why $2 - i$ is also a root of the equation.

(ii) Give a reason why $-(2 + i)$ is also a root of the equation.

(iii) Find the other three roots, giving reasons for each step in your argument (it is not intended that long division be used).

6 (c)



The shaded semicircle in the diagram above is rotated about the line $x = 2$.

(i) Using the method of cylindrical shells, show that the volume V of the resulting solid is given by:

$$V = \int_0^1 4\pi(2 - x)\sqrt{1 - x^2}\, dx.$$

(ii) Hence find the volume of the solid.

QUESTION FOUR (Start a new answer booklet)

Marks

$\boxed{4}$ (a) Give reasons why each of the following statements is true or false. You need not evaluate the integrals:

(i) $\displaystyle\int_{-1}^{1} \frac{e^x - e^{-x}}{2}\, dx = 0,$

(ii) $\displaystyle\int_0^1 x^8\, dx < \int_0^1 x^9\, dx,$

(iii) $\displaystyle\int_0^{\pi} \sin^8 x\, dx > \int_0^{\pi} \sin 8x\, dx.$

$\boxed{6}$ (b) A projectile is fired vertically upwards with initial speed V. It experiences air resistance proportional to the square of its speed, as well as gravitational acceleration g, so that in its upwards flight, the equation of motion is:

$$\ddot{x} = -g - kv^2, \quad \text{for some constant } k > 0.$$

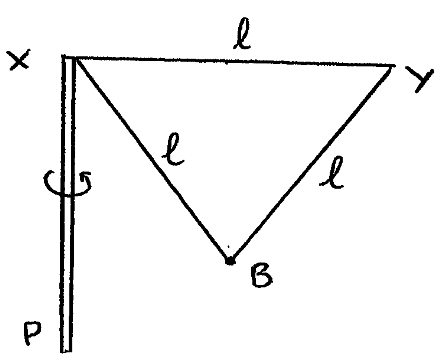
(i) By replacing $\ddot{x}$ by $\dfrac{dv}{dt}$ and integrating, find the time t when the velocity is v.

(ii) Hence find the time T taken to reach maximum height.

(iii) By replacing $\ddot{x}$ by $v\,\dfrac{dv}{dx}$ and integrating, find the height x when the velocity is v.

(iv) Hence find the maximum height H.

$\boxed{5}$ (c)



The diagram above shows a rod XY of length ℓ metres projecting horizontally from a rotating vertical pole XP. A basket B of mass m kg is held by two strings XB and YB, each of length ℓ. The pole, rod and basket are rotating with angular speed ω.

Let T be the tension in the string XB, and let U be the tension in the string YB.

(i) Show that the radius of the basket's circular motion about the pole is $\frac{1}{2}\ell$.

(ii) Resolve forces at B radially and vertically.

(iii) Show that the string BY will remain taut provided $\omega^2 \leq \dfrac{2g}{\ell\sqrt{3}}$.

(iv) Find the ratio of the tensions in the two strings, assuming they are both taut.

QUESTION FIVE (Start a new answer booklet)

Marks

8 (a) Sketch, showing the exact values of all turning points, intercepts and endpoints:

(i) $y = \cos \pi x$, for $-1 \le x \le 3$,

(ii) $y^2 = \cos \pi x$, for $-1 \le x \le 3$,

(iii) $y = e^{\cos \pi x}$, for $0 \le x \le 2$ (you need not find inflexions here),

(iv) $y = \cos \pi e^x$, for $x \le 1$ (again, you need not find inflexions).

7 (b) Let $z = \cos\theta + i\sin\theta$ be any complex number of modulus 1.

(i) Show that $\dfrac{z^2-1}{z} = 2i\sin\theta$.

(ii) Using the formula for the sum to n terms of a GP, and part (i), prove that:

$$z + z^3 + z^5 + z^7 + z^9 = \frac{\sin 10\theta + i(1-\cos 10\theta)}{2\sin\theta}.$$

(iii) Hence show that $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \dfrac{\sin 10\theta}{2\sin\theta}$.

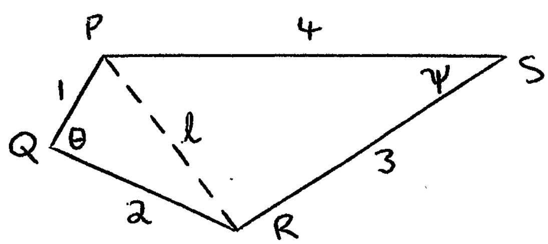
(iv) Find the first two positive solutions of:

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \tfrac{1}{2}.$$

QUESTION SIX (Start a new answer booklet)

Marks

7 (a)



The quadrilateral $PQRS$ above has sides $PQ = 1$, $QR = 2$, $RS = 3$ and $SP = 4$. Let $\theta = \angle Q$ and $\psi = \angle S$, and let $\ell = PR$.

(i) Show that the area of $PQRS$ is given by $A = \sin\theta + 6\sin\psi$.

(ii) By equating expressions for ℓ^2, show that $6\cos\psi - \cos\theta = 5$.

(iii) Differentiate the identity in part (ii) implicitly to find an expression for $\dfrac{d\psi}{d\theta}$.

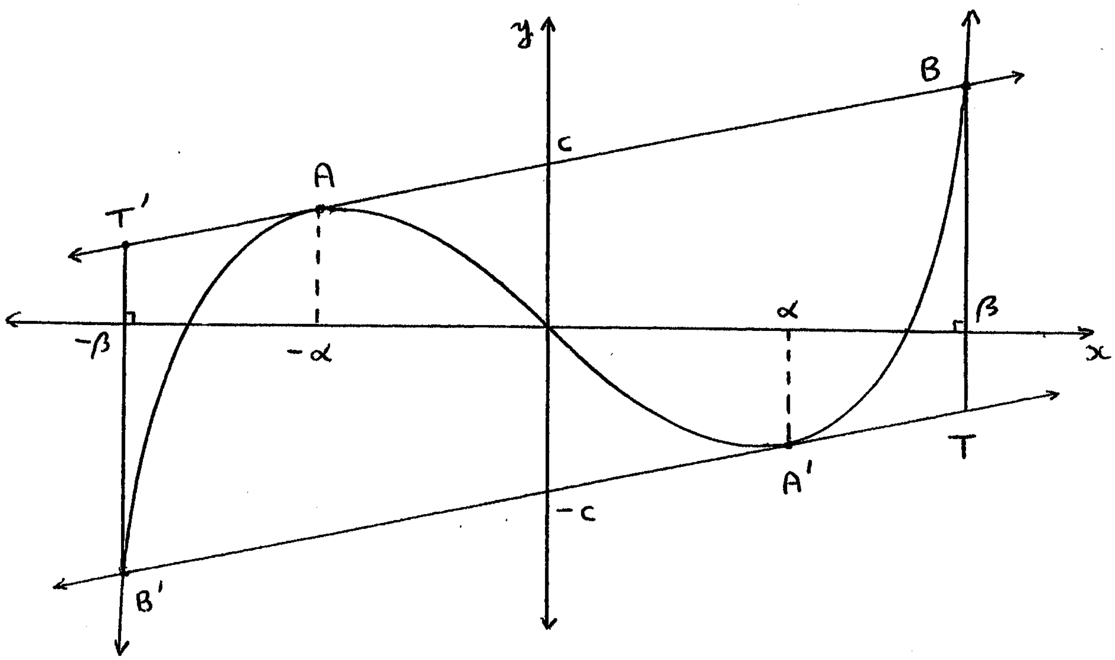
(iv) Prove that the area of the quadrilateral is maximum when it is cyclic (you need not prove that the relevant stationary point is a maximum).

(v) Find the area of the quadrilateral when it is cyclic.

QUESTION SIX (Continued)

2 (b) Use the table of standard integrals to find $\int_a^{2a} \frac{dx}{\sqrt{x^2 - a^2}}$, and show that the integral is independent of a.

6 (c)



The diagram above shows the cubic curve $y = x^3 - x$. The tangent AB with gradient m touches the cubic at a point A to the left of the y-axis and crosses the curve again at B.

Let the point of contact A have x-coordinate $-\alpha$, and let B have x-coordinate β. Let the tangent have equation $y = mx + c$.

Because the cubic is an odd function, the other tangent with this same gradient m will have equation $y = mx - c$, will touch the cubic at the point A' with x-coordinate α, and will cross the curve at the point B' with x-coordinate $-\beta$.

Vertical lines at B and B' each meet the other tangent at T and T' respectively.

(i) By solving simultaneously the cubic and the tangent AB, explain why:

$$x^3 - (m+1)x - c = 0$$

has roots $-\alpha$, $-\alpha$ and β.

(ii) Show that $\beta = 2\alpha$.

(iii) Show that $\alpha^2 = \frac{1}{3}(m+1)$.

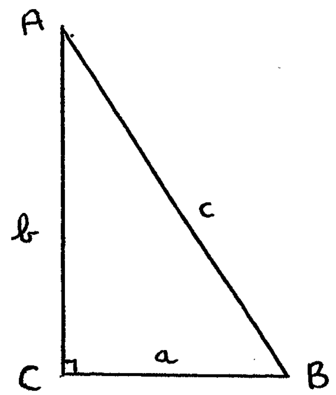
(iv) Show that the parallelogram $TBT'B'$ has area $\frac{16}{9}(m+1)^2$.

(v) Give a geometrical explanation as to why the area is zero when $m = -1$.

QUESTION SEVEN (Start a new answer booklet)

arks

4 (a)

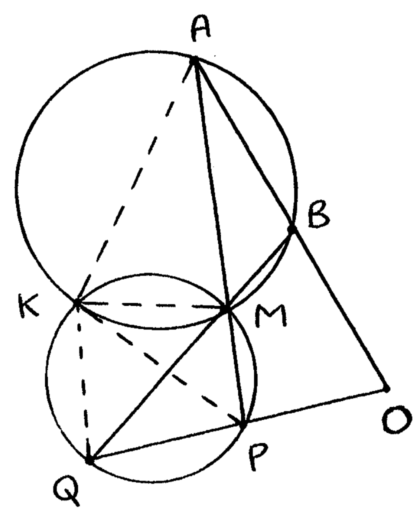


The hypotenuse of the right triangle above has length c, and the other two sides have lengths a and b respectively.

(i) If b is the arithmetic mean of a and c, find the ratio $a : c$.

(ii) If b is the geometric mean of a and c, find the ratio $a : c$.

5 (b) NOTE: The diagram printed below for this question has been reprinted on Page 11 so that working can be done on the diagram. Tear out Page 11, write your candidate number at the top of that sheet, and place the sheet inside your answer booklet for Question Seven.



The diagram above shows two circles intersecting at K and M. From points A and B on the outer arc of one circle, lines are drawn through M to meet the other circle at P and Q respectively. The lines AB and QP meet at O.

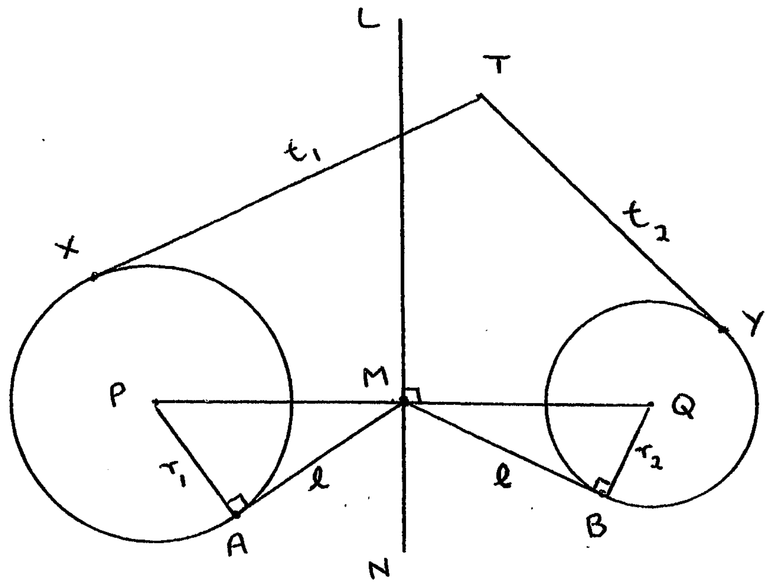
(i) Let $\theta = \angle KAB$, and give a reason why $\angle KMQ = \theta$.

(ii) Prove $AKPO$ is a cyclic quadrilateral.

(iii) Let $\psi = \angle AKM$. Show that if $OBMP$ is a cyclic quadrilateral, then the points A, K and Q are collinear.

QUESTION SEVEN (Continued)

(c) NOTE: The diagram printed below for this question has been reprinted on Page 11 so that working can be done on the diagram. Tear out Page 11, write your candidate number at the top of that sheet, and place the sheet inside your answer booklet for Question Seven.



The diagram above shows two non-intersecting circles with centres P and Q respectively and radii r_1 and r_2 respectively. The point M is the point on PQ where the tangents MA and MB respectively to the two circles have equal length ℓ. The line LMN is drawn through M perpendicular to PQ.

Let T be any point outside the two circles, and let the tangents TX and TY to the two circles have lengths t_1 and t_2 respectively.

(i) Show that $PM^2 - QM^2 = {r_1}^2 - {r_2}^2$.

(ii) Show that $TP^2 - TQ^2 = ({t_1}^2 - {t_2}^2) + ({r_1}^2 - {r_2}^2)$.

(iii) Prove that the tangents TX and TY are equal if and only if T lies on LMN.

(iv) A third circle is drawn tangent to the first circle at a point J and to the second circle at a point K. Prove that LMN and the tangents at J and K are concurrent.

QUESTION EIGHT (Start a new answer booklet)

arks

3 (a) Solve $|x^2 - 2x - 3| < 3x - 3$.

7 (b) Consider the integral $I_n = \displaystyle\int_0^1 x^{2n+1}\, e^{-x^2}\, dx$.

(i) Use integration by parts to show that $I_n = -\dfrac{1}{2e} + nI_{n-1}$, for $n \geq 1$.

(ii) Show that $I_0 = \dfrac{1}{2} - \dfrac{1}{2e}$.

(iii) Prove by mathematical induction that for all $n \geq 1$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = e - \frac{2e\, I_n}{n!}.$$

(iv) It is clear that $0 \leq I_n \leq 1$, because $0 \leq x^{2n+1}\, e^{-x^2} \leq 1$, for $0 \leq x \leq 1$.
Use this fact to deduce from the previous part that:

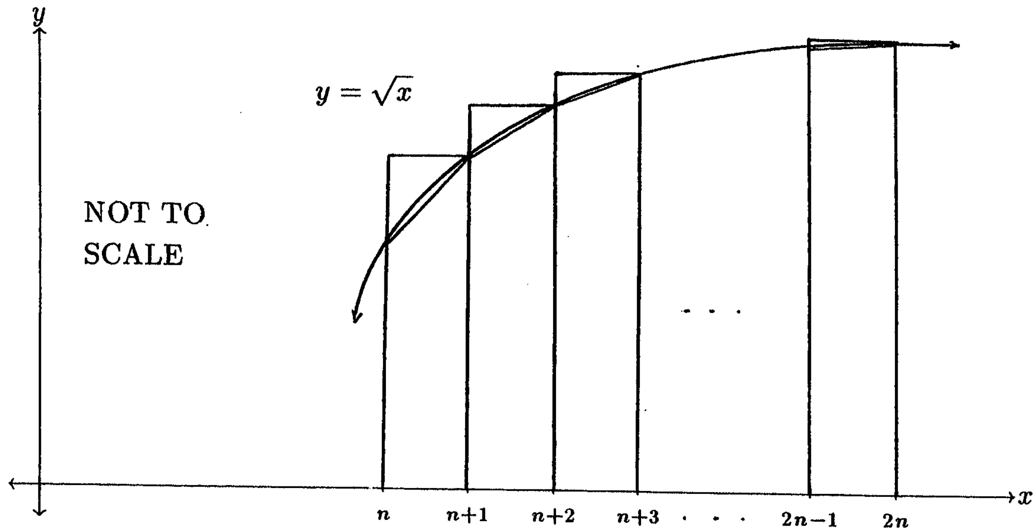
$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e.$$

CARE: QUESTION EIGHT CONTINUES OVERLEAF

<u>QUESTION EIGHT</u> (Continued)

(c) Show that $\displaystyle\int_n^{2n} \sqrt{x}\,dx = \tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right)$, where n is a positive integer.

1 (d)



The diagram above shows part of the graph of $y = \sqrt{x}$, drawn quite distorted so that the chords can be seen more clearly. For some positive integer n, ordinates have been drawn at $x = n$, $x = n+1$, $\ldots$, $x = 2n$. Upper rectangles and trapezia have been constructed as shown.

(i) Using the upper rectangles and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \cdots \sqrt{2n} > \tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \sqrt{n}$$

(you may assume that $y = \sqrt{x}$ is an increasing function).

(ii) Using the trapezia and part (c), show that:

$$\sqrt{n} + \sqrt{n+1} + \cdots \sqrt{2n} < \tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \tfrac{1}{2}\left(\sqrt{n} + \sqrt{2n}\right)$$

(you may assume that $y = \sqrt{x}$ is concave down).

(iii) Suppose it is claimed that the average of the numbers:

$$\sqrt{1\,000\,000}, \quad \sqrt{1\,000\,001}, \quad \ldots, \quad \sqrt{2\,000\,000}$$

is about 1218·9512. If one relies on the bounds established in parts (i) and (ii), what is the maximum possible error in this claim?

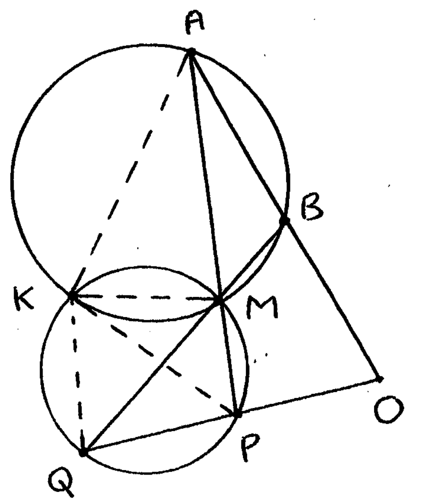
WMP

Candidate Number: ....

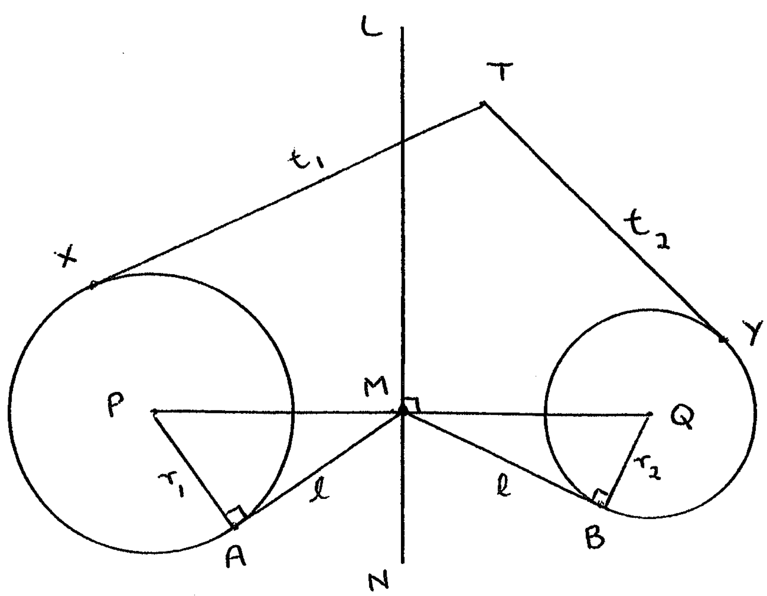
Tear out this sheet of paper, write your candidate number at the top, and place the sheet inside your answer booklet for Question Seven.

QUESTION SEVEN

(b) This is a reprint of the diagram for part (b) of Question Seven.



(c) This is a reprint of the diagram for part (c) of Question Seven.



QUESTION ONE

arks

2 (a) $\displaystyle\int_0^1 \frac{4}{\sqrt{3x+1}}\,dx = \int_0^1 4(3x+1)^{-\frac{1}{2}}\,dx$

$\displaystyle = 4 \times \tfrac{2}{1} \times \tfrac{1}{3}\Big[(3x+1)^{\frac{1}{2}}\Big]_0^1 \quad \boxed{\checkmark}$

$= \tfrac{8}{3}. \quad \boxed{\checkmark}$

2 (b) $\displaystyle\int_1^e \log_e x\,dx = \Big[x\log_e x\Big]_1^e - \int_1^e x \times \frac{1}{x}\,dx \quad \boxed{\checkmark}$

$\displaystyle = (e\log_e e - 1\log_e 1) - \Big[x\Big]_1^e$

$= e - e + 1$

$= 1. \quad \boxed{\checkmark}$

3 (c) $\displaystyle\int_0^\pi \sin^3 x\,dx = \int_0^\pi \sin x(1-\cos^2 x)\,dx.$

$\displaystyle = \Big[-\cos x + \tfrac{1}{3}\cos^3 x\Big]_0^\pi, \quad \boxed{\checkmark}$ since $\dfrac{d}{dx}(-\cos x) = \sin x, \quad \boxed{\checkmark}$

$= \tfrac{4}{3}. \quad \boxed{\checkmark}$

4 (d) $\displaystyle\int \frac{1}{2+\sin x}\,dx = \int \frac{1}{2+\dfrac{2t}{1+t^2}} \times \frac{2dt}{1+t^2} \quad \boxed{\checkmark}$

$\displaystyle = \int \frac{dt}{t^2+1+t}$

$\displaystyle = \int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} \quad \boxed{\checkmark}$

$\displaystyle = \frac{2}{\sqrt{3}}\tan^{-1}\frac{t+\frac{1}{2}}{\sqrt{3}/2} + C,$ for some constant $C \quad \boxed{\checkmark}$

$\displaystyle = \tfrac{2}{3}\sqrt{3}\tan^{-1}\frac{2t+1}{\sqrt{3}}.$

Let $t = \tan\frac{1}{2}x$,

then $dt = \frac{1}{2}\sec^2\frac{1}{2}x\,dx$

$2dt = 1+\tan^2\frac{1}{2}x\,dx$

$\displaystyle dx = \frac{2dt}{1+t^2}. \quad \boxed{\checkmark}$

4 (e) (i) Given

$$\frac{x^2+x+1}{x^3+3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

$$x^2+x+1 = Ax(x+3) + B(x+3) + Cx^2.$$

Put $x=0$, then $1 = 3B$, and $B = \frac{1}{3}. \quad \boxed{\checkmark}$

Put $x=-3$, then $9-3+1 = 9C$, and $C = \frac{7}{9}. \quad \boxed{\checkmark}$

Equate coefficients of x^2, $1 = A + C$,

$A = \frac{2}{9}. \quad \boxed{\checkmark}$

So

$$\frac{x^2+x+1}{x^3+3x^2} = \frac{2/9}{x} + \frac{1/3}{x^2} + \frac{7/9}{x+3}.$$

(ii) $\displaystyle\int \frac{x^2+x+1}{x^3+3x^2}\,dx = \tfrac{2}{9}\log|x| - \tfrac{1}{3}x^{-1} + \tfrac{7}{9}\log|x+3| + C,$ for some constant $C. \quad \boxed{\checkmark}$

QUESTION TWO

arks

2 (a) $(2-5i)^3 = 2^3 - 3\times 2^2 \times 5i + 3\times 2\times 5^2 i^2 - 5^3 i^3$
$= 8 - 60i - 150 + 125i$
$= -142 + 65i.$ $\boxed{\checkmark\checkmark \text{ (one off for each mistake)}}$

2 (b) Given $z^2 = 5 - 12i$, let $z = x + iy$.
Then $x^2 - y^2 = 5$ and $xy = -6$. $\boxed{\checkmark}$
So $x = 3$ and $y = -2$, or $x = -3$ and $y - 2$.
So $z = 3 - 2i$ or $-3 + 2i$. $\boxed{\checkmark}$

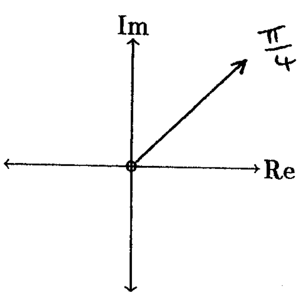
4 (c) (i) $-1 + i = \sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$. $\boxed{\checkmark \text{ (modulus) } \checkmark \text{ (argument)}}$

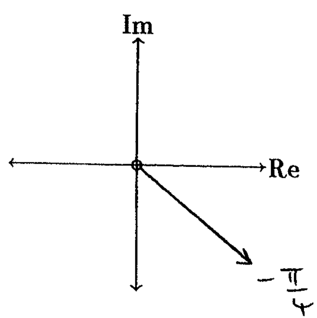
(ii) $(-1+i)^{10} = (\sqrt{2}\,)^{10} \times (\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})^{10}$
$= 2^5 \times (\cos\frac{15\pi}{2} + i\sin\frac{15\pi}{2})$
$= 32 \times (\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})$
$= -32i.$ $\boxed{\checkmark \text{ (for 32) } \checkmark\text{(for } -i)}$

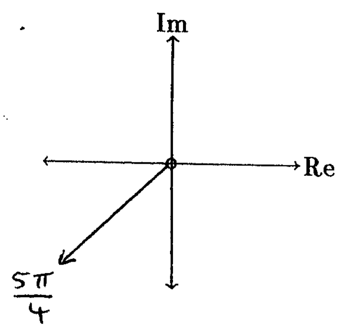
3 (d) (i) $\arg z = \frac{\pi}{4}$. $\boxed{\checkmark}$

(ii) If $\arg(\overline{z}) = \frac{\pi}{4}$, then $\arg z = -\frac{\pi}{4}$. $\boxed{\checkmark}$

(iii) If $\arg(-z) = \frac{\pi}{4}$, then $\arg z = \frac{5\pi}{4}$. $\boxed{\checkmark}$







] (e) (i) $OIZP$ is a rhombus because $OI = OZ = 1$ and $OIZP$ is a parallelogram. $\boxed{\checkmark \text{ (proof)}}$

(ii) The diagonals IZ and OP of the rhombus are perpendicular. So the complex numbers they represent, $z - 1$ and $z + 1$, have arguments differing by $\frac{\pi}{2}$. So $\dfrac{z-1}{z+1}$ is purely imaginary. $\boxed{\checkmark \text{ (proof)}}$

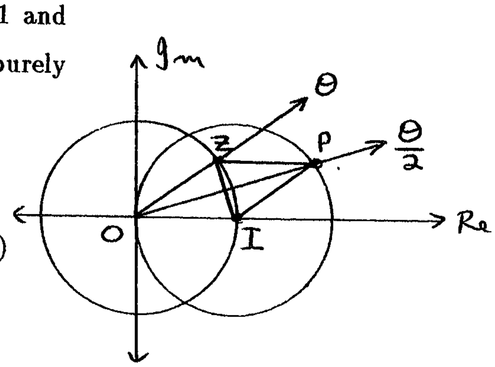
(iii) The diagonal OP bisects the vertex angle $\angle O$,

so $\arg(z+1) = \frac{1}{2}\theta$. $\boxed{\checkmark}$

By the cosine rule, $|z+1|^2 = 1^2 + 1^2 - 2\cos(\pi - \theta)$
$= 2 + 2\cos\theta$
$= 4\cos^2 \frac{1}{2}\theta,$

so $|z+1| = 2\cos\frac{1}{2}\theta,$ $\boxed{\checkmark}$

since $\cos\frac{1}{2}\theta > 0$ because $\frac{1}{2}\theta$ is acute.



QUESTION THREE

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(a) (i) Since ω is a root of $z^7 - 1$, $\quad \omega^7 - 1 = 0.$

Factorising, $\quad (\omega - 1)(\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0.$

Since ω is not real, $\omega - 1 \neq 0$, so $\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0.$ $\boxed{\checkmark \text{ (proof)}}$

(ii) Expanding, $(1+\omega)(1+\omega^2)(1+\omega^4) = 1 + \omega + \omega^2 + \omega^4 + \omega^3 + \omega^6 + \omega^5 + \omega^7$

$= 1,$ by part (i), and since $\omega^7 = 1$. $\boxed{\checkmark \text{ (proof)}}$

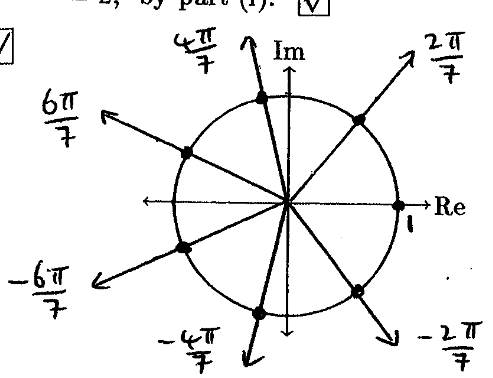
(iii) Expanding, $(\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3)$

$= \omega^7 + \omega^6 + \omega^4 + \omega^8 + \omega^7 + \omega^5 + \omega^{10} + \omega^9 + \omega^7$

$= 1 + \omega^6 + \omega^4 + \omega + 1 + \omega^5 + \omega^3 + \omega^2 + 1,$ since $\omega^7 = 1$, $\boxed{\checkmark}$

$= 2,$ by part (i). $\boxed{\checkmark}$

(iv) $\boxed{\checkmark}$



The black dots are the seven roots of $z^7 - 1 = 0$.

(b) (i) We are given that $2+i$ is a zero of $f(x) = x^6 - 7x^4 + 31x^2 - 25$.

Since all coefficients are real, the conjugate $2 - i$ is also a root. $\boxed{\checkmark}$

(ii) Since the polynomial is even, $-2-i$ is also a root. $\boxed{\checkmark}$

(iii) Since all coefficients are real, the conjugate $-2+i$ is also a root.

Since $f(1) = 0$, 1 is a root. $\boxed{\checkmark}$

Since $f(x)$ is even, -1 is also a root.

So the roots are $2+i$, $2-i$, $-2+i$, $-2-i$, 1 and -1. $\boxed{\checkmark}$

(c) (i) Slice the shaded region into vertical strips, each of width dx.

A typical strip is distant x from the origin, and so has height $2\sqrt{1-x^2}$. $\boxed{\checkmark}$

The strip is distant $2-x$ from the axis of rotation $x = 2$. $\boxed{\checkmark}$

So the area of the cylindrical shell generated by the strip is $4\pi(2-x)\sqrt{1-x^2}$, $\boxed{\checkmark}$

giving a volume of $4\pi(2-x)\sqrt{1-x^2}\,dx$ for the strip.

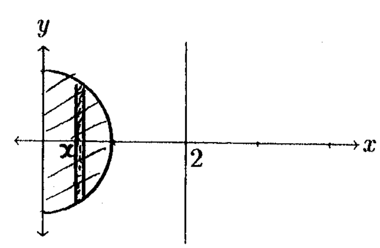
So the total volume is $V = \int_0^1 4\pi(2-x)\sqrt{1-x^2}\,dx$.

(ii) $V = 8\pi \int_0^1 \sqrt{1-x^2}\,dx + 2\pi \int_0^1 (-2x)\sqrt{1-x^2}\,dx$

$= 8\pi \times \frac{\pi}{4} + 2\pi \times \frac{2}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^1$ $\boxed{\checkmark\checkmark}$

$= 2\pi^2 + \frac{4\pi}{3}(0-1)$ $\boxed{\checkmark}$

$= 2\pi^2 - \frac{4\pi}{3}.$



QUESTION FOUR

Marks

4 (a) (i) $\displaystyle\int_{-1}^{1} \frac{e^x - e^{-x}}{2}\, dx = 0$ is true, because the integrand is odd. ☑

(ii) $\displaystyle\int_0^1 x^8\, dx < \int_0^1 x^9\, dx$ is false.

For $0 < x < 1$, $x^8 > x^9$, so $\displaystyle\int_0^1 x^8\, dx > \int_0^1 x^9\, dx$. ☑

(iii) $\displaystyle\int_0^{\pi} \sin^8 x\, dx > \int_0^{\pi} \sin 8x\, dx$ is true.

First, $\displaystyle\int_0^{\pi} \sin 8x\, dx = 0$, because the integral is over four complete periods. ☑

Secondly, $\displaystyle\int_0^{\pi} \sin^8 x\, dx > 0$, because $\sin^8 x > 0$ for $0 < x < \pi$. ☑

6 (b) (i) Given

$$\frac{dv}{dt} = -g - kv^2$$

$$\frac{dt}{dv} = -\frac{1}{g + kv^2}$$

$$t = -\frac{1}{\sqrt{kg}} \tan^{-1} \frac{v\sqrt{k}}{\sqrt{g}} + C, \text{ for some constant } C. \ ☑$$

When $t = 0$, $v = V$, so $\displaystyle 0 = -\frac{1}{\sqrt{kg}} \tan^{-1} \frac{V\sqrt{k}}{\sqrt{g}} + C$

and $\displaystyle t = \frac{1}{\sqrt{kg}} \left(\tan^{-1} \frac{V\sqrt{k}}{\sqrt{g}} - \tan^{-1} \frac{v\sqrt{k}}{\sqrt{g}} \right).$ ☑

(ii) The time T to reach the maximum height is found by substituting $v = 0$, so:

$$T = \frac{1}{\sqrt{kg}} \tan^{-1} \frac{V\sqrt{k}}{\sqrt{g}}. \ ☑$$

(iii) Given

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{dx}{dv} = -\frac{1}{2k} \times \frac{2kv}{g + kv^2}$$

$$x = -\frac{1}{2k} \log(g + kv^2) + C, \text{ for some constant } C. \ ☑$$

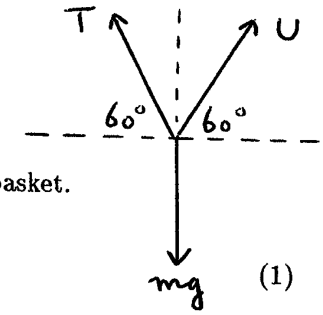
When $t = 0$, $v = V$, so $\displaystyle 0 = -\frac{1}{2k} \log(g + kV^2) + C$

so $\displaystyle x = \frac{1}{2k} \log \frac{g + kV^2}{g + kv^2}.$ ☑

(iv) The maximum height H is found by substituting $v = 0$, so:

$$H = \frac{1}{2k} \log\left(1 + \frac{kV^2}{g}\right). \ ☑$$

(c) (i) The basket B is vertically below the midpoint of XY, and since $xy = \ell$, B is distant $\frac{1}{2}\ell$ from the pole. (or use simple trigonometry on equilateral $\triangle XYB$). $\boxed{\checkmark}$



(ii) The diagram on the right shows the three forces acting on the basket.

Resolving radially, $T\cos 60^\circ - U\cos 60^\circ = mr\omega^2$ $\boxed{\checkmark}$

and since $r = \frac{1}{2}\ell$, $T - U = m\ell\omega^2\,.$ (1)

Resolving vertically, $T\cos 30^\circ + U\cos 30^\circ - mg = 0$ $\boxed{\checkmark}$

$$T + U = \frac{2mg}{\sqrt{3}}\,. \qquad (2)$$

(iii) Subtracting (1) from (2),

$$2U = \frac{2mg}{\sqrt{3}} - m\ell\omega^2$$
$$U = \frac{m}{2\sqrt{3}}(2g - \ell\omega^2\sqrt{3}\,).$$

The string remains taut provided

$$U \ge 0$$
$$2g - \ell\omega^2\sqrt{3} \ge 0$$
$$\omega^2 \le \frac{2g}{\ell\sqrt{3}}\,. \quad \boxed{\checkmark\text{ (proof)}}$$

(iv) Adding (1) and (2), $2T = \dfrac{2mg}{\sqrt{3}} + m\ell\omega^2$

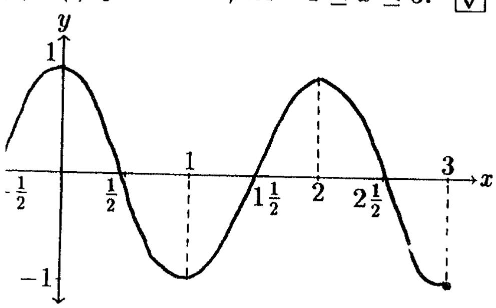
$$T = \frac{m}{2\sqrt{3}}(2g + \ell\omega^2\sqrt{3}\,).$$

So

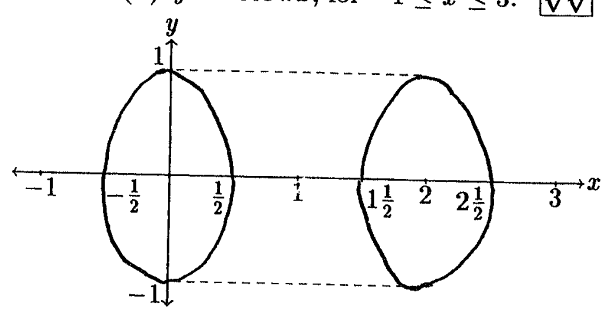
$$\frac{T}{U} = \frac{2g + \ell\omega^2\sqrt{3}}{2g - \ell\omega^2\sqrt{3}}\,. \quad \boxed{\checkmark}$$

QUESTION FIVE

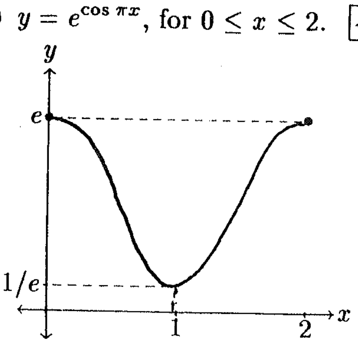
(a) (i) $y = \cos \pi x$, for $-1 \le x \le 3$. $\boxed{\checkmark}$



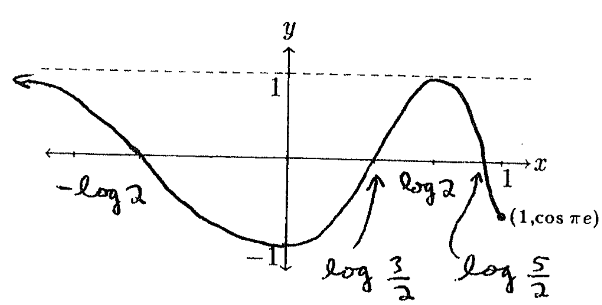
(ii) $y^2 = \cos \pi x$, for $-1 \le x \le 3$. $\boxed{\checkmark\checkmark}$



(iii) $y = e^{\cos \pi x}$, for $0 \le x \le 2$. $\boxed{\checkmark\checkmark}$



(iv) $y = \cos \pi e^x$, for $x \le 1$. $\boxed{\checkmark\checkmark\checkmark}$



$\boxed{7}$ (b) (i) $\dfrac{z^2-1}{z} = z - \dfrac{1}{z}$

$$\begin{aligned} &= \cos\theta + i\sin\theta - \frac{1}{\cos\theta + i\sin\theta} \quad \boxed{\checkmark} \\ &= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) \\ &= 2i\sin\theta. \quad \boxed{\checkmark\text{ (proof)}} \end{aligned}$$

(ii) The series $z + z^3 + z^5 + z^7 + z^9$ is a GP with $a = z$ and $r = z^2$ and 5 terms,

so
$$\begin{aligned} z + z^3 + z^5 + z^7 + z^9 &= \frac{z(z^{10}-1)}{z^2-1} \quad \boxed{\checkmark} \\ &= \frac{z^{10}-1}{2i\sin\theta}, \text{ by part (i),} \\ &= \frac{-i(\cos 10\theta + i\sin 10\theta - 1)}{2\sin\theta} \\ &= \frac{\sin 10\theta + i(1-\cos 10\theta)}{2\sin\theta}. \quad \boxed{\checkmark\text{ (proof)}} \end{aligned}$$

(iii) Equating real parts of both sides:

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \frac{\sin 10\theta}{2\sin\theta}. \quad \boxed{\checkmark\text{ (proof)}}$$

(iv) Given $\quad \cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \frac{1}{2}.$

Applying part (iii) to the LHS,
$$\begin{aligned} \frac{\sin 10\theta}{2\sin\theta} &= \frac{1}{2} \\ \sin 10\theta &= \sin\theta. \end{aligned}$$

The first two positive solutions are given by $\quad 10\theta = \pi - \theta$ and $10\pi = 2\pi + \theta$

$11\theta = \pi$ and $9\theta = 2\pi$

So the first two positive solutions are $\theta = \frac{\pi}{11}$ and $\theta = \frac{2\pi}{9}$ $\boxed{\checkmark\checkmark}$ (or apply sums to products).

QUESTION SIX

Marks

$\boxed{7}$ (a) (i) Using the area formula in $\triangle ABC$ and $\triangle ADC$:

$$\begin{aligned} A &= \tfrac{1}{2} \times 1 \times 2 \times \sin\theta + \tfrac{1}{2} \times 3 \times 4 \times \sin\psi \\ &= \sin\theta + 6\sin\psi. \quad \boxed{\checkmark\text{ (proof)}} \end{aligned}$$

(ii) Using the cosine rule in $\triangle ABC$ and $\triangle ADC$:

$$\ell^2 = 1^2 + 2^2 - 2 \times 1 \times 2 \times \cos\theta \quad \text{and} \quad \ell^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos\psi.$$

Equating these,
$$\begin{aligned} 5 - 4\cos\theta &= 25 - 24\cos\psi \\ 24\cos\psi - 4\cos\theta &= 20 \\ 6\cos\psi - \cos\theta &= 5. \quad \boxed{\checkmark\text{ (proof)}} \end{aligned}$$

(iii) Differentiating implicitly with respect to θ:

$$\begin{aligned} -6\sin\psi\frac{d\psi}{d\theta} + \sin\theta &= 0 \quad \boxed{\checkmark} \\ \frac{d\psi}{d\theta} &= \frac{\sin\theta}{6\sin\psi}. \end{aligned}$$

(iv) Differentiating the expression for area with respect to θ:

$$\begin{aligned}\frac{dA}{d\theta} &= \cos\theta + 6\cos\psi\,\frac{d\psi}{d\theta} \\ &= \cos\theta + 6\cos\psi \times \frac{\sin\theta}{6\sin\psi} \quad \boxed{\surd} \\ &= \frac{\sin\psi\cos\theta + \cos\psi\sin\theta}{\sin\psi} \\ &= \frac{\sin(\psi+\theta)}{\sin\psi},\end{aligned}$$

which is zero when $\psi + \theta = \pi$, that is, when $PQRS$ is a cyclic quadrilateral. $\boxed{\surd \text{ (proof)}}$

(v) Since the quadrilateral is cyclic, $\theta = \pi - \psi$,

so $\cos\theta = -\cos\psi$ and $\sin\theta = \sin\psi$,

so from part (ii), $7\cos\psi = 5$

$\cos\psi = \frac{5}{7}$. $\boxed{\surd}$

Hence $\sin\psi = \frac{2}{7}\sqrt{6}$,

so from part (ii), $A = 7\sin\psi$

$= 2\sqrt{6}$. $\boxed{\surd}$

(b) (i)
$$\begin{aligned}\int_a^{2a} \frac{dx}{\sqrt{x^2-a^2}} &= \left[\log\left(x + \sqrt{x^2-a^2}\right)\right]_a^{2a} \\ &= \log\left(2a + a\sqrt{3}\right) - \log(a+0) \quad \boxed{\surd} \\ &= \log a + \log\left(2+\sqrt{3}\right) - \log a \\ &= \log\left(2+\sqrt{3}\right), \quad \boxed{\surd} \text{ which is independent of } a.\end{aligned}$$

(c) (i) Solving cubic and line simultaneously, $x^3 - x = mx + c$

$$x^3 - (m+1)x - c = 0.$$

Then $-\alpha$ is a double root of this equation, because AB is a tangent, and β is a single root. $\boxed{\surd}$

(ii) Using sum of roots, $-2\alpha + \beta = 0$

$$\beta = 2\alpha. \quad \boxed{\surd \text{ (proof)}}$$

(iii) Taking products of pairs of roots, $\alpha^2 - \alpha\beta - \alpha\beta = -(m+1)$

Substituting $\beta = 2\alpha$, $-3\alpha^2 = -(m+1)$

$$\alpha^2 = \tfrac{1}{3}(m+1). \quad \boxed{\surd \text{ (proof)}}$$

(iv) Taking the product of the roots, $\alpha^2\beta = c$.

So area of parallelogram
$$\begin{aligned}TBT'B' &= 2\beta \times 2c \quad \boxed{\surd} \\ &= 4\alpha^2\beta^2 \\ &= 16\alpha^4, \text{ since } \beta = 2\alpha, \\ &= \tfrac{16}{9}(m+1)^2. \quad \boxed{\surd \text{ (proof)}}\end{aligned}$$

(v) Since $y' = 3x^2 - 1$, the gradient of the cubic at the origin is -1.

So for $m = -1$, the points A, A', B, B', T, T' and O all coincide. $\boxed{\surd}$

QUESTION SEVEN

Marks

$\boxed{4}$ (a) We know by Pythagoras' theorem that $a^2 + b^2 = c^2$. (1)

(i) Given that $$2b = a + c,$$
then substituting into $4 \times (1)$,
$$4a^2 + (a+c)^2 = 4c^2 \quad \boxed{\surd}$$
$$5a^2 + 2ac - 3c^2 = 0$$
$$(5a - 3c)(a + c) = 0.$$
Since a and c are both positive, $5a = 3c$, and $a : c = 3 : 5$ $\boxed{\surd}$ (the $3 : 4 : 5$ triangle).

(ii) Given that $$b^2 = ac$$
then substituting into (1), $a^2 + ac - c^2 = 0$. $\boxed{\surd}$
Since $\Delta = 5c^2$, $$a = \tfrac{1}{2}\big(-c + c\sqrt{5}\big) \text{ or } \tfrac{1}{2}\big(-c - c\sqrt{5}\big),$$
but a and c are positive, so $a : c = \big(-1 + \sqrt{5}\big) : 2$ $\boxed{\surd}$ (the golden mean)

$\boxed{5}$ (b) (i) $\angle KMQ = \theta$ (exterior angle of cyclic quadrilateral $AKMB$). $\boxed{\surd}$

(ii) Also $\angle KPQ = \theta$ (angles on the same arc KQ) $\boxed{\surd}$
so $AKPO$ is a cyclic quadrilateral, because the interior angle $\angle KAO$
equals the opposite exterior angle $\angle KPQ$. $\boxed{\surd \text{ (proof)}}$

(iii) Let $\angle AKM = \psi$,
then $\angle MBO = \psi$ (opposite exterior angle of cyclic quadrilateral $ABMK$).
Suppose now that $BMPO$ is a cyclic quadrilateral.
Then $\angle MPQ = \psi$ (opposite exterior angle of cyclic quadrilateral $BMPO$),
so $\angle MKQ = \pi - \psi$ (opposite angles of cyclic quadrilateral $KMPQ$)
so $\angle MKQ$ and $\angle MKA$ are supplementary, and QKA is a straight line. $\boxed{\surd\surd \text{ (proof)}}$

$\boxed{6}$ (c) (i) Since $$\angle MAP = \angle MBQ = \tfrac{\pi}{2} \quad \text{(radius and tangent)},$$
$$PM^2 - QM^2 = ({r_1}^2 + \ell^2) - ({r_2}^2 + \ell^2) \quad \text{(Pythagoras)}$$
$$= {r_1}^2 - {r_2}^2, \text{ as required.} \quad \boxed{\surd \text{ (proof)}}$$

(ii) Again, $$\angle TXP = \angle TYQ = \tfrac{\pi}{2} \quad \text{(radius and tangent)},$$
so $$TP^2 - TQ^2 = ({t_1}^2 + {r_1}^2) - ({t_2}^2 + {r_2}^2) \quad \text{(Pythagoras)}$$
$$= ({t_1}^2 - {t_2}^2) + ({r_1}^2 - {r_2}^2), \text{ as required.} \quad \boxed{\surd \text{ (proof)}}$$

(iii) Suppose first that T lies on LMN, then by Pythagoras' theorem:

$$\begin{aligned} TP^2 - TQ^2 &= (PM^2 + TM^2) - (QM^2 + TM^2) \\ &= {r_1}^2 - {r_2}^2, \text{ by part (i)}, \end{aligned}$$

and so by part (ii), ${t_1}^2 - {t_2}^2 = 0$. as required. $\boxed{\surd \text{ (proof)}}$

Conversely, suppose $\quad t_1 = t_2$.

Then
$$\begin{aligned} TP^2 - QT^2 &= ({r_1}^2 - {r_2}^2), \text{ by part (ii)}, \\ &= PM^2 - QM^2, \text{ by part (i)}. \end{aligned}$$

Let $\angle TMP = \theta$, then by the cosine rule:

$$\begin{aligned} TP^2 - QT^2 = (&PM^2 + TM^2 - 2 \times PM \times TM \cos\theta) \\ &- (QM^2 + TM^2 + 2 \times QM \times TM \cos\theta), \end{aligned}$$

so if (1) holds, then $\cos\theta = 0$, and $\theta = \frac{\pi}{2}$, as required. $\boxed{\surd\surd \text{ (proof)}}$

Alternatively, and probably more elegantly, one can construct a perpendicular from T to PQ, and show that assuming M is not the foot of that perpendicular leads to a contradiction.

(iv) Let the tangents at J and K meet at F.

Then $JF = KF$, since they are both tangents to the third circle,

so by the part (iii), F must lie on the line LMN. $\boxed{\surd \text{ (proof)}}$

NOTE: The line LMN is called the *radical axis* of the two circles. It can be constructed whether the circles overlap, are tangential, or do not intersect, either as shown or with one inside the other. When the circles overlap, the radical axis is their common chord. When they are tangent, the radical axis is their common tangent at the point of contact.

Examine the various cases, and particularly to what extent the proof given here can be made to hold without major modifications. Prove also the more general result that the three radical axes of any three circles taken in pairs are concurrent.

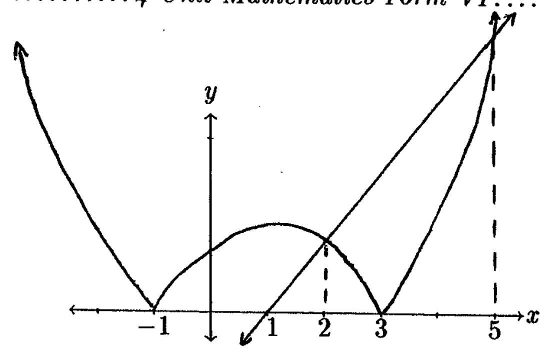
Given a point M, a line LMN through M, and a length ℓ, there is a corresponding family of circles whose centres all lie on the perpendicular to LMN through M, and so that the tangent from M to any of the circles has length ℓ. Then LMN is the radical axis of any two distinct circles in the family. The two points distant ℓ from M on the perpendicular to LMN through M are called the *limiting centres*, and are the points of convergence of the centres of the circles in the family as their radii approach zero.

What are the conditions for the family of circles to be mutually tangent? If the point M, the line LMN and the distance ℓ are given, how can circles in the family be constructed using straight edge and compasses? Can one construct the circle in the family: (a) with given centre P on the perpendicular through M to LMN, (b) passing through a given point X?

QUESTION EIGHT

Marks

$\boxed{3}$ (a)



The graph above shows $y = |x^2 - 2x - 3|$ and $y = 3x - 3$ on the one set of axes. $\boxed{\surd}$

The curves meet at the points with x-coordinates $x = 2$ and $x = 5$, $\boxed{\surd}$

so from the graph, the inequality has solution $2 < x < 5$. $\boxed{\surd}$

$\boxed{4}$ (b) (i)
$$\begin{aligned}
I_n &= \int_0^1 x^{2n+1}\, e^{-x^2}\, dx \\
&= -\tfrac{1}{2} \int_0^1 x^{2n} (-2x\, e^{-x^2})\, dx \\
&= -\tfrac{1}{2} \int_0^1 x^{2n}\, d(e^{-x^2}) \quad \boxed{\surd} \\
&= -\tfrac{1}{2} \left[x^{2n}\, e^{-x^2}\right]_0^1 + \tfrac{1}{2} \int_0^1 e^{-x^2}\, d\left(x^{2n}\right) \\
&= -\tfrac{1}{2}(e^{-1} - 0) + n \int_0^1 x^{2n-1}\, e^{-x^2}\, dx, \text{ since } n \geq 1, \\
&= -\frac{1}{2e} + nI_{n-1}, \text{ as required. } \boxed{\surd \text{ (proof)}}
\end{aligned}$$

(ii)
$$\begin{aligned}
I_0 &= \int_0^1 x\, e^{-x^2}\, dx \\
&= -\tfrac{1}{2} \int_0^1 (-2x) e^{-x^2}\, dx \\
&= -\tfrac{1}{2} \left[e^{-x^2}\right]_0^1 \\
&= -\tfrac{1}{2}(e^{-1} - 1) \\
&= \frac{1}{2} - \frac{1}{2e}. \quad \boxed{\surd}
\end{aligned}$$

(iii) A. When $n = 1$,
$$\begin{aligned}
I_1 &= -\frac{1}{2e} + I_0 \\
&= -\frac{1}{2e} + \frac{1}{2} - \frac{1}{2e} \\
&= \frac{1}{2} - \frac{1}{e}.
\end{aligned}$$

So $\quad \text{LHS} = 1 + 1,$

and
$$\begin{aligned}
\text{RHS} &= e - 2e\left(\frac{1}{2} - \frac{1}{e}\right) \\
&= e - e + 2 \\
&= \text{LHS}. \quad \boxed{\surd}
\end{aligned}$$

B. Suppose k is a positive integer for which the result is true.

That is, $$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}=e-\frac{2e\,I_k}{k!}\,. \qquad (**)$$

We now prove the result for $n=k+1$,

That is we prove $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\cdots+\dfrac{1}{(k+1)!}=e-\dfrac{2e\,I_{k+1}}{(k+1)!}$.

By the induction hypothesis (**), LHS $=e-\dfrac{2e\,I_k}{k!}+\dfrac{1}{(k+1)!}$ $\boxed{\checkmark}$

By the reduction formula,
$$\begin{aligned}\text{RHS}&=e-\frac{2e}{(k+1)!}\left(-\frac{1}{2e}+(k+1)I_k\right)\\&=e+\frac{1}{(k+1)!}-\frac{2e\,I_k}{k!}\quad\boxed{\checkmark\text{ (proof)}}\\&=\text{LHS}.\end{aligned}$$

C. It follows from A and B by mathematical induction that the result is true for all positive integers n.

(iv) We know $$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=e-\frac{2e\,I_n}{n!}\,.$$

Since $0\le I_n\le 1$, $$0\le\frac{2e\,I_n}{n!}\le\frac{2e}{n!}$$

But $\dfrac{2e}{n!}\to 0$ as $n\to\infty$, so $$\lim_{n\to\infty}\frac{2e\,I_n}{n!}=0.$$

So $$\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)=e.\quad\boxed{\checkmark\text{ (proof)}}$$

NOTE: By working harder on bounding the integral, one can prove quickly from part (iii) that the number e is irrational:

First, $$0<x^{2n+1}e^{-x^2}<x^{2n+1}\text{ over the interval }0<x<1,$$

so $$0<I_n<\int_0^1 x^{2n+1}\,dx$$
$$0<I_n<\frac{1}{2n+2}\,.$$

Secondly, from part (iii), $$e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{2e\,I_n}{n!}\,,$$

$\boxed{\times\, n!}$ $$e\times n!=\left(n!+\frac{n!}{1!}+\frac{n!}{2!}+\frac{n!}{3!}+\cdots+\frac{n!}{n!}\right)+2e\,I_n.$$

Thus $$e\times n!=N+K_n,$$

where N is an integer, and $0<K_n<\dfrac{2e}{2n+2}$, that is $0<K_n<\dfrac{e}{n+1}$.

Since $e<3$, it follows that $0<K_n<1$ provided $n\ge 2$,

Now suppose by way of contradiction that e were rational,

and let $e=\dfrac{p}{q}$, where p and q are integers and $q\ge 2$.

Put $n=q$, then $e\times q!=N+K_q$.

But $e\times q!$ and N are integers, and $0<K_q<1$.

This is a contradiction. So e cannot be rational.

$\boxed{1}$ (c) $\displaystyle\int_n^{2n} \sqrt{x}\,dx = \tfrac{2}{3}\left[x^{\frac{3}{2}}\right]_n^{2n}$

$\quad = \tfrac{2}{3}\left(2n\sqrt{2n} - n\sqrt{n}\right)$

$\quad = \tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2} - 1\right)$. $\boxed{\surd \text{ (proof)}}$

$\boxed{4}$ (d) (i) Because the upper rectangles have area greater than the integral:

$$\int_n^{2n} \sqrt{x}\,dx < \sqrt{n+1} + \sqrt{n+2} + \cdots + \sqrt{2n},$$

then adding $\sqrt{n}$ to both sides and using the value of the integral in (a):

$$\tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \sqrt{n} < \sqrt{n} + \sqrt{n+1} + \cdots + \sqrt{2n}. \quad \boxed{\surd \text{ (proof)}}$$

(ii) Because the curve is concave down, the trapezia have area less than the integral:

$$\tfrac{1}{2}\left(\sqrt{n} + \sqrt{n+1}\right) + \tfrac{1}{2}\left(\sqrt{n+1} + \sqrt{n+2}\right) + \cdots + \tfrac{1}{2}\left(\sqrt{2n-1} + \sqrt{2n}\right) < \int_n^{2n} \sqrt{x}\,dx,$$

then collecting like terms and adding $\tfrac{1}{2}\left(\sqrt{n} + \sqrt{2n}\right)$ to both sides:

$$\sqrt{n} + \sqrt{n+1} + \cdots \sqrt{2n} < \tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \tfrac{1}{2}\left(\sqrt{n} + \sqrt{2n}\right). \quad \boxed{\surd \text{ (proof)}}$$

(iii) Let $A_n = \dfrac{\sqrt{n} + \sqrt{n+1} + \cdots + \sqrt{2n}}{n+1}$ be the average of the numbers,

then it follows from part (i) and (ii) that:

$$\frac{\tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \sqrt{n}}{n+1} < A_n < \frac{\tfrac{2}{3}n\sqrt{n}\left(2\sqrt{2}-1\right) + \tfrac{1}{2}\left(\sqrt{n} + \sqrt{2n}\right)}{n+1}.$$

Substituting $n = 1{,}000{,}000$ into this:

$$\frac{\tfrac{2}{3}\left(2\sqrt{2}-1\right) \times 10^9 + 10^3}{1\,000\,001} < A_{1\,000\,000} < \frac{\tfrac{2}{3}\left(2\sqrt{2}-1\right) \times 10^9 + 500\left(1+\sqrt{2}\right)}{1\,000\,001}$$

and to the limit of the calculator:

$$1218{\cdot}951\,198 < A_{1\,000\,000} < 1218{\cdot}951\,405,$$

so the maximum error of 1218·9512 is about 0·000 205.

$\boxed{\surd\surd \text{ (accept 0·0002, but both bounds must be calculated)}}$

WMP