Student Name: ______________________

Teacher: ___________________________

Sydney Technical High School

2025 | HIGHER SCHOOL CERTIFICATE **TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless work or illegible writing
- **Begin each question on a new page**

Total marks: 100

Section I — 10 marks

- Attempt Questions 1 - 10
- Allow about 15 minutes for this part

Section II — 90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section.

Section I

10 Marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1. Let $z = 3 + 2i$. What is the value of $\overline{iz}$, expressed in the form $x + iy$?

(A) $2 + 3i$

(B) $-2 + 3i$

(C) $-2 - 3\text{i}$

(D) $2 - 3i$

2. Which of the following is a square root of $7 - 24i$?

(A) $-4 - 3i$

(B) $-4 + 3i$

(C) $-3 - 4i$

(D) $-3 + 4i$

3. Given $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, find $(\text{proj}_{\underset{\sim}{\mathrm{b}}}\underset{\sim}{a}) \cdot \underset{\sim}{k}$.

(A) 1

(B) -1

(C) $\frac{2}{3}$

(D) $-\frac{2}{3}$

4. Given $A(3, 6, 8)$ and $B(6, 3, 5)$, find the point that divides AB internally in the ratio $2: 1$?

(A) $(4, 5, 6)$

(B) $(6, 3, 5)$

(C) $(5, 4, 6)$

(D) $(7, 2, 4)$

5. Consider the statement "If a quadrilateral is a parallelogram, then it is a rhombus". Which of the following is true?

(A) The statement and its converse are true.

(B) Neither the statement nor its converse is true.

(C) The statement is true and its converse is false.

(D) The statement is false and its converse is true.

6. Which of the following is the negation of $\exists x: P(x) \Longrightarrow \neg Q(x)$?

(A) $\exists x: P(x) \wedge \neg Q(x)$

(B) $\exists x: P(x) \wedge Q(x)$

(C) $\forall x: P(x) \wedge \neg Q(x)$

(D) $\forall x: P(x) \wedge Q(x)$

7. Consider the vectors $\underset{\sim}{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$, $\underset{\sim}{c} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\underset{\sim}{d} = \underset{\sim}{0}$, where $x, y, z \in \mathbb{Z}^{+}$.
Which of the following statements is true?

(A) All of these vectors could be perpendicular to one another.

(B) Only $\underset{\sim}{a}$ and $\underset{\sim}{b}$ cannot be perpendicular to each other.

(C) You can have both $\underset{\sim}{a} \perp \underset{\sim}{c}$ and $\underset{\sim}{b} \perp \underset{\sim}{c}$, but no other vectors can be perpendicular.

(D) You can have exactly one of $\underset{\sim}{a} \perp \underset{\sim}{c}$ and $\underset{\sim}{b} \perp \underset{\sim}{c}$, and no other vectors can be perpendicular.

8. The square has vertex O at the origin. $\overrightarrow{OA}$ is represented by the complex number $2 + 3i$ and B is in the second quadrant. Which of the following is the complex number representing the diagonal $\overrightarrow{AC}$?

(A) $-5 - i$

(B) $-1 - 5i$

(C) $-5 + i$

(D) $1 - 5i$

9. A particle is moving vertically in a resistive medium under the influence of gravity. The resistive force is proportional to the velocity of the particle.

Which of the following is false?

(A) If the particle is initially moving downwards, then its speed could either increase or decrease.

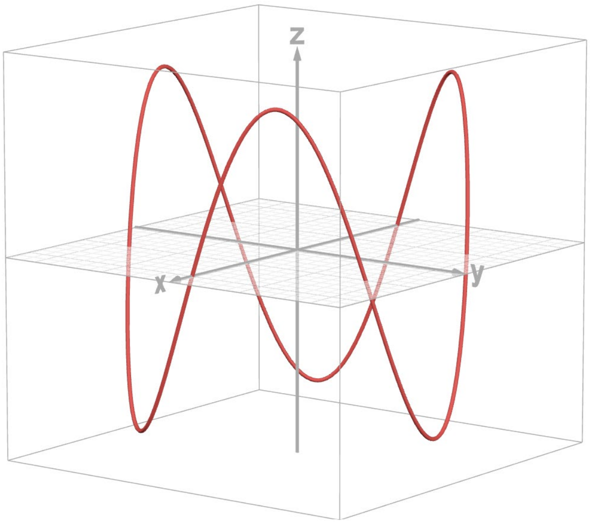
(B) If the particle is initially moving upwards, then its speed could either increase or decrease.

(C) If the particle is initially moving upwards, it will eventually reach a terminal velocity.

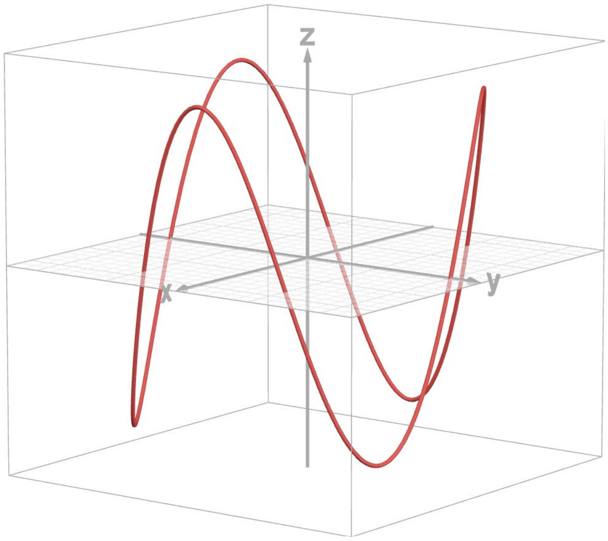
(D) If the particle is initially moving downwards, it will eventually reach a terminal velocity.

10. Which of the following graphs represents the vector equation $\underset{\sim}{v} = \begin{bmatrix} \cos t \\ \sin t \\ \cos 3t \end{bmatrix}$, $t \in \mathbb{R}$.

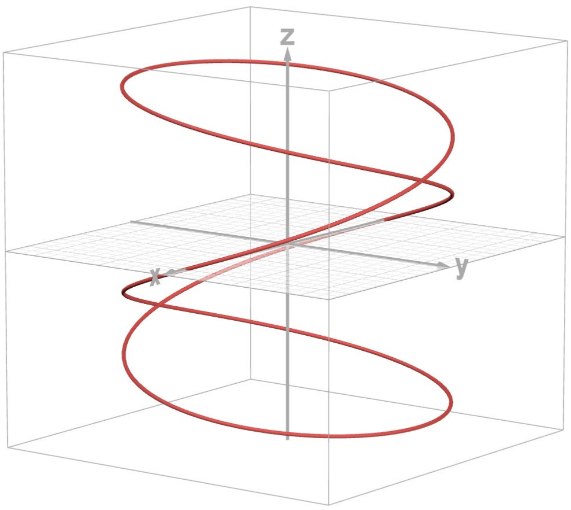
(A)



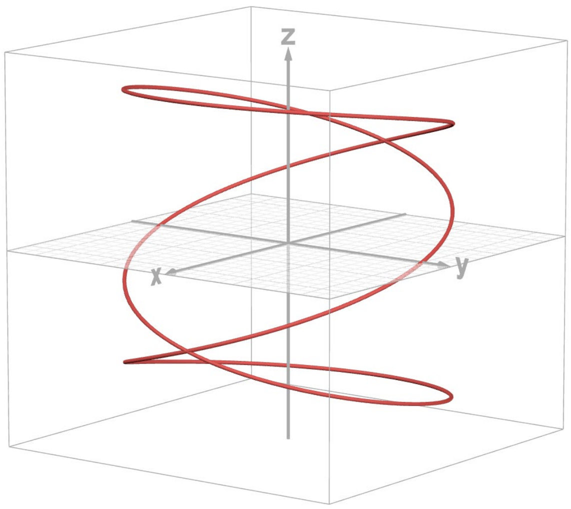
(B)



(C)



(D)



Section II

Total marks – 90
Attempt Question 11-16
Allow about 2 hour and 45 minutes for this section

Begin each question on a NEW page

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Begin a NEW page.

a) For the complex number $z = -\sqrt{3} + i$

i) Express z in the form $r(\cos\theta + i\sin\theta)$. **2**

ii) Hence, find n, the smallest positive integer such that $z^n \in \mathbb{R}$. **2**

b) Find the vector parallel to $\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ that has a magnitude of 5. **2**

c) Find:

i) $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$ **2**

ii) $\int x^7\sqrt{x^4-3}\,dx$ **3**

d) The polynomial $P(x) = x^3 - 5x^2 + 28x + k$, where $k \in \mathbb{R}$, has a zero of $3 + 5i$.

i) Find the other two zeros of $P(x)$. **2**

ii) Find the value of k. **2**

End of Question 11

Question 12 (15 marks) Begin a NEW page.

a) Find the following integrals

i) $\int \frac{1}{1+\cos x} dx$ **2**

ii) $\int \cos^5 x \sin^6 x \, dx$ **3**

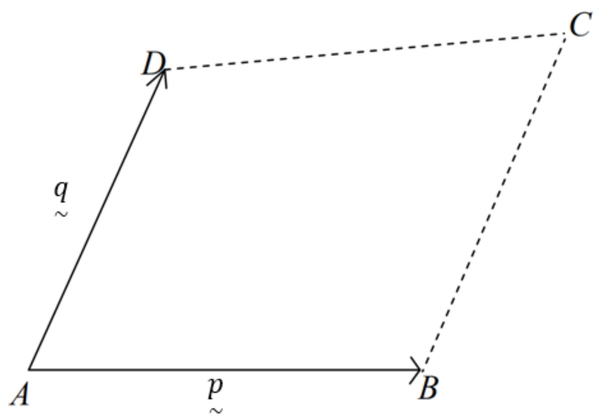
b) For $x \in \mathbb{Z}$, prove that if x^3 is even, then x is even. **3**

c) The line l passes through $A(-2,11,2)$ and $B(2,8,2)$. The sphere S has centre $(3,1,2)$ with radius 5.

i) Find the vector equations of the line and sphere. **2**

ii) Find any points of intersection between the sphere and the line. **2**

d) Consider the generic quadrilateral $ABCD$ below, where $\overrightarrow{AB} = \underset{\sim}{p}$, and $\overrightarrow{AD} = \underset{\sim}{q}$. **3**



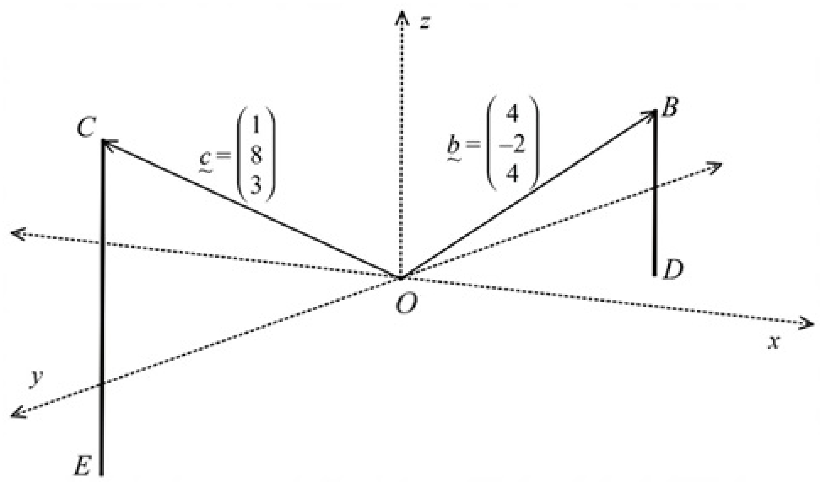
Use vector methods to prove that if the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

End of Question 12

Question 13 (15 marks) Begin a NEW page.

a) Prove by contradiction that there are an infinite number of prime numbers. **3**

b) Two vertical posts CE and BD stand on a horizontal plane. Use origin O and position vectors $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$, as shown on the diagram below:



i) Find the size of $\angle OED$ to the nearest degree. **2**

ii) Find the area of ΔOBC. **2**

Question 13 Continues on Page 9

c) Given that ω is a non-real fifth root of unity.

i) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. **1**

ii) Show that $(1 + \omega + \omega^4)(1 + \omega^2 + \omega^3) = -1$. **2**

d) A particle moves in a straight line experiencing an acceleration of $\ddot{x} = 4x + 12$, where x m is the displacement from the origin and t is time in seconds.
Initially, the particle is at the origin and has a velocity of 6 m/s to the right of the origin.

i) Find an expression for the velocity of the particle in terms of displacement. **3**

ii) Find the time taken for the particle to travel 1 metre from the origin. **2**

End of Question 13

Question 14 (15 marks) Begin a NEW page.

a)

i) Prove, by mathematical induction for $n \in \mathbb{Z}^{+}$: **4**

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_n}\right) \geq n^2$$

ii) Hence show that $\sec^2 x + \operatorname{cosec}^2 x + \cot^2 x \geq 9\cos^2 x$. **1**

b)

i) Prove that $2\cos(n\theta) = e^{in\theta} + e^{-in\theta}$. **1**

ii) Hence, find $\int \cos^6 \theta \, d\theta$. **4**

c) Use partial fractions to find **5**

$$\int \frac{2x - 18}{(x - 3)(x^2 + 3)} dx$$

End of Question 14

Question 15 (15 marks) Begin a NEW page.

a) Sketch on the Argand diagram the values of z such that $\frac{z}{z-2}$ is purely imaginary. **3**

b) An object of mass 2 kg is projected vertically upwards from the ground at a speed of 30 m/s. The object is subject to acceleration due to gravity of 10 m/s^2 and an air resistance force of $\frac{1}{15}v^2$, where v is the magnitude of the particle's velocity at that time.

i) Show that while the object is travelling upwards the equation of motion is **1**

$$\ddot{x} = -\left(10 + \frac{1}{30}v^2\right)$$

ii) Calculate the greatest height reached by the object. **2**

iii) After reaching the greatest height, the object drops back down to the ground under the same forces of gravity and resistance. Calculate the speed of the particle as it returns to the ground. **3**

c) For this question, you may assume the arithmetic mean-geometric mean inequality for three real numbers:

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

For $x, y, z \in \mathbb{R}$

i) Prove $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{x+y+z}$.

3

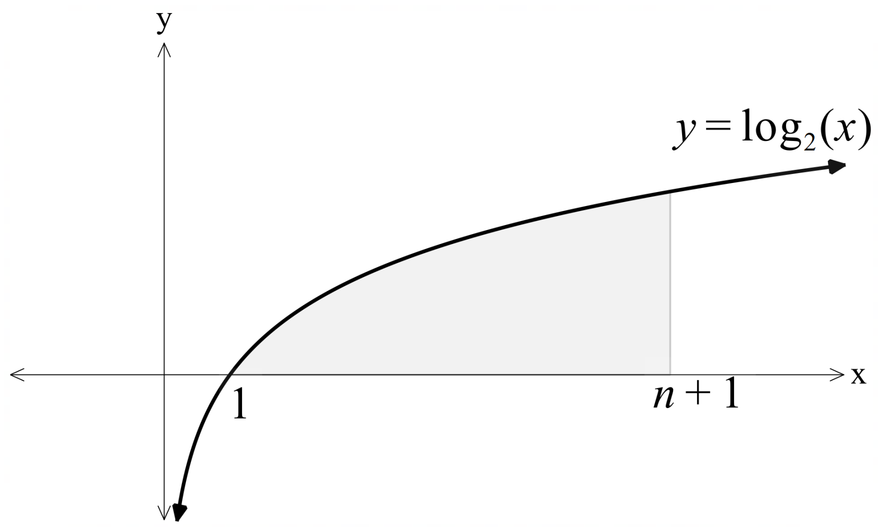
ii) Hence, given $x, y, z > 0$ and $x + y + z = 1$, prove **3**

$$\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right)\left(\frac{1}{z} - 1\right) \geq 8$$

End of Question 15

Question 16 (15 marks) Begin a NEW page.

a) Consider the graph of $y = \log_2 x$ below, where n is some integer.



i) Explain why $\log_2(n!) \le \int_1^{n+1} \log_2 x \, dx \le \log_2\big((n+1)!\big)$ **2**

ii) Hence show **4**

$$\left(\frac{(n+1)^n}{n!}\right)^{\ln 2} \le 2^n \le \left(\frac{(n+1)^{n+1}}{n!}\right)^{\ln 2}$$

Question 16 Continues on Page 13

b) Given $I_n = \int_{-3}^{0} x^n \sqrt{x+3}\ dx$

i) Show that $I_n = -\frac{6n}{2n+3} I_{n-1}$. **3**

ii) Hence, find I_3. **2**

c)

i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

ii) Hence, evaluate I, where **3**

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + (\sin x - \cos x)^2}\,dx$$

End of Examination

Multiple Choice CBCCD DCABB

1. $z=3+2i$

$iz = -2+3i$ C.

$\overline{iz} = -2-3i$

2. $(x+iy)^2 = 7-24i$

$x^2-y^2+2xyi = 7-24i$

$x^2-y^2 = 7$, $xy=-12$

$x=\pm 4$, $y=\mp 3$ B

3. $\left(\mathrm{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}\right) \cdot \underset{\sim}{k}$

$$= \left(\frac{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{2-2-4}{1+4+1} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{-4}{6} \times -1 \times 1 = \frac{2}{3}$$ C.

4. $(3,6,8)$ $(6,3,5)$

$2:1$

$$P: \left(\frac{3+2\times6}{3}, \frac{6+2\times3}{3}, \frac{8+2\times5}{3}\right)$$

$$= (5,4,6)$$ C.

5. Statement: Parallelogram $\Rightarrow$ Rhombus F

Converse: Rhombus $\Rightarrow$ Parallelogram T

$\therefore$ D.

6. $\neg(\exists x (P(x) \Rightarrow \neg Q(x)))$

$= \forall x (\neg(P(x) \Rightarrow \neg Q(x)))$

$= \forall x (P(x) \wedge \neg\neg Q(x))$ D.

$= \forall x\ P(x) \wedge Q(x)$

7. $\underset{\sim}{0}$ not perpendicular to anything:

Not A. or B.

If $\underset{\sim}{a} \perp \underset{\sim}{c}$: $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

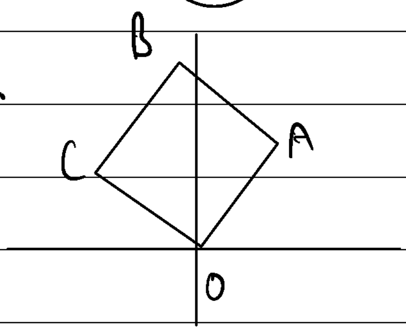
$$x + y - z = 0$$

$$\therefore -x - y + z = 0$$

$$\therefore \underset{\sim}{b} \cdot \underset{\sim}{c} = 0, \ \underset{\sim}{b} \perp \underset{\sim}{c}$$

C.

8.



$$\overrightarrow{OA} = 2 + 3i$$

$$\overrightarrow{OC} = i(2+3i)$$

$$= -3 + 2i$$

$$\therefore \overrightarrow{AC} = -3 + 2i - (2 + 3i)$$

$$= -5 - i$$

A.

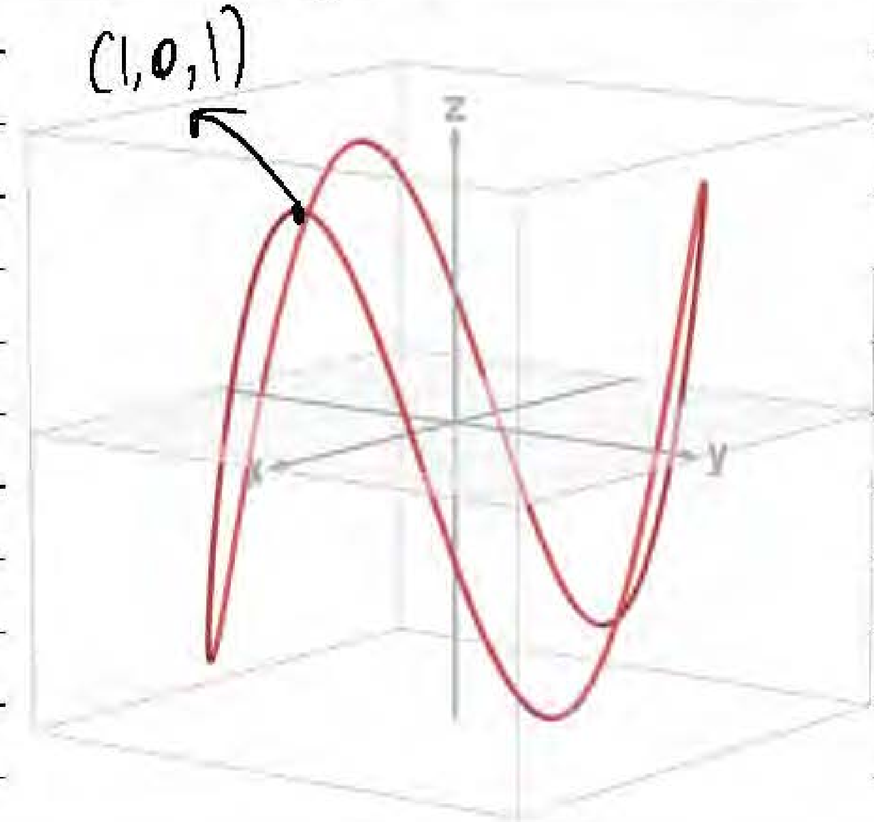
9.

B. is false as an upwards moving particle is only experiencing downwards acceleration: speed must decrease.

(D. z-axis has higher frequency: Not C or D.

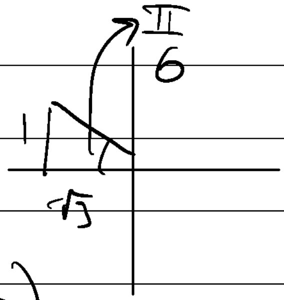
At $t=0$, $\underline{x} = \begin{bmatrix} \cos 0 \\ \sin 0 \\ \cos 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{B.}$$



11.

a) i) $z = -\sqrt{3} + i$



$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

ii) $z^n = 2^n\left(\cos\frac{5n\pi}{6} + i\sin\frac{5n\pi}{6}\right)$

Real if $\sin\frac{5n\pi}{6} = 0$

$\frac{5n\pi}{6} = k\pi$

$\therefore$ $n = 6$ is the smallest positive integer

b) $\hat{\underset{\sim}{n}} = \sqrt{2^2+1^2+2^2}$

$= 3$

$\therefore$ Vector $\frac{5}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

c) i) $\displaystyle\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{5-(x^2+4x)}}\,dx$$

$$= \int \frac{1}{\sqrt{5-(x^2+4x+4)+4}}$$

$$= \int \frac{1}{\sqrt{9-(x+2)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x+2}{3}\right) + C$$

ii) $\displaystyle\int x^7\sqrt{x^4-3}\,dx$ $\qquad u = x^4-3$

$du = 4x^3\,dx$

and $x^4 = u+3$

$$= \frac{1}{4}\int x^4 \times 4x^3 \sqrt{x^4-3}\,dx$$

$$= \frac{1}{4}\int (u+3)\,u^{\frac{1}{2}}\,du$$

$$= \frac{1}{4}\int u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\,du$$

$$= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} + 3\times\frac{2}{3}u^{\frac{3}{2}}\right] + C$$

$$= \frac{1}{10}(x^4-3)^{\frac{5}{2}} + \frac{1}{2}(x^4-3)^{\frac{3}{2}} + C$$

d) $P(x) = x^3 - 5x^2 + 28x + k$

i) Zeros $3+5i$, $3-5i$, α

Sum of roots: $3+5i+3-5i+\alpha = \frac{5}{1}$

$$6+\alpha = 5$$

$$\alpha = -1$$

$\therefore$ Zeros are $3+5i$, $3-5i$, -1

ii) Product of roots:

$$(3-5i)(3+5i)\times -1 = -k$$

$$(3^2+5^2) = k$$

$$k = 34$$

12.

$t = \tan\frac{x}{2}$

a) i) $\displaystyle\int \frac{1}{1+\cos x}\,dx$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dt = \frac{1}{2}(1+t^2)\,dx$

$dx = \dfrac{2dt}{1+t^2}$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$

$$= \int \frac{2}{1+t^2+1-t^2}\,dt$$

$$= \int 1\,dt$$

$$= t + c$$

$$= \tan\frac{x}{2} + c$$

ii) $\displaystyle\int \cos^5 x \sin^6 x\,dx$

$u = \sin x$

$du = \cos x\,dx$

$$= \int \cos x \times \cos^4 x \sin^6 x\,dx$$

$$= \int \cos x\,(1-\sin^2 x)^2 \sin^6 x\,dx$$

$$= \int (1-u^2)^2 u^6\,du$$

$$= \int (1-2u^2+u^4)u^6\,du$$

$$= \int u^6 - 2u^8 + u^{10}\,du$$

$$= \frac{1}{7}u^7 - \frac{2}{9}u^9 + \frac{1}{11}u^{11} + c$$

$$= \frac{1}{7}\sin^7 x - \frac{2}{9}\sin^9 x + \frac{1}{11}\sin^{11} x + c$$

b) $x \in \mathbb{Z}$ if x^3 is even, then x is even.

Take the contropositive:

If x is odd then x^3 is odd.

Assume x is odd $\quad x = 2k+1 \quad , \quad x \in \mathbb{Z}$.

$$
\begin{aligned}
x^3 &= (2k+1)^3 \\
&= (2k)^3 + 3(2k)^2(1) + 3(2k)(1)^2 + (1)^3 \\
&= 8k^3 + 12k^2 + 6k + 1 \\
&= 2(4k^3 + 6k + 3k) + 1 \\
&= 2M + 1, \qquad M \in \mathbb{Z}
\end{aligned}
$$

$\therefore$ If x is odd, x^3 is odd

Contropositively, if x^3 is even, x is even.

c) i) $\vec{AB} = \begin{bmatrix} 2+2 \\ 8-11 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$

$\therefore \; l = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$

or $l = \begin{bmatrix} -2 \\ 11 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}$

direction vector $\begin{bmatrix} -4 \\ 5 \\ 0 \end{bmatrix}$

Sphere: $\left| \underset{\sim}{v} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right| = 5$

ii) Sub line into sphere

$$\left| \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right| = 5$$

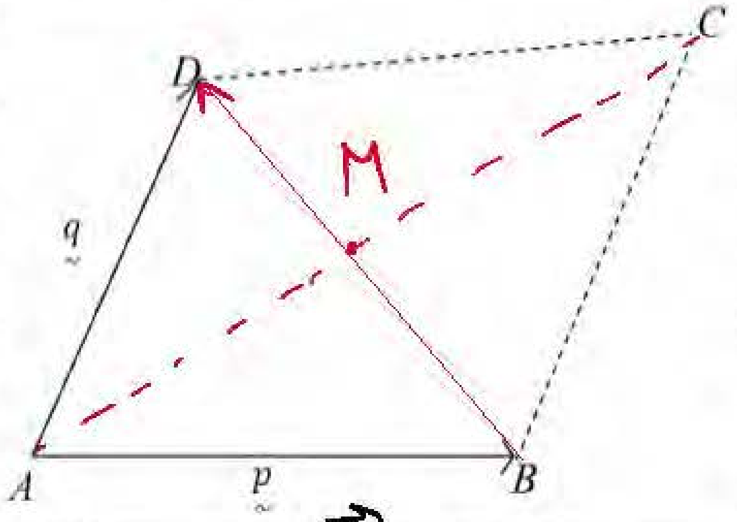
$$\left| \begin{bmatrix} -1+4\lambda \\ 7-3\lambda \\ 0 \end{bmatrix} \right| = 5$$

$$(-1+4\lambda)^2 + (7-3\lambda)^2 = 25$$
$$1-8\lambda+16\lambda^2+49-42\lambda+9\lambda^2 = 25$$
$$25\lambda^2-50\lambda+25=0$$
$$\lambda^2-2\lambda+1=0$$
$$(\lambda-1)^2=0$$
$$\lambda=1$$

$\therefore$ Point from $\begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$

$\therefore$ Point of intersection: $(6, 5, 2)$

d)



Diagonals $\vec{AC}$ and $\vec{BD}$ both pass M, the midpoint.

i.e. $\frac{1}{2}\vec{AC} = \vec{AB} + \frac{1}{2}\vec{BD}$

$\vec{AC} = 2\vec{AB} + \vec{BD}$

$\vec{AB} + \vec{BC} = 2p + q - p$

$p + \vec{BC} = p + q$

$\therefore \vec{BC} = q$.

So $\vec{AD} = \vec{BC}$

$\therefore$ ABCD is a parallelogram

(one pair of sides is equal and parallel).

13. a) Prove there are infinitely many prime numbers.

For contradiction, assume there are finite prime numbers: $p_1, p_2, p_3, \dots p_k$, where p_k is the largest prime.

Now consider

$p_1 p_2 p_3 \dots p_k + 1$.

This is 1. Not divisible by $p_1, p_2, p_3, \dots,$ or p_k. It is hence prime.

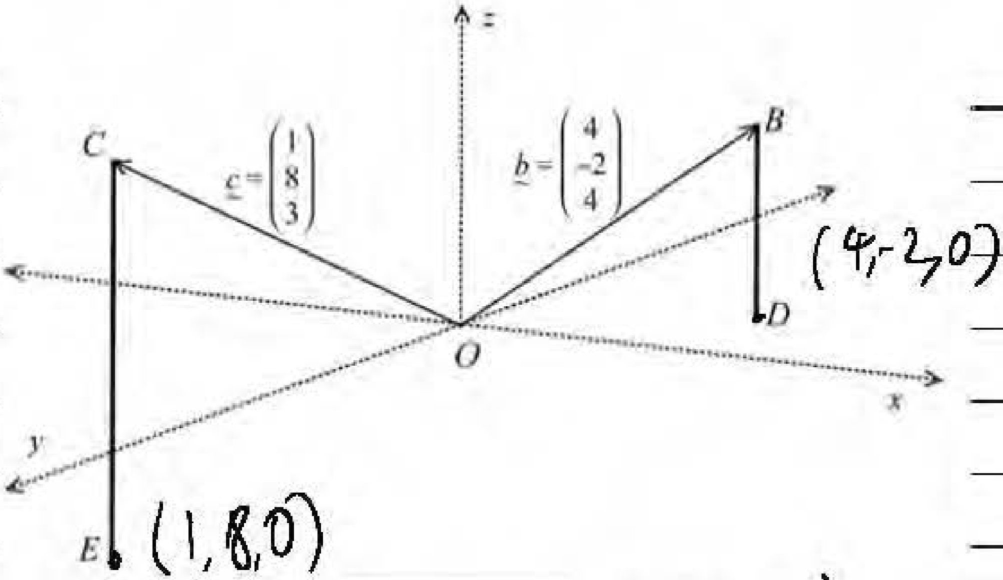
2. Larger than p_k.

This contradicts the fact that p_k is the largest prime.

$\therefore$ Assumption was false.

$\therefore$ By contradiction there are infinitely many prime numbers.

b)



i) $\vec{EO} = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$ $\qquad \vec{EB} = \begin{bmatrix} 4-1 \\ -2-8 \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$

$|\vec{EO}| = \sqrt{65}$ $\qquad |\vec{EB}| = \sqrt{109}$

$\vec{EO} \cdot \vec{EB} = -3 + 80$

$= 77$

$\therefore 77 = \sqrt{65}\sqrt{109} \cos\theta$

$\cos\theta = \dfrac{77}{\sqrt{65}\sqrt{109}}$

$= 24°$ (nearest degree)

ii) $\underset{\sim}{c} \cdot \underset{\sim}{b} = 4 - 16 + 12$

$= 0$

$\therefore \underset{\sim}{c} \perp \underset{\sim}{b}$

$\therefore A = \frac{1}{2} |\underset{\sim}{c}| |\underset{\sim}{b}|$

$= \frac{1}{2} \sqrt{1+64+9} \sqrt{16+4+16}$

$= 3\sqrt{74}$ units^2

c) i) w solves $z^5-1=0$

$$w^5-1=0$$

$$(w-1)(w^4+w^3+w^2+w+1)=0$$

As $w \notin \mathbb{R}$, $w \neq 1$ $\therefore w^4+w^3+w^2+w+1=0$

ii) $LHS=(1+w+w^4)(1+w^2+w^3)$

$$=1+w^2+w^3$$
$$+w+w^3+w^4$$
$$+w^4+w^6+w^7$$

$$=1+w+w^2+2w^3+2w^4+w\times w^5+w^2\times w^5$$

as $w^5=1$

$$=1+2w+2w^2+2w^3+2w^4+1\cdot 1$$

$$=2(1+w+w^2+w^3+w^4)-1$$

$$=2\times 0-1 \quad \text{from i)}$$

$$=-1$$

$$=RHS$$

d) $\ddot{x} = 4x+12$

i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x+12$

$$\frac{1}{2}v^2 = 2x^2+12x+c$$

At $t=0$, $x=0$, $v=6$

$$\frac{1}{2}\times 6^2 = 0+0+c$$
$$c = 18$$

$$\therefore \frac{1}{2}v^2 = 2x^2+12x+18$$

$$v^2 = 4x^2+24x+36$$

$$v^2 = 4(x^2+6x+9)$$

$$v^2 = \left(2(x+3)\right)^2$$

$v = 2(x+3)$, as initially $v>0$

ii) $\dfrac{dx}{dt} = 2(x+3)$

$\dfrac{dt}{dx} = \dfrac{1}{2(x+3)}$

$t = \dfrac{1}{2}\displaystyle\int \dfrac{1}{x+3}\, dx$

$t = \dfrac{1}{2}\ln|x+3| + c$

At $t=0,\ x=0$

$0 = \frac{1}{2}\ln|3| + c$

$c = -\frac{1}{2}\ln 3$

$\therefore t = \frac{1}{2}\ln|x+3| - \frac{1}{2}\ln 3$

At $x=1$

$t = \frac{1}{2}\ln|4| - \frac{1}{2}\ln 3$

$= \frac{1}{2}\ln\left(\frac{4}{3}\right)$ seconds

14.

a) i) Test for $n=1$

$$a_1 \times \frac{1}{a_1} = 1$$
$$\geq 1^2$$

$\therefore$ Statement is true for $n=1$.

Assume true for $n \leq k$, i.e. assume

$$\left(a_1 + a_2 + \dots a_k\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_k}\right) \geq k^2$$

Hence show true for $n=k+1$, RTP:

$$\left(a_1 + a_2 + \dots a_k + a_{k+1}\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_k} + \frac{1}{a_{k+1}}\right) \geq (k+1)^2$$

$$LHS = \left(a_1 + a_2 + \dots a_k + a_{k+1}\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_k} + \frac{1}{a_{k+1}}\right)$$

$$= \left(a_1 + a_2 + \dots a_k\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_k}\right)$$

$$+ a_{k+1}\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_k}\right)$$

$$+ \frac{1}{a_{k+1}}\left(a_1 + a_2 + \dots a_k\right)$$

$$+ a_{k+1} + \frac{1}{a_{k+1}}$$

$$\geq k^2 + a_{n+1}\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots\frac{1}{a_k}\right) \quad \text{by assumption}$$

$$+\frac{1}{a_{n+1}}(a_1+a_2+\dots a_k)+1$$

$$= k^2 + \frac{a_{k+1}}{a_n}+\frac{a_{n+1}}{a_2}+\dots\frac{a_{n+1}}{a_n}$$

$$+\frac{a_1}{a_{n+1}}+\frac{a_2}{a_{n+1}}+\dots\frac{a_n}{a_{n+1}}+1$$

Now note from assumption:

$$(a_m+a_n)\left(\frac{1}{a_m}+\frac{1}{a_n}\right)\geq 2^2$$

$$\frac{a_m}{a_n}+\frac{a_n}{a_m}+1+1\geq 4$$

$$\frac{a_m}{a_n}+\frac{a_n}{a_m}\geq 2$$

$$\therefore LHS \geq k^2+(\underbrace{2+2+\dots 2}_{k\text{ times}})+1$$

$$= k^2+2k+1$$

$$=(k+1)^2$$

$$\therefore LHS\geq (k+1)^2$$

$\therefore$ If the statemet is true for $n=k$, then it is true for $n=k+1$.

As it is true for $n=1$, by mathematical induction it is true for $n \in \mathbb{Z}^+$

ii) By i) For $n=3$

$$(a_1+a_2+a_3)\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right) \geq 3^2$$

Let $a_1=\cos^2 x$, $a_2=\sin^2 x$, $a_3=\tan^2 x$

$$\therefore (\cos^2 x+\sin^2 x+\tan^2 x)(\sec^2 x+\operatorname{cosec}^2 x+\cot^2 x) \geq 3^2$$

$$(1+\tan^2 x)(\sec^2 x+\operatorname{cosec}^2 x+\cot^2 x) \geq 9$$

$$\sec^2 x(\sec^2 x+\operatorname{cosec}^2 x+\cot^2 x) \geq 9$$

$$\sec^2 x+\operatorname{cosec}^2 x+\cot^2 x \geq 9\cos^2 x$$

b) i)

RHS $= e^{in\theta} - e^{in\theta}$

$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$ cosx is even, sinx is odd

$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

$= 2\cos(n\theta)$

= LHS

ii)

$(2\cos\theta)^6 = (e^{i\theta} + e^{-i\theta})^6$

$64\cos^6\theta$

$e^{6i\theta} + 6e^{5i\theta}e^{-i\theta} + 15e^{4i\theta}e^{-2i\theta} + 20e^{3i\theta}e^{-3i\theta} + 15e^{2i\theta}e^{-4i\theta} + 6e^{i\theta}e^{-5i\theta} + e^{-6i\theta}$

$(e^{6i\theta} + e^{-6i\theta}) + 6(e^{4i\theta} - e^{-4i\theta}) + 20 + 15(e^{2i\theta} + e^{-2i\theta})$

$= 2\cos 6\theta + 12\cos 4\theta + 20 + 30\cos 2\theta$

$\int \cos^6\theta \, d\theta = \int \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{5}{16} + \frac{15}{32}\cos 2\theta \, d\theta$

$= \quad = \frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{5\theta}{16} + \frac{15}{64}\sin 2\theta + c$

$$c)\quad \frac{2x-18}{(x-3)(x^2+3)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+3}$$

$$2x-18 = A(x^2+3) + (Bx+C)(x-3)$$

At $x=3$

$$2\times3-18 = A(9+3) + 0$$

$$-12 = 12A$$

$$A = -1$$

At $x=0$

$$2\times0-18 = -1(0+3) + (B\times0+C)(0-3)$$

$$-18 = -3 - 3C$$

$$-15 = -3C$$

$$C = 5$$

At $x=4$

$$2\times4-18 = -1(4^2+3) + (4B+5)(4-3)$$

$$-10 = -19 + 4B + 5$$

$$4 = 4B$$

$$B = 1$$

$$\therefore \int \frac{2x-18}{(x-3)(x^2+3)}\,dx = \int \frac{-1}{x-3} + \frac{x+5}{x^2+3}\,dx$$

$$= -\int \frac{1}{x-3}\,dx + \frac{1}{2}\int \frac{2x}{x^2+3}\,dx + 5\int \frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= -\ln|x-3| + \frac{1}{2}\ln|x^2+3| + \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

15.

a) $\dfrac{z}{z-2}$ is purely imaginary

Let $z = x + iy$

$\therefore \dfrac{x+iy}{(x-2)+iy} = 0 + bi \quad , \quad b \in \mathbb{R}, \; b \neq 0$

$\therefore 0 + bi = \dfrac{x+iy}{(x-2)+iy} \times \dfrac{(x-2)-iy}{(x-2)-iy}$

$= \dfrac{x(x-2) - ixy + iy(x-2) + y^2}{(x-2)^2 + y^2}$

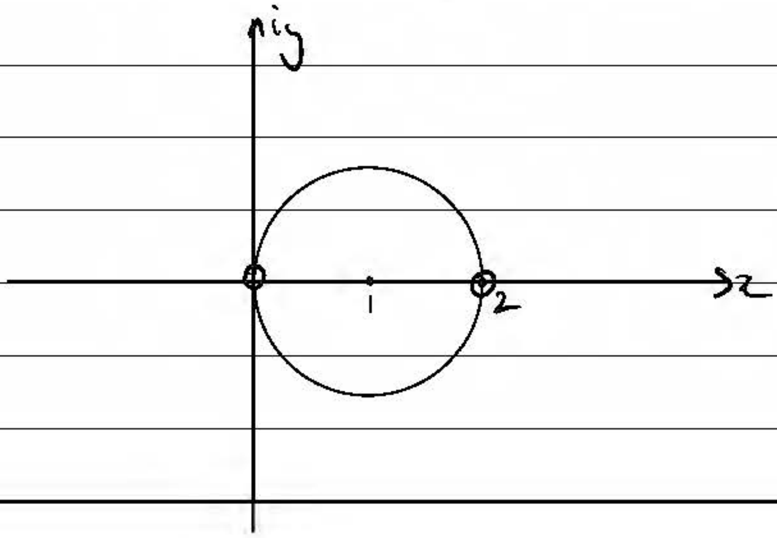
$\therefore x(x-2) + y^2 = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

Note: $z \neq 2 + 0i$: Dividing by 0

$z \neq 0 + 0i$: Not $\in \mathbb{I}$



b)

$t=0$
$v=30$
$x=0$

Forces:

$\downarrow -10\times 2$ $\downarrow \frac{-1}{15}$

$F = ma$

$\therefore -10\times 2 - \frac{1}{15}v^2 = 2\ddot{x}$

$$\ddot{x} = -10 - \frac{1}{30}v^2$$

$$= -\left(10 + \frac{1}{30}v^2\right)$$

ii) $v \times \frac{dv}{dx} = -\left(10 + \frac{1}{30}v^2\right)$

$$\frac{dv}{dx} = \frac{1}{v} \times -\left(\frac{300+v^2}{30}\right)$$

$$= -\left(\frac{300+v^2}{30v}\right)$$

$$\frac{dx}{dv} = -\frac{30v}{300+v^2}$$

$$x = \frac{-30}{2}\int \frac{2v}{300+v^2}\,dv$$

$$= -15\ln|300+v^2| + c$$

At $t=0$, $x=0$, $v=30$

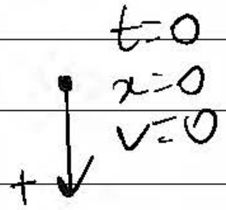
$$0 = -15\ln|300+30^2| + c_1$$
$$c_1 = 15\ln|1200|$$

$$\therefore \quad x = -15\ln|300+v^2| + 15\ln|1200|$$

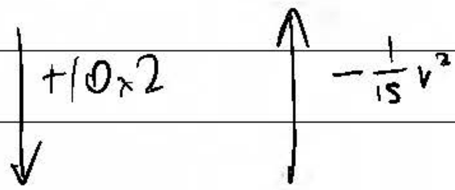
Greatest height at $v=0$

$$x = -15\ln|300+0^2| + 15\ln|1200|$$
$$= 15\ln\left(\frac{1200}{300}\right)$$
$$= 15\ln 4 \text{ m}$$

ii)



Forces:



$F = ma$

$$10\times2 - \frac{1}{15}v^2 = 2\ddot{x}$$

$$\ddot{x} = 10 - \frac{1}{30}v^2$$

$$v\times\frac{dv}{dx} = 10 - \frac{1}{30}v^2$$

$$\frac{dv}{dx} = \frac{300-v^2}{30v}$$

$$\frac{dx}{dv} = \frac{30v}{300-v^2}$$

$$x = \frac{30}{-2}\int \frac{-2v}{300-v^2}\,dv$$

$$= -15\ln|300-v^2| + C_2$$

At $t=0$, $v=0$, $x=0$

$$0 = -15\ln|300-0| + C_2$$

$$C_2 = 15\ln|300|$$

$\therefore x = -15\ln|300 - v^2| + 15\ln|300|$

Hits ground at $x = 15\ln 4$ (we know $v > 0$)

$$15\ln 4 = -15\ln(300 - v^2) + 15\ln(300)$$

$$\ln 4 = \ln\left(\frac{300}{300 - v^2}\right)$$

$$4 = \frac{300}{300 - v^2}$$

$$1200 - 4v^2 = 300$$

$$-4v^2 = -900$$

$$v^2 = 225$$

$$v = 15 \text{ m/s} \quad (v > 0 \text{ on impact})$$

$\therefore$ Speed of impact of 15 m/s.

c) i) We have $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$

So $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ —— (1)

and:

$$\frac{\left(\frac{1}{x}\right)+\left(\frac{1}{y}\right)+\left(\frac{1}{z}\right)}{3} \geq \sqrt[3]{\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)\left(\frac{1}{z}\right)}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{3}{\sqrt[3]{xyz}} \quad \text{—— (2)}$$

Using (1) in (2) (Dividing by a larger value)

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{3}{\left(\frac{x+y+z}{3}\right)}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{x+y+z}$$

ii) Prove $\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right) \geq 8$,

given $x+y+z=1$, $x,y,z>0$.

$$LHS = \left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right)$$

$$= \frac{(1-x)(1-y)(1-z)}{xyz}$$

$$= \frac{1-z-y+yz-x+xz+xy-xyz}{xyz}$$

$$= \frac{1-(x+y+z)+(xy+xz+yz)-(xyz)}{xyz}$$

$$= \frac{1-1+(xy+xz+yz)-(xyz)}{xyz}$$

$$= \frac{(xy+xz+yz)}{xyz} - \frac{(xyz)}{xyz}$$

$$= \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$$

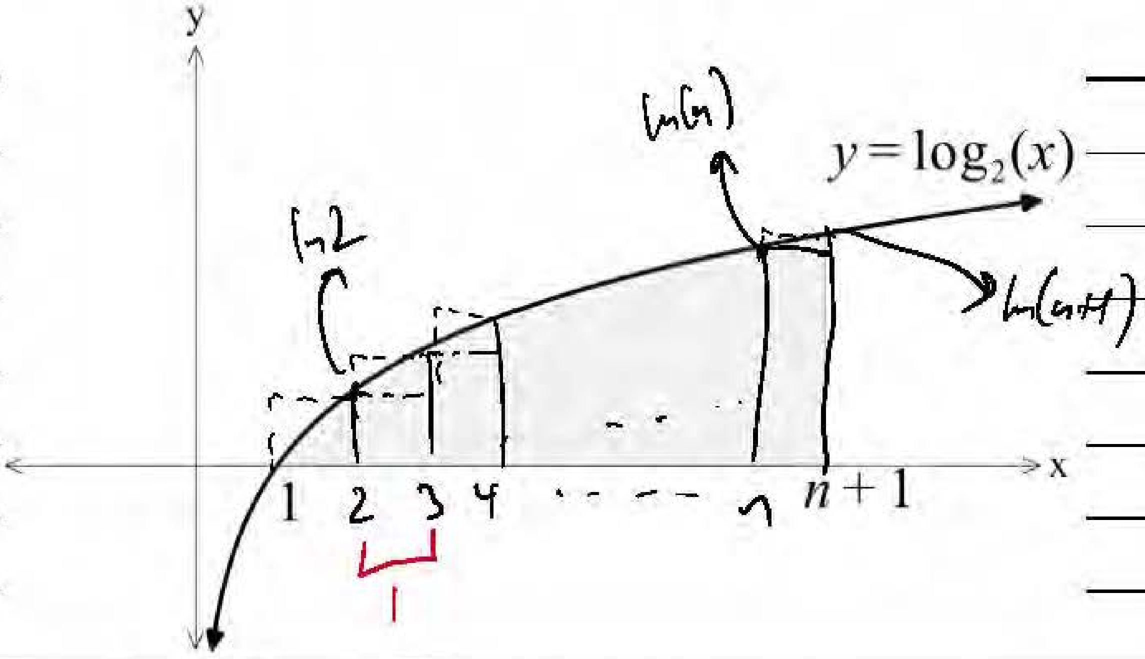
$$\geq \frac{9}{x+y+z}-1 \quad \text{by i)}$$

$$= \frac{9}{1}-1$$

$$= 8$$

$\therefore \left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right) \geq 8$

16.



Area by lower rectangles

$= 1\times\log_2 1 + 1\times\log_2 2 + 1\times\log_2 3 + \dots 1\times\log_2 n$

$= \log_2(1\times2\times3\times\dots - n)$

$= \log_2(n!)$

Area by upper rectangles

$= 1\times\log_2 2 + 1\times\log_2 3 + \dots + 1\times\log_2(n) + 1\times\log_2(n+1)$

$= \log_2(2\times3\times\dots\times n\times(n+1))$

$= \log_2(1\times2\times3\times\dots\times n(n+1))$

$= \log_2((n+1)!)$

$\therefore \log_2(n!) \leq \int_1^{n+1} \log_2 x \, dx \leq \log_2((n+1)!)$

Now $\int_1^{n+1} 1 \times \log_2 x \, dx$ $\quad u = \log_2 x \quad v' = 1$

$u' = \frac{1}{x \ln 2} \quad v = x$

$$= \left[x \log_2 x\right]_1^{n+1} - \frac{1}{\ln 2}\int_1^{n+1} 1 \, dx$$

$$= (n+1)\log_2(n+1) - 1 \times \log_2 1 - \frac{1}{\ln 2}\left[x\right]_1^{n+1}$$

$$= (n+1)\log_2(n+1) - \frac{1}{\ln 2}(n+1-1)$$

$$= (n+1)\log_2(n+1) - \frac{n}{\ln 2}$$

So we have

1. $(n+1)\log_2(n+1) - \frac{n}{\ln 2} \geq \log_2(n!)$

$$\ln 2\,(n+1)\log_2(n+1) - n \geq \ln 2 \log_2(n!)$$

$$\log_2 (n+1)^{(n+1)\ln 2} - \log_2 2^n \geq \log_2 (n!)^{\ln 2}$$

$$\log_2 2^n \leq \log_2\left(\frac{(n+1)^{(n+1)\ln 2}}{(n!)^{\ln 2}}\right)$$

$$2^n \leq \left(\frac{(n+1)^{n+1}}{n!}\right)^{\ln 2}$$

2.

$$(n+1)\log_2(n+1) - \frac{n}{\ln 2} \le \log_2((n+1)!)$$

$$(n+1)\ln 2 \log_2(n+1) - n \le \ln 2 \log_2((n+1)!)$$

$$\log_2 (n+1)^{(n+1)\ln 2} - \log_2 2^n \le \log_2((n+1)!)^{\ln 2}$$

$$\log_2 2^n \ge \log_2 \left(\frac{(n+1)^{(n+1)\ln 2}}{((n+1)!)^{\ln 2}}\right)$$

$$2^n \ge \left(\frac{(n+1)^{n+1}}{(n+1)!}\right)^{\ln 2}$$

$$2^n \ge \left(\frac{(n+1)^n}{n!}\right)^{\ln 2}$$

1. and 2. together:

$$\left(\frac{(n+1)^n}{n!}\right)^{\ln 2} \le 2^n \le \left(\frac{(n+1)^{n+1}}{n!}\right)^{\ln 2}$$

b) i)

$$I_n = \int_{-3}^{0} x^n (x+3)^{\frac{1}{2}}\, dx \qquad u = x^n \quad v' = (x+3)^{\frac{1}{2}}$$

$$u' = nx^{n-1} \quad v = \frac{2}{3}(x+3)^{\frac{3}{2}}$$

$$= \left[\frac{2}{3} x^n (x+3)^{\frac{3}{2}}\right]_{-3}^{0} - \frac{2n}{3}\int x^{n-1}(x+3)^{\frac{3}{2}}\, dx$$

$$= 0 - 0 - \frac{2n}{3}\int x^{n-1}(x+3)^{\frac{1}{2}}(x+3)^{1}\, dx$$

$$= -\frac{2n}{3}\left[\int x^n (x+3)^{\frac{1}{2}}\, dx + 3\int x^{n-1}(x+3)^{\frac{1}{2}}\, dx\right]$$

$$\frac{-3}{2n} I_n = I_n + 3I_{n-1}$$

$$I_n\left(\frac{-3}{2n} - 1\right) = 3I_{n-1}$$

$$I_n\left(\frac{-3-2n}{2n}\right) = 3I_{n-1}$$

$$I_n \times -(3+2n) = 6nI_{n-1}$$

$$I_n = \frac{-6n}{2n+3} I_{n-1}$$

ii)

Hence

$$I_3 = \frac{-6\times 3}{2\times 3+3} I_2$$

$$= -2 \times \frac{-6\times 2}{2\times 2+3} \times I_1$$

$$= \frac{24}{7} \times \frac{-6\times 1}{2\times 1+3} I_0$$

$$= -\frac{144}{35}\int_{-3}^{0} (x+3)^{\frac{1}{2}}\, dx$$

$$= -\frac{144}{35}\left[\frac{2}{3}(x+3)^{\frac{3}{2}}\right]_{-3}^{0}$$

$$= -\frac{96}{35}\left[(0+3)^{\frac{3}{2}} - 0\right]$$

$$= -\frac{96}{35}\sqrt{3^3}$$

$$= -\frac{288}{35}\sqrt{3}$$

i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$LHS = \int_0^a f(x)\,dx$$

$u = a - x$

$x = a - u$

$dx = -du$

$$= \int_a^0 f(a-u) \times -du$$

$x = 0 \rightarrow u = a$

$x = a \rightarrow u = 0$

$$= \int_0^a f(a-u)\,du$$

$$= \int_0^a f(a-x)\,dx$$ (dummy variable)

$= RHS$

ii) $$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+(\sin x - \cos x)^2}\,dx \quad \text{——} \ (1)$$

by i)

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\left(\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\cos x - \sin x)^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+(\sin x - \cos x)^2}\,dx \quad \text{——} \ (2)$$

(1)+(2):

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{1+(\sin x - \cos x)^2}\,dx$$

$u = \sin x - \cos x$

$du = \cos x + \sin x \, dx$

$x = 0 \rightarrow u = \sin 0 - \cos 0$

$= -1$

$x = \frac{\pi}{2} \rightarrow u = \sin\frac{\pi}{2} - \cos\frac{\pi}{2}$

$= 1$

$$2I = \int_{-1}^{1} \frac{1}{1+u^2}\,du \quad \text{(even function)}$$

$$2I = 2\left[\tan^{-1} u\right]_0^1$$

$$I = \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4}$$