SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

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CANDIDATE NUMBER

**2025** **Trial HSC Examination**

# Form VI Mathematics Extension 2

## Thursday 14th August 2025
## 8:40am

**General Instructions**

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

**Total Marks: 100**

**Section I (10 marks)** Questions 1 – 10
- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

**Section II (90 marks)** Questions 11 – 16
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

**Collection**

- Your name and master should only be written on this page.
- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.

**Checklist**
- Reference sheet
- Multiple-choice answer sheet
- 6 booklets per boy
- Candidature: 85 pupils

**Writer:** RR

# Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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1. A line has Cartesian equation $x+2y=3$. Which of these could be its vector equation?

   (A) $\underset{\sim}{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (B) $\underset{\sim}{r} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

   (C) $\underset{\sim}{r} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

   (D) $\underset{\sim}{r} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2. A particle is moving in simple harmonic motion with a period of motion of 2 seconds. When it passes through its centre of motion, it has a speed of 1 metre per second. What is the maximum displacement of this particle from its centre of motion?

   (A) $\frac{1}{\pi}$ metres

   (B) 1 metre

   (C) $\pi$ metres

   (D) $\pi^2$ metres

3. Which of the following is a correct integral for $\displaystyle\int x \cos x \; dx$?

   (A) $-x \sin x - \cos x + C$

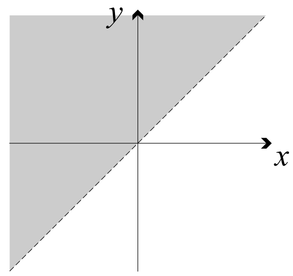
   (B) $-x \cos x + \sin x + C$

   (C) $x \sin x + \cos x + C$

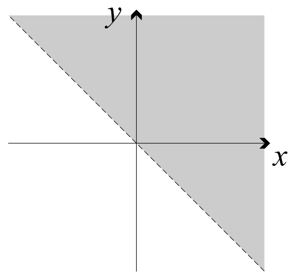
   (D) $x \cos x - \sin x + C$

4. Which of the following best represents the region of the complex plane that satisfies the inequation $\mathrm{Re}(\bar{z}) > \mathrm{Im}(\bar{z})$ ?

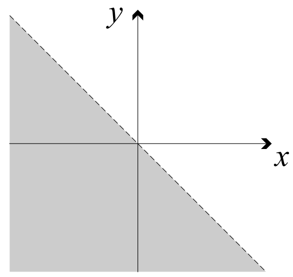
(A)



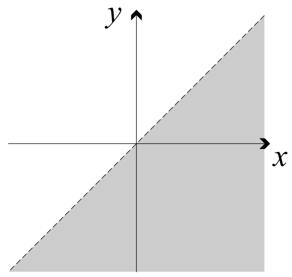

(B)




(C)




(D)




5. Consider the two vectors $\underset{\sim}{u} = \begin{bmatrix} 3-t \\ t+1 \\ 2t-1 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} t+5 \\ 3t+3 \\ 3 \end{bmatrix}$.

For which value of $t$ will the two vectors be parallel?

(A) $t = -1$

(B) $t = 0$

(C) $t = 1$

(D) $t = 2$

6. Consider the following statement:

"For all even numbers $n$, there is an odd number between $n$ and $2n$."

What is the negation of this statement?

(A) For all odd numbers $n$, there is an even number between $n$ and $2n$.

(B) For all even numbers $n$, there are no odd numbers between $n$ and $2n$.

(C) There exists some even number $n$, such that there are no odd numbers between $n$ and $2n$.

(D) There exists some even number $n$, such that there exists an even number between $n$ and $2n$.

7. Let $z$ be a complex number. Which of the following will always be real?

   (A) $(z+\bar{z})(z-\bar{z})$

   (B) $(z+i\bar{z})(z-i\bar{z})$

   (C) $(z+i\bar{z})(\bar{z}+iz)$

   (D) $(z-i\bar{z})(\bar{z}-iz)$

8. Let $\omega$ be a complex cube root of unity.

   Which of these is equivalent to $\dfrac{1}{1-\omega}+\dfrac{1}{1-\omega^2}$ ?

   (A) $0$

   (B) $1$

   (C) $\omega$

   (D) $\omega^2$

9. Given that $x$ and $y$ are real numbers, which of the following is true?

   (A) If $x<y$ then $x-y<|x|-|y|$

   (B) If $x<y$ then $x-y>|x|-|y|$

   (C) If $x-y<|x|-|y|$ then $x<y$

   (D) If $x-y<|x|-|y|$ then $x>y$

10. Which of the following is a correct integral for $\displaystyle\int \frac{\tan^{-1}\left(\frac{1}{x}\right)}{1+x^2}\,dx$?

   (A) $\frac{1}{2}(\tan^{-1}x)^2+C$

   (B) $-\frac{1}{2}(\tan^{-1}x)^2+C$

   (C) $\frac{1}{2}(\tan^{-1}x)^2+\frac{\pi}{2}\tan^{-1}x+C$

   (D) $-\frac{1}{2}(\tan^{-1}x)^2+\frac{\pi}{2}\tan^{-1}x+C$

**End of Section I**

**The paper continues in the next section**

# Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

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**QUESTION ELEVEN** (15 marks) Start a new answer booklet. **Marks**

(a) Consider the complex numbers $z = 1 + \sqrt{3}i$ and $w = 1 - i$.

(i) Express both $z$ and $w$ in modulus-argument form. 2

(ii) Find the product $zw$ in two different ways, and hence give the exact value of $\cos \frac{\pi}{12}$. 3

(b) Consider the two vectors $\underset{\sim}{u} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 4 \\ -5 \\ 3 \end{bmatrix}$.

(i) Find the lengths $|\underset{\sim}{u}|$ and $|\underset{\sim}{v}|$. 1

(ii) Hence find the angle between the two vectors. Give your answer correct to the nearest degree. 2

(c) Suppose $z = e^{i\theta} = \cos\theta + i\sin\theta$.

(i) Show that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$. 1

(ii) Hence show that $1 + z^2 = 2z\cos\theta$. 1

(d) Find the following integrals:

(i) $\displaystyle\int_1^e x \ln x \, dx$ 2

(ii) $\displaystyle\int x\sqrt{1-x}\, dx$ 3

**The paper continues on the next page**

**QUESTION TWELVE** (15 marks) Start a new answer booklet. **Marks**

(a) (i) Find the two square roots of $5 - 12i$. 2

(ii) Hence solve $z^2 + 3z + 1 + 3i = 0$. 2

(b) (i) Determine values of $A, B$, and $C$ such that 2

$$\frac{x-5}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}\,.$$

(ii) Hence determine $\displaystyle\int \frac{x-5}{(x-1)(x^2+1)}\,dx$. 2

(c) A particle has velocity $v = 2 - x$, and is initially at the origin.

(i) Show that $t = \ln\left(\dfrac{2}{2-x}\right)$. 2

(ii) Describe what happens to the position of the particle as $t \to \infty$. 1

(d) Consider the following statement:

"If $\sin 2\theta$ is irrational, then $\sin\theta + \cos\theta$ is irrational."

(i) Write down the contrapositive to this statement. 1

(ii) Hence prove that the original statement is true. 2

(iii) Is the converse of the original statement true? Justify your answer. 1

**The paper continues on the next page**

**QUESTION THIRTEEN** (15 marks) Start a new answer booklet. **Marks**

(a) Consider the points $A(0,-1,6)$, $B(2,3,5)$ and $C(1,4,4)$. Let $\underset{\sim}{a} = \overrightarrow{CA}$ and $\underset{\sim}{b} = \overrightarrow{CB}$.

(i) Show that $\left|\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}\right| = 2\sqrt{3}$. $\boxed{2}$

(ii) Hence determine the minimum distance from $A$ to the line $BC$. $\boxed{1}$

(b) Prove using induction that $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$, for all positive integers $n$. $\boxed{3}$

(c) (i) Use DeMoivre's Theorem to find the solutions to $z^5 + 1 = 0$, and hence show that $\boxed{3}$

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}\,.$$

(ii) Use the fact that $\cos\dfrac{3\pi}{5} = -\sin\dfrac{\pi}{10}$ to show that $4\sin^2\dfrac{\pi}{10} + 2\sin\dfrac{\pi}{10} - 1 = 0\,.$ $\boxed{1}$

(iii) Hence determine the exact value of $\sin\dfrac{\pi}{10}$. $\boxed{2}$

(d) Consider the cubic equation $z^3 + az^2 + bz + c = 0$, where $a, b,$ and $c$ are real numbers. It is known that the points represented by the three solutions to the equation: $z_1, z_2,$ and $z_3$, form a triangle with area 9 square units in the Argand diagram. $\boxed{3}$

Given $z_1 = -1 + 3i$, find the possible values of $c$.

**The paper continues on the next page**

**QUESTION FOURTEEN** (15 marks) Start a new answer booklet. **Marks**

(a) A firework of mass 2 kg is launched vertically with an initial velocity of 120 m/s. It experiences a resistive force equal to $0.8v$ newtons, where $v$ is the velocity in metres per second. Assume $g = 10\,\text{m/s}^2$.

(i) Show that the motion of the firework satisfies the equation **3**

$$t = \frac{5}{2}\ln\left(\frac{145}{25+v}\right).$$

(ii) If the firework explodes at its maximum height, determine the height at which it explodes and how long it takes to reach this height. Give your answers correct to three significant figures. **3**

(b) Consider the equation $z^2 = \dfrac{az+w}{\overline{z}}$, where $a$ is real, and both $w$ and $z$ are complex.

(i) Show that $\text{Im}\left(\dfrac{z}{w}\right) = 0$. **1**

(ii) Hence, or otherwise, find the solutions to the equation $z^2 = \dfrac{7z+6i}{\overline{z}}$. **3**

(c) The Fermat numbers are a sequence of (not necessarily prime) integers which satisfy the equation $F_n = 2^{2^n} + 1$ for $n \geq 0$.

(i) Use induction to prove that $F_n = F_0 \times F_1 \times F_2 \times \ldots \times F_{n-1} + 2$ for $n \geq 1$. **3**

(ii) Hence prove using contradiction that no two Fermat numbers share a common factor other than 1. **2**

**The paper continues on the next page**

**QUESTION FIFTEEN** (15 marks) Start a new answer booklet. **Marks**

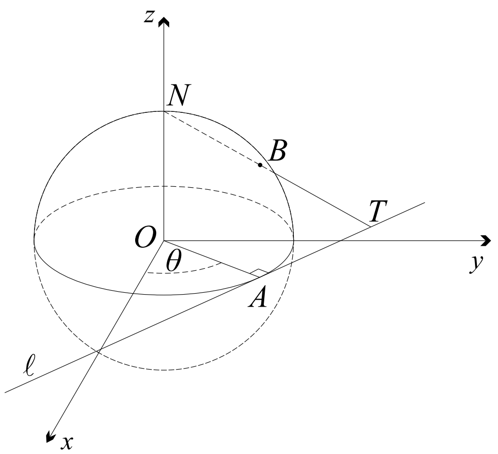
(a) Consider the integral $I_n = \displaystyle\int_0^1 \left(1 - x^2\right)^n dx$ for integers $n \geq 0$.

(i) Show that the integral satisfies the reduction formula $I_n = \dfrac{2n}{2n+1} I_{n-1}$. **3**

(ii) Hence show that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$. **3**

(b) In the diagram below, two points $A(\cos\theta, \sin\theta, 0)$ and $N(0,0,1)$ are on a sphere of radius 1 centred at the origin. The line $\ell$ is constructed in the $xy$-plane such that it is tangent to the sphere at the point $A$, hence for each point $T$ on $\ell$ distinct from $A$, it follows that $\overrightarrow{OA} \perp \overrightarrow{AT}$.

For each point $T$, let the line $NT$ intersect the sphere again at $B$.




(i) Given that $\overrightarrow{OT} = \begin{bmatrix} \cos\theta - \lambda\sin\theta \\ \sin\theta + \lambda\cos\theta \\ 0 \end{bmatrix}$ for some $\lambda \neq 0$, show that **4**

$$\overrightarrow{OB} = \frac{2}{2+\lambda^2}\begin{bmatrix} \cos\theta - \lambda\sin\theta \\ \sin\theta + \lambda\cos\theta \\ \frac{1}{2}\lambda^2 \end{bmatrix}.$$

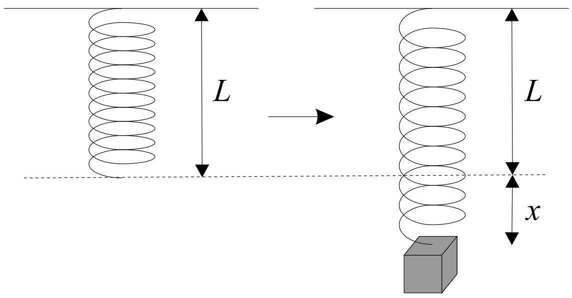
(ii) Hence, show that $\angle NBA = 90^\circ$. **2**

(c) Show that $\displaystyle\int_0^\pi \sqrt{1 - \sin x}\, dx = 4\sqrt{2} - 4$. **3**

**The paper continues on the next page**

**QUESTION SIXTEEN** (15 marks) Start a new answer booklet. **Marks**

(a) A spring of length $L$ metres is first attached to the ceiling from one of its ends and then an object of mass $m$ kilograms is suspended from the other end, causing the spring to stretch. The object experiences two forces: a force due to gravity equal to $mg$ newtons, and a resistive force equal to $-mkx$ newtons, where $k$ is some positive constant, and $x$ is the displacement in metres of the bottom end of the spring from its original resting position.




The object is then pulled down so that the spring is twice its original length, and released. When it reaches its maximum height, the spring is half its original length.

(i) Show that the object moves in simple harmonic motion about some fixed centre. 1

(ii) Hence, or otherwise, prove that the maximum speed of the object is $\frac{3}{2}\sqrt{g\,L}$ m/s. 3

(b) (i) Let $ABC$ be a triangle, and let $O$ be the centre of the unique circle which passes through the three vertices. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, and $\overrightarrow{OC} = \underset{\sim}{c}$. 1

Let $H$ be the point which satisfies

$$\overrightarrow{OH} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}.$$

Show that $\overrightarrow{BH} \perp \overrightarrow{AC}$.

(ii) Let $n$ and $k$ be positive integers, and let $z_k = e^{i\frac{2k\pi}{n}}$. 4

Prove that three distinct complex numbers $z_a$, $z_b$, and $z_c$ satisfy the equation

$$\frac{z_b}{z_a} = \frac{z_c}{z_b}$$

if and only if the points represented by $0, z_b$, and $z_a + z_b + z_c$ are collinear.

**The paper continues on the next page**

**QUESTION SIXTEEN** (Continued)

(c) (i) Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers which satisfy $\boxed{2}$

$$x_1 + x_2 + \ldots + x_n = M\,,$$

for some fixed value of $M$.

Prove by contradiction that the product $x_1 \times x_2 \times \ldots \times x_n$ is maximised when $x_1 = x_2 = \ldots = x_n$.

(ii) Hence prove the AM/GM inequality for $n$ terms, i.e. $\boxed{1}$

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 \times x_2 \times \ldots \times x_n}\,.$$

(iii) By considering the graph of $y = \sqrt{x}$ and the result in part (ii), prove that $\boxed{3}$

$$\sqrt{x} + \sqrt{1 + x^2} + \sqrt{2 + x^3} + \ldots + \sqrt{(n-1) + x^n} > \frac{2n}{3}\sqrt{nx}\,,$$

for all $x > 0$ and positive integers $n$.

**—————— END OF PAPER ——————**

# SGS MX2 Trial 2025 - Solutions

Q1 D  Only D has correct direction vector.

Q2 A  $\frac{2\pi}{n} = 2$ so $n = \pi$, $|v_{max}| = na = 1$
so $a = \frac{1}{\pi}$

Q3 C  IBP or check by differentiating.

Q4 B  $x > -y$ so $y > -x$

Q5 C  If $t=1$, $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\underset{\sim}{x} = \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix} = 3\underset{\sim}{u}$.

Q6 C  Only the quantifiers are negated.

Q7 B  $(z+i\bar{z})(z-i\bar{z}) = z^2 + \bar{z}^2 = 2\text{Re}(z^2)$

Q8 B  $\frac{1}{1-\omega} + \frac{1}{1-\omega^2} = \frac{1}{1-\omega} - \frac{\omega}{\omega^3-\omega} = \frac{1-\omega}{1-\omega} = 1$

Q9 C  A & B false for $0 < x < y$.
C: $x - |x| < y - |y|$ is true if $x < 0$ & $x < y$.

Q10 D  Note $\tan^{-1}\frac{1}{x} = \frac{\pi}{2} - \tan^{-1}x$, and

$$\int \frac{\pi}{2}\left(\frac{1}{1+x^2}\right) - \frac{\tan^{-1}x}{1+x^2}\,dx$$

$$= \frac{\pi}{2}\tan^{-1}x - \frac{1}{2}(\tan^{-1}x)^2 + C$$

# Question Eleven

a i) $z = 2 \operatorname{cis} \frac{\pi}{3}$, $w = \sqrt{2} \operatorname{cis} \frac{-\pi}{4}$ ✓ ✓ (pay 1 for both moduli or both arguments correct)

ii) $zw = 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right)$ ✓

and

$zw = (1+\sqrt{3}i)(1-i) = (1+\sqrt{3}) + (\sqrt{3}-1)i$ ✓

Equating real parts: $2\sqrt{2}\cos\frac{\pi}{12} = 1+\sqrt{3}$

so $\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$ ✓

b i) $|\underset{\sim}{u}| = \sqrt{2^2+(-1)^2+(-5)^2} = \sqrt{30}$

$|\underset{\sim}{v}| = \sqrt{4^2+(-5)^2+(3)^2} = \sqrt{50}$ ✓ both

ii) $\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{8+5-15}{\sqrt{1500}}$ ✓

$= \frac{-1}{5\sqrt{15}}$

$\theta = 93°$ ✓ (nearest degree)

c i) $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ (as $\cos\theta$ is even & $\sin\theta$ is odd)

so $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ ✓

$\Rightarrow \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ as required.

ii) $1 + z^2 = 1 + e^{i2\theta}$

$= 2e^{i\theta}\left(\frac{1}{2}(e^{-i\theta} + e^{i\theta})\right)$ ✓

$= 2z\cos\theta$ as required.

Question Eleven (cont.)

d i) Let $I = \int_1^e x \ln x \, dx$ ; $u = \ln x$ $dv = x \, dx$
$du = \frac{1}{x} dx$ $v = \frac{1}{2}x^2$

$= \left[\frac{1}{2}x^2 \ln x\right]_1^e - \int_1^e \frac{1}{2}x \, dx$ ✓ attempts IBP

$= \frac{1}{2}e^2 - \left[\frac{1}{4}x^2\right]_1^e$

$= \frac{1}{4}e^2 + \frac{1}{4}$ ✓

ii) Let $I = \int x\sqrt{1-x} \, dx$ ; $u = 1-x \Rightarrow x = 1-u$
$dx = -du$

$= \int (1-u)\sqrt{u} \cdot (-du)$ ✓ valid substitution

$= \int u\sqrt{u} - \sqrt{u} \, du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$ ✓ correct integral

$= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C$ ✓ answer in terms of x.

# Question Twelve

a i) Let $(x+iy)^2 = 5-12i$

Then $(x^2-y^2) + 2ixy = 5-12i$

$$\text{so} \quad x^2-y^2 = 5$$
$$2xy = -12 \quad ✓$$

by inspection, $x = \pm 3$, $y = \mp 2$

So the two roots are $3-2i$ and $-3+2i$ ✓

ii) Let $z^2+3z+1+3i=0$

then $\Delta = (3)^2 - 4(1)(1+3i) = 5-12i$ ✓

$$\text{so} \quad z = \frac{-3 \pm (3-2i)}{2}$$

$$z = -i \quad \text{or} \quad z = -3+i \quad ✓$$

b i) $$x-5 = A(x^2+1) + (Bx+C)(x-1)$$

Letting $x=1$, $-4 = 2A \Rightarrow A = -2$.

Equating coefficients of $x^2$: $0 = -2+B \Rightarrow B = 2$.

Equating constants, $-5 = -2-C \Rightarrow C = 3$

✓ Any one correct

✓ All three correct

ii) $$\int \frac{x-5}{(x-1)(x^2+1)}\,dx = \int \frac{-2}{x-1} + \frac{2x+3}{x^2+1}\,dx$$

$$= \int \frac{-2}{x-1} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\,dx$$

$$= -2\ln|x-1| ✓ + \ln|x^2+1| + 3\tan^{-1}x ✓ + C$$

Question Twelve (cont.)

c i) $\frac{dx}{dt} = 2-x$ so $\frac{dt}{dx} = \frac{1}{2-x}$

integrating, $t = -\ln|2-x| + C$ ✓

when $t=0$, $x=0$ (so $2-x>0$) and

$0 = -\ln 2 + C \Rightarrow C = \ln 2$ ✓

so $t = \ln 2 - \ln(2-x)$

$= \ln\left(\frac{2}{2-x}\right)$

ii) Note $e^{-t} = \frac{2-x}{2} \Rightarrow x = 2-2e^{-t}$;

as $t \to \infty$, $x \to 2^-$ ✓

d i) If $\sin\theta + \cos\theta$ is rational, then $\sin 2\theta$ is rational. ✓

ii) Suppose, by way of contraposition, that $\sin\theta+\cos\theta$ is rational.

Let $\sin\theta+\cos\theta = \frac{p}{q}$ ✓ for $p, q \in \mathbb{Z}$.

Then $\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \frac{p^2}{q^2}$

$\Rightarrow \quad \sin 2\theta + 1 = \frac{p^2}{q^2}$

$\Rightarrow \quad \sin 2\theta = \frac{p^2-q^2}{q^2}$ ✓ which is rational.

Hence, if $\sin 2\theta$ is irrational, $\sin\theta+\cos\theta$ is also irrational.

iii) No. The converse says that

$\sin 2\theta$ is irrational whenever $\sin\theta+\cos\theta$ is irrational,

but this is clearly false for certain angles. E.g. $\theta = \frac{\pi}{4}$,

then $\sin 2\theta = 1$, $\sin\theta + \cos\theta = \sqrt{2}$. ✓ correct answer with an explicit counterexample.

# Question Thirteen

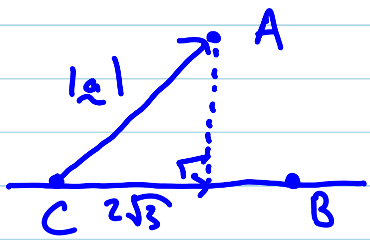
ai) $\underset{\sim}{a} = \begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ✓ correctly finds both $\underset{\sim}{a}$ & $\underset{\sim}{b}$

$$\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\,\underset{\sim}{b} = \frac{6}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} ✓$$

$$= 2\underset{\sim}{b}$$

so $\left|\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a}\right| = 2\sqrt{1^2+(-1)^2+1^2} = 2\sqrt{3}$ as required.

ii) min distance $= \sqrt{|\underset{\sim}{a}|^2 - (2\sqrt{3})^2}$




$$= \sqrt{30-12}$$

$$= 3\sqrt{2} \text{ units} ✓$$

b) $\boxed{A}$ When $n=1$, LHS $= \frac{1}{1^2}$ and RHS $= 2 - \frac{1}{1}$

$$= 1 \qquad\qquad = 1$$

Since LHS = RHS, the result holds for $n=1$. ✓

$\boxed{B}$ Suppose the result is true for $n=k$, for some $k \geq 1$.

I.e. $\displaystyle\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$ $\quad (**)$

Then $\displaystyle\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2}$

$$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} ✓ \quad (\text{by } **)$$

$$< 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)}$$

$$= 2 - \frac{k}{k(k+1)}$$

$$= 2 - \frac{1}{k+1} ✓$$

so the result holds for $n = k+1$.

$\boxed{C}$ By Parts A & B, the result holds for all $n \geq 1$.

## Question Thirteen (cont.)

c i) Let $z = re^{i\theta}$, then $r^5 e^{i5\theta} = -1$

$= e^{i\pi}$

so $r^5 = 1 \implies r = 1$

$5\theta = \pi + 2k\pi \implies \theta = \frac{2k+1}{5}\pi$, $-2 \le k \le 2$

so $z = -1$, $z = e^{i\frac{3\pi}{5}}$, $z = e^{i\frac{\pi}{5}}$, $z = e^{-i\frac{\pi}{5}}$, $z = e^{-\frac{3\pi}{5}}$

From sum of roots: $e^{i\frac{\pi}{5}} + e^{-i\frac{\pi}{5}} + e^{i\frac{3\pi}{5}} + e^{-i\frac{3\pi}{5}} - 1 = 0$

$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 1 = 0$

$\implies \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

ii) Note $\cos\frac{\pi}{5} = 1 - 2\sin^2\frac{\pi}{10}$ and $\cos\frac{3\pi}{5} = -\sin\frac{\pi}{10}$

so $\left(1 - 2\sin^2\frac{\pi}{10}\right) - \sin\frac{\pi}{10} = \frac{1}{2}$

$-2\sin^2\frac{\pi}{10} - \sin\frac{\pi}{10} + \frac{1}{2} = 0$

$\implies 4\sin^2\frac{\pi}{10} + 2\sin\frac{\pi}{10} - 1 = 0$

iii) $\sin\frac{\pi}{10} = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$

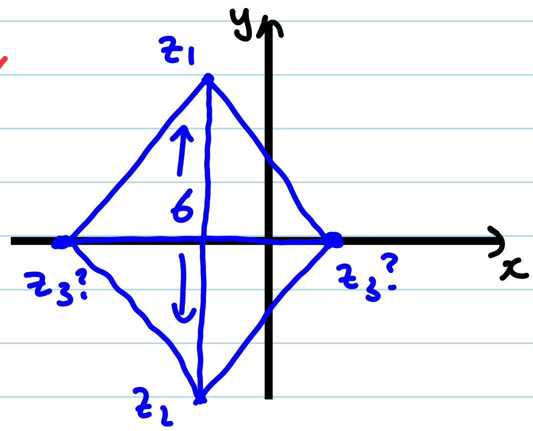
but $\frac{\pi}{10}$ is acute, so $\sin\frac{\pi}{10} = \frac{-1+\sqrt{5}}{4}$ only.

d) By conjugate root theorem, roots occur in conjugate pairs

so $z_2 = -1 - 3i$ and $z_3 = k$ $(k \in \mathbb{R})$

Area $= 9$ so $|k+1| = 3$

$k = -4$ or $k = 2$




Then $c = -$ product of roots

$= -k(-1-3i)(-1+3i)$

$= -10k$

$c = 40$ or $c = -20$

# Question Fourteen

ai) $F = m\ddot{y} = -mg - 0.8v$

so $\ddot{y} = -10 - 0.4v$ ✓

hence $\dfrac{-1}{10+0.4v} \cdot \dfrac{dv}{dt} = 1$

$\Rightarrow -\dfrac{5}{2}\displaystyle\int \dfrac{1}{25+v}\,dv = \int dt$ ✓ separates variables

$\Rightarrow -\dfrac{5}{2}\ln|25+v| = t + C$

when $t=0$, $v=120$, so $C = -\dfrac{5}{2}\ln(145)$ ✓ finds $C$

hence $t = \dfrac{5}{2}\ln(145) - \dfrac{5}{2}\ln(25+v)$

$= \dfrac{5}{2}\ln\left(\dfrac{145}{25+v}\right)$ as required

ii) Maximum height occurs when $v=0$,

i.e. $t = \dfrac{5}{2}\ln\left(\dfrac{145}{25}\right) \doteqdot 4.39$ seconds ✓

Also note $\dfrac{2}{5}t = \ln\left(\dfrac{145}{25+v}\right)$

so $\dfrac{145}{25+v} = e^{0.4t}$

$v = 145e^{-0.4t} - 25$ ✓

$x = \dfrac{-725}{2}e^{-0.4t} - 25t + C$

when $t=0$, $x=0$ so $C = \dfrac{725}{2}$

subbing $t = \dfrac{5}{2}\ln\left(\dfrac{29}{5}\right)$ gives $x \doteqdot 190$ metres ✓

Question Fourteen (cont.)

b i) Note $z^2 \bar{z} = az + w$

$\Rightarrow |z|^2 z = az + w$

so $(|z|^2 - a)z = w$

hence $\frac{z}{w} = \frac{1}{|z|^2 - a}$ ✓ which is real

so $\text{Im}\left(\frac{z}{w}\right) = 0.$

ii) From i) $\text{Im}\left(\frac{z}{6i}\right) = 0$ so $z = ki$ ✓ for some $k \in \mathbb{R}$.

Then $(ki)^2 = \frac{7k\cancel{i} + 6\cancel{i}}{-k\cancel{i}}$

$\Rightarrow k^3 = 7k + 6$

$\Rightarrow k^3 - 7k - 6 = 0$ ✓

$(k+1)(k+2)(k-3) = 0$

Hence $z = -i$, $z = -2i$ and $z = 3i$ are the solutions ✓

c i) RTP: $F_n = F_0 \times F_1 \times \ldots \times F_{n-1} + 2$ for $n \geq 1$

$\boxed{A}$ When $n = 1$, LHS $= F_1$ RHS $= F_0 + 2$

$= 2^{2^1} + 1$ $= 2^{2^0} + 1 + 2$

$= 5$ $= 5$ ✓

LHS = RHS so the result holds for $n = 1$

$\boxed{B}$ Suppose the result holds for $n = k$ for some $k \geq 1$,

i.e. $F_k = F_0 \times F_1 \times \ldots \times F_{k-1} + 2$

whence $F_0 \times F_1 \times \ldots \times F_{k-1} = F_k - 2$ $(**)$

So $(F_0 \times F_1 \times \ldots \times F_{k-1}) \times F_k = (F_k)^2 - 2F_k$ ✓ some progress

$= (2^{2^k} + 1)^2 - 2(2^{2^k} + 1)$

Question Fourteen (cont.)

ci) (cont.) $(F_0 \times F_1 \times \dots \times F_{k-1}) \times F_k = 2^{2^k \times 2} + \cancel{2 \times 2^{2^k}} + 1 - \cancel{2 \times 2^{2^k}} - 2$

$$= \left(2^{2^{k+1}} + 1\right) - 2$$

$$= F_{k+1} - 2$$

✓ complete proof

hence $F_{k+1} = F_0 \times F_1 \times \dots \times F_k + 2$ as required.

$\boxed{C}$ By Parts A & B, the result holds for all $n \geq 1$

ii) Suppose, by way of contradiction, $F_a$ & $F_b$ share a common factor other than 1 for some $a > b$.

Note $F_a = F_0 \times F_1 \times \dots \times F_b \times \dots \times F_{a-1} + 2$

so the common factor must divide 2 as well, i.e. the common factor must be 2. ✓ show 2 is a common factor

But $F_k = 2^{2^k} + 1$ is always odd. ✓ show $F_n$ odd

This is a contradiction, so no two Fermat numbers share a common factor.

Question Fifteen

ai) Let $I_n = \int_0^1 (1-x^2)^n \, dx$  $u = (1-x^2)^n$  $dv = dx$

$du = (-2nx)(1-x^2)^{n-1} dx$  $v = x$

$$= \underbrace{\left[x(1-x^2)^n\right]_0^1}_{=0} + \int_0^1 2nx^2 (1-x^2)^{n-1} dx$$

$$\text{so } I_n = 2n\int_0^1 (x^2-1+1)(1-x^2)^{n-1} dx$$

$$= 2n\int_0^1 (1-x^2)^{n-1} - (1-x^2)^n \, dx$$

$$= 2n I_{n-1} - 2n I_n$$

$$\text{so } (2n+1)I_n = 2nI_{n-1} \Rightarrow I_n = \frac{2n}{2n+1} I_{n-1}$$

ii) Note $I_0 = \int_0^1 (1-x^2)^0 dx = 1$

and $I_n = \dfrac{2n}{2n+1} I_{n-1}$

$$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} I_{n-2}$$

$$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \dots \times \frac{2}{1}$$

$$= \frac{\left(2n \times (2n-2) \times \dots \times 4 \times 2\right)^2}{(2n+1) \times (2n) \times (2n-1) \times (2n-2) \times \dots \times 2 \times 1}$$

$$= \frac{(2^n \times n!)^2}{(2n+1)!}$$

$$= \frac{2^{2n} \times (n!)^2}{(2n+1)!} \quad \text{as required}$$

Question Fifteen (cont.)

b i) First note $\vec{NT} = \vec{OT} - \vec{ON} = \begin{pmatrix} \cos\theta - \lambda\sin\theta \\ \sin\theta + \lambda\cos\theta \\ -1 \end{pmatrix}$

so $\vec{OB} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} \cos\theta - \lambda\sin\theta \\ \sin\theta + \lambda\cos\theta \\ -1 \end{pmatrix}$ ✓

Now $|\vec{OB}| = 1$

so $(\cos\theta - \lambda\sin\theta)^2\mu^2 + (\sin\theta + \lambda\cos\theta)^2\mu^2 + (1-\mu)^2 = 1$ ✓

$$\mu^2\cos^2\theta - \cancel{2\mu^2\lambda\cos\theta\sin\theta} + \mu^2\lambda^2\sin^2\theta$$
$$+ \mu^2\sin^2\theta + \cancel{2\mu^2\lambda\cos\theta\sin\theta} + \mu^2\lambda^2\cos^2\theta + 1 - 2\mu + \mu^2 = 1$$

so $\mu^2 + \mu^2\lambda^2 - 2\mu + \mu^2 = 0$

$\mu(\mu\lambda^2 + 2\mu - 2) = 0$

$\mu = 0$ or $2\mu + \lambda^2\mu = 2$ ✓

$\mu = \dfrac{2}{2+\lambda^2}$

Hence $\vec{OB} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \dfrac{2}{2+\lambda^2}\begin{pmatrix} \cos\theta - \lambda\sin\theta \\ \sin\theta + \lambda\cos\theta \\ -1 \end{pmatrix}$

$= \dfrac{2}{2+\lambda^2}\begin{pmatrix} \cos\theta - \lambda\sin\theta \\ \sin\theta + \lambda\cos\theta \\ \left(\frac{2+\lambda^2}{2}\right) - 1 \end{pmatrix}$

$= \dfrac{2}{2+\lambda^2}\begin{pmatrix} \cos\theta - \lambda\sin\theta \\ \sin\theta + \lambda\cos\theta \\ \frac{1}{2}\lambda^2 \end{pmatrix}$ ✓ as required

ii) Let $\vec{ON} = \underset{\sim}{n}$, $\vec{OB} = \underset{\sim}{b}$, $\vec{OA} = \underset{\sim}{a}$

$\vec{NB} \cdot \vec{BA} = (\underset{\sim}{b} - \underset{\sim}{n}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$

$= \underset{\sim}{b}\cdot\underset{\sim}{a} - \underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{n}\cdot\underset{\sim}{a} + \underset{\sim}{n}\cdot\underset{\sim}{b}$

$= \underset{\sim}{b}\cdot\underset{\sim}{a} - |\underset{\sim}{b}|^2 - 0 + \underset{\sim}{n}\cdot\underset{\sim}{b}$

Question Fifteen (cont.)

b ii) $\vec{NB} \cdot \vec{BA} = \underset{\sim}{b} \cdot \underset{\sim}{a} - 1 + \underset{\sim}{n} \cdot \underset{\sim}{b}$ ✓

$= \frac{2}{2+\lambda^2}\left(\cos^2\theta - \cancel{\lambda \sin\theta\cos\theta} + \sin^2\theta + \cancel{\lambda\sin\theta\cos\theta}\right)$

$-1 + \frac{2}{2+\lambda^2}\left(\frac{1}{2}\lambda^2\right)$

$= \frac{2}{2+\lambda^2} + \frac{2}{2+\lambda^2}\left(\frac{1}{2}\lambda^2\right) - 1$

$= \frac{2}{2+\lambda^2}\left(\frac{2+\lambda^2}{2}\right) - 1$ ✓

$= 0$

so $\angle NBA = 90°$

c) $\int_0^{\pi} \sqrt{1-\sin x}\, dx = \int_0^{\pi} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1+\sin x}}\, dx$ ✓

$= \int_0^{\pi} \frac{|\cos x|}{\sqrt{1+\sin x}}\, dx$

$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin x}}\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{-\cos x}{\sqrt{1+\sin x}}\, dx$ ✓

$= \left[2\sqrt{1+\sin x}\right]_0^{\frac{\pi}{2}} - \left[2\sqrt{1+\sin x}\right]_{\frac{\pi}{2}}^{\pi}$

$= (2\sqrt{2}-2) - (2-2\sqrt{2})$ ✓

$= 4\sqrt{2} - 4$

(Alternative method)

Note $\int_0^{\pi} \sqrt{1-\sin x}\, dx = 2 \times \int_0^{\frac{\pi}{2}} \sqrt{1-\sin x}\, dx$ ✓

$= 2 \times \int_0^{\frac{\pi}{2}} \sqrt{\sin^2 \tfrac{1}{2}x + \cos^2 \tfrac{1}{2}x - 2\sin\tfrac{x}{2}\cos\tfrac{x}{2}}\, dx$

$= 2 \times \int_0^{\frac{\pi}{2}} \sqrt{\left(\cos\tfrac{x}{2} - \sin\tfrac{x}{2}\right)^2}\, dx$

$= 2 \times \int_0^{\frac{\pi}{2}} \cos\tfrac{x}{2} - \sin\tfrac{x}{2}\, dx$ ✓

...

## Question Sixteen

a i) Let downwards be positive. Then

$$F = mg - mkx$$
$$m\ddot{x} = mg - mkx$$
$$\ddot{x} = g - kx$$
$$= -k\left(x - \frac{g}{x}\right) \checkmark$$

and hence the motion is simple harmonic with centre $\frac{g}{k}$
and period $\frac{2\pi}{\sqrt{k}}$

ii) Let $x = A\cos(nt) + \frac{g}{k}$

then $\dot{x} = -An\sin(nt)$ so max speed $= An$

We note extremes of $+L$ and $-\frac{1}{2}L$, so centre $= \frac{L}{4}$

and hence $\frac{g}{k} = \frac{L}{4} \Rightarrow k = \frac{4g}{L}$ ✓.

and $n^2 = k = \frac{4g}{L}$ ✓
so $n = 2\sqrt{\frac{g}{L}}$

Further, we have $A = \frac{3L}{4}$
hence max speed $= \frac{3L}{4} \times 2\sqrt{\frac{g}{L}}$
$$= \frac{3}{2}\sqrt{\frac{gL^2}{L}} \checkmark$$
$$= \frac{3}{2}\sqrt{gL}$$ as required

Question Sixteen (cont.)

b i) First note: since $O$ is the centre of the circle, we have

$$|\underset{\sim}{a}| = |\underset{\sim}{b}| = |\underset{\sim}{c}|$$

Now $\overrightarrow{BH} = (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) - \underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{c}$

so $\overrightarrow{BH} \cdot \overrightarrow{AC} = (\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{a})$

$$= \cancel{\underset{\sim}{a} \cdot \underset{\sim}{c}} - \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \cancel{\underset{\sim}{c} \cdot \underset{\sim}{a}}$$

$$= -|\underset{\sim}{a}|^2 + |\underset{\sim}{c}|^2 \quad \checkmark$$

$$= 0 \quad \text{as} \quad |\underset{\sim}{a}| = |\underset{\sim}{c}|$$

so $\overrightarrow{BH} \perp \overrightarrow{AC}$

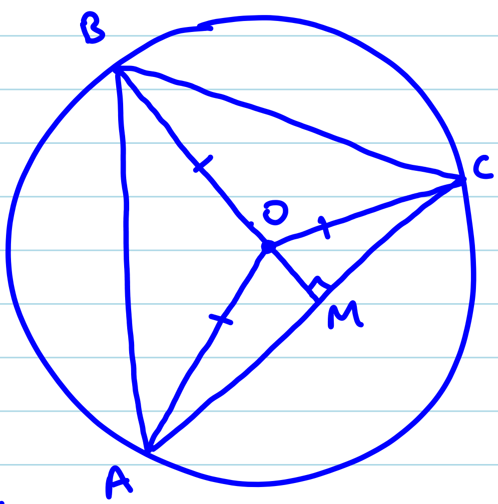
ii) Let $z_a$, $z_b$, and $z_c$ represent points $A, B, C$ respectively

and note $|z_a| = |z_b| = |z_c| = 1$, so
$z_a + z_b + z_c$ corresponds to $H$ in part i)

$\boxed{A}$ Suppose $O, B$, and $H$ are collinear.

Since $BH \perp AC$ (from i), we have $OB \perp AC$. ✓

Let $OB$ meet $AC$ at $M$.




Since $\Delta AOC$ is isosceles
$\angle AOM = \angle COM$
$\Rightarrow \angle BOA = \angle BOC$ ✓

so $\arg\left(\dfrac{z_a}{z_b}\right) = \arg\left(\dfrac{z_b}{z_c}\right)$

and $\left|\dfrac{z_a}{z_b}\right| = \left|\dfrac{z_b}{z_c}\right| = 1$

hence $\dfrac{z_a}{z_b} = \dfrac{z_b}{z_c}$ as required.

Question Sixteen (cont.)

bii) $\boxed{B}$ Conversely, suppose $\frac{z_a}{z_b} = \frac{z_b}{z_c} = e^{i\theta}$

for some angle $\theta$.

Then $z_a + z_b + z_c = z_b e^{i\theta} + z_b + z_b e^{-i\theta}$ ✓

$= z_b(1 + e^{i\theta} + e^{-i\theta})$

$= z_b(1 + 2\cos\theta)$

$= k z_b$ for some $k \in \mathbb{R}$ ✓

so $z_a + z_b + z_c$, $z_b$, and $0$ are collinear.

ci) Suppose, by way of contradiction, that the maximum product occurs with some $x_i \neq x_j$.

But note $x_i + x_j = \left(\frac{x_i + x_j}{2}\right) + \left(\frac{x_i + x_j}{2}\right)$ ✓ considers how to make product bigger

and $\left(\frac{x_i + x_j}{2}\right)^2 - x_i \times x_j$

$= \frac{x_i^2 - 2x_i x_j + x_j^2}{4}$

$= \left(\frac{x_i - x_j}{2}\right)^2 > 0$ (as $x_i \neq x_j$) ✓ complete proof

so replacing $x_i$ and $x_j$ with their average increases the product while preserving the sum.

This is a contradiction of the product being maximal.

Hence the maximum product occurs when $x_1 = x_2 = \dots = x_n$.

Question Sixteen (cont.)

cii) Consider the set $x_1, x_2, \dots, x_n$ and let $M = x_1 + x_2 + \dots + x_n$.

From i) the set $\frac{M}{n}, \frac{M}{n}, \dots, \frac{M}{n}$ has maximal product,

so $\left(\frac{M}{n}\right)^n \geqslant x_1 x_2 \dots x_n$ ✓ with justification

i.e. $\frac{M}{n} \geqslant \sqrt[n]{x_1 x_2 \dots x_n}$

$\Rightarrow \frac{x_1 + x_2 + \dots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \dots x_n}$

iii) From ii) $\frac{1+x^2}{2} \geqslant \sqrt{1 \cdot x^2} = x$ so $1 + x^2 \geqslant 2x$

similarly $\frac{1+1+x^3}{3} \geqslant \sqrt{1 \cdot 1 \cdot x^3} = x$ so $2 + x^3 \geqslant 3x$

and $\frac{(k-1) + x^k}{k} \geqslant \sqrt[k]{x^k} = x$ so $(k-1) + x^k \geqslant kx$

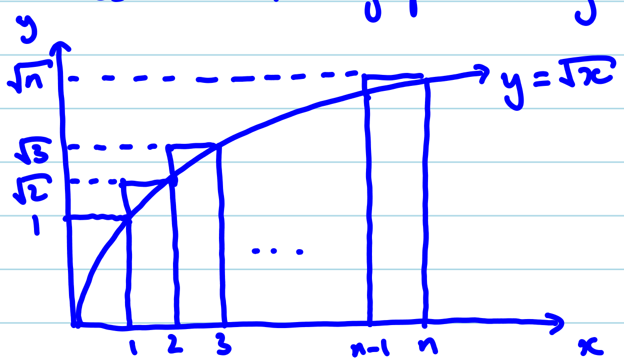
✓ Attempts to use AM-GM on inner terms

Hence $\sqrt{x} + \sqrt{1+x^2} + \sqrt{2+x^3} + \dots + \sqrt{(n-1)+x^n}$

$\geqslant \sqrt{x} + \sqrt{2x} + \sqrt{3x} + \dots + \sqrt{nx}$ ✓

$= (\sqrt{1} + \sqrt{2} + \dots + \sqrt{n})\sqrt{x}$ $(*)$

Now consider the graph of $y = \sqrt{x}$




From the area of the rectangles,

$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} > \int_0^n \sqrt{x}\, dx = \frac{2}{3} n\sqrt{n}$ ✓ $(**)$

Combining $(*)$ & $(**)$ LHS $\geqslant \left(\frac{2}{3} n\sqrt{n}\right)\sqrt{x} = \frac{2}{3} n\sqrt{nx}$

as required.

# SGS Mathematics Extension 2 Trial 2025 - Marker's Comments

## Question 11

### a

- **i**: Done well by all. Some boys are conflating modulus-argument form with Euler form, but were not penalised.
- **ii**: Done well by all. Everyone recognised that they had to equate components, but should take care to specifically reference that this result comes from equating the real parts.

### b

- **i**: No issues here.
- **ii**: Generally fine, but some candidates think that two vectors cannot form an obtuse angle. They are likely conflating the convention for two lines where both an acute and an obtuse angle are formed, and we only give the acute angle.

### c

- **i**: Some laziness here. Many candidates wrote $\frac{\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)}{2} = \cos\theta$, without any justification. This is simply rewriting the question and was not awarded the mark.
- **ii**: Generally well done. A fairly even split of candidates beginning with the LHS vs. those who began with RHS. Candidates should note that this is a HENCE question, and thus should be specifically using the Euler result found in part (i), but were not penalised in this instance.

### d

- **i**: Very well done by all. All candidates recognised to use integration by parts and showed clear working. The most common error was evaluating the double negative at the end.
- **ii**: Many varied approaches. The intended solution was a substitution using $u = 1 - x$, but other successful candidates used $u^2 = 1 - x$ or $x = \sin\theta$. A sizeable proportion of candidates used integration by parts which was efficient for some, and clunky for others. The most common errors included issues with fractions and not substituting back in terms of $x$.

## Question 12

### (a)

- **(i):** This part was generally well done.
- **(ii):** This part was also generally well done. The most common error was miscalculating the discriminant or failing to use the result from part (a)(i) to find the square root of the discriminant.

### (b)

- **(i):** Some pupils did not set up the initial equations correctly and skipped too many steps, leading to incorrect answers. It is crucial to show sufficient working, such as equating coefficients, to justify the values of $A$, $B$, and $C$.
- **(ii):** Few mistakes with signs were noted. It is important to be careful with signs, especially when integrating terms like $\frac{-2}{x-1}$ or those involving inverse tangent.

### (c)

- **(i):** This part was generally done well.
- **(ii):** Most pupils correctly noted that $x$ was approaching 2, which was awarded the mark. However, it is also useful to note that the position approaches $x = 2^-$ (from the left) as $t \to \infty$.

### (d)

- **(i):** Most pupils were able to correctly state the contrapositive.
- **(ii):** Some pupils showed weakness in their logical reasoning. While many started correctly by letting $\sin\theta + \cos\theta = p/q$ where $p$ and $q$ are integers and coprimes, those who tried to use alternative methods, such as the auxiliary angle formula, often had difficulty completing the proof. Many lacked sufficient logical steps and formal proof to back their claims.
- **(iii):** The converse states that "if $\sin 2\theta$ is irrational, then $\sin\theta + \cos\theta$ is irrational." This statement is false, and this part was generally done well because pupils were able to provide correct counterexamples to justify their answer. For example, a correct counterexample would be $\theta = \frac{\pi}{12}$. Here, $\sin 2\theta = \sin(\frac{\pi}{6}) = \frac{1}{2}$ (rational), but $\sin\theta + \cos\theta = \sin(\frac{\pi}{12}) + \cos(\frac{\pi}{12}) = \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{2\sqrt{6}}{4} = \frac{\sqrt{6}}{2}$ (irrational), showing the converse is false.

## Question 13

### a

- Generally very well done for both parts.

### b

- Generally well done. The most common mistakes were simple algebraic mistakes.

### c

- **i**: Very well done.
- **ii**: Generally well done. Candidates who began with the expression $4\sin^2(\frac{\pi}{10}) + 2\sin(\frac{\pi}{10}) - 1$ and tried to show that it simplified to 0 were less successful than those who used the result from part (i) to prove the required result in part (ii).
- **iii**: Generally well done. The most common mistake was incorrect use of the quadratic formula.

### d

- Most candidates correctly used the complex conjugate root theorem to find $z_2 = -1 - 3i$. The most common mistake was only considering one of the two possible values of $z_3$. Candidates who did not deduce that $z_3$ is a real number and attemtped to use other methods instead were generally unsuccessful.

## Question 14

### a

- **i**: A few pupils forgot to include mass when relating force and acceleration, i.e. did not correctly use Newton's first law ($F = ma$).
- **ii**: Although this was generally well done, about half the cohort did not realise that they could rearrange the result in (i) to get a simpler integral instead of restarting the question and applying $a = v\frac{dv}{dx}$.

### b

- **i**: Generally well done, but a number of pupils proved $Im(\frac{w}{z}) = 0$ instead of what the question actually asked. These candidates were not penalised in this instance.
- **ii**: Most pupils noticed $z = ki$ for some value $k$, but did not know to substitute it into the expression and then solve a polynomial. Pupils who used long division and factorised the polynomial had more success than pupils who used sums and product of roots.

### c

- **i**: A number of pupils were confused about what they needed to prove. The question requires you to prove $F_{k+1} = F_0 F_1 \dots F_k + 2$, not $F_{k+1} = 2^{2^{k+1}}$. A few also made the mistake of letting $2^{2^{k+1}} = (2^2)(2^k)$ which is not correct, it should instead be $2^{2^{k+1}} = (2^{2^k})^2$.
- **ii**: The essential part of the proof requires an explanation of why the only possible common factor must be 2 which is a contradiction because all Fermat numbers are odd. Any claims that a contradiction exists without those two points were not rewarded.

## Question 15

This was a challenging question and many boys found it difficult. Across all parts, common issues included insufficient working, algebraic carelessness, and poor examination technique.

### a

- **i**: The main difficulties observed were:

  1. Failure to apply the correct integration by parts strategy.
  2. Careless algebraic mistakes which prevented further progress.
  3. Not recognising the need for algebraic manipulation after completing the integration by parts.

- **ii**: Several recurring issues were evident:

  1. Insufficient working shown, making it unclear how the solution was obtained.
  2. Failure to identify the final $I_0$ term.
  3. Attempting to "force" the result with fabricated or logically flawed steps.
  4. Extremely untidy and unclear working.
  5. Use of methods from outside the course without appropriate justification.

### b

- **i**: Performance was generally poor. Common mistakes included:

  1. Misuse of notation, particularly confusing vector $OB$ and line $NT$.
  2. Algebraic and logical mistakes due to insufficient clarity / structure.
  3. Failure to substitute the value of $mu$ back into the expression after determining its value.

A number of boys attempted alternative methods (e.g. projection, equations of spheres) but were unable to progress to a valid solution.

- **ii**: Many boys did not think to expand the dot product using the distributive law, and instead arrived at unnecessarily cumbersome algebra. These candidates generally had error-prone work which was often incomplete.

**c**

- This part was also poorly done. The most common problems were:

  1. Not recognising the need for an absolute value after carrying out algebraic manipulation or substitution, ultimately arriving at a result of zero.
  2. Attempting multiple substitutions consecutively and becoming lost in lengthy algebraic steps.
  3. Multiple unrelated unsuccessful attempts left uncrossed, reflecting weak examination technique. Certain candidates who persisted with six separate approaches should have moved on and allocated time more effectively.

# Question 16

**a**

- **i**: Generally well done. Many candidates knew to express acceleration in the form $\ddot{x} = -n^2(x-c)$. Boys should be careful to clearly note which direction they are assuming to be positive, as this affected the use of $mg$ vs $-mg$, and also affected the position of the centre: ($\frac{g}{k}$ vs. $-\frac{g}{k}$).

- **ii**: Surprisingly few candidates considered modelling the motion using the time equation, instead opting for a displacement-velocity approach. All candidates needed to be careful interpretting the extremes of motion: $x = -\frac{L}{2}$ and $x = L$, NOT $x = \frac{L}{2}$ and $x = 2L$; but especially those who integrated and needed to substitute in to evaluate their constants. Those who arrived at a valid displacement-velocity equation (without a +C) were awarded a mark.

**b**

- **i**: Generally done well, and was an easier mark in this question. Many boys skipped steps when expanding their dot product, but were not penalised if they used the fact that $|\underset{\sim}{a}| = |\underset{\sim}{c}|$ since they are radii.

- **ii**: Most candidates did not know how to begin either direction of this proof. Geometric proofs were generally more successful, and algebraic proofs which noted that $z_a$, $z_b$, and $z_c$ were rotations of each other by the same angle were more successful than other algebraic attempts.

**c**

- **i**: Many candidates incorrectly set up the contradiction, either assuming that all $x_i$ had to be different, or that exactly two of the $x_i$ varied symmetrically. In the latter case, the essence of the proof was still correct (any two distinct terms vary symmetrically about their average), and this results in the necessary contradiction of maximality.

- **ii**: Many proofs were quite hand-wavy, but correctly argued that the maximum value of the RHS is at most the value of the LHS.

- **iii**: Few candidates had time to attempt this problem. Some boys were guided by their intuition that $\frac{2}{3}n\sqrt{n}$ comes from the integral of $y = \sqrt{x}$ and attained a mark for this recognition. A few others noted that $x > 0$, and so dividing both sides of the inequality by $\sqrt{x}$ results in a more familiar result. Some candidates were misled to apply the AM-GM to the $n$ terms on the LHS, which unfortunately just leads to a mess.