

Student Name _____

Teacher's Name: _____

Extension 2 Mathematics

TRIAL HSC

August 2021

**General
Instructions**

- Reading time – 10 minutes
- Working time – 180 minutes
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In questions 11-16, show relevant mathematical reasoning and/or calculations

**Total marks:
100**

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I 10 Marks

Attempt Questions 1-10

Allow approximately 15 minutes for this section.

Use the Multiple Choice answer sheet for questions 1-10.

1. Which vector is perpendicular to

$$\begin{pmatrix} 6 \\ -4 \\ 8 \end{pmatrix}?$$

- A) $\begin{pmatrix} 12 \\ -8 \\ 16 \end{pmatrix}$ B) $\begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$ C) $\begin{pmatrix} 6 \\ 12 \\ 2 \end{pmatrix}$ D) $\begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix}$

2. A particle is describing SHM in a straight line with an amplitude of 2 metres. Its speed is 3m/s when the particle is 1 metre from the centre of motion.

What is the period of the motion?

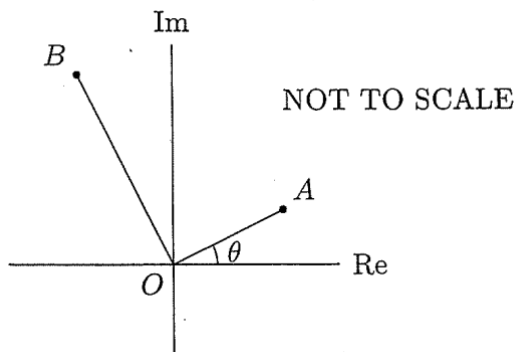
- A) $\frac{\pi\sqrt{3}}{2}$ B) $\frac{2\pi\sqrt{3}}{3}$ C) $\pi\sqrt{3}$ D) $\frac{2\pi\sqrt{2}}{3}$

3. Consider the following statement for $n \in \mathbb{Z}$.

If $n^2 + 4n$ is odd, then n is odd. Which of the following statements is the contrapositive of this statement for $n \in \mathbb{Z}$?

- A) If n is even, then $n^2 + 4n$ is even
B) If $n^2 + 4n$ is even then n is even
C) If n is odd then $n^2 + 4n$ is odd
D) If $n^2 + 4n$ is odd then n is even

4. The points A and B in the diagram represent the complex numbers z_1 and z_2 respectively, where $|z_1| = 1$ and $\arg(z_1) = \theta$ and $z_2 = \sqrt{3}iz_1$



Which of the following represents $z_2 - z_1$?

- A) $2e^{i(\frac{2\pi}{3}+\theta)}$ B) $3e^{i(\frac{2\pi}{3}+\theta)}$ C) $2e^{i(\frac{2\pi}{3}-\theta)}$ D) $3e^{i(\frac{2\pi}{3}-\theta)}$

5. Which of the following is an expression for $\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x} dx$ after using the substitution $t = \tan \frac{x}{2}$?

- A) $\int_0^1 \frac{1}{1+2t} dt$ B) $\int_0^1 \frac{1}{1+t^2} dt$ C) $\int_0^1 dt$ D) $\int_0^1 \frac{2}{(1+t)^2} dt$

6. Which of the following is false?

- A) $\int_{-3}^3 x^3 e^{-x^2} dx = 0$ B) $\int_{-4}^4 \frac{x^2}{x^2+4} dx = 2 \int_0^4 \frac{x^2}{x^2+4} dx$
 C) $\int_0^\pi \sin^4 \theta d\theta > \int_0^\pi \sin 4\theta d\theta$ D) $\int_0^1 x^4 dx < \int_0^1 x^5 dx$

7. The points A, B and C are collinear where

$$\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}, \quad \overrightarrow{OB} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}, \quad \overrightarrow{OC} = 3\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}.$$

What are the values of a and b ?

- A) $a = 3, b = 2$ B) $a = -3, b = 2$
 C) $a = 3, b = -2$ D) $a = -3, b = -2$

8. If z is any complex number satisfying $|z - 2| = 1$, then which of the following is correct?

A) $\arg(z - 2) = 2\arg(z - 1)$

B) $2\arg z = \frac{2}{3}\arg(z^2 + 2z)$

C) $\arg z = 2\arg(z - 2)$

D) $\arg(z + 2) = \arg(z - 2)$

9. Let $\tilde{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\tilde{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

The angle between the vectors $\tilde{u}$ and $\tilde{v}$ is

A) 30°

B) 22.5°

C) 0°

D) 45°

10. Given that $w^7 = 1$ and w is a complex number, what is the value of

$$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7?$$

A) 1

B) w

C) $-w$

D) 0

Section II 90 Marks

Attempt Questions 11-16.

Allow approximately 2 hours and 45 minutes for this section.

Your responses should include relevant mathematical reasoning and/or calculations.

Begin each question on a NEW page.

Question 11	(15 marks)	Marks
a)	Let $z = 1 + \sqrt{3}i$. Find the complex numbers for	
(i)	$z\bar{z}$	2
(ii)	z^2	2
b)	Express $-2 + 2\sqrt{3}i$ in the form $re^{i\theta}$	2
c)	Find	
(i)	$\int \frac{dx}{x(\ln x)}$	2
(ii)	$\int \frac{1}{x^2 + 6x + 13} dx$	2
d)	(i) Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to find a similar simplified result for $e^{-i\theta}$	1
	(ii) Hence show $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$	1
	(iii) Use the result from part (ii) to show that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$	3

End of Question 11

Question 12	(15 marks)	Begin a NEW page	Marks
a)	The velocity v m/s of a particle moving in simple harmonic motion along the x axis is given by $v^2 = 8 - 2x - x^2$.		
(i)	Between which two points is the particle oscillating?		2
(ii)	Find the acceleration of the particle in terms of x		1
(iii)	Find the period of the motion		1
(iv)	What is the time taken by the particle to travel the first 60m of its motion (leave in exact form)		1
b)	Given that a , b and c are positive real numbers, prove that $(a + b + c)^2 \geq 3(ab + ac + bc)$		3
c)	If $\underline{a} = 3\underline{i} + 2\underline{j} + 2\underline{k}$ and $\underline{b} = \underline{i} - \underline{j} - \underline{k}$, find the unit vector(s) perpendicular to both $\underline{a}$ and $\underline{b}$		2
d)	The points P , Q and R have position vectors $\underline{i} + 2\underline{j} - \underline{k}$, $\underline{i} + 2\underline{j}$ and $2\underline{i} - 3\underline{j} - 2\underline{k}$ respectively.		
(i)	Find the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$		1
(ii)	Find the magnitude of $\angle PQR$ correct to the nearest degree		2
e)	(i) Find the vector equation of line vector $\underline{r}$ given it passes through $(1, 3, -2)$ and $(2, -1, 2)$		1
(ii)	Determine if $\underline{r}$ passes through $(4, -9, 10)$		1

End of Question 12

a) Find $\int \frac{dx}{x\sqrt{x^2-4}}$ using the substitution $u = \sqrt{x^2-4}$ 2

b) Evaluate $\int x^2 \sin x dx$ 2

c) Given that for $k > 0$, $\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$ use Mathematical Induction to prove that for all integers $n \geq 2$, 3

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

d) If $P = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $R = 9\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k}$ find the point Q on $\overrightarrow{PR}$ such that $PQ : QR = 2 : 3$ 3

e) The rise and fall of tides can be approximated to simple harmonic motion. At 11am the depth of water in a tidal lagoon is lowest at 1m. The following high tide is at 5:21pm with a depth of 5m.

- | | | |
|-------|--|---|
| (i) | Find the period of this motion in minutes | 1 |
| (ii) | Express the motion using a cosine function | 2 |
| (iii) | Calculate between which times a yacht could safely cross the lagoon if a minimum depth of 2m is required | 2 |

End of Question 13

- a) Prove the following statement using a proof by contradiction. 3

“For each irrational number s , the number $2s + 1$ is also irrational”

- b) (i) If $I_n = \int_0^{\frac{\pi}{4}} \sec^n \theta d\theta$ show that 3

$$I_n = \frac{1}{n-1} [(n-2)I_{n-2} + (\sqrt{2})^{n-2}]$$

for $n \geq 2$

- (ii) Evaluate $\int_0^{\frac{\pi}{4}} \sec^4 \theta d\theta$ 2

- c) (i) Find real numbers a, b and c such that 2

$$\frac{1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}$$

- (ii) Hence find $\int \frac{1}{x^2(x-1)} dx$ in simplest form. 2

- d) Find a point of intersection of the line with parametric equation 3

$r = \tilde{i} + 3\tilde{j} - 4\tilde{k} + t(\tilde{i} + 2\tilde{j} + 2\tilde{k})$ and the sphere with equation

$$(x-1)^2 + (y-3)^2 + (z+4)^2 = 81$$

End of Question 14

Question 15	(15 marks)	Begin a NEW page	Marks
a)	The point $P(x, y)$ representing the complex number z moves in the Argand diagram so that $ z - 6 = z + 2i $.		
(i)	Show that the path P traces out, has equation $3x + y - 8 = 0$		2
(ii)	Find the minimum value of $ z $ as P moves on this path		2
b)	The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has zeros $a + bi$ and $a + 2bi$ where a and $b \in \mathbb{R}$ and $b \neq 0$. By evaluating a and b , find all the zeros of $P(x)$.		3
c)	A particle moves along the x – axis, starting at $x = 0.1$ at time $t = 0$. The velocity of the particle is described by $v = \sqrt{2x} e^{-x^2} \quad x \geq 0.1$ where x is the displacement of the particle from the origin.		
(i)	Show that the acceleration of the particle is given by $a = e^{-2x^2}(1-4x^2)$		2
(ii)	Hence find the fastest speed attained by the particle.		2
(iii)	Show that T , the time taken to travel from $x = 1$ to $x = 2$ can be expressed as $T = \int_1^2 \frac{1}{\sqrt{2x}} e^{x^2} dx$		2
(iv)	Use the trapezoidal rule with three function values to give an approximate value of T correct to the nearest whole number.		2

End of Question 15

a) Given $\overrightarrow{AP} = \begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} -2 \\ 5 \\ -7 \end{bmatrix}$

- | | | |
|------|---|---|
| (i) | Find $proj_{\tilde{b}} \tilde{p}$ where $\overrightarrow{AP} = \tilde{p}$ and $\overrightarrow{AB} = \tilde{b}$ | 2 |
| (ii) | Find the perpendicular distance d from P to the line AB . | 2 |

- | | | | |
|----|-----|--|---|
| b) | (i) | Using a suitable substitution, show that | 2 |
|----|-----|--|---|

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

- | | | |
|------|--|---|
| (ii) | A function $f(x)$ has the property that $f(x) + f(a-x) = f(a)$ | 2 |
| | Using part (i), or otherwise, show that | |

$$\int_0^a f(x)dx = \frac{a}{2}f(a)$$

- | | | | |
|----|-----|---|---|
| c) | (i) | Given $f(t) = \sin(a+nt) \sin b - \sin a \sin(b-nt)$ | 2 |
| | | where a, b and n are constants with $a > 0, b > 0$ and $a + b < \pi$ and $n \neq 0$, | |
| | | show that $f(t) = \sin(a+b) \sin nt$ | |

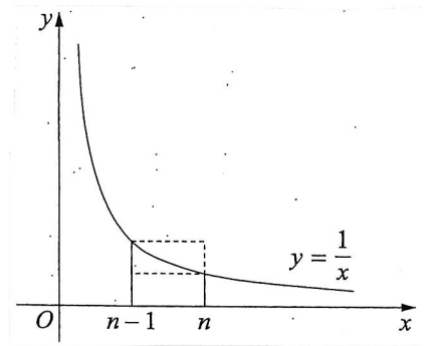
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|------|----------------------------------|---|
| (ii) | Find all values of t for which | 3 |
|------|----------------------------------|---|

$$\frac{\sin(a+nt)}{\sin(b-nt)} = \frac{\sin a}{\sin b}$$

Question 16 continues.....

d)

2



Let n be a positive integer greater than 1. The area of the region under the curve

$y = \frac{1}{x}$ from $x = n - 1$ to $x = n$ is between the areas of two rectangles, as shown in the diagram.

Show that $e^{\frac{-n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$

End of Exam

2021 Ext. 2 Trial HSC solutions

1. only one where
dot product = 0
is D.

$$2. v^2 = n^2(a^2 - x^2)$$

$$3^2 = n^2(2^2 - 1^2)$$

$$9 = 3n^2$$

$$n^2 = 3 \quad n = \sqrt{3}$$

$$T = \frac{2\pi}{\sqrt{3}} = \frac{2\sqrt{3}\pi}{3}$$

B

3. A

$$4. z_2 - z_1$$

$$= \sqrt{3}iz_1 - z_1$$

$$= \text{cis } \theta (-1 + \sqrt{3}i)$$

$$= \text{cis } \theta \times 2 \text{cis} \left(-\frac{\pi}{3} \right)$$

$$= 2 \text{cis} \left(\theta - \frac{2\pi}{3} \right)$$

$$= 2e^{i(\theta - \frac{2\pi}{3})}$$

A

$$5. \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x} dx$$

$$t = \tan \frac{x}{2}$$

$$x = 0 \Rightarrow t = 0$$

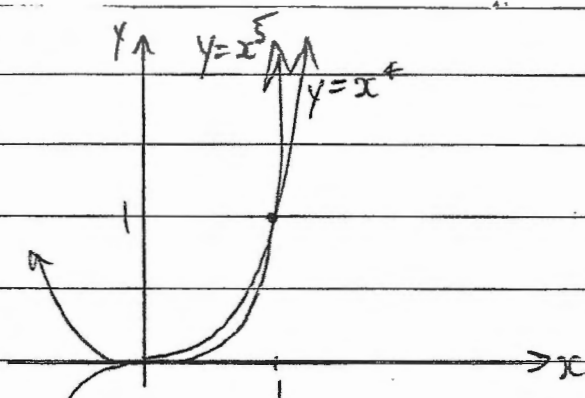
$$x = \frac{\pi}{2} \Rightarrow t = 1$$

$$\int_0^1 \frac{1}{1 + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$\int_0^1 \frac{2dt}{1+t^2+1-t^2} = \int_0^1 dt$$

C

6. D



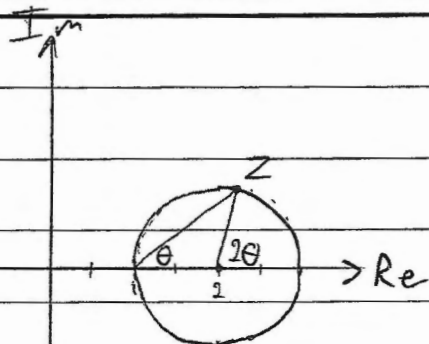
Area under $y = x^5$ is
lower between $x=0$
and $x=1$.

$$7. \frac{2}{1} = \frac{a-1}{-2} = \frac{b}{1}$$

$$\therefore a = -3, b = 2$$

B

8.



A is true as exterior
angle equals twice interior
opp. angle in isosceles Δ
A

$$9. \cos \theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}| \times |\underline{v}|}$$

$$= \frac{1 \times 1 + 1 \times 2 + 0 \times 2}{\sqrt{1^2 + 1^2 + 0^2} \times \sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = 45^\circ$$

D

$$10. \omega^7 - 1 = 0$$

$$(\omega - 1)(\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$$

If ω is complex then 2nd bracket = 0.

$$\text{but } \omega^7 = 1$$

$$\therefore \underbrace{1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7}_{0 + 1}$$

$$= 1$$

A

$$1) a) z = 1 + \sqrt{3}i$$

$$\begin{aligned} \text{ci) } z\bar{z} &= (1 + \sqrt{3}i)(1 - \sqrt{3}i) \quad \text{①} \\ &= 1^2 + (\sqrt{3})^2 \\ &= 4 \quad \text{②} \end{aligned}$$

$$\begin{aligned} \text{cii) } z^2 &= (1 + \sqrt{3}i)^2 \\ &= 1 + 2\sqrt{3}i + 3i^2 \quad \text{①} \\ &= -2 + 2\sqrt{3}i \quad \text{②} \end{aligned}$$

$$\begin{aligned} b) \quad -2 + 2\sqrt{3}i \\ r &= \sqrt{(-2)^2 + (2\sqrt{3})^2} \\ &= \sqrt{4 + 12} \\ &= 4 \quad \text{①} \end{aligned}$$

$$\begin{aligned} \arg &= \tan^{-1} \frac{2\sqrt{3}}{-2} \\ &= \tan^{-1}(-\sqrt{3}) \end{aligned}$$

$$\begin{aligned} \text{Working angle is } \frac{\pi}{3} \\ \text{2nd quadrant } \therefore \frac{2\pi}{3} \quad \text{①} \end{aligned}$$

$$\text{so } -2 + 2\sqrt{3}i = 4e^{\frac{2\pi i}{3}} = 4e^{i(\frac{2\pi}{3})} \quad \text{②}$$

$$\text{c) ci) } \int \frac{dx}{x(\ln x)}$$

$$\text{cii) } \int \frac{1}{x^2 + 6x + 13} dx$$

$$\begin{aligned} \text{Let } u &= \ln x \\ \therefore \frac{du}{dx} &= \frac{1}{x} \\ \therefore du &= \frac{dx}{x} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{(x+3)^2 + 2^2} dx \quad \text{①} \\ = \frac{1}{2} \tan^{-1} \left(\frac{x+3}{2} \right) + C \quad \text{②} \end{aligned}$$

$$\int \frac{du}{u} \quad \text{①}$$

$$= \log_e u$$

$$= \log_e |\log_e x| + C \quad \text{②}$$

$$\text{dc) } e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta)$$

$$e^{-i\theta} = \cos \theta - i \sin \theta \quad \text{①}$$

$$\begin{aligned} \text{cii) } e^{i\theta} - e^{-i\theta} &= (\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta) \\ &= 2i \sin \theta \end{aligned}$$

$$\therefore \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \quad \text{①}$$

$$\begin{aligned}
 \text{ciii) } \sin \theta &= \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \\
 \sin^3 \theta &= \frac{1}{8i^3} (e^{i\theta} - e^{-i\theta})^3 \quad (1) \\
 &= \frac{-1}{8i} (e^{3i\theta} - 3e^{2i\theta} \times e^{-i\theta} + 3e^{i\theta} e^{-2i\theta} - e^{-3i\theta}) \\
 &= \frac{-1}{8i} (e^{3i\theta} - e^{-3i\theta} - 3(e^{i\theta} - e^{-i\theta})) \quad (2) \\
 &= \frac{-1}{8i} (2i \sin 3\theta - 3 \times 2i \sin \theta) \\
 &= \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta \quad (3) \text{ as required.}
 \end{aligned}$$

$$\begin{aligned}
 \text{12a) i) } v^2 &= 8 - 2x - x^2 & \text{cii) } \frac{1}{2} v^2 &= 4 - x - \frac{x^2}{2} \\
 \text{Let } v &= 0 & \frac{d}{dx} \left(\frac{1}{2} v^2 \right) &= -1 - \frac{2x}{2} \\
 0 &= x^2 + 2x - 8 \quad (1) & a &= -(x+1) \quad (1) \\
 0 &= (x-2)(x+4) \\
 x &= 2 \text{ and } -4 \quad (2) & \text{ciii) Period} &= \frac{2\pi}{n} \\
 \text{oscillates between these.} & & n &= 1 \text{ from above} \\
 & & \therefore \text{Period is } &2\pi \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{civ) Travels } 12\text{m in one} \\
 \text{period } \therefore 5 \text{ periods} \\
 = 5 \times 2\pi = 10\pi \text{ seconds} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } (a-b)^2 &\geq 0 \Rightarrow a^2 + b^2 \geq 2ab \\
 (b-c)^2 &\geq 0 \Rightarrow b^2 + c^2 \geq 2bc \\
 (a-c)^2 &\geq 0 \Rightarrow a^2 + c^2 \geq 2ac \quad (1) \\
 2a^2 + 2b^2 + 2c^2 &\geq 2(ab + bc + ac) \\
 a^2 + b^2 + c^2 &\geq ab + bc + ac \quad (2) \\
 (a+b+c)^2 - 2ab - 2bc - 2ac &\geq ab + bc + ac \\
 \therefore (a+b+c)^2 &\geq 3ab + 3bc + 3ac \quad (3)
 \end{aligned}$$

12c) Let the unit vector be $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0 \quad \text{gives}$$

$$\textcircled{1} \quad 3a + 2b + 2c = 0 \quad \textcircled{1}$$

$$a - b - c = 0 \quad \textcircled{2}$$

$$\textcircled{1} + 2 \times \textcircled{2}$$

$$5a = 0$$

$$\therefore a = 0 \quad \therefore b = -c$$

unit vector means

$$a^2 + b^2 + c^2 = 1$$

$$\therefore 2b^2 = 1$$

$$b = \pm \frac{1}{\sqrt{2}} \quad c = \mp \frac{1}{\sqrt{2}}$$

$$\therefore \underline{v} = \pm \left(\frac{1}{\sqrt{2}} \underline{j} - \frac{1}{\sqrt{2}} \underline{k} \right) \quad \textcircled{2}$$

$$\begin{aligned} \text{d) (i) } \underline{QP} &= \underline{i} + 2\underline{j} - \underline{k} - (\underline{i} + 2\underline{j}) & \underline{QR} &= 2\underline{i} - 3\underline{j} - 2\underline{k} \\ &= -\underline{k} & & - (\underline{i} + 2\underline{j}) \\ & & & = \underline{i} - 5\underline{j} - 2\underline{k} \end{aligned}$$

① for both

$$\begin{aligned} \text{c) (ii) } \underline{QP} \cdot \underline{QR} &= |\underline{QP}| \times |\underline{QR}| \cos \theta \\ 2 &= 1 \times \sqrt{1^2 + 5^2 + 2^2} \cos \theta \quad \textcircled{1} \\ \cos \theta &= \frac{2}{\sqrt{30}} \\ \theta &= 69^\circ \quad \textcircled{2} \end{aligned}$$

$$\begin{aligned} \text{e) (i) } \underline{r} &= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2-1 \\ -1-3 \\ 2--2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} \quad \textcircled{1} \end{aligned}$$

cii) Let $\vec{r}$ be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ 4 \end{pmatrix} \Rightarrow \begin{aligned} 4 &= 1 + \lambda \Rightarrow \lambda = 3 \\ -9 &= 3 - 4\lambda \Rightarrow \lambda = 3 \quad (1) \\ 10 &= -2 + 4\lambda \Rightarrow \lambda = 3 \end{aligned}$$

$\therefore$ Line can pass through $(4, -9, 10)$

13 a) $\int \frac{dx}{x\sqrt{x^2-4}}$

Let $v = \sqrt{x^2-4}$

$$dv = \frac{x}{\sqrt{x^2-4}} dx$$

$$\int \frac{1}{x} \times \frac{dv}{x}$$

$$\frac{dv}{x} = \frac{dx}{\sqrt{x^2-4}}$$

$$\int \frac{dv}{x^2}$$

$$\int \frac{dv}{v^2+4} \quad (1)$$

Since $v^2 = x^2 - 4$

$$= \frac{1}{2} \tan^{-1} \frac{v}{2} + C$$

$$= \frac{1}{2} \tan^{-1} \frac{\sqrt{x^2-4}}{2} + C \quad (2)$$

b) $\int \frac{a}{x^2 \sin x} dx$

$$\int x^2 \frac{d}{dx} (-\cos x) dx$$

$$= -x^2 \cos x - \int 2x \times -\cos x dx$$

$$= -x^2 \cos x + 2 \int x \frac{d}{dx} (\sin x) dx \quad (1)$$

$$= -x^2 \cos x + 2 [x \sin x - \int \sin x dx]$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C \quad (2)$$

c) Show true for $n=2$

$$\frac{1}{1^2} + \frac{1}{2^2} < 2 - \frac{1}{2}$$

$$1 + \frac{1}{4} < \frac{3}{2} \checkmark$$

so true for $n=2$

Assume true for $n=k$ where $k > 2$

ie: $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k}$

Now to show true for $n=k+1$

Using assumption

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \quad (1)$$

$$< 2 - \frac{1}{k+1} + \left(\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} \right)$$

$$< 2 - \frac{1}{k+1} \text{ given}$$

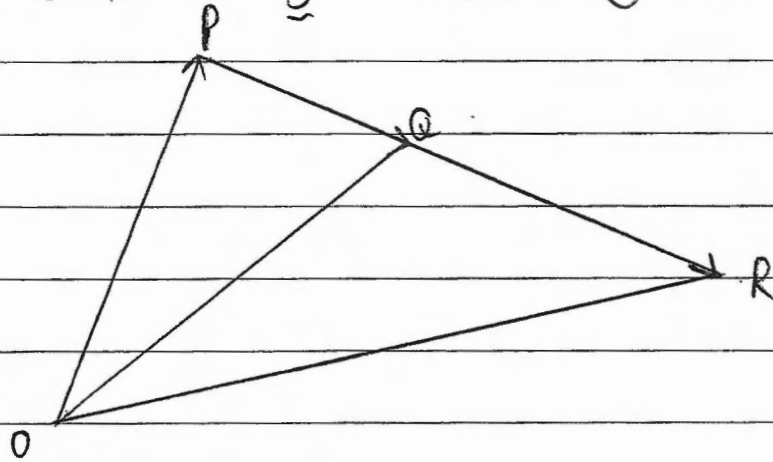
$$\frac{1}{(k+1)^2} \geq \frac{1}{k} + \frac{1}{k+1} < 0$$

$\therefore$ True for $n=k+1$ (2)

Since result is true for $n=2$, it must also be true for $n=2+1=3$,

$n=3+1=4$ and by induction for all positive integral values of $n \geq 2$. (3)

$$\begin{aligned}
 d) \quad \vec{PR} &= (9\hat{i} + 3\hat{j} + 8\hat{k}) - (\hat{i} + \hat{j} + \hat{k}) \\
 &= 8\hat{i} + 2\hat{j} + 7\hat{k} \\
 \vec{PQ} &= \frac{2}{5}(8\hat{i} + 2\hat{j} + 7\hat{k}) \quad (1)
 \end{aligned}$$

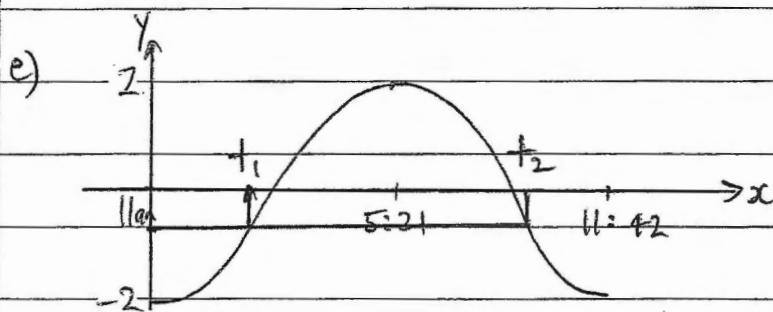


$$\text{Now } \vec{OP} + \vec{PQ} = \vec{OQ}$$

$$\therefore \vec{OQ} = \hat{i} + \hat{j} + \hat{k} + \frac{2}{5}(8\hat{i} + 2\hat{j} + 7\hat{k}) \quad (2)$$

$$\vec{OQ} = \frac{21}{5}\hat{i} + \frac{9}{5}\hat{j} + \frac{19}{5}\hat{k}$$

$$\therefore Q = \frac{21}{5}\hat{i} + \frac{9}{5}\hat{j} + \frac{19}{5}\hat{k} \quad (3) \quad \text{or } \left(\frac{21}{5}, \frac{9}{5}, \frac{19}{5}\right)$$



$$\begin{aligned}
 \text{ci) Period} &= 12 \text{ hrs } 42 \text{ min} \\
 &= 762 \text{ mins.} \quad (1)
 \end{aligned}$$

$$\text{cii) } y = -2 \cos n t \quad (1)$$

$$T = 762 = \frac{2\pi}{n}$$

$$\therefore n = \frac{\pi}{381}$$

$$\therefore y = -2 \cos \frac{\pi t}{381} \quad (2)$$

and 1 m above low tide $\Rightarrow y = -1$ on our graph.

$$-1 = -2 \cos \frac{\pi}{381} \quad (1)$$

$$\frac{1}{2} = \cos \frac{\pi}{381}$$

$$\therefore \frac{\pi}{381} = \frac{\pi}{3} \cdot \frac{5\pi}{3}$$

$$+ = \frac{381}{3}, \frac{5 \times 381}{3}$$

$$= 127, 635$$

= 2 hrs 7 min and 10 hrs 35 min after
11 am i.e. safe between 1:07 pm and 9:35 pm. (2)

14.a) Assume the statement is false
that for an irrational number s ,
there exists a number $2s+1$ which is
rational.

i.e. $2s+1 = \frac{a}{b}$ (1) where a and b are
integers with no common
factors.

$$\therefore 2s = \frac{a}{b} - 1$$

$$= \frac{a}{b} - \frac{b}{b}$$

$$2s = \frac{a-b}{b}$$

$$s = \frac{a-b}{2b}$$

$\therefore s = \frac{c}{d}$ where c and d are
(2) integers since a and b
are.

$\therefore s$ is rational

CONTRADICTION (3)

$\therefore$ Original statement is true.

$$b) i) I_n = \int_0^{\frac{\pi}{4}} \sec^n \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^2 \theta \sec^{n-2} \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{d}{d\theta} (\tan \theta) \sec^{n-2} \theta d\theta \quad (1)$$

$$= [\tan \theta \sec^{n-2} \theta]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan \theta \times (n-2) \sec^{n-3} \theta \times \sec \theta \tan \theta d\theta$$

$$= (\sqrt{2})^{n-2} - \int_0^{\frac{\pi}{4}} (n-2) \tan^2 \theta \sec^{n-2} \theta d\theta$$

$$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\frac{\pi}{4}} (\sec^2 \theta - 1) \sec^{n-2} \theta d\theta$$

$$= (\sqrt{2})^{n-2} - (n-2) \int_0^{\frac{\pi}{4}} \sec^n \theta d\theta + (n-2) \int_0^{\frac{\pi}{4}} \sec^{n-2} \theta d\theta$$

$$I_n = (\sqrt{2})^{n-2} - (n-2) I_n + (n-2) I_{n-2} \quad (2)$$

$$I_n + (n-2) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$$

$$(n-1) I_n = (n-2) I_{n-2} + (\sqrt{2})^{n-2}$$

$$I_n = \frac{1}{n-1} [(n-2) I_{n-2} + (\sqrt{2})^{n-2}] \quad (3)$$

$$ii) I_4 = \frac{2}{3} I_2 + \frac{1}{3} (\sqrt{2})^2 \quad (1)$$

$$= \frac{2}{3} (0 + 1) + \frac{2}{3}$$

$$I_4 = \frac{4}{3} \quad (2)$$

c) ci) $1 = ax(x-1) + b(x-1) + cx^2$ d)

Let $x=1 \therefore 1=c$ ① for
 $x=0 \therefore 1=-b$ ^{one of these}

Let $x=2 \therefore 1=2a+b+4c$
 $1=2a-1+4$

So $a=-1, b=-1, c=1$ ②

ii) $\int \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x-1} dx$
 $= -\log_e |x| + x^{-1} + \log_e |x-1|$ ①

$= \log_e \left| \frac{x-1}{x} \right| + \frac{1}{x} + C$ ②

$x = 1+t$

$y = 3+2t$

$z = -4+2t$ ①

Sub. these into sphere

$(1+t-1)^2 + (3+2t-3)^2$

$+ (-4+2t+4)^2 = 81$

$t^2 + (2t)^2 + (2t)^2 = 81$

$9t^2 = 81$

$t^2 = 9$

$t = \pm 3$ ②

sub. into ① ② ③

gives $(4, 9, 2)$ or

$(-2, -3, -10)$

(one of) ③

15. a) ci) $|z-6| = |z+2i|$

Let $x+iy = z$

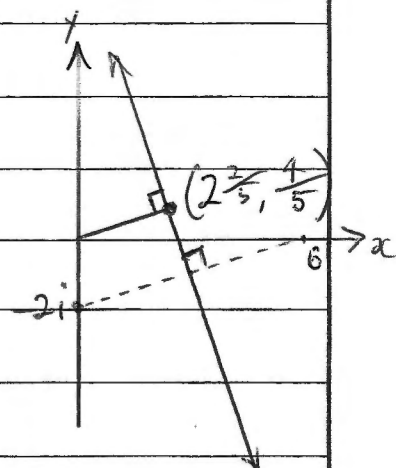
$|x-6+iy| = |x+i(y+2)|$

$\sqrt{(x-6)^2+y^2} = \sqrt{x^2+(y+2)^2}$ ①

$x^2 - 12x + 36 + y^2 = x^2 + y^2 + 4y + 4$

$0 = 12x + 4y - 32$

$0 = 3x + y - 8$ ②



ii) Min. value of $|z|$ is $\perp$ distance of this line from $(0,0)$.

Need to find perpendicular line to P passing through origin.

Gradient of z locus is -3

$\therefore$ Gradient of line through origin is $(0,0)$ is $\frac{1}{3}$. $\therefore$ Eq'n is $y = \frac{2x}{3}$

Solve simultaneously:

$$3x + \frac{2x}{3} - 8 = 0$$

$$\frac{10x}{3} = 8$$

$$10x = 24$$

$$x = 2\frac{2}{5} \therefore y = \frac{2 \times 2\frac{2}{5}}{3} = \frac{4}{5} \quad (1)$$

$$\therefore \min |z| \text{ is } \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \frac{\sqrt{160}}{5} = \frac{4\sqrt{10}}{5} \quad (2)$$

(2.53)

b) $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10 = (x^2 - 2ax + (a^2 + b^2))x$
 $(x^2 - 2ax + (a^2 + 4b^2)) \quad (1)$

Equating coefficients of x^3

$$-4ax^3 = -4x^3 \therefore a = 1 \quad (2)$$

Equate constants

$$10 = (a^2 + b^2)(a^2 + 4b^2)$$

$$10 = (b^2 + 1)(4b^2 + 1)$$

$$0 = 4b^4 + 5b^2 - 9$$

$$0 = 4b^4 + 9b^2 - 4b^2 - 9$$

$$0 = b^2(4b^2 + 9) - (4b^2 + 9)$$

$$0 = (b^2 - 1)(4b^2 + 9)$$

$$\therefore b = \pm 1$$

$$\begin{array}{r} + | 5 \\ \times | -36 \\ \hline 9 - 4 \end{array}$$

$\therefore$ Roots/zeros are $1 \pm i, 1 \pm 2i \quad (3)$

Q(i) $V = \sqrt{2x} e^{-x^2}$, $x \geq 0.1$

$$\frac{dv}{dx} = \sqrt{2x} \times -2x \times e^{-x^2} + e^{-x^2} \times \sqrt{2} \times \frac{1}{2} x^{-\frac{1}{2}}$$

$$= -2\sqrt{2} x^{\frac{3}{2}} e^{-x^2} + \frac{\sqrt{2} e^{-x^2}}{\sqrt{2x}} \quad (1)$$

$$a = V \frac{dv}{dx} = \sqrt{2x} e^{-x^2} \left[-2\sqrt{2} x^{\frac{3}{2}} e^{-x^2} + \frac{e^{-x^2}}{\sqrt{2x}} \right]$$

$$= -4x^2 e^{-2x^2} + e^{-2x^2} \quad (2)$$

$$= e^{-2x^2} (1 - 4x^2) \text{ as req'd}$$

cii) When $x = 0.1$, $a = V \frac{dv}{dx} > 0$. Particle increases speed until $a = 0$ i.e. when $1 - 4x^2 = 0$.

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = 0.5 \text{ (since } x \geq 0.1) \quad (1)$$

Sub. this into V gives

$$V = \sqrt{2 \times 0.5} \times e^{-0.5^2}$$

$$V = e^{-0.25} \text{ is max. speed.} \quad (2)$$

ciii) $V = \frac{dx}{dt} = \sqrt{2x} e^{-x^2}$

$$\therefore \int_1^2 \frac{e^{x^2} dx}{\sqrt{2x}} = \int dt \quad (1)$$

$$T = \int_1^2 \frac{e^{x^2} dx}{\sqrt{2x}} \quad (2)$$

(iv) x	1	1.5	2
$f(x)$	$\frac{e}{\sqrt{2}}$	$\frac{e^{2.25}}{\sqrt{3}}$	$\frac{e^4}{2}$

$$\therefore T = \frac{h}{2} [y_0 + 2y_1 + y_2]$$

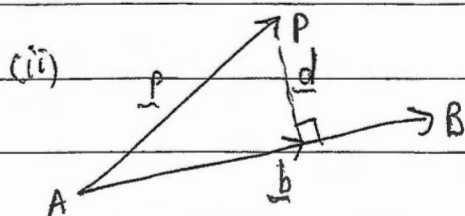
$$= \frac{0.5}{2} \left[\frac{e}{\sqrt{2}} + 2 \frac{e^{2.25}}{\sqrt{3}} + \frac{e^4}{2} \right] \quad (1)$$

$$= 10.04. \quad (2)$$

16.a) (i)

$$\text{proj}_{\underline{b}} \underline{p} = \frac{-1 \times (-2) + 6 \times (5) - 1 \times (-7)}{2^2 + 5^2 + 7^2} (-2\underline{i} + 5\underline{j} - 7\underline{k}) \quad (1)$$

$$= \frac{1}{2} (-2\underline{i} + 5\underline{j} - 7\underline{k}) \quad (2)$$



$$d = |\text{proj}_{\underline{b}} \underline{p} - \underline{p}| \quad (1)$$

$$= \left| \frac{1}{2} (-2\underline{i} + 5\underline{j} - 7\underline{k}) - (-\underline{i} + 6\underline{j} - \underline{k}) \right|$$

$$= \left| 0\underline{i} - \frac{7}{2}\underline{j} - \frac{5}{2}\underline{k} \right|$$

$$= \frac{\sqrt{74}}{2} \quad (2)$$

$$\text{or } \sqrt{\frac{37}{2}}$$

$$b) \text{ (i) } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

In RHS, let $u = a - x$ when $x=0$, $u=a$
 $du = -dx$ when $x=a$, $u=0$ (1)

$$\therefore \text{RHS becomes } \int_a^0 f(u) \times -du$$

$$= \int_0^a f(u) du \quad (2)$$

$$= \text{LHS (different variable irrelevant)}$$

$$a) \text{ (i) } f(x) + f(a-x) = f(a)$$

$$\int_0^a f(x) dx + \int_0^a f(a-x) dx = \int_0^a f(a) dx$$

$$2 \int_0^a f(x) dx = [f(a)x]_0^a \quad (1)$$

$$2 \int_a^a f(x) dx = a f(a)$$

$$\therefore \int_a^a f(x) dx = \frac{a}{2} f(a) \quad (2)$$

$$\begin{aligned} \text{c) i) } f(t) &= \sin(a+nt) \sin b - \sin a \sin(b-nt) \\ &= (\sin a \cos nt + \sin nt \cos a) \sin b - \\ &\quad \sin a (\sin b \cos nt - \sin nt \cos b) \quad (1) \\ &= \cos a \sin b \sin nt + \sin a \cos b \sin nt \\ &= (\cos a \sin b + \sin a \cos b) \sin nt \\ &= \sin(a+b) \sin nt \quad (2) \end{aligned}$$

$$\text{c) ii) } \frac{\sin(a+nt)}{\sin(b-nt)} = \frac{\sin a}{\sin b}$$

$$\begin{aligned} \therefore \sin b \sin(a+nt) &= \sin a \sin(b-nt) \\ \sin b \sin(a+nt) - \sin a \sin(b-nt) &= 0 \\ \Rightarrow \sin(a+b) \sin nt &= 0 \quad \text{using c) i) } (1) \\ \text{as } \sin(a+b) &\neq 0, \end{aligned}$$

$$\sin nt = 0 \quad (2)$$

$$nt = k\pi$$

$$t = \frac{k\pi}{n}$$

(3)

$$k = 0, \pm 1, \pm 2, \dots$$

$$k = 0, \pm 1, \pm 2, \pm 3, \dots$$

d) Area of lower rectangle = $\frac{1}{n}$

Area of upper rectangle = $\frac{1}{n-1}$

$$\therefore \frac{1}{n} < \int_{n-1}^n \frac{dx}{x} < \frac{1}{n-1}$$

$$\frac{1}{n} < [\ln x]_{n-1}^n < \frac{1}{n-1}$$

$$\frac{1}{n} < \ln \frac{n}{n-1} < \frac{1}{n-1}$$

$$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}} \quad (1)$$

$$e < \left(\frac{n}{n-1}\right)^n < e^{\frac{n}{n-1}}$$

$$\frac{1}{e} > \left(\frac{n-1}{n}\right)^n > \frac{1}{e^{\frac{n}{n-1}}}$$

$$\frac{1}{e^{\frac{n}{n-1}}} < \left(\frac{n-1}{n}\right)^n < \frac{1}{e}$$

$$e^{\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1} \quad \text{as req'd.} \quad (2)$$

