



SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
TRIAL EXAMINATIONS 2007

FORM VI

MATHEMATICS EXTENSION 2

Examination date

Wednesday 1st August 2007

Time allowed

3 hours (plus 5 minutes reading time)

Instructions

- All eight questions may be attempted.
- All eight questions are of equal value.
- All necessary working must be shown.
- Marks may not be awarded for careless or badly arranged work.
- Approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the eight questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

- SGS booklets: 8 per boy. A total of 750 booklets should be sufficient.
- Candidature: 71 boys.

Examiner

DS

QUESTION ONE (15 marks) Use a separate writing booklet.**Marks**

(a) Show that $\int_0^{\frac{\pi}{6}} x \cos x \, dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$. 3

(b) Find $\int \frac{1}{2 + \sqrt{x}} \, dx$ by using the substitution $\sqrt{x} = u$. 3

(c) Find $\int \tan^4 x \, dx$. 2

(d) (i) Show that $\int_0^1 \frac{1}{(5x+3)(x+1)} \, dx = \frac{1}{2} \ln \frac{4}{3}$. 3

(ii) Hence find $\int_0^{\frac{\pi}{2}} \frac{1}{4 \sin x - \cos x + 4} \, dx$ using the substitution $t = \tan \frac{x}{2}$. 4

QUESTION TWO (15 marks) Use a separate writing booklet.

Marks

(a) Given that $z = \frac{2+i}{1-i}$, find $z + \frac{1}{z}$ in the form $a + bi$, where a and b are real. 3

(b) Find the two square roots of $8i$ in the form $a + bi$, where a and b are real. 3

(c) Let $z = 1 + i \tan \theta$, where $0 < \theta < \frac{\pi}{2}$.
Find, in simplest form, expressions for:

(i) $|z|$ 2

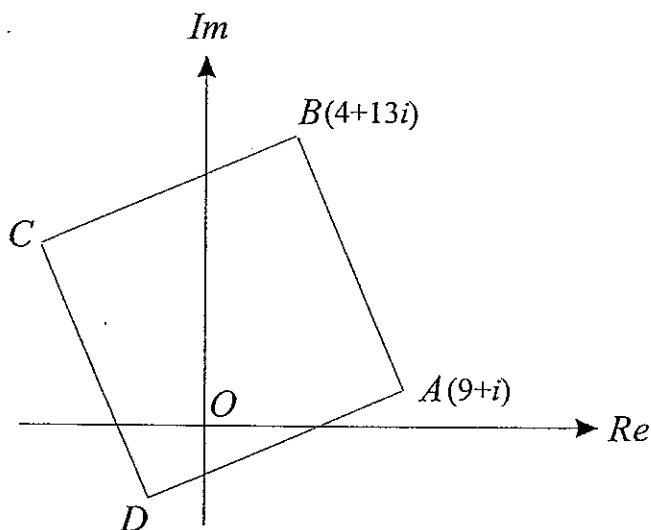
(ii) $\arg z$ 1

(d) The locus of the complex number z is defined by the equation $\arg(z + 1) = \frac{\pi}{4}$.

(i) Sketch the locus of z . 1

(ii) Find the least value of $|z|$. 2

(e)



The diagram above shows a square $ABCD$ in the complex plane. The vertices A and B represent the complex numbers $9 + i$ and $4 + 13i$ respectively. Find the complex numbers represented by:

(i) the vector AB , 1

(ii) the vertex D . 2

QUESTION THREE (15 marks) Use a separate writing booklet.

Marks

- (a) (i) Use the formulae for $\cos(A + B)$ and $\cos(A - B)$ to prove that

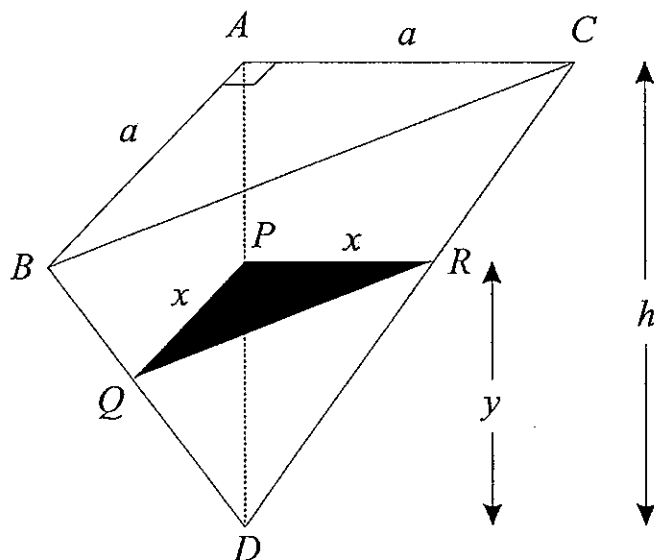
2

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}.$$

- (ii) Hence, or otherwise, solve the equation $\cos 7x + \cos 3x = 0$, for $0 \leq x \leq \frac{\pi}{2}$.

3

(b)



In the diagram above, $ABCD$ is a triangular pyramid. Its base ABC is a right-angled isosceles triangle with equal sides AB and AC of length a units, and its perpendicular height AD is h units. The typical triangular cross-section PQR shown is parallel to the base and y units above D . Let $PQ = PR = x$ units.

- (i) Find x in terms of a , h and y .

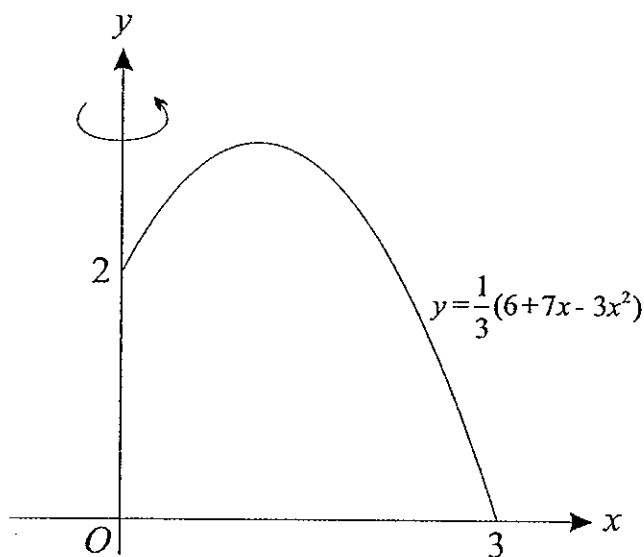
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- (ii) Use integration to find the volume of the pyramid.

4

(c)

4



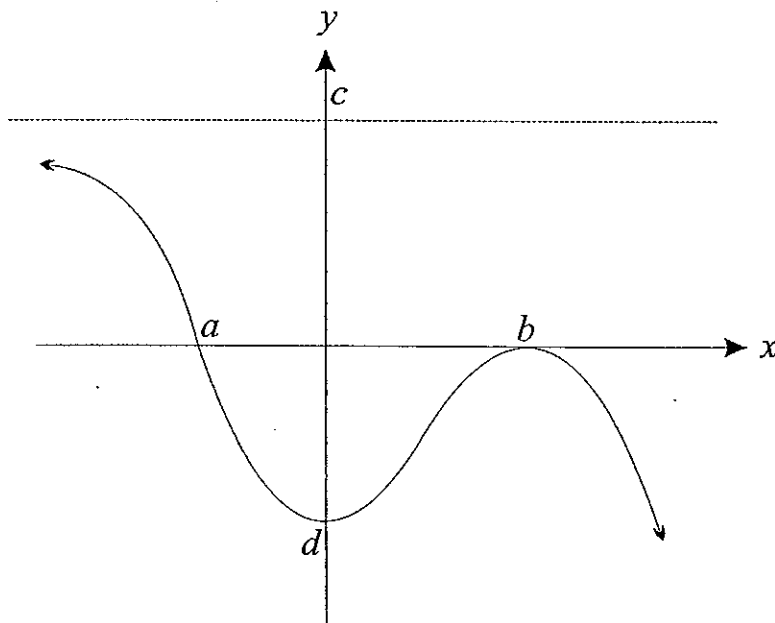
The diagram above shows the region in the first quadrant bounded by the parabola $y = \frac{1}{3}(6 + 7x - 3x^2)$ and the x and y axes. This region is rotated through 360° about the y -axis to form a solid. Use the method of cylindrical shells to find the exact volume of the solid.

QUESTION FOUR (15 marks) Use a separate writing booklet.

Marks

- (a) (i) Expand $(\sqrt{3} + 1)^2$. **1**
- (ii) The polynomial equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$ has roots α, β, γ and δ . **3**
Find the polynomial equation whose roots are $\alpha + 1, \beta + 1, \gamma + 1$ and $\delta + 1$.
- (iii) Hence, or otherwise, solve the equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$. **3**

(b)



The diagram above shows the graph of the function $y = f(x)$. Note that $c > |d| > 1$.
On separate diagrams of roughly one-third of a page, sketch the graphs of:

- (i) $y = (f(x))^2$ **2**
- (ii) $y = \frac{1}{f(x)}$ **2**
- (c) (i) Sketch the graphs of $y = x^3$ and $y = e^{-x}$ on a number plane. **1**
- (ii) Hence, on the same diagram as part (i), carefully sketch the graph of $y = x^3 e^{-x}$ without any use of calculus. **3**

QUESTION FIVE (15 marks) Use a separate writing booklet.

Marks

(a) The polynomial $P(x) = x^3 + ax + b$ has zeroes α , β and $2(\alpha - \beta)$.

(i) Show that $a = -13\alpha^2$.

2

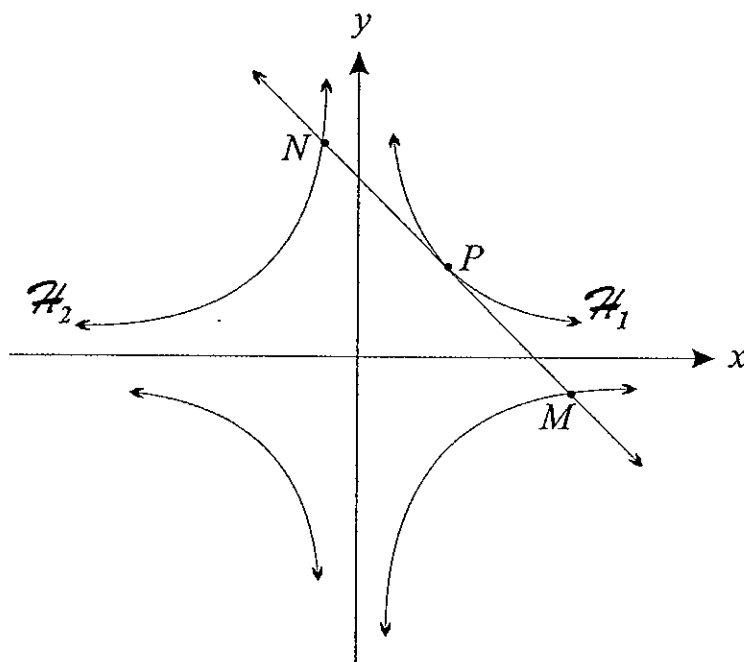
(ii) Show that $b = 12\alpha^3$.

1

(iii) Deduce that the zeroes of $P(x)$ are $-\frac{13b}{12a}$, $-\frac{13b}{4a}$ and $\frac{13b}{3a}$.

2

(b)



In the diagram above, $\mathcal{H}_1$ is the rectangular hyperbola $xy = c^2$, while $\mathcal{H}_2$ is the rectangular hyperbola $xy = -c^2$. The tangent to $\mathcal{H}_1$ at the variable point $P\left(ct, \frac{c}{t}\right)$ intersects $\mathcal{H}_2$ at M and N , as shown in the diagram. Let M and N be the points $\left(cp, -\frac{c}{p}\right)$ and $\left(cq, -\frac{c}{q}\right)$ respectively, and let T be the point of intersection of the tangents to $\mathcal{H}_2$ at M and N .

(i) Show that the tangent to $\mathcal{H}_1$ at P has equation $x + t^2y = 2ct$.

2

(ii) Use the fact that M and N lie on the tangent at P to show that $p^2 + 6pq + q^2 = 0$.

3

(iii) Find the equations of the tangents to $\mathcal{H}_2$ at M and N ,

3

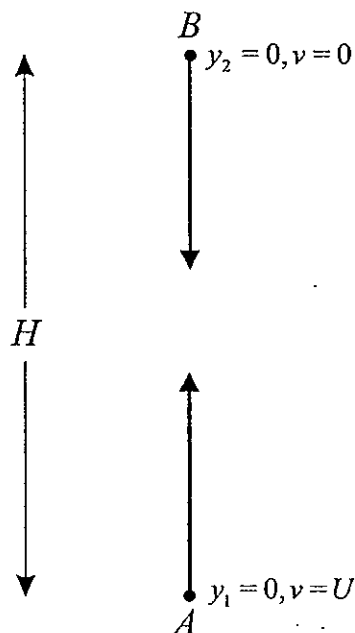
and hence show that T has coordinates $\left(\frac{2cpq}{p+q}, \frac{-2c}{p+q}\right)$.

(iv) Deduce that T lies on $\mathcal{H}_1$.

2

QUESTION SIX (15 marks) Use a separate writing booklet.**Marks**

(a)



A particle P_1 of mass m is projected vertically upwards from a point A with initial velocity U . At the same instant, a second particle P_2 , also of mass m , is dropped from a point B directly above A . The distance H between A and B is equal to the maximum height that P_1 would reach were it not to collide with P_2 . As the particles P_1 and P_2 move, they each experience air resistance of magnitude mkv^2 , where k is a positive constant and v is velocity. At the instant the particles collide, P_2 has reached 50% of its terminal velocity V . Let y_1 be the distance of P_1 above A , and y_2 the distance of P_2 below B .

(i) Show that $V = \sqrt{\frac{g}{k}}$. 2

(ii) Show that $y_1 = \frac{1}{2k} \ln \left(\frac{g + kU^2}{g + kv^2} \right)$, where v is the velocity of P_1 . 3

(iii) Hence show that $H = \frac{1}{2k} \ln \left(1 + \frac{U^2}{V^2} \right)$. 2

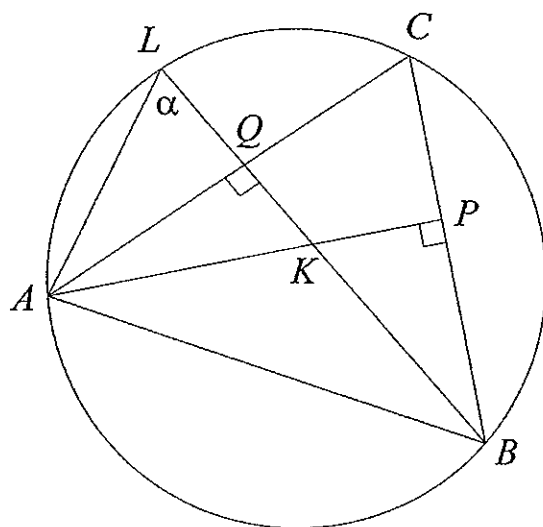
(iv) Assuming that $y_2 = \frac{1}{2k} \ln \left| \frac{g}{g - kv^2} \right|$, show that at the instant the particles collide, 2

$$y_2 = \frac{1}{2k} \ln \frac{4}{3}.$$

(v) Deduce that the speed of P_1 at the instant the particles collide is $\frac{V}{\sqrt{3}}$. 2

(b)

4



The points A , B and C lie on a circle, as shown in the diagram above. The altitudes AP and BQ of $\triangle ABC$ intersect at K . The interval BQ produced meets the circle at L . Let $\angle ALQ = \alpha$.

Prove that $AK = AL$.

QUESTION SEVEN (15 marks) Use a separate writing booklet.

Marks

(a) Let $z = \cos \theta + i \sin \theta$.

(i) Show that $\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$ and that $\sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$.

3

(ii) Hence prove that

4

$$\cos^3 \theta \sin^4 \theta = \frac{1}{64} (\cos 7\theta - \cos 5\theta - 3 \cos 3\theta + 3 \cos \theta).$$

(b) (i) Use the substitution $u = \pi - x$ to show that, for any function $f(x)$,

3

$$\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx.$$

(ii) Hence show that

5

$$\int_0^\pi \frac{x \sin^3 x}{1 + \cos^2 x} dx = \frac{\pi}{2} (\pi - 2).$$

QUESTION EIGHT (15 marks) Use a separate writing booklet.

Marks

- (a) The complex numbers ω_1 and ω_2 have modulus 1, and arguments α_1 and α_2 respectively, where $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$.

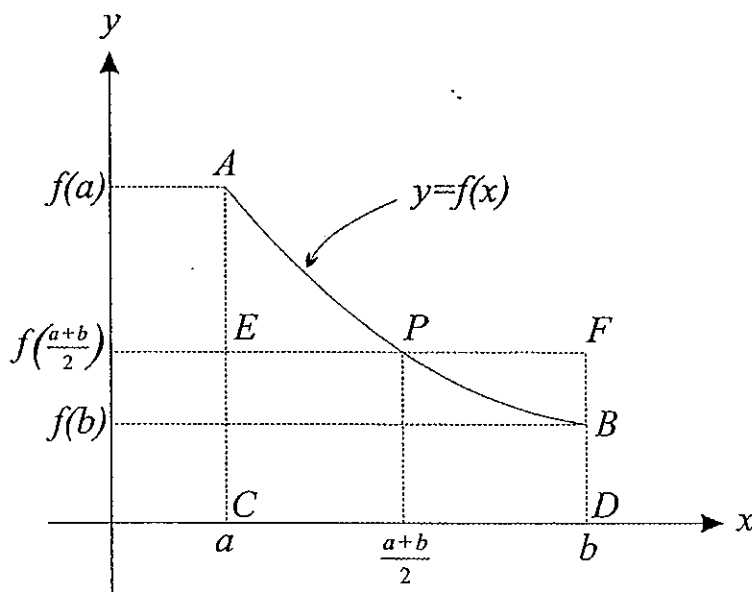
(i) Draw a diagram showing all the given information.

2

(ii) Show that $\arg(\omega_1 - \omega_2) = \frac{1}{2}(\alpha_1 + \alpha_2 - \pi)$.

3

(b)



The diagram above shows the curve $y = f(x)$ for $a \leq x \leq b$. Note that $f''(x)$ is positive for $a \leq x \leq b$.

- (i) Copy the diagram, and then use areas to explain briefly why

3

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x) dx < (b-a)\frac{f(a)+f(b)}{2}.$$

- (ii) Use the result in part (i) with $f(x) = \frac{1}{x^2}$, $a = n-1$ and $b = n$, where n is an integer greater than 1, to show that

2

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2} \left(\frac{1}{(n-1)^2} + \frac{1}{n^2} \right).$$

- (iii) Deduce that

2

$$4 \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \right).$$

- (iv) Show that

1

$$\frac{1}{2} \left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

- (v) Hence show that $\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}$.

2

END OF EXAMINATION

$$\begin{aligned}
 (1)(a) \quad & \int_0^{\frac{\pi}{6}} x \cos x \, dx \\
 &= [x \sin x]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x \, dx \quad \checkmark \\
 &= \frac{\pi}{6} \cdot \frac{1}{2} - 0 + [\cos x]_0^{\frac{\pi}{6}} \quad \checkmark \\
 &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1
 \end{aligned}$$

$$\begin{aligned}
 & \text{Let } u = x \\
 & \therefore u' = 1 \\
 & \text{Let } v' = \cos x \\
 & \therefore v = \sin x
 \end{aligned}
 \quad \checkmark$$

$$\begin{aligned}
 (b) \quad & \int \frac{1}{2+\sqrt{x}} \, dx \\
 &= \int \frac{2u}{2+u} \, du \quad \checkmark \\
 &= 2 \int \frac{(2+u)-2}{2+u} \, du \\
 &= 2 \int 1 \, du - 4 \int \frac{1}{2+u} \, du \quad \checkmark \\
 &= 2u - 4 \ln|2+u| + c \\
 &= 2\sqrt{x} - 4 \ln|2+\sqrt{x}| + c \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 & \text{Let } \sqrt{x} = u \\
 & \therefore x = u^2 \\
 & \therefore dx = 2u \, du
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad & \int \tan^4 x \, dx \\
 &= \int \tan^2 x (\sec^2 x - 1) \, dx \quad \checkmark \\
 &= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\
 &= \frac{1}{3} \tan^3 x - \tan x + x + c \quad \checkmark
 \end{aligned}$$

$$(d)(i) \text{ Let } \frac{1}{(5x+3)(x+1)} = \frac{A}{5x+3} + \frac{B}{x+1}$$

$$\therefore 1 = A(x+1) + B(5x+3)$$

$$\text{Let } x = -1.$$

$$\therefore 1 = -2B$$

$$\therefore B = -\frac{1}{2}$$

$$\text{Let } x = -\frac{3}{5}.$$

$$\therefore 1 = \frac{2}{5}A$$

$$\therefore A = \frac{5}{2}$$

$$\begin{aligned} \therefore \int_0^1 \frac{1}{(5x+3)(x+1)} dx &= \frac{1}{2} \int_0^1 \frac{5}{5x+3} - \frac{1}{2} \int_0^1 \frac{1}{x+1} dx \\ &= \frac{1}{2} \left[\ln |5x+3| \right]_0^1 - \frac{1}{2} \left[\ln |x+1| \right]_0^1 \\ &= \frac{1}{2} \left[\ln \left| \frac{5x+3}{x+1} \right| \right]_0^1 \\ &= \frac{1}{2} (\ln 4 - \ln 3) \\ &= \frac{1}{2} \ln \frac{4}{3} \end{aligned}$$

$$(ii) \int_0^{\frac{\pi}{2}} \frac{1}{4\sin x - \cos x + 4} dx$$

$$= \int_0^1 \frac{1}{4 \cdot \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 4} \cdot \frac{2}{1+t^2} dt$$

$$= \int_0^1 \frac{2}{8t - 1 + t^2 + 4 + 4t^2} dt$$

$$= \int_0^1 \frac{2}{5t^2 + 8t + 3} dt$$

$$= 2 \int_0^1 \frac{1}{(5t+3)(t+1)} dt$$

$$= 2 \cdot \frac{1}{2} \ln \frac{4}{3} \quad (\text{using (i)})$$

$$= \ln \frac{4}{3}$$

$$\text{Let } t = \tan \frac{x}{2}$$

$$\therefore x = 2 \tan^{-1} t$$

$$\therefore dx = \frac{2}{1+t^2} dt$$

x	0	$\frac{\pi}{2}$
t	0	1

$$\begin{aligned}
 (2)(a) \quad z + \frac{1}{z} &= \frac{2+i}{1-i} + \frac{1-i}{2+i} \\
 &= \frac{(2+i)(1+i)}{2} + \frac{(1-i)(2-i)}{5} \quad \checkmark \\
 &= \frac{1+3i}{2} + \frac{1-3i}{5} \quad \checkmark \\
 &= \frac{5+15i+2-6i}{10} \quad \checkmark \\
 &= \frac{7}{10} + \frac{9}{10}i
 \end{aligned}$$

(b) Let $(a+bi)^2 = 8i$.

$$\therefore (a^2 - b^2) + 2abi = 0 + 8i \quad \checkmark$$

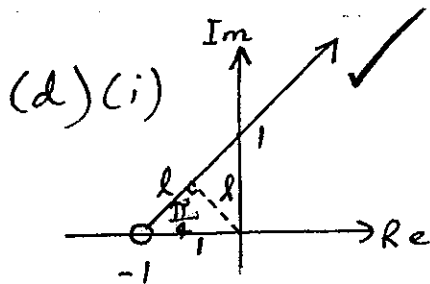
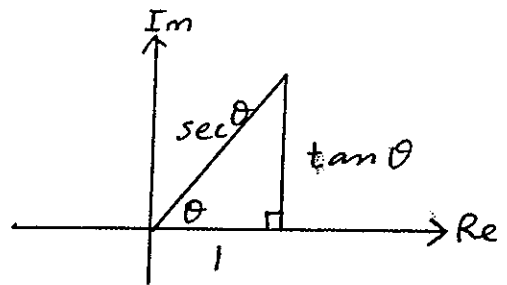
$$\therefore a^2 - b^2 = 0 \text{ and } ab = 4 \quad \checkmark$$

$$\therefore (a, b) = (2, 2) \text{ or } (-2, -2) \quad \checkmark$$

So the square roots of $8i$ are $2+2i$ and $-2-2i$.

$$\begin{aligned}
 (c)(i) \quad |z| &= \sqrt{1 + \tan^2 \theta} \quad \checkmark \\
 &= \sec \theta \quad \checkmark
 \end{aligned}$$

$$(ii) \arg z = \theta \quad \checkmark$$



$$(ii) \quad l^2 + l^2 = 1 \quad (\text{Pythagoras}) \quad \checkmark$$

$$\therefore l = \frac{1}{\sqrt{2}} \quad \checkmark$$

So the least value of $|z|$ is $\frac{1}{\sqrt{2}}$.

$$(e)(i) \quad \vec{AB} = \vec{OB} - \vec{OA}$$

$$\begin{aligned}
 \therefore \vec{AB} &\text{ represents } (4+13i) - (9+i) \\
 &= -5 + 12i \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \vec{AD} &\text{ represents } (-5+12i)i \\
 &= -12 - 5i \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \vec{OD} &= \vec{OA} + \vec{AD}, \\
 \text{so } \vec{OD} &\text{ represents } (9+i) + (-12-5i) \\
 &= -3 - 4i, \\
 \text{so } D &\text{ represents } -3 - 4i. \quad \checkmark
 \end{aligned}$$

$$(3)(a)(i) \quad \cos(A+B) = \cos A \cos B - \sin A \sin B \quad (1)$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \quad (2)$$

$$(1) + (2) : \cos(A+B) + \cos(A-B) = 2\cos A \cos B \quad (3)$$

Let $A+B = P$ and let $A-B = Q$.

$$\therefore A = \frac{P+Q}{2} \text{ and } B = \frac{P-Q}{2}$$

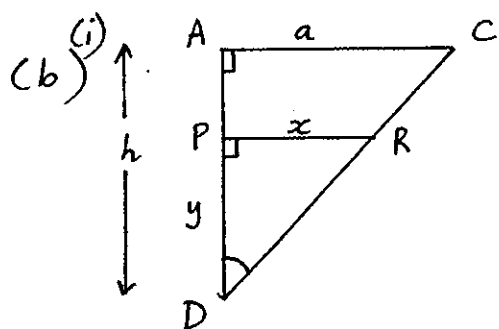
$$\text{Substitute into (3)} : \cos P + \cos Q = 2\cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$

$$(ii) \quad \cos 7x + \cos 3x = 0, \quad 0 \leq x \leq \frac{\pi}{2}$$

$$\therefore 2\cos 5x \cos 2x = 0, \quad 0 \leq 5x \leq \frac{5\pi}{2} \text{ and } 0 \leq 2x \leq \pi$$

$$\therefore 5x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } 2x = \frac{\pi}{2}$$

$$\therefore x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2} \text{ or } \frac{\pi}{4}$$



$\triangle PRD \parallel \triangle ACD$ (equiangular)

$$\therefore \frac{x}{a} = \frac{y}{h} \quad (\text{matching sides in equal ratios})$$

$$\therefore x = \frac{ay}{h}$$

$$(ii) \quad V = \int_{y=0}^{y=h} \frac{1}{2} x^2 dy$$

$$= \int_0^h \frac{1}{2} \left(\frac{ay}{h} \right)^2 dy$$

$$= \frac{a^2}{2h^2} \int_0^h y^2 dy$$

$$= \frac{a^2}{2h^2} \left[\frac{y^3}{3} \right]_0^h$$

$$= \frac{a^2}{2h^2} \cdot \frac{h^3}{3}$$

$$= \frac{1}{6} a^2 h$$

$$(c) \quad V = \pi \int_0^3 2\pi r h \, dx, \text{ where } r = x \text{ and } h = y$$

$$= 2\pi \int_0^3 xy \, dx \quad \checkmark$$

$$= \frac{2\pi}{3} \int_0^3 (6x + 7x^2 - 3x^3) \, dx \quad \checkmark$$

$$= \frac{2\pi}{3} \left[3x^2 + \frac{7x^3}{3} - \frac{3x^4}{4} \right]_0^3 \quad \checkmark$$

$$= \frac{2\pi}{3} \left(27 + 63 - \frac{243}{4} \right) \quad \checkmark$$

$$= \frac{39\pi}{2} \, u^3 \quad \checkmark$$

(4)(a)(i) $(\sqrt{3}+1)^2 = 4+2\sqrt{3}$ ✓

(ii) Replace x with $x-1$.

The required equation is

$$(x-1)^4 + 4(x-1)^3 - 2(x-1)^2 - 12(x-1) - 3 = 0$$
 ✓

$$x^4 - 4x^3 + 6x^2 - 4x + 1 + 4x^3 - 12x^2 + 12x - 4 - 2x^2 + 4x - 2 - 12x + 12 - 3 = 0$$
 ✓

$$x^4 - 8x^2 + 4 = 0$$
 ✓

(iii) $x^2 = \frac{8 \pm \sqrt{48}}{2}$

$$= 4 \pm 2\sqrt{3}$$
 ✓

So the new equation has roots

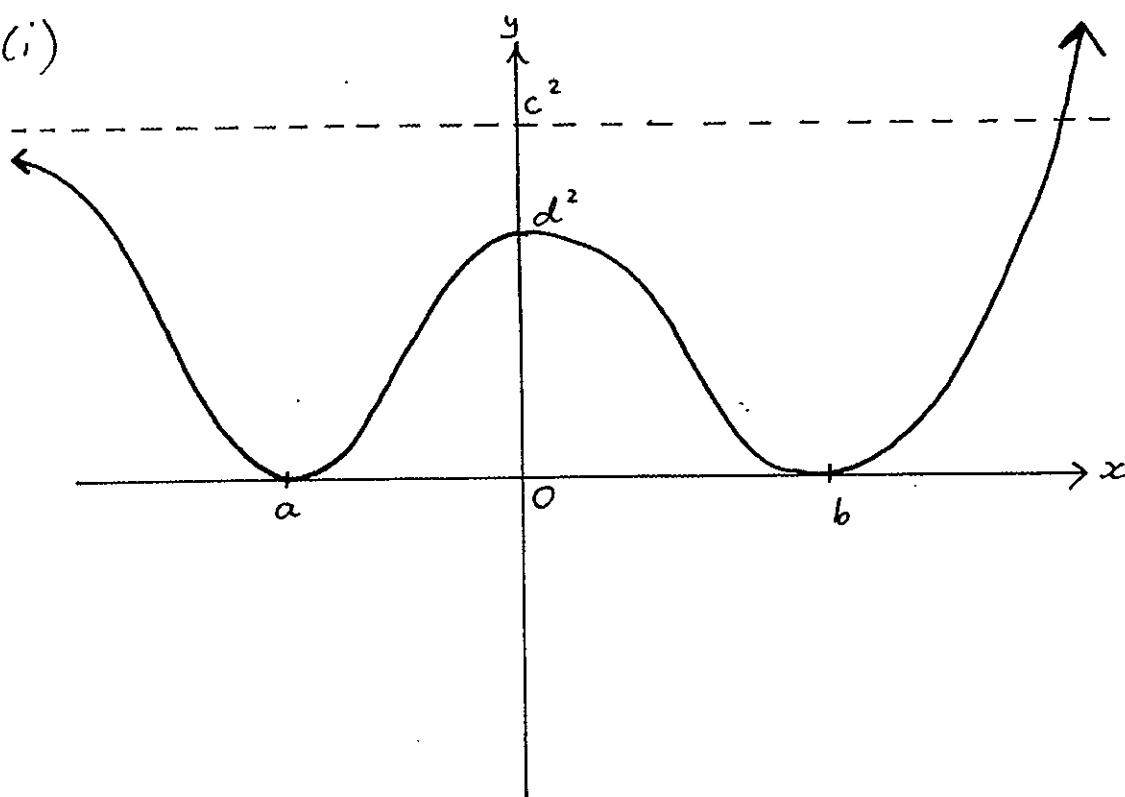
$$x = \sqrt{3}+1, -\sqrt{3}-1, \sqrt{3}-1 \text{ or } -\sqrt{3}+1$$
 ✓

(using part (i)).

So the original equation has roots

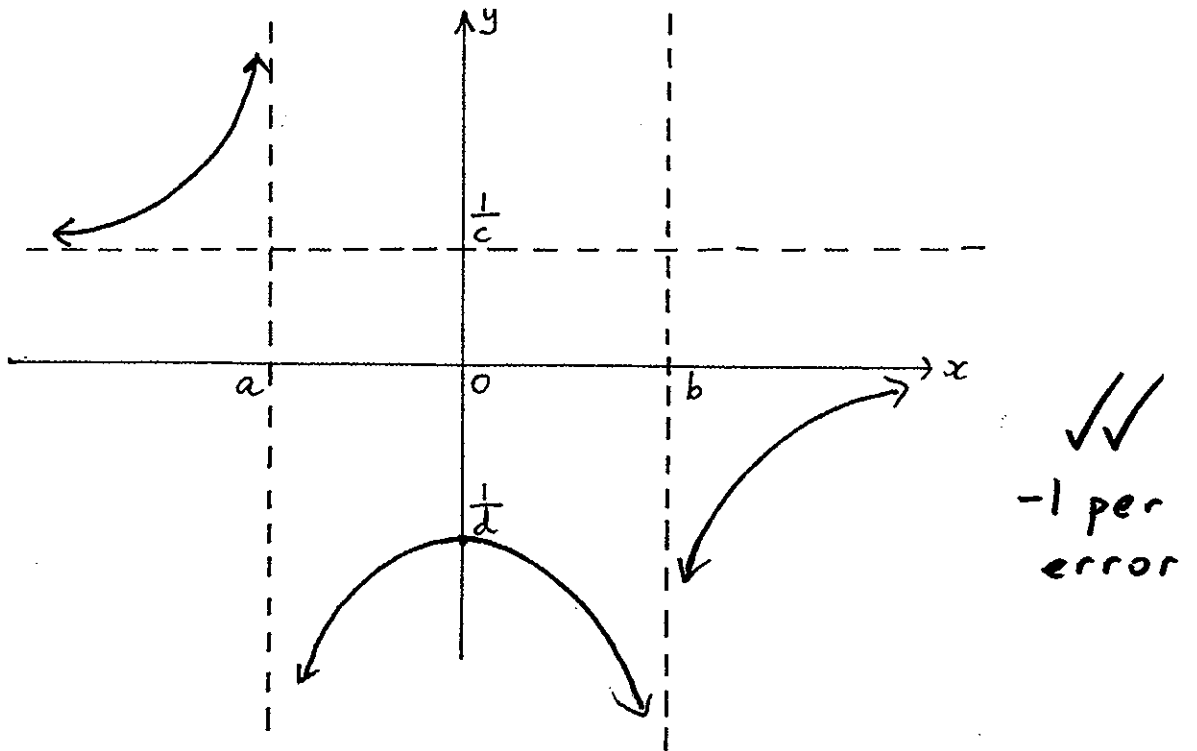
$$x = \sqrt{3}, -\sqrt{3}-2, \sqrt{3}-2 \text{ or } -\sqrt{3}.$$
 ✓

(b)(i)



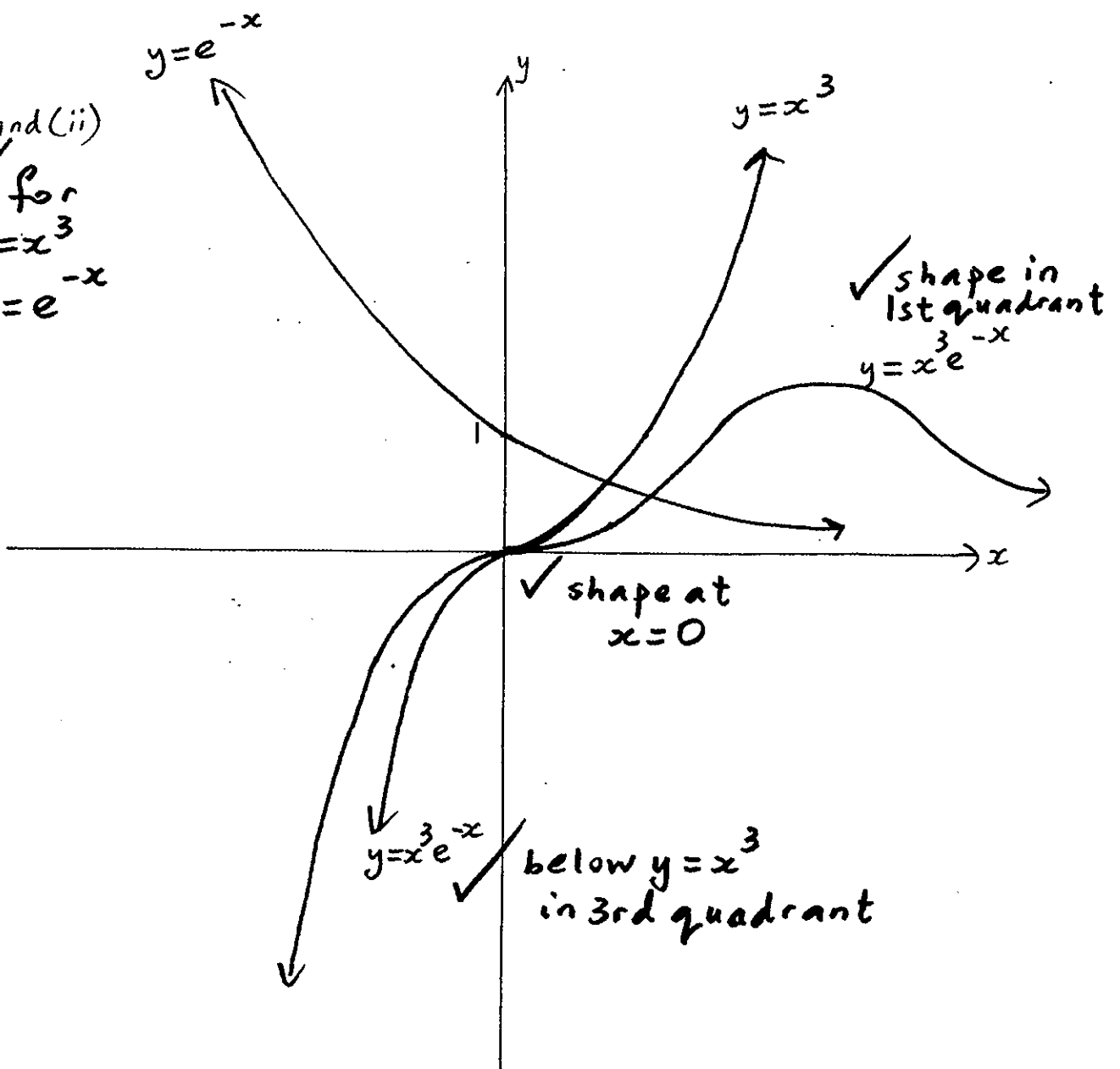
✓✓
-1 per
error

(ii)



✓✓
-1 per
error

(c) (i) and (ii)
✓ for
 $y=x^3$
and $y=e^{-x}$



✓ shape in
1st quadrant

✓ shape at
 $x=0$

below $y=x^3$
in 3rd quadrant

(5)(a)(i) Sum of zeros = 0

$$\therefore \alpha + \beta + 2\alpha - 2\beta = 0$$

$$\therefore 3\alpha = \beta \quad (1) \quad \checkmark$$

Sum of zeros multiplied in pairs = a

$$\therefore \alpha\beta + 2\alpha(\alpha - \beta) + 2\beta(\alpha - \beta) = a$$

$$\alpha\beta + 2\alpha^2 - \cancel{2\alpha\beta} + \cancel{2\alpha\beta} - 2\beta^2 = a$$

Substitute $\beta = 3\alpha$ from (1):

$$\therefore 3\alpha^2 + 2\alpha^2 - 18\alpha^2 = a$$

$$\therefore a = -13\alpha^2 \quad \checkmark$$

(ii) Product of zeros = -b

$$\therefore 2\alpha\beta(\alpha - \beta) = -b$$

Substitute $\beta = 3\alpha$ from (1):

$$6\alpha^2 \cdot (-2\alpha) = -b$$

$$\therefore b = 12\alpha^3 \quad \checkmark$$

(iii) Dividing the results in (ii) and (i),

$$\frac{b}{a} = -\frac{12\alpha}{13}$$

$$\therefore \alpha = -\frac{13b}{12a}$$

$$\text{Using (1), } \beta = 3 \cdot \left(-\frac{13b}{12a}\right)$$

$$= -\frac{13b}{4a}$$

$$\text{Finally, } 2(\alpha - \beta) = 2\left(-\frac{13b}{12a} + \frac{13b}{4a}\right)$$

$$= 26b \cdot \frac{-1 + 3}{12a}$$

$$= \frac{26b}{6a}$$

$$= \frac{13b}{3a} \quad \checkmark$$

So the zeros of $P(x)$ are $-\frac{13b}{12a}$, $-\frac{13b}{4a}$ and $\frac{13b}{3a}$.

(b) (i) $\mathcal{H}_1$ has equation $y = c^2 x^{-1}$

$$\therefore y' = -c^2 x^{-2} \\ = -\frac{c^2}{x^2}$$

$$\therefore \text{at } P, \text{ gradient is } -\frac{c^2}{c^2 t^2} = -\frac{1}{t^2}$$

$\therefore$ tangent at P has equation

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2 y - ct = -x + ct$$

$$x + t^2 y = 2ct$$

(ii) $M(cp, -\frac{c}{p})$ and $N(cq, -\frac{c}{q})$ lie on the line $x + t^2 y = 2ct$.

$$\therefore cp - \frac{ct^2}{p} = 2ct \quad \text{and} \quad cq - \frac{ct^2}{q} = 2ct$$

$$\therefore cp^2 - ct^2 = 2cpt \quad (*) \quad \text{and} \quad cq^2 - ct^2 = 2cqt$$

Subtracting, we get $c(p^2 - q^2) = 2ct(p - q)$

$$\therefore t = \frac{1}{2}(p + q)$$

Substitute into (*):

$$cp^2 - c \cdot \frac{1}{4}(p+q)^2 = 2cp \cdot \frac{1}{2}(p+q)$$

$$\frac{1}{4}(p+q)^2 + p(p+q) - p^2 = 0$$

$$\frac{1}{4}(p^2 + 2pq + q^2) + pq = 0$$

$$p^2 + 2pq + q^2 + 4pq = 0$$

$$p^2 + 6pq + q^2 = 0$$

(iii) $\mathcal{H}_2$ has equation $y = -c^2 x^{-1}$

$$\therefore y' = +c^2 x^{-2} \\ = \frac{c^2}{x^2}$$

$$\therefore \text{at } M, \text{ gradient is } \frac{c^2}{c^2 p^2} = \frac{1}{p^2}$$

$\therefore$ tangent at M has equation

$$y + \frac{c}{p} = \frac{1}{p^2}(x - cp)$$

$$p^2 y + cp = x - cp$$

$$x - p^2 y = 2cp \quad (1)$$

Similarly, the tangent at N has equation

$$x - q^2 y = 2cq \quad (2)$$

$$(2) - (1): (p^2 - q^2)y = -2c(p - q)$$

$$\therefore y = \frac{-2c}{p+q}$$

Substitute into (1):

$$x = 2cp + p^2 \cdot \frac{-2c}{p+q}$$

$$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$$

$$= \frac{2cpq}{p+q}$$

$$\therefore T \text{ is the point } \left(\frac{2cpq}{p+q}, \frac{-2c}{p+q} \right)$$

(iv) T lies on $\mathcal{H}_1: xy = c^2$

if its coordinates satisfy $xy = c^2$.

$$\text{That is, if } \frac{-4c^2 pq}{(p+q)^2} = c^2$$

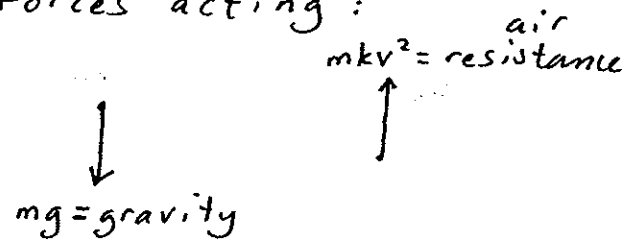
$$-4pq = (p+q)^2$$

$$p^2 + 6pq + q^2 = 0$$

We know from part (ii) that this condition is satisfied, so T lies on $\mathcal{H}_1$.

(b)(a)(i) For P_2 :

Forces acting :



$$\therefore m\ddot{y}_2 = mg - mkv^2 \quad \left\{ \begin{array}{l} \text{(taking downwards as positive)} \\ \ddot{y}_2 = g - kv^2 \end{array} \right. \checkmark$$

When $\ddot{y}_2 = 0$, $v = V$

$$\therefore 0 = g - kV^2$$

$$\therefore V = \sqrt{\frac{g}{k}} \quad (V > 0)$$

(ii) For P_1 :

Forces acting :



$$\therefore m\ddot{y}_1 = -mg - mkv^2 \quad \text{(taking upwards as positive)}$$
$$\ddot{y}_1 = -g - kv^2$$

$$\therefore v \cdot \frac{dv}{dy_1} = -g - kv^2 \quad \checkmark$$

$$\frac{dy_1}{dv} = \frac{-v}{g + kv^2}$$

$$\therefore y_1 = -\frac{1}{2k} \int \frac{2kv}{g + kv^2} dv$$

$$= -\frac{1}{2k} \ln |g + kv^2| + c \quad \checkmark$$

When $t=0$, $y_1=0$ and $v=U$.

$$\therefore c = \frac{1}{2k} \ln |g + kU^2|$$

$$\therefore y_1 = \frac{1}{2k} \left(\ln |g + kU^2| - \ln |g + kv^2| \right) \quad \checkmark$$

$$\therefore y_1 = \frac{1}{2k} \ln \left| \frac{g + kU^2}{g + kv^2} \right|$$

(or $\frac{1}{2k} \ln \left(\frac{g + kU^2}{g + kv^2} \right)$, since numerator and denominator are both positive.)

(iii) when $v=0$, $y_1 = H$ (if no collision occurs).

$$\begin{aligned}\therefore H &= \frac{1}{2k} \ln \left(\frac{g + kU^2}{g} \right) \checkmark \\ &= \frac{1}{2k} \ln \left(1 + \frac{kU^2}{g} \right) \\ &= \frac{1}{2k} \ln \left(1 + \frac{U^2}{g/k} \right) \\ &= \frac{1}{2k} \ln \left(1 + \frac{U^2}{V^2} \right) \end{aligned} \quad \left. \vphantom{\begin{aligned} \end{aligned}} \right\} \checkmark$$

(iv) At the instant the particles collide, $y_1 + y_2 = H$, and the speed of P_2 is 50% of $V = \frac{V}{2}$.

$$\begin{aligned}\text{So } y_2 &= \frac{1}{2k} \ln \left| \frac{g}{g - k \cdot \frac{V^2}{4}} \right| \checkmark \\ &= \frac{1}{2k} \ln \left| \frac{4g}{4g - kV^2} \right| \\ &= \frac{1}{2k} \ln \left| \frac{4g}{4g - k \cdot \frac{g}{k}} \right| \quad (\text{using (i)}) \\ &= \frac{1}{2k} \ln \frac{4}{3} \end{aligned} \quad \left. \vphantom{\begin{aligned} \end{aligned}} \right\} \checkmark$$

(v) Find v (the speed of P_1) when $y_1 + y_2 = H$.

$$\frac{1}{2k} \ln \left(\frac{g + kU^2}{g + kv^2} \right) + \frac{1}{2k} \ln \frac{4}{3} = \frac{1}{2k} \ln \left(1 + \frac{U^2}{V^2} \right) \quad (\text{using (iii)})$$

$$\therefore \frac{4}{3} \cdot \frac{g/k + U^2}{g/k + v^2} = 1 + \frac{U^2}{V^2} \quad \checkmark$$

$$\frac{4}{3} \cdot \frac{V^2 + U^2}{V^2 + v^2} = \frac{V^2 + U^2}{V^2}$$

$$\frac{4}{3V^2 + 3v^2} = \frac{1}{V^2}$$

$$3V^2 = V^2 + 3v^2$$

$$\therefore v^2 = \frac{V^2}{3}$$

$$\therefore v = \frac{V}{\sqrt{3}} \quad (v > 0)$$

[illegible]

✓ But quadrilateral $QCPK$ is cyclic (opposite angles CQK and CPK are supplementary).

✓ { $\therefore \triangle AKL$ is isosceles (since two of its angles $\angle ALK$ and $\angle AKL$ are equal).

{ $\therefore AK = AL$ (sides opposite equal angles).

(Don't be too strict with the reasons.)

$$\begin{aligned}
 (7) (a) (i) \quad z + \frac{1}{z} &= \cos \theta + i \sin \theta + (\cos \theta + i \sin \theta)^{-1} \\
 &= \cos \theta + i \sin \theta + \cos(-\theta) + i \sin(-\theta) \quad \checkmark \\
 &\quad \text{(de Moivre's theorem)} \\
 &= \cos \theta + i \sin \theta + \cos \theta - i \sin \theta \quad \checkmark \\
 &= 2 \cos \theta
 \end{aligned}$$

$$\therefore \cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

$$\begin{aligned}
 \text{Similarly, } z - \frac{1}{z} &= \cos \theta + i \sin \theta - (\cos \theta - i \sin \theta) \quad \checkmark \\
 &= 2i \sin \theta
 \end{aligned}$$

$$\therefore \sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$$

$$\begin{aligned}
 (ii) \quad \cos^3 \theta \sin^4 \theta &= \frac{1}{8} \left(z + \frac{1}{z} \right)^3 \cdot \frac{1}{16} \left(z - \frac{1}{z} \right)^4 \\
 &= \frac{1}{128} \left(z^2 - \frac{1}{z^2} \right)^3 \left(z - \frac{1}{z} \right) \quad \checkmark \\
 &= \frac{1}{128} \left(z - \frac{1}{z} \right) \left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6} \right) \\
 &= \frac{1}{128} \left(z^7 - 3z^3 + \frac{3}{z} - \frac{1}{z^5} - z^5 + 3z - \frac{3}{z^3} + \frac{1}{z^7} \right) \quad \checkmark \\
 &= \frac{1}{128} \left(\left(z^7 + \frac{1}{z^7} \right) - \left(z^5 + \frac{1}{z^5} \right) - 3 \left(z^3 + \frac{1}{z^3} \right) + 3 \left(z + \frac{1}{z} \right) \right) \quad \checkmark \\
 &= \frac{1}{128} \left(2 \cos 7\theta - 2 \cos 5\theta - 3(2 \cos 3\theta) + 3(2 \cos \theta) \right) \\
 &= \frac{1}{64} \left(\cos 7\theta - \cos 5\theta - 3 \cos 3\theta + 3 \cos \theta \right) \quad \checkmark
 \end{aligned}$$

$$(b)(i) \int_0^{\pi} x \cdot f(\sin x) dx$$

$$= - \int_{\pi}^0 (\pi - u) \cdot f(\sin(\pi - u)) du \checkmark$$

$$\checkmark \left\{ \begin{aligned} &= \pi \int_0^{\pi} f(\sin u) du - \int_0^{\pi} u \cdot f(\sin u) du \\ &= \pi \int_0^{\pi} f(\sin x) dx - \int_0^{\pi} x \cdot f(\sin x) dx \end{aligned} \right.$$

$$\text{Let } u = \pi - x$$

$$\therefore dx = -du$$

x	0	π
u	π	0

$$\therefore 2 \int_0^{\pi} x \cdot f(\sin x) dx = \pi \int_0^{\pi} f(\sin x) dx$$

$$\therefore \int_0^{\pi} x \cdot f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx \quad \checkmark$$

$$(ii) \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx$$

$$= \int_0^{\pi} x \cdot \frac{\sin^3 x}{2 - \sin^2 x} dx$$

$$= \int_0^{\pi} x \cdot f(\sin x) dx, \text{ where } f(\sin x) = \frac{\sin^3 x}{2 - \sin^2 x}$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{\sin^3 x}{2 - \sin^2 x} dx \quad (\text{using (i)}) \checkmark$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{1 - \cos^2 x}{1 + \cos^2 x} \cdot \sin x dx \checkmark$$

$$= -\frac{\pi}{2} \int_{-1}^{+1} \frac{1 - u^2}{1 + u^2} du \checkmark$$

$$= \frac{\pi}{2} \int_{-1}^{+1} \frac{-(1 + u^2) + 2}{1 + u^2} du \checkmark$$

$$= \frac{\pi}{2} \left[-u + 2 \tan^{-1} u \right]_{-1}^{+1}$$

$$= \frac{\pi}{2} \left[-1 + \frac{\pi}{2} - \left(1 - \frac{\pi}{2} \right) \right]$$

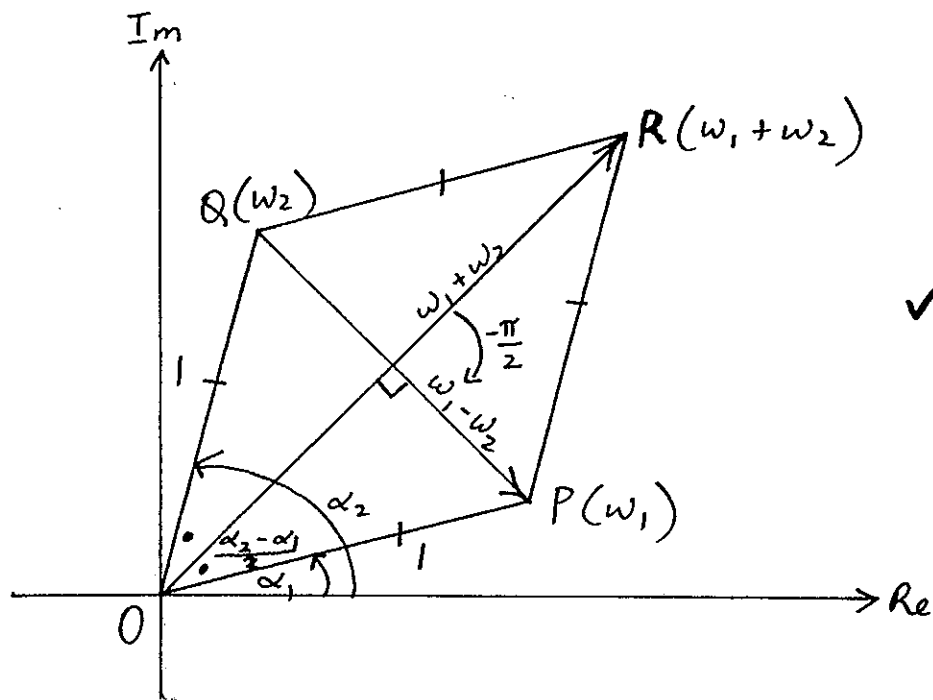
$$= \frac{\pi}{2} (\pi - 2)$$

$$\text{Let } u = \cos x$$

$$\therefore \sin x dx = -du$$

x	0	π
u	1	-1

(8)(a)(i)



(ii) $OPRQ$ is a rhombus, since $|w_1| = |w_2|$.

✓ $\therefore \angle QOR = \angle POR = \frac{\alpha_2 - \alpha_1}{2}$ (diagonal OR of rhombus bisects $\angle QOP$)

✓ { $\therefore \arg(w_1 + w_2) = \alpha_1 + \angle POR$

$$= \alpha_1 + \frac{\alpha_2 - \alpha_1}{2}$$

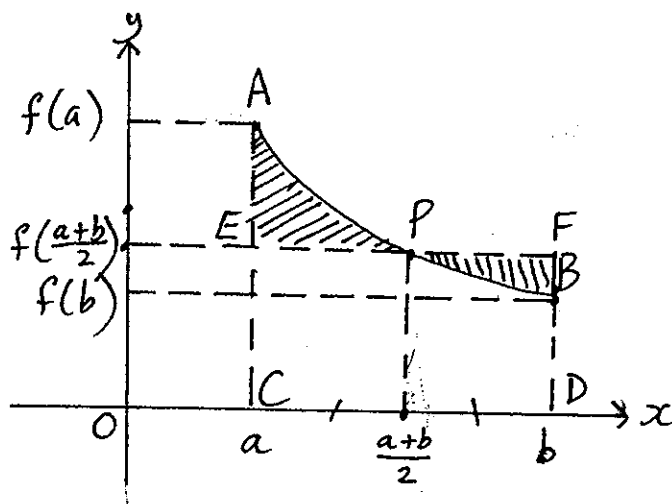
$$= \frac{\alpha_1 + \alpha_2}{2}$$

✓ { $\therefore \arg(w_1 - w_2) = \arg(w_1 + w_2) - \frac{\pi}{2}$ (the diagonals OR and QP are perpendicular)

$$= \frac{\alpha_1 + \alpha_2}{2} - \frac{\pi}{2}$$

$$= \frac{1}{2}(\alpha_1 + \alpha_2 - \pi)$$

(b)



(i) Exact area between curve < area of trapezium ACDB and x-axis

$$\therefore \int_a^b f(x) dx < \frac{b-a}{2} (f(a) + f(b))$$

Also, area of portion PFB < area of portion PEA, since the arc AP is steeper than the arc PB.

$\therefore$ area of rectangle EFDC < exact area between curve and x-axis

$$\therefore (b-a) \cdot f\left(\frac{a+b}{2}\right) < \int_a^b f(x) dx$$

(ii) Let $f(x) = \frac{1}{x^2}$, $a = n-1$, $b = n$ in (i), so that $\frac{a+b}{2} = \frac{2n-1}{2}$.

$$\therefore \frac{4}{(2n-1)^2} < \int_{n-1}^n x^{-2} dx < \frac{1}{2} \left(\frac{1}{(n-1)^2} + \frac{1}{n^2} \right)$$

$$\therefore \frac{4}{(2n-1)^2} < \left[-\frac{1}{x} \right]_{n-1}^n < \frac{1}{2} \left(\frac{1}{(n-1)^2} + \frac{1}{n^2} \right)$$

$$\therefore \frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2} \left(\frac{1}{(n-1)^2} + \frac{1}{n^2} \right)$$

(iii) Put $n = 2, 3, 4, \dots$ into the result in (ii) and add: ✓ (for the idea)

$$\frac{4}{3^2} + \frac{4}{5^2} + \frac{4}{7^2} + \dots < \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots$$

$$< \frac{1}{2} \left(\left(\frac{1}{1^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \dots \right)$$

$$\therefore 4 \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$$

(iv) LHS = $\frac{1}{2} \left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \right)$

$$< \frac{1}{2} \left(\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \dots \right) \quad \checkmark$$

$$= \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$= \text{RHS}$$

(v) $\left\{ \begin{array}{l} \text{From (iv), } 2 \left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \right) < 4 \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right) \\ \text{So using (iii),} \end{array} \right.$

$$2 \left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$$

$$\therefore \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots < \frac{1}{2} \text{ and } \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots > \frac{1}{2}$$

$$\therefore \left. \begin{array}{l} 1 + \frac{1}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{2} \\ \text{i.e. } \frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4} \end{array} \right\} \quad \checkmark$$