Trinity Grammar School

Mathematics Department

2013

TRIAL EXAMINATION

HSC ASSESSMENT TASK 4

Year 12

Mathematics Extension 2

Board of Studies Student Number
(Year 12 only)

Class Teacher:

………………………………………………………………

Name:
(Year 11 only)

………………………………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination: 5/08/2013

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Only approved calculators for this course are allowed in this task
- A table of Standard Integrals is supplied
- Show all necessary working
- Write your Board of Studies Student Number (Year 12 HSC) or Name (Year 11) and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 40%

Total marks – 100

Section I

10 marks

- Attempt all Questions
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt all Questions
- Allow about 2 hours 45 minutes for this section

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Section I **10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Shade the correct response on the multiple-choice answer sheet for Questions 1 – **10**
- Each question is worth 1 mark

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1 Let $z = 3 - i$ and $w = 2 + i$. What is the value of $z \times \overline{w}$?

(A) $-5 + 5i$

(B) $5 - 5i$

(C) $-7 + 5i$

(D) $7 - 5i$

2 Which of the following points is a focus of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$?

(A) $\sqrt{5}, 0$

(B) $-\sqrt{13}, 0$

(C) $\left(\dfrac{9\sqrt{13}}{13}, 0\right)$

(D) $\left(-\dfrac{9\sqrt{5}}{5}, 0\right)$

Question 3 commences on the next page

3 Which integral has the smallest value?

(A) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$

(B) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \cos x\, dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \tan x\, dx$

4 What is the gradient of the tangent to the circle $x^2 - 2x + y^2 = 9$ at the point $0, -3$?

(A) $-\dfrac{1}{3}$

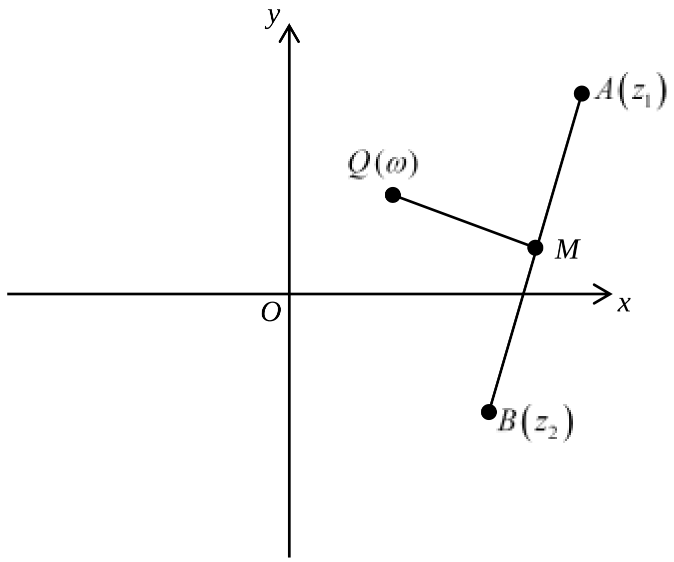
(B) $-\dfrac{11}{6}$

(C) $\dfrac{1}{3}$

(D) $\dfrac{1}{6}$

Question 5 commences on the next page

5 On the Argand diagram below, points A and B correspond to the complex numbers z_1 and z_2 respectively. M is the mid-point of the interval AB and QM is drawn perpendicular to AB. Given $QM = AM = BM$ and if Q corresponds to the complex number ω, then $\omega =$



(A) $i\left(\dfrac{z_1 - z_2}{2}\right)$

(B) $i\left(\dfrac{z_1 + z_2}{2}\right)$

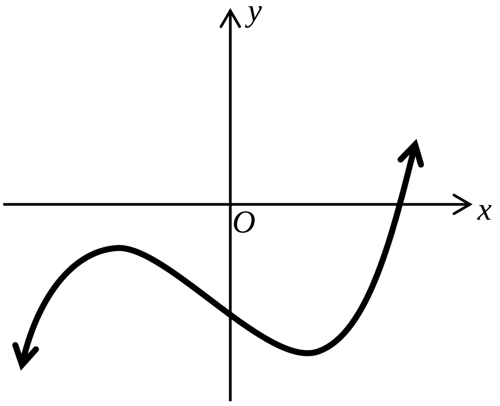
(C) $\dfrac{z_1 + z_2}{2} + i\left(\dfrac{z_1 + z_2}{2}\right)$

(D) $\dfrac{z_1 + z_2}{2} + i\left(\dfrac{z_1 - z_2}{2}\right)$

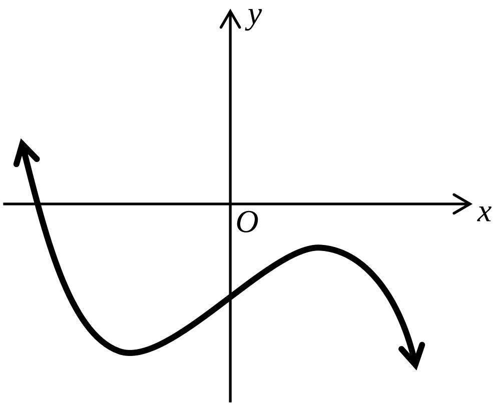
Question 6 commences on the next page

6 The polynomial $P(x)=x^3-11x-20$ has a zero at $x=-2+i$. Which of the graphs shown below could be $y=P(x)$?

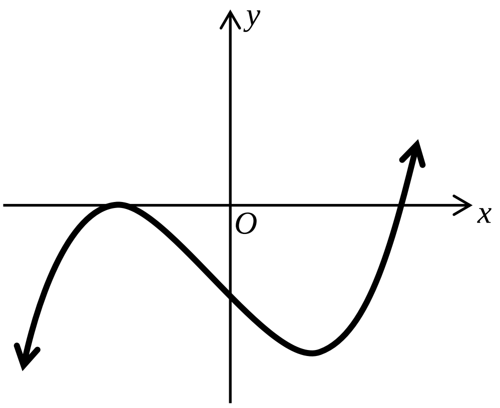
(A)



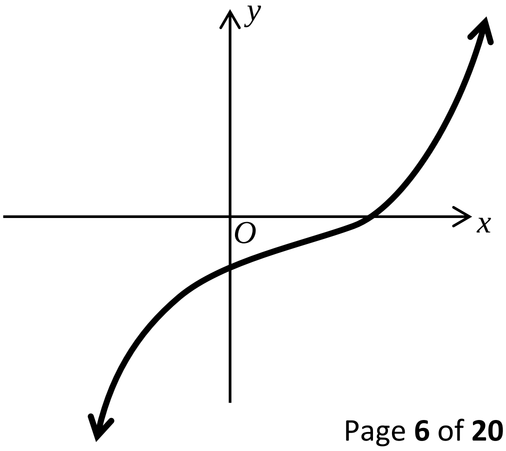
(B)



(C)



(D)



7 The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β and γ. Which of the following polynomial equations have roots $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$ and $\alpha + \beta + 2\gamma$?

(A) $x^3 - 6x^2 + 44x - 49 = 0$

(B) $x^3 - 12x^2 + 44x - 49 = 0$

(C) $x^3 + 3x^2 + 36x + 5 = 0$

(D) $x^3 + 6x^2 + 36x + 5 = 0$

8 Using the binomial theorem $(1+x)^n = {}^nC_0 + {}^nC_1x^1 + {}^nC_2x^2 + \ldots + {}^nC_nx^n = \sum_{k=0}^{n} {}^nC_kx^k$ which of the following expressions is correct?

(A) $\left(1+\dfrac{1}{n}\right)^n = \sum_{k=0}^{n} \dfrac{n(n-1)(n-2)\ldots(n-k)}{n^k} \times \dfrac{1}{k!}$

(B) $\left(1+\dfrac{1}{n}\right)^n = \sum_{k=0}^{n} \dfrac{n(n-1)(n-2)\ldots(n-k)}{n^k} \times \dfrac{1}{(k+1)!}$

(C) $\left(1+\dfrac{1}{n}\right)^n = \sum_{k=0}^{n} \dfrac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} \times \dfrac{1}{k!}$

(D) $\left(1+\dfrac{1}{n}\right)^n = \sum_{k=0}^{n} \dfrac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} \times \dfrac{1}{(k+1)!}$

9 The equation $2x^3 - 7x + 1 = 0$ has roots α, β and γ. What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

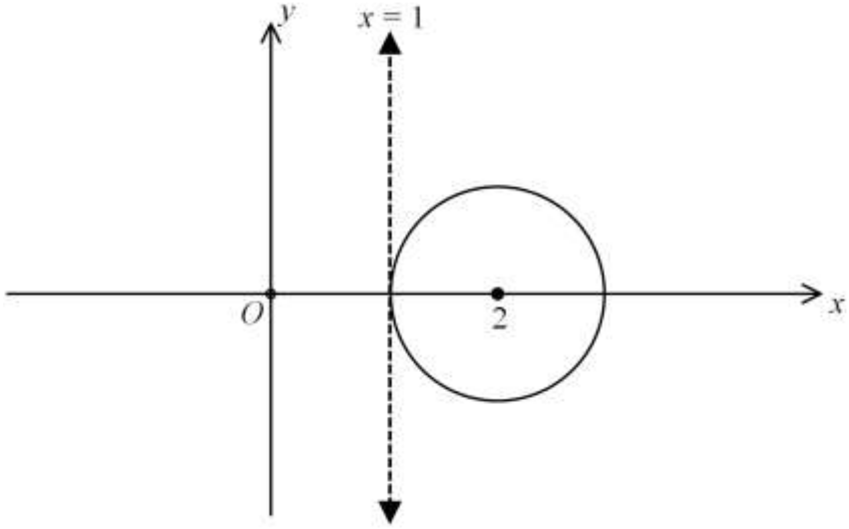
(A) 0

(B) $-\dfrac{1}{2}$

(C) $\dfrac{43}{4}$

(D) $-\dfrac{3}{2}$

10 The diagram below shows the circle $(x-2)^2+y^2=1$. The circle is rotated around the line $x=1$ to form a ***torus***.



Which integral represents the volume of the torus?

(A) $2\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$

(B) $4\pi\int_1^3 x\sqrt{1-(x-2)^2}\,dx$

(C) $2\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$

(D) $4\pi\int_1^3 (x-1)\sqrt{1-(x-2)^2}\,dx$

End of Section I
Section II commences on the next page

Section II **90 marks**

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section
- Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.
- In Questions 11 – 16, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 **(15 marks)** Use a SEPARATE writing booklet

(a) Express $\dfrac{4+i}{1+i}$ in the form $x+yi$ where x and y are real. **2**

(b) (i) Factorise $e^{3x}+8$. **1**

(ii) Evaluate $\displaystyle\int_0^{\ln 2} \frac{e^{3x}+8}{e^x+2}\,dx$. **2**

(c) (i) Find A and B such that $\dfrac{2}{x^2-4x+3} \equiv \dfrac{A}{x-1}+\dfrac{B}{x-3}$. **2**

(ii) Hence find $\displaystyle\int \frac{2}{x^2-4x+3}\,dx$. **2**

(d) Evaluate $\displaystyle\int_0^3 u\sqrt{u+1}\,du$. **3**

(e) Using the substitution $t=\tan\dfrac{x}{2}$, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x-\sin x}\,dx$. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) (i) Write $z = -\sqrt{3} + i$ in modulus-argument form. **2**

(ii) Hence express $\dfrac{1}{z^3}$ in the form $x + yi$, where x and y are real. **2**

(b) Shade the region on the Argand diagram where the two inequalities **3**

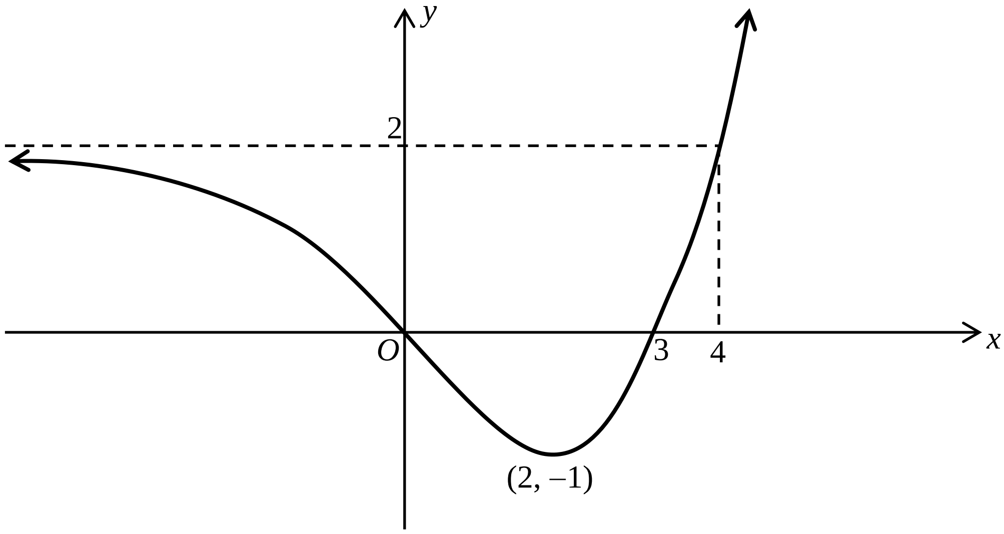
$|\, z \,| \leq 2$ and $|\, z - 2 \,| \leq |\, z \,|$ both hold.

(c) $P \;\; a\sec\theta, b\tan\theta$ is a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. The tangent at P meets at the asymptotes of the hyperbola at the points Q and R.

(i) Prove that the tangent at P has equation $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$. **2**

(ii) Show that $PQ = PR$. **3**

(d) The diagram below shows the graph of $y = f(x)$.



Draw **separate diagrams** of the graphs of the following carefully showing any horizontal or vertical asymptotes and any intercepts with the coordinate axes.

(i) $y = |f(x)|$ **1**

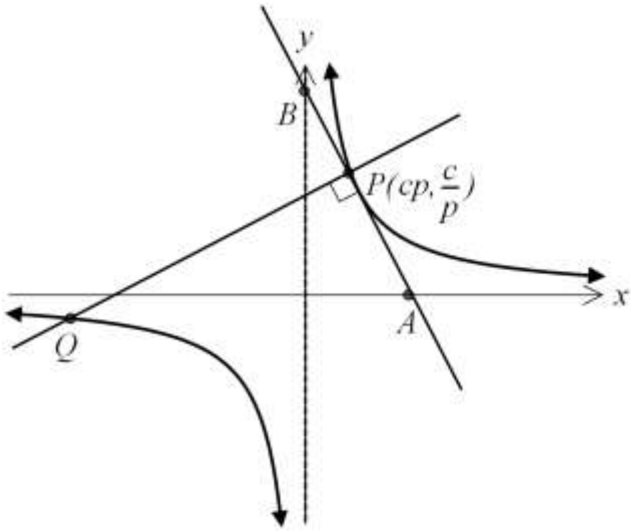
(ii) $y = \;\; f(x) \;\; ^2$ **2**

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) (i) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ (for $n = 0, 1, 2\ldots.$), show that $I_n = \frac{n-1}{n} I_{n-2}$. **4**

(ii) Hence, or otherwise, find the value of $\int_0^{\frac{\pi}{2}} \cos^4 x \, dx$. **2**

(b) The point $P\left(cp, \frac{c}{p}\right)$ is a point on the hyperbola $xy = c^2$. The tangent to the hyperbola at P cuts the x and y axes at A and B respectively and the normal to the hyperbola at P cuts the hyperbola again at Q. The tangent at P has equation $x + p^2 y = 2cp$ (DO NOT PROVE THIS).



(i) Show that the length of the interval AB is $2c\sqrt{p^2 + \frac{1}{p^2}}$ units. **2**

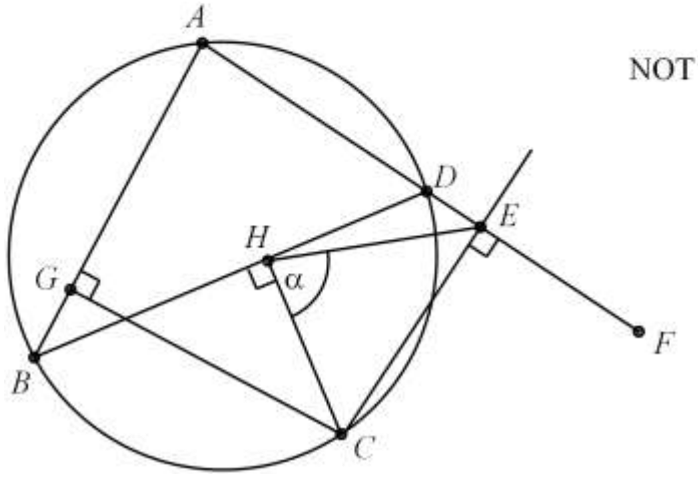
(ii) Given that the equation of the normal at P is $py - c = p^3 \; x - cp$ find the coordinates of Q. **2**

(iii) Show that the area of triangle ABQ is given by $c^2\left(p^2 + \frac{1}{p^2}\right)^2$ units^2 **2**

(iv) Given that $p^2 + \frac{1}{p^2} \geq 2$ (DO NOT PROVE THIS), find the minimum area of triangle ABQ. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet

(a)



The diagram above shows a circle passing through points *A, B, C* and *D.* *AD* is produced to *F*. *CG* is perpendicular to *AB, CH* is perpendicular to *BD* and *CE* is perpendicular to *AF* as shown. Let $\angle CHE = \alpha$.

Copy this diagram into your writing booklet.

(i) Explain why *C, H, D* and *E* are concyclic points. **1**

(ii) Explain why *C, H, G* and *B* are concyclic points. **1**

(iii) Show that $\angle ABC = \alpha$. **2**

(iv) Hence show that points *G, H* and *E* are collinear. **2**

(b) A rock is projected to just clear two poles of height *h* metres at distances of *b* and *c* metres from the point of projection. Suppose v is the velocity of the projection at an angle θ to the horizontal. Assume *g* is the acceleration due to gravity and ignore air resistance).

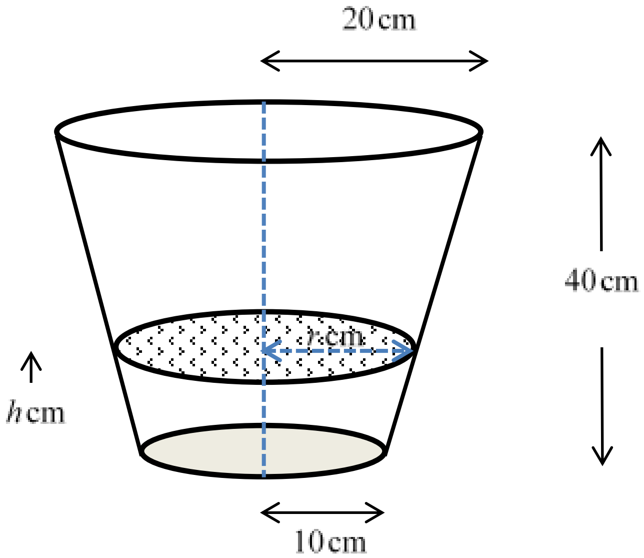
(i) Prove that the Cartesian equation of the path of the rock is given by $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2v^2}$. **2**

(ii) Hence show that $v^2 = \dfrac{(b+c)g\sec^2\theta}{2\tan\theta}$. **2**

Question 14 continues on the next page

Question 14 (continued)

(c)



The diagram above shows a flowerpot with a circular base and top. Cross-sections parallel to the base are circular. The circular base has a radius of 10 cm and the top has a radius of 20cm. The flowerpot has a height of 40 cm.

Cross-sections taken at a height h cm above the base have a radius of r cm.

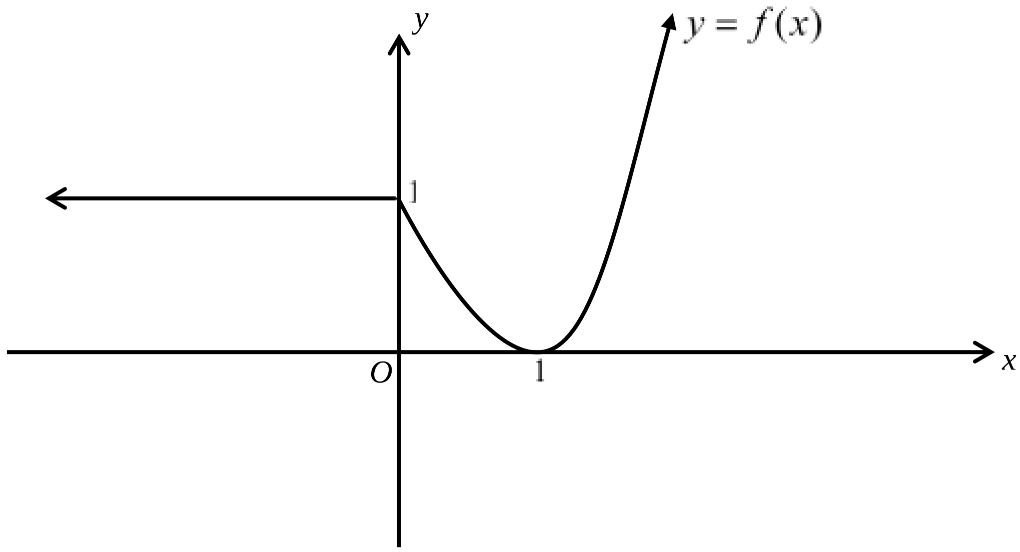
(i) Show that $r = \dfrac{h}{4} + 10$. **2**

(ii) Find the volume of the flowerpot. **3**

Question 15 commences on the next page

Question 15 (15 marks) Use a SEPARATE writing booklet

(a)



Copy this diagram into your writing booklet.

The graph of $y = f(x)$ is shown above. **2**

Draw a sketch of $y = F(x)$ where $F(x) = f(x) + f(-x)$

On your sketch clearly show any intercepts with the coordinate axes.

(b) (i) Write down the quadratic expression $2x^2 + x - 3$ as the product of two linear factors. **1**

(ii) Hence, or otherwise, find the coefficient of x in the expansion of $(2x^2 + x - 3)^8$. **2**

(c) Solve the inequality $\binom{n}{3} - \binom{2n}{2} > 32n$, where $n \geq 3$. **3**

Question 15 continues on the next page

Question 15 (continued)

(d) Let $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

(i) Show, using de Moivre's theorem that ω is a solution to $z^5 - 1 = 0$. **2**

(ii) Show that $(\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = \omega^5 - 1$. **2**

(iii) Deduce that $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$. **1**

(iv) Show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ **2**

Question 16 commences on the next page

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) (i) Write down the coefficient of x^r in the expansion of $(1+x)^{n+1}$. **1**

(ii) By considering the coefficient of x^r in the expansion of $(1+x)(1+x)^n$, **2**
show that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$ for integers r and n such that $1 \le r \le n$.

(iii) From the expansion of $(1+x)^n$, write down the values of ${}^nC_0$, ${}^nC_1$, and ${}^nC_n$. **2**

(iv) Hence, prove using mathematical induction that **3**

$$ {}^nC_r = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...3\times 2\times 1}, $$

where $1 \le r \le n$.

(b) Let $z_1 = (1+i\sqrt{3})^m$ and $z_2 = (1-i)^n$ for some positive integer m and n.

(i) Find the modulus and the arguments of z_1 and z_2 in terms of m and n respectively. **4**

(ii) Hence, find the smallest positive integers m and n such that $z_1 = z_2$. **3**

End of Section II
End of Examination

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Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \qquad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \;, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \qquad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

Note $\ln x = \log_e x, \quad x > 0$

Trinity Grammar School
Mathematics Department
2013
Year 12 Mathematics Extension 2
TRIAL EXAMINATION

HSC Assessment Task 4
Section I
Multiple Choice Answer Sheet

NSW Board of Studies Student Number
(Year 12 only)

Name: ____
(Year 11 only)

Class Teacher: ____________________________________

- Write your answers for **Section I** on this answer sheet.
- ***Shade completely only ONE answer for each question.***
- To change an answer, erase your previous mark completely and then record your new answer.

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

A table of Standard Integrals, if required, is printed on the reverse side of this answer sheet

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \qquad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \;, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \qquad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

Note $\ln x = \log_e x, \quad x > 0$

2013 TGS Year 12 Mathematics Extension 2 TRIAL HSC Examination
Suggested Marking Criteria

SECTION I (1 mark each)

Question 1

Solution	**Answer**
$z\times\overline{w}=(3-i)(2-i)$ $=6-5i+i^2$ $=6-5i-1$ $=5-5i$	B

Question 2

Solution	**Answer**
$a=3, b=2$ $b^2=a^2\left(1-e^2\right)$ $4=9\left(1-e^2\right)$ $e^2=\frac{5}{9}$ $e=\frac{\sqrt{5}}{3}$ $ae=\sqrt{5}$	**A**

Question 3

Solution	**Answer**
For $0<x<\frac{\pi}{4},\ 0<\sin x<\cos x<1$ $\therefore \sin^2 x<\cos^2 x$ $\sin x\times\sin x<\sin x\times\cos x$ $\therefore \sin^2 x<\sin x\cos x$ *because* $0<\cos x<1$ $\sin x<\frac{\sin x}{\cos x}$ $\sin x<\tan x$ $\sin^2 x<\sin x\tan x$ $\therefore \int_0^{\frac{\pi}{4}}\sin^2 x\,dx$ has the smallest value	**A**

Question 4

Solution	Answer
$x^2-2x+y^2=9$ Differentiate w.r.t x $2x-2+2y\dfrac{dy}{dx}=0$ $\dfrac{dy}{dx}=\dfrac{1-x}{y}$ At $(0,-3)$, $\dfrac{dy}{dx}=-\dfrac{1}{3}$	**A**

Question 5

Solution	Answer	Comment
$\overrightarrow{OQ}=\overrightarrow{OM}+\overrightarrow{MQ}$ $=\overrightarrow{OM}+i\times\overrightarrow{MA}$ $=\dfrac{z_1+z_2}{2}+i\left(\dfrac{z_1-z_2}{2}\right)$	**D**	$\overrightarrow{BA}=z_1-z_2$ $\therefore \overrightarrow{MA}=\dfrac{z_1-z_2}{2}$ $M=$ midpoint of AB.

Question 6

Solution	Answer	Comment
$x=-2+i$ is a zero and coefficients are real $\therefore\ x=-2-i$ is also a zero $\therefore\ P(x)$ has one real zero α Sum of roots: $(-2+i)+(-2-i)+\alpha=0$ $\alpha=4$ $P'(x)=3x^2-11$ $P'(x)=0$ gives $x=\pm\sqrt{\dfrac{11}{3}}$ $\therefore\ y=P(x)$ has two stationary points. $\therefore$ The graph of $y=P(x)$ has only one x-intercept (which is greater than 0) and two stationary points.	**A**	$P(x)$ can only have one real root because of the 2 complex roots already established and since the sign of the leading coefficient of $P(x)$ is positive, therefore B and C are not choices respectively. D implies a triple real root. So by elimination A is only possible answer.

Question 8

$(1+\dfrac{1}{n})^n=\sum\limits_{k=0}^{n}{}^nC_k(\dfrac{1}{n})^k$ $=\sum\limits_{k=0}^{n}\dfrac{n!}{(n-k)!k!}(\dfrac{1}{n})^k$ $=\sum\limits_{k=0}^{n}\dfrac{n(n-1)(n-2)...(n-k+1)}{n^k}\dfrac{1}{k!}$	**C**

Question 7 is on the next page…

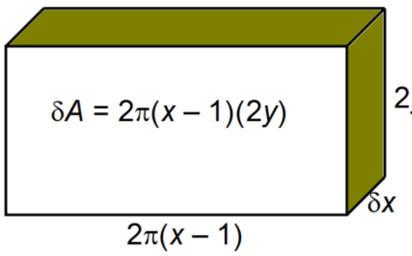
Question 7

The equation $x^3-3x^2-x+2=0$ has roots α,β,γ. $\alpha+\beta+\gamma=-\dfrac{b}{a}=3$. That is $\alpha+\beta=3-\lambda$, $\alpha+\lambda=3-\beta$, and $\beta+\lambda=3-\alpha$. This suggests then that each of the new roots are: $x_1=2\alpha+\beta+\gamma=2\alpha+3-\alpha=\alpha+3$ $x_2=\alpha+2\beta+\gamma=2\beta+3-\beta=\beta+3$ $x_3=\alpha+\beta+2\gamma=2\gamma+3-\lambda=\lambda+3$ $Let\ y=x+3\Rightarrow x=y-3$ substituting in $x^3-3x^2-x+2=0$ we get: $(y-3)^3-3(y-3)^2-(y-3)+2=0$ $\therefore y^3-12y^2+44y-49=0$ is the equation. This equation will have the same roots as $x^3-12x^2+44x-49=0$	**B**

Question 9

Solution	Answer
$2\alpha^3-7\alpha+1=0$ $2\beta^3-7\beta+1=0$ $2\gamma^3-7\gamma+1=0$ *adding* $2(\alpha^3+\beta^3+\gamma^3)-7(\alpha+\beta+\gamma)+3=0$ $2(\alpha^3+\beta^3+\gamma^3)-7(0)+3=0$ $\alpha^3+\beta^3+\gamma^3=-\dfrac{3}{2}$	**D**

Question 10

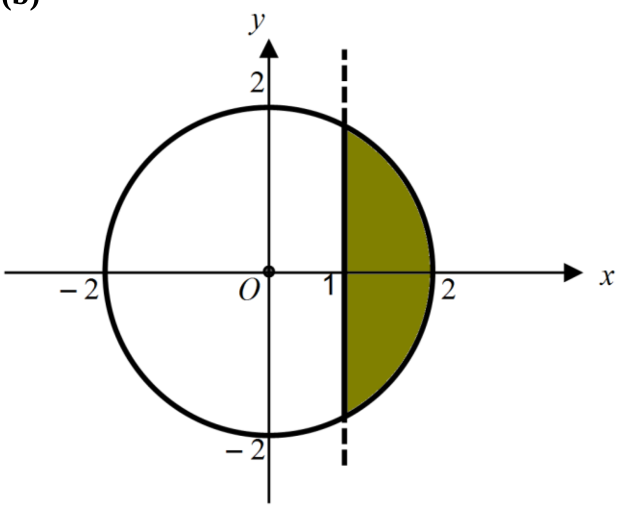
Solution	Answer	COMMENT
$V=\int_1^3 2\pi rh\,dx$ $=2\pi\int_1^3(x-1)2y\,dx$ $=4\pi\int_1^3(x-1)\sqrt{1-(x-2)^2}\,dx$	**D**	 $V=\lim_{\delta x\to 0}\sum_{x=1}^{3}(x-1)\sqrt{1-(x-2)^2}\ \delta x$

SECTION II: Question 11: (15 Marks)

Suggested Solution	Marking Criteria	Comments
(a) $\frac{4+i}{1+i}=\frac{4+i}{1+i}\times\frac{1-i}{1-i}$ $=\frac{4-4i+i-i^2}{1-i^2}$ $=\frac{5-3i}{2}$ $=\frac{5}{2}-\frac{3}{2}i$	• **[2]** Correct solution $\left(\text{full marks for }\frac{5-3i}{2}\right)$ • **[1]** Multiplies by $\frac{1-i}{1-i}$	
(b) **(i)** $e^{3x}+8=(e^x)^3+2^3$ $=(e^x+2)(e^{2x}-2e^x+4)$	• **[1]** correct (full) factorisation	
(b) **(ii)** $\int_0^{\ln 2}\frac{e^{3x}+8}{e^x+2}\,dx=\int_0^{\ln 2}\frac{\left(e^x+2\right)\left(e^{2x}-2e^x+4\right)}{e^x+2}\,dx$ $=\int_0^{\ln 2}\left(e^{2x}-2e^x+4\right)dx$ $=\left[\frac{1}{2}e^{2x}-2e^x+4x\right]_0^{\ln 2}$ $=4\ln 2-\frac{1}{2}$	• **[2]** correct solution and uses (i) • **[1]** correct integration	
(c) **(i)** $\frac{2}{x^2-4x+3}\equiv\frac{A}{x-1}+\frac{B}{x-3}$ $2\equiv A(x-3)+B(x-1)$ $2\equiv\left(A+B\right)x-(3A+B)$ Equating $A+B=0,\ 3A+B=-2$ $A=-1, B=1$	• **[2]** Finds correct values of A and B • **[1]** Makes some progress in partial fraction decomposition	
(c) **(ii)** $\int\frac{2}{x^2-4x+3}\,dx=\int\left(\frac{1}{x-3}-\frac{1}{x-1}\right)dx$ $=\ln\lvert x-3\rvert-\ln\lvert x-1\rvert+c$ $=\ln\left\lvert\frac{x-3}{x-1}\right\rvert+c$	• **[2]** Correct integration of partial fractions from (i). No penalty for no absolute value or no $+c$ or no simplifying of logarithmic expression. • **[1]** Some progress towards answer must use (i) somehow.	

Suggested Solution	Marking Criteria	Comments
(d) $Let\ x=u+1$ $\int_0^3 u\sqrt{u+1}\,du=\int_1^4 (x-1)\sqrt{x}\,dx$ $=\int_1^4 x^{\frac{3}{2}}-x^{\frac{1}{2}}\,dx$ $=\left[\frac{2}{5}x^{\frac{5}{2}}-\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$ $=\frac{116}{15}$	• **[3]** Correct solution • **[2]** Some further progress towards evaluation of integral • **[1]** choosing a correct substitution e.g. *x = u + 1*	
(e) $\text{Let}\ t=\tan\frac{x}{2}$ $\therefore \frac{dx}{dt}=\frac{2}{1+t^2}$ $\int_0^{\frac{\pi}{3}}\frac{1}{1+\cos x-\sin x}\,dx$ $=\int_0^{\frac{\sqrt{3}}{3}}\frac{1}{1+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}}\times\frac{2}{1+t^2}\,dt$ $=\int_0^{\frac{\sqrt{3}}{3}}\frac{2}{1+t^2+1-t^2-2t}\,dt$ $=\int_0^{\frac{\sqrt{3}}{3}}\frac{2}{1+t^2+1-t^2-2t}\,dt$ $=\int_0^{\frac{\sqrt{3}}{3}}\frac{1}{1-t}\,dt$ $=\left[-\ln(1-t)\right]_0^{\frac{\sqrt{3}}{3}}$ $=-\ln\left(1-\frac{\sqrt{3}}{3}\right)$ $=\ln\left(\frac{3+\sqrt{3}}{2}\right)$ *OR equivalent*	• **[3]** Correct solution (integration and substitution of limits). No penalty for not simplifying the logarithmic expression • **[2]** Correct simplification of integral expression from one in terms of *x* to one in terms of *t* • **[1]** Correct substitution of $t=\tan\frac{x}{2}$ and use of $\frac{dx}{dt}=\frac{2}{1+t^2}$ in the integral expression	

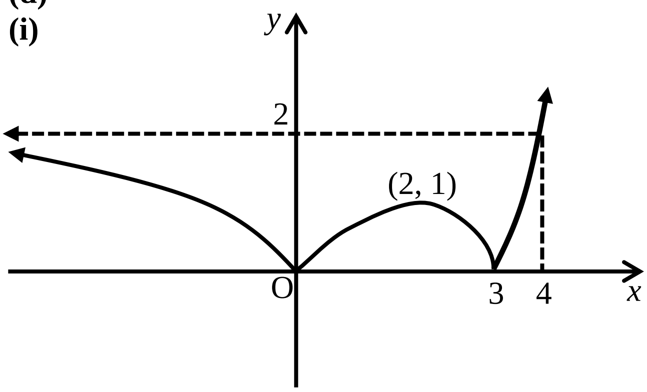
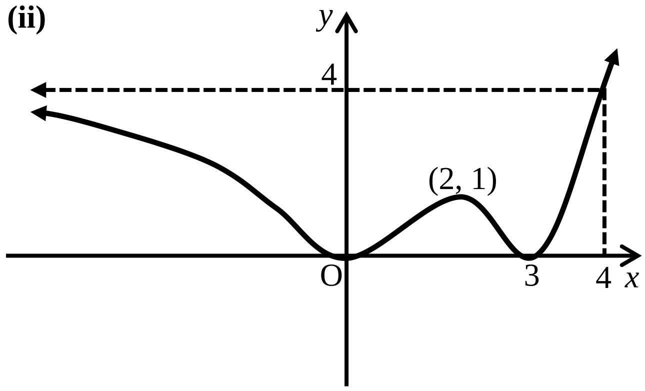
SECTION II: Question 12: (15 Marks)

<table>
<tr><th>Suggested Solution</th><th>Marking Criteria</th><th>Comments</th></tr>
<tr><td>(a)
(i)
$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$</td><td>• [2] Both modulus and argument correct and written in correct form
• [1] Modulus or argument correct</td><td></td></tr>
<tr><td>(a)
(ii)
$\frac{1}{z^3} = \left[2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^{-3}$
$= 2^{-3}\left(\cos\left(\frac{-5\pi}{2}\right) + i\sin\left(\frac{-5\pi}{2}\right)\right)$
by De-Moivre's Theorem
$= \frac{1}{8}(0 - i)$
$= 0 - \frac{1}{8}i$</td><td>• [2] Correct use of De-Moivre's Theorem and writing in Cartesian form.
• [1] Attempt to correctly use De-Moivre's Theorem</td><td>Accepted $-\frac{1}{8}i$.</td></tr>
<tr><td>(b)


Note that $|z| = 2$ is a circle, centre at O and radius 2 units. The locus of z such that $|z - 2| = |z|$ is the straight line $x = 1$ since the distance that z is from O is the same as the distance that z is from 2. This can be shown algebraically as well.</td><td>• [3] Both loci correct and correct area is shaded
• [2] Both loci correct
• [1] Some progress e.g. recognise that $|z| = 2$ is a circle or $|z-2| = |z|$ is a vertical line through $x = 1$ on the x-axis.</td><td>To qualify for two marks, must have both loci correct. If one is correct and the other is not, no follow through allowed as the main aim is in sketching correct boundaries and region.</td></tr>
</table>

Question 12 continued on the next page

(c) (i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ differentiate w.r.t x $\frac{2x}{a^2}-\frac{2y}{b^2}\times\frac{dy}{dx}=0$ $\frac{dy}{dx}=\frac{b^2x}{a^2y}$ $At\ P\left(a\sec\theta, b\tan\theta\right)$ $\frac{dy}{dx}=\frac{b\sec\theta}{a\tan\theta}$ $m_{\text{tangent}}=\frac{b\sec\theta}{a\tan\theta}$ equation of tangent $y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}\left(x-a\sec\theta\right)$ $ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$ $bx\sec\theta-ay\tan\theta=ab\left(\sec^2\theta-\tan^2\theta\right)$ $bx\sec\theta-ay\tan\theta=ab$ $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$	• **[2]** Correctly derives the equation of the tangent using calculus and coordinate geometry methods • **[1]** Correctly finds gradient of tangent or correctly finds $\frac{dy}{dx}=\frac{b^2x}{a^2y}$ at P.	Instead of using implicit differentiation, one can use $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, that is the chain rule on the given parametric coordinates.
(c) (ii) For Q, $y=\frac{b}{a}x$ $\frac{x_Q\sec\theta}{a}-\frac{\frac{b}{a}x_Q\tan\theta}{b}=1$ $\frac{x_Q}{a}\left(\sec\theta-\tan\theta\right)=1$ $x_Q=\frac{a}{\sec\theta-\tan\theta}$ For R, $y=-\frac{b}{a}x$, gives $x_R=\frac{a}{\sec\theta+\tan\theta}$ If M is the midpoint of QR $x_M=\frac{x_Q+x_R}{2}\quad=\frac{1}{2}\left(\frac{a}{\sec\theta-\tan\theta}+\frac{a}{\sec\theta+\tan\theta}\right)$ $=\frac{1}{2}\left(\frac{2a\sec\theta}{\sec^2\theta-\tan^2\theta}\right)$ $=a\sec\theta$ $=x_P$ And P, Q, R, M lie on on tangent $\therefore P$ is the midpoint of QR $ie\ PQ=PR$	• **[3]** Correctly establishes the result • **[2]** Makes substantial progress towards establishing the result (e.g. shows that P is the midpoint of QR) • **[1]** Some progress towards establishing the result (e.g. correctly finds the x coordinate of Q or R)	

(d) (i) 	• **[1]** correct graph, with asymptote, (2, 1) shown as well.	
(d) (ii) 	• **[2]** correct graph with all supporting or important features as shown • **[1]** some progress towards forming a correct graph for example showing new asymptote of $y = 4$, points of minima at O and at $x = 3$.	

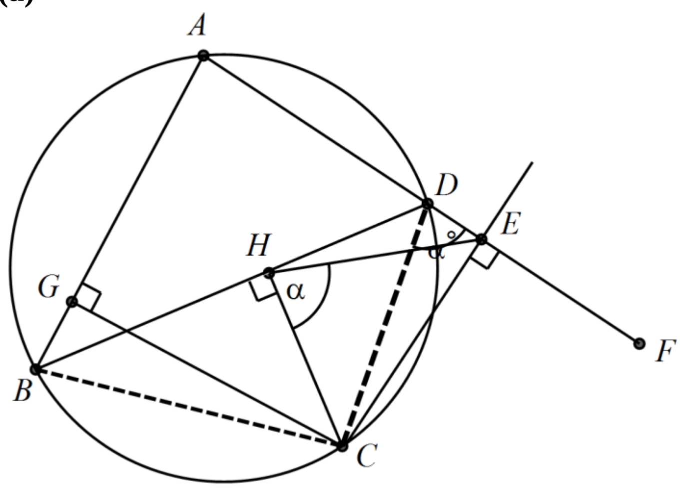
Question 13 continued on the next page

SECTION II: Question 13: (15 Marks)

Suggested Solution	Marking Criteria	Comments
(a)(i) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$ $= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\,dx$ $= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \frac{d}{dx}(\sin x)\,dx$ $= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \frac{d}{dx}\left(\cos^{n-1} x\right)dx$ $= 0 - \int_0^{\frac{\pi}{2}} \sin x\left[(n-1)\cos^{n-2} x(-\sin x)\right]dx$ $= (n-1)\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x\,dx$ $= (n-1)\int_0^{\frac{\pi}{2}} \left(1-\cos^2 x\right)\cos^{n-2} x\,dx$ $= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x - \cos^n x\,dx$ $\therefore I_n = (n-1)\left(I_{n-2} - I_n\right)$ $I_n + (n-1)I_n = (n-1)I_{n-2}$ $nI_n = (n-1)I_{n-2}$ $I_n = \frac{n-1}{n} I_{n-2}$	• **[4]** Correctly establishes result. • **[3]** Makes some subsequent progress towards establishing the result e.g. use of $\sin^2 x = 1-\cos^2 x$ after Integration by Parts. • **[2]** Correct use of Integration by Parts method once. • **[1]** Correctly rewrites integral in a form to use Integration by Parts e.g. $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$ $= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\,dx$	
(a) (ii) $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx = I_4 = \frac{3}{4} I_2$ $= \frac{3}{4}\left(\frac{1}{2} I_0\right)$ $= \frac{3}{8}\int_0^{\frac{\pi}{2}} (\cos x)^0 dx$ $= \frac{3}{8}\int_0^{\frac{\pi}{2}} 1\,dx$ $= \frac{3}{8}\left[\frac{\pi}{2} - 0\right]$ $= \frac{3\pi}{16}$	• **[2]** Correctly solution including the correct value of the integral. • **[1]** Some correct use of the recurrence relation from part (i)	
(b) (i)	• **[2]** correctly establishes the result • **[1]** some progress towards the answer, for example find the coordinates of A or of B	

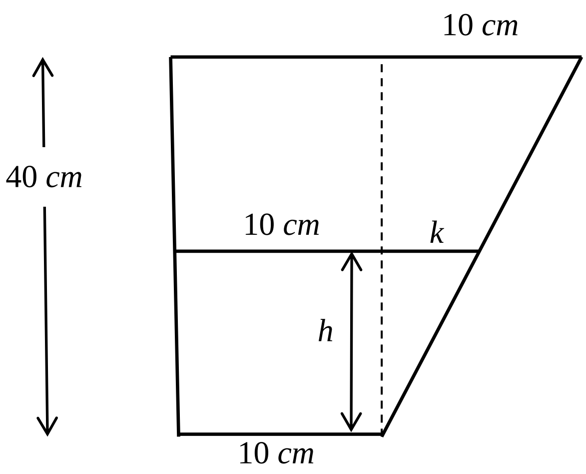
$x+p^2y=2cp$ (*given*) $\therefore A(2cp,0)$ and $B\left(0,\frac{2c}{p}\right)$ so $AB=\sqrt{(2cp)^2+\left(\frac{2c}{p}\right)^2}$ $=2c\sqrt{p^2+\frac{1}{p^2}}$		
(b) (ii) $xy=c^2 \Rightarrow y=\frac{c^2}{x}$ For Q $p\left(\frac{c^2}{x}\right)-c=p^3(x-cp)$ $p^3x^2-(cp^4-c)x-c^2p=0$ $(p^3x+c)(x-cp)=0$ Q is $\left(-\frac{c}{p^3},-cp^3\right)$	• **[2]** find the correct coordinates of Q • **[1]** some progress towards the answer , for example, forms a quadratic equation to find the point(s) of intersection or recognises on the roots of such a quadratic equation is the x ordinate at P and then attempts to use the result of $\alpha+\beta=-\frac{b}{a}$ to find the other root (x-ordinate at Q)	
(b) (iii) $PQ=\sqrt{\left(cp+\frac{c}{p^3}\right)^2+\left(\frac{c}{p}+cp^3\right)^2}$ $=c\sqrt{\left(p^2+\frac{1}{p^2}\right)^3}$ Area of triangle $ABQ=\frac{1}{2}\times PQ\times AB$ $=\frac{1}{2}\times c\sqrt{\left(p^2+\frac{1}{p^2}\right)^3}\times 2c\sqrt{p^2+\frac{1}{p^2}}$ $=c^2\left(p^2+\frac{1}{p^2}\right)^2$	• **[2]** Correctly shows the result • **[1]** Some progress towards answer (e.g. finds the length of PQ)	
(b) (iv) From *the given result* : $p^2+\frac{1}{p^2}\geq 2$ $\therefore$ Area of triangle $ABQ\geq 4c^2$ $\therefore$ Minimum area $=4c^2$	• **[3]** correct solution **incorporating** the fact • **[2]** replacing $p^2+\frac{1}{p^2}\geq 2$ in the expression $A=c^2\left(p^2+\frac{1}{p^2}\right)^2$ and forming an appropriate inequality • **[1]** correct answer without working	

SECTION II: **Question 14: (15 Marks)**

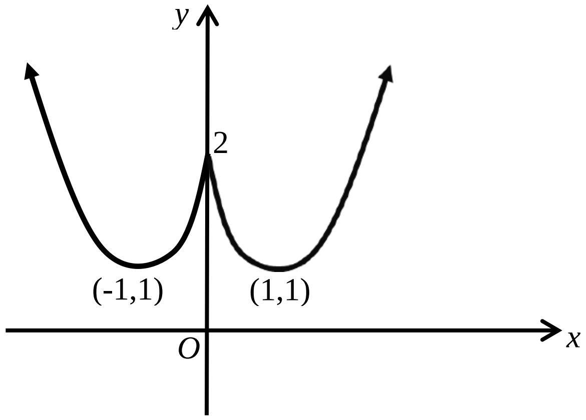
<table>
<tr><th>Suggested Solution</th><th>Marking Criteria</th><th>Comments</th></tr>
<tr><td>(a)


(i)
$\angle CHD + \angle CED = 180^\circ$
$\therefore C, H, D$ and E are concyclic
(opposite angles are supplementary).</td><td>• [1] gives a correct reason for concyclic points based on the data given or derived</td><td></td></tr>
<tr><td>(a) (ii)
$\angle CHB = \angle CGB = 90^\circ$
$\therefore C, H, G$ and B are concyclic
(converse of angles in the same segment).</td><td>• [1] gives a correct reason for concyclic points based on the data given or derived</td><td></td></tr>
<tr><td>(a) (iii)
$CHDE$ is a cyclic quadrilateral
$\therefore \angle CDE = \angle CHE = \alpha$
(angles standing on the same arc DE)
$ABCD$ is a cyclic quadrilateral
$\therefore \angle ABC = \angle CDE = \alpha$
(exterior angle of a cyclic quadrilateral is equal to the interior opposite angle)</td><td>• [2] correctly establishes the result
• [1] Some progress towards the required result (e.g. shows $\angle CDE = \angle CHE$)</td><td>There may exist alternative methods</td></tr>
<tr><td>(a) (iv)</td><td>• [2] Correctly proves that G, H and E are collinear
• [1] Some valid progress</td><td></td></tr>
</table>

$From\,(iii)\ \angle GBC = \alpha$ $\therefore \angle GHC = 180^\circ - \alpha$ (opposite angles of cyclic quadrilateral $BGHC$ are supplementary) $\angle GHE = \angle GHC + \angle EHC$ $= 180^\circ - \alpha + \alpha$ $= 180^\circ$ $\therefore G, H$ and E are collinear.	towards required result with reason(s).	
(b) (i) This question indicates a proof is required therefore the equation must be derived. However, we may assume that $x = (v\cos\theta)t \; and \; y = -\frac{1}{2}gt^2 + (v\sin\theta)t$ from the initial conditions. Substituting $t = \frac{x}{v\cos\theta}$ into $y = -\frac{1}{2}gt^2 + (v\sin\theta)t$ the result is then established. $y = -\frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2 + (v\sin\theta).\frac{x}{v\cos\theta}$ and simplifying to get the intended result.	• **[2]** correct derivation of the result from $x = (v\cos\theta)t \; and$ $y = -\frac{1}{2}gt^2 + (v\sin\theta)t$ • **[1]** attempts to derive the result by correctly replacing *t* with a correct expression in terms of *x*.	
(b) (ii) Cartesian equation of path $y = x\tan\theta - \frac{gx^2}{2v^2}.\sec^2\theta$ from (i) Passes through (b,c) and (c,h). $h = b\tan\theta - \frac{gb^2}{2v^2}\sec^2\theta \qquad (1)$ $h = c\tan\theta - \frac{gc^2}{2v^2}\sec^2\theta \qquad (2)$ $\therefore b\tan\theta - \frac{gb^2}{2v^2}\sec^2\theta = c\tan\theta - \frac{gc^2}{2v^2}\sec^2\theta$ $(b-c)\tan\theta = (b^2 - c^2)\frac{g}{2v^2}\sec^2\theta$ $\tan\theta = (b+c)\frac{g}{2v^2}\sec^2\theta$ $v^2 = \frac{(b+c)g\sec^2\theta}{2\tan\theta}$	• **[2]** correct solution • **[1]** substitutes correctly the coordinates of (b,c) and (c,h) into the Cartesian equation given in part (i) to form two equations as shown in the solution	

(c) **(i)**  Let the radius of the slice at *h* units from the bottom (or base) be $r = (k + 10)$ cm. Using similar figures and ratios of corresponding sides: $\frac{k}{10} = \frac{h}{40}$ $k = \frac{h}{4}$ $\therefore r = \frac{h}{4} + 10$	• **[2]** Correctly establishes the result • **[1]** some progress towards establishing the result	
(c) **(ii)** Consider a slice of thickness δh at height h above the base. $V_{slice} \approx \pi\left(\frac{h}{4}+10\right)^2 \delta h$ $V_{solid} = \lim_{\delta h \to 0} \sum_{h=0}^{40} \pi\left(\frac{h}{4}+10\right)^2 \delta h$ $= \pi\int_0^{40}\left(\frac{h}{4}+10\right)^2 dh$ $= \pi\left[\frac{4}{3}\left(\frac{h}{4}+10\right)^3\right]_0^{40}$ $= \frac{28000\pi}{3} cm^3$	• **[3]** correct solution (with answer) including setting up expressions for $\delta A, \delta V$ and incorporating or forming an integral completely in terms of *h* • **[2]** Establishes correct expression for the volume of the solid. • **[1]** Some progress towards an answer, for example finds δA or δV (maybe in terms of *r* and/or *h*)	Must see a correct limit before an integral statement to qualify for 3 marks.

SECTION II: Question 15: (15 Marks)

Suggested Solution	Marking Criteria	Comments
(a) 	• **[2]** correct graph, turning points, symmetry, intercept(s) • **[1]** symmetry about the *y*-axis, intersecting at *y* = 2 on the *y*-axis; or correct turning points concave up.	
(b) **(i)** $2x^2 + x - 3 = (2x + 3)(x - 1)$	• **[1]** a correct factorisation with two linear factors	NB: $2\left(x+\frac{3}{2}\right)(x-1)$ *is acceptable.*
(b) **(ii)** $(2x^2 + x - 3)^8 = (2x + 3)^8(x - 1)^8$ $= (3^8 + 8(3^7)(2x) + \ldots)((-1)^8 + 8(-1)^7(x) + \ldots)$ coefficient of $x = 3^8 \times 8 \times (-1)^7 + 3^7 \times 8 \times 2 \times (-1)^8$ $= -17\ 496$ **<u>Note</u>:** Under FT, only an integer answer is acceptable. **OR** $(2x^2 + x - 3)^8 = (3 - (x - 2x^2))^8$ $= 3^8 + 8(-(x - 2x^2)(3^7) + \ldots)$ coefficient of $x = 8 \times (-1) \times 3^7$ $= -17\ 496$ **<u>Note</u>:** Under FT, only an integer answer is acceptable.	• **[2]** correct solution • **[1]** attempt to expand the given expression using the binomial theorem or general term extracting the terms in *x*	There are various ways to set out a solution to this problem, all considered on merit
(c) Given: $\binom{n}{3} - \binom{2n}{2} > 32n$ $\frac{n!}{(n-3)!3!} - \frac{(2n)!}{(2n-2)!2!}$ $\frac{n(n-1)(n-2)}{6} - \frac{2n(2n-1)}{2}$ $= \frac{n(n^2-15n+8)}{6} \quad \left(= \frac{n^3-15n+8n)}{6}\right)$ **Now** $\frac{n^3-15n^2+8n}{6} > 32n$ Solve the cubic inequality $n^3 - 15n^2 - 184n > 0$ $n(n - 23)(n + 8) > 0$ $n > 23 \quad (\text{or } n \geq 24)$	• **[3]** correct solution • **[2]** simplifying $\binom{n}{3} - \binom{2n}{2}$ • **[1]** answer only or an attempt to write the given inequality with factorial notation.	

(d) (i)	The question did say using deMoivre's Theorem and the following were acceptable in this instance: **EITHER** $w^5 = \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^5$ $= \cos 2\pi + \mathrm{i}\sin 2\pi$ $= 1$ Hence ω is a root of $z^5 - 1 = 0$ **OR** Solving $z^5 = 1$ $z = \cos\frac{2\pi}{5}n + \mathrm{i}\sin\frac{2\pi}{5}n, \quad n = 0, 1, 2, 3, 4.$ $n = 1$ gives $\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}$ which is ω.	• **[2]** correct solution • **[1]** attempt to use de Moivre's theorem correctly	
(d) (ii)	$(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4)$ $= \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 - 1 - \omega - \omega^2 - \omega^3 - \omega^4$ $= \omega^5 - 1$	• **[2]** correct solution • **[1]** attempt to expand the LHS of the expression to be proven	
(d) (iii)	Since $\omega^5 - 1 = 0$ **and** $\omega \neq 1$, $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$.	• **[1]** correct solution or reasoning	
(d) (iv)	$1 + \omega + \omega^2 + \omega^3 + \omega^4$ $= 1 + \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5} + \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^2 +$ $\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^3 + \left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)^4$ $1 + \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5} +$ $\cos\frac{6\pi}{5} + \mathrm{i}\sin\frac{6\pi}{5} + \cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}$ $1 + \cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5} +$ $\cos\frac{4\pi}{5} - \mathrm{i}\sin\frac{4\pi}{5} + \cos\frac{2\pi}{5} - \mathrm{i}\sin\frac{2\pi}{5}$ $= 1 + 2\cos\frac{4\pi}{5} + 2\cos\frac{2\pi}{5}$ **Hence:** $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ **since** $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$.	• **[2]** correct solution • **[1]** significant progress, including substitution or changing arguments to principle argument form (s)	There are alternative methods **Note:** Correct methods involving equating real parts, use of conjugates or reciprocals are also accepted. **Note:** Use of cis notation is acceptable throughout this question.

SECTION II: Question 16: (15 Marks)

Suggested Solution	Marking Criteria	Comments
(a) (i) $(1+x)^{n+1}$ $=\left[{}^{n+1}C_0+{}^{n+1}C_1x+{}^{n+1}C_2x^2+\ldots+{}^{n+1}C_{r-1}x^{r-1}+{}^{n+1}C_rx^r+\ldots+{}^{n+1}C_nx^n\right]$ Therefore the coefficient of x^r in the expansion of $(1+x)^{n+1}$ is ${}^{n+1}C_r$	• **[1]** correct answer	
(a) (ii) $(1+x)(1+x)^n$ $=(1+x)\left[1+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\ldots+{}^nC_{r-1}x^{r-1}+{}^nC_rx^r+\ldots+{}^nC_nx^n\right]$ From this expansion, the terms in x^r are $1\times{}^nC_rx^r$ *and* $x\times{}^nC_{r-1}x^{r-1}$. The coefficient of x^r is therefore ${}^nC_r+{}^nC_{r-1}$. *This must equal* ${}^{n+1}C_r$ $\because$ *the* expression used is the same as $(1+x)^{n+1}$ *and we equating matching coefficients.*	• **[2]** correct solution using the expansion noted in the question (there are alternative methods) • **[1]** recognising that $1\times{}^nC_rx^r$ *or* $x\times{}^nC_{r-1}x^{r-1}$ are the required terms for consideration	
(a) (iii) $(1+x)^n$ $=\left[{}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\ldots+{}^nC_{r-1}x^{r-1}+{}^nC_rx^r+\ldots+{}^nC_nx^n\right]$ **Let *x* = 0** Therefore 1^n = ${}^nC_0$ If $(1+x)^n$ $=\left[{}^nC_n+{}^nC_{n-1}x+{}^nC_{n-2}x^2+{}^nC_{n-3}x^3+\ldots+{}^nC_{n-r+1}x^{r-1}+{}^nC_{n-r}x^r+\ldots+{}^nC_0x^n\right]$ **Let *x* = 0** Therefore 1^n = ${}^nC_n$ Returning to $(1+x)^n$ $=\left[{}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\ldots+{}^nC_{r-1}x^{r-1}+{}^nC_rx^r+\ldots+{}^nC_nx^n\right]$ *Differentiating both sides*: $n(1+x)^{n-1}={}^nC_1+2\,{}^nC_2x^1+3\,{}^nC_3x^2+\ldots+(r-1)\,{}^nC_{r-1}x^{r-2}+r\,{}^nC_rx^{r-1}+$ $\ldots+n\,{}^nC_nx^{n-1}$ *Substituting* $x=0$ $\Rightarrow n={}^nC_1$	• **[2]** correct values for ${}^nC_1$ and either ${}^nC_0$ or ${}^nC_n$ • **[1]** any one correct value of ${}^nC_1$ or ${}^nC_0$ ${}^nC_n$	

(a) (iv) We have already established ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$ In addition, ${}^{n}C_0 = 1,\ {}^{n}C_1 = n$ and ${}^{n}C_n = 1\ \therefore\ {}^{1}C_0 = 1$ and ${}^{1}C_1 = 1$ from above. Suppose $S(n,r): {}^{n}C_r = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...3\times2\times1}$ $= \frac{n(n-1)(n-2)...(n-(r-1))(n-r)!}{\left[r(r-1)(r-2)...3\times2\times1\right](n-r)!}$ $= \frac{n!}{(n-r)!r!}, 1\le r\le n$ Prove $S(1,1)$ and $S(2,1)$ and S(2,2) are true: If n = 1, then r = 1 as well. $S(1,1) \Rightarrow \frac{1!}{(1-1)!1!}$ $= \frac{1}{0!1!}$ $=1$ $= {}^{1}C_1$ If n = 2, then r = 1, 2 We know ${}^{2}C_1 = 2,\ {}^{2}C_2 = 1$ from (iii). Now $S(2,1) \Rightarrow \frac{2!}{(2-1)!1!} = \frac{2}{1} = 2 = {}^{2}C_1$ and $S(2,2) \Rightarrow \frac{2!}{(2-2)!2!} = \frac{2}{2} = 1 = {}^{2}C_2$ Note this all agrees with the expansion of : Consider $(1+x)^2 = 1+2x+x^2 = {}^{2}C_0 + {}^{2}C_1x + {}^{2}C_2x^2$ (Equating corresponding coefficients (in the quadratic expressions above)) Assume the statement is true for $n = k$ $(1\le r\le k\le n)$. ${}^{k}C_r = \frac{k(k-1)(k-2)...(k-(r-1))}{r(r-1)(r-2)...3\times2\times1} == \frac{k!}{(k-r)!r!}$ from above! Required to prove the statement is true for $n = k+1$ *i.e.* ${}^{k+1}C_r = \frac{(k+1)(k)(k-1)...(k-(r-2))}{r(r-1)(r-2)...3\times2\times1}$ $= \frac{(k+1)!}{(k-(r-1))!r!}$ We already know from (a)(ii) that ${}^{k+1}C_r = {}^{k}C_r + {}^{k}C_{r-1},\quad 1\le(r-1)\le r\le k$ *i.e.* on using the assumption on the RHS as well:	•	

$^{k+1}C_r = \frac{k(k-1)(k-2)...(k-(r-1))}{r(r-1)(r-2)...3\times2\times1} + \frac{k(k-1)(k-2)(k-3)...(k-(r-2))}{(r-1)(r-2)...3\times2\times1}$ $= \frac{k(k-1)(k-2)...(k-(r-2))}{(r-1)(r-2)...3\times2\times1}\left[\frac{1+k-r}{r}+1\right]$ $= \frac{k(k-1)(k-2)...(k-r)}{(r-1)(r-2)...3\times2\times1}\left[\frac{1+k}{r}\right]$ $= \frac{(k+1)!}{(k-(r-1))!r!}$ The above method is not entirely correct as the induction is on *r* not *n*. The following is a preferred proof by induction. Let: $S(n,r): {}^nC_r = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...3\times2\times1}$ $= \frac{n(n-1)(n-2)...(n-(r-1))(n-r)!}{\left[r(r-1)(r-2)...3\times2\times1\right](n-r)!}$ $= \frac{n!}{(n-r)!r!}, 1\le r\le n$ Prove $S(n,1)$ is true: LHS = ${}^nC_1 = n$ (already stated above) RHS = $\frac{n}{1} = n$. The statement is true for $r = 1$. Assume $S(n, k)$ is true and Prove $S(n, k+1)$ is true. That is: *Assume* ${}^nC_k = \frac{n(n-1)(n-2)...(n-k+1)}{k(k-1)(k-2)...3\times2\times1}$ $= \frac{n(n-1)(n-2)...(n-k+1)}{k!}$ $= \frac{n(n-1)(n-2)...(n-k+1)}{k!}\left[\frac{(n-k)!}{(n-k)!}\right]$ $= \frac{n!}{(n-k)!k!}, 1\le k\le r\le n$ *To prove* ${}^nC_{k+1} = \frac{n(n-1)(n-2)...(n-k)}{(k+1)k(k-1)(k-2)...3\times2\times1}$ $= \frac{n(n-1)(n-2)...(n-k)}{(k+1)!}$ $= \frac{n(n-1)(n-2)...(n-k+1)}{(k+1)!}\left[\frac{(n-(k+1))!}{(n-(k+1))!}\right]$ $= \frac{n!}{(n-(k+1))!(k+1)!}$		

$${}^{n+1}C_{k+1} = {}^{n}C_{k+1} + {}^{n}C_{k}$$ $$i.e. {}^{n}C_{k+1} = {}^{n+1}C_{k+1} - {}^{n}C_{k}$$ $$= \frac{(n+1)!}{[(n+1)-(k+1)]!(k+1)!} - \frac{n!}{(n-k)!k!}$$ $$= \frac{(n+1)!}{(n-k)!(k+1)!} - \frac{n!}{(n-k)!k!}$$ From (ii) $$= \frac{n!}{(n-k)!}\left[\frac{n+1}{(k+1)!} - \frac{1}{k!}\right]$$ $$= \frac{n!}{(n-k)!}\left[\frac{n+1}{(k+1)!} - \frac{k+1}{(k+1)!}\right]$$ $$= \frac{n!(n-k)}{(n-k)!(k+1)!}$$ $$= \frac{n!}{(n-(k+1))!(k+1)!}$$ By the principle of mathematical induction, the statement is true for all positive integers n.		
(b) **(i)** $$\mid z_1 \mid^m = \left[\sqrt{1^2 + \left(\sqrt{3}\right)^2}\right]^m$$ $$= 2^m$$ $$\mid z_2 \mid^n = \left[\sqrt{1^2 + (-1)^2}\right]^n$$ $$= \left[\sqrt{2}\right]^n$$ $$\arg\left(z_1\right)^m = \frac{m\pi}{3} \textit{ and } \arg\left(z_2\right)^n = -\frac{n\pi}{4}$$	• **[4]** correct answers for each moduli and argument in terms of *m* and *n* as required. • **[3] three** correct of moduli or arguments • **[2] two** correct of moduli and arguments • **[1] one** correct of moduli and arguments	Equivalent forms for arguments are acceptable (need not be principle argument form)
(b) (ii) We require $$2^m = (\sqrt{2})^n$$ *i.e.* $$2^m = 2^{\frac{n}{2}}$$ $\Rightarrow$ $n = 2m$ (*an even positive* integer)	• **[3]** correct solution • **[2]** establishing $n = 2m$ or equivalent with working • **[1]** answers only without working.	

$\frac{m\pi}{3} = -\frac{n\pi}{4} + 2k\pi$ *i.e.* $\frac{m\pi}{3} + \frac{n\pi}{4} = 2k\pi$ Also: Since $n = 2m$ $\frac{m\pi}{3} + \frac{2m\pi}{4} = 2k\pi$ $\Rightarrow m = \frac{12k}{5}$ (*m must be a positive* int) The smallest positive value of k such that m is an integer is 5. Hence m = 12 and n = 24.		

End of solutions