**2014**

**TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

# Mathematics Extension 2

## General Instructions

- Reading time – 5 minutes
- Working time –3 hours
- Write using black or blue pen
  Black pen is preferred
- Board- approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

## Total marks – 100

- Section I Pages 2-5
  **10 Marks**
- Attempt Questions 1-10
- Allow about 15 minutes for this section
- Section II Pages 6-9
  **90 Marks**
- Attempt Questions 11-16
- Allow about 2 hours 45 minutes for this section

*Care has been taken to ensure that this paper is free of errors and that it mirrors the format and style of past HSC papers. The questions have been adapted from various sources, in an attempt to provide students with exposure to a broad range of questions. However, there is no guarantee whatsoever that the HSC examination will have similar content, style or format. This paper is intended only as a trial for the HSC examination or as revision leading to the examination.*

## Section I

**10 marks**
**Attempt Questions 1 - 10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1-10

**1.** If $z = 2 - 5i$ find $z^{-1}$ expressed with a real denominator.

(A) $\frac{1}{24}(2+5i)$

(B) $\frac{1}{29}(2+5i)$

(C) $\frac{1}{29}(2-5i)$

(D) $\frac{1}{24}(-2+5i)$

**2.** The equation $x^3 + 2x - 1 = 0$ has roots $\alpha, \beta$ and $\gamma$. Find the value of $\alpha^2 + \beta^2 + \gamma^2$

(A) -4

(B) -2

(C) 6

(D) 8

**3.** Find $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

(A) $4\sin\sqrt{x} + c$

(B) $2\cos\sqrt{x} + c$

(C) $\frac{1}{2}\sin\sqrt{x} + c$

(D) $2\sin\sqrt{x} + c$

**4.** Find $\int \frac{dx}{\sqrt{4x^2+36}}$.

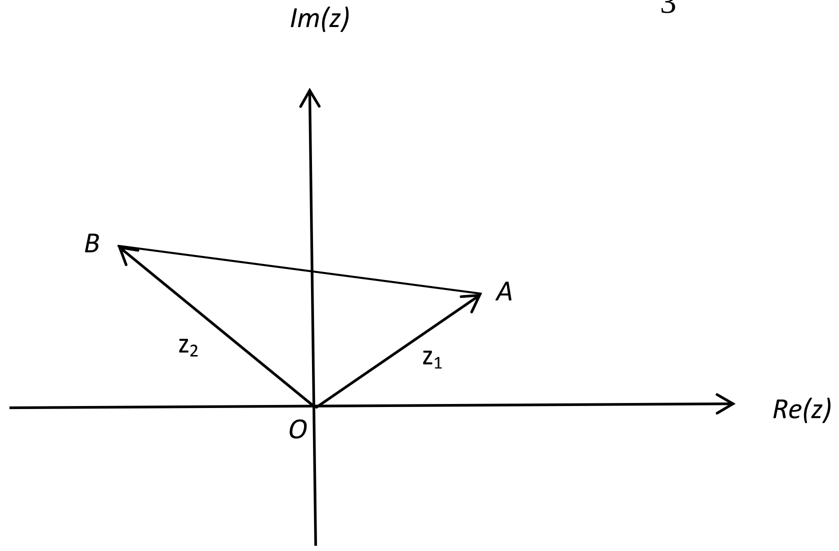
(A) $\frac{1}{2}\ln\left(x+\sqrt{x^2+9}\right)+c$

(B) $\ln\left(x+\sqrt{x^2+9}\right)+c$

(C) $x+(\ln\sqrt{x})+c$

(D) $\sqrt{x}+\sqrt{x}(\ln x+1)+c$

**5.** In the Argand diagram below, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1$ and $z_2$ respectively where $|z_2| = 2|z_1|$ and $\angle AOB = \frac{2\pi}{3}$.




Vector $\overrightarrow{AB}$ represents?

(A) $\left(-2+i\sqrt{3}\right)z_1$

(B) $\left(2-i\sqrt{3}\right)z_1$

(C) $\left(-\sqrt{3}+2i\right)z_1$

(D) $\left(\sqrt{3}-2i\right)z_1$

**6.** Find real numbers *a*, *b* and *c* such that

$$\frac{8-2x}{(1+x)(4+x^2)}=\frac{a}{1+x}+\frac{bx+c}{4+x^2}$$

(A) $a=-2,\ b=2$ and $c=1$

(B) $a=3,\ b=-2$ and $c=2$

(C) $a=2,\ b=-2$ and $c=0$

(D) $a=3,\ b=-3$ and $c=0$

**7.** An ellipse has its centre at the origin and its foci on the *x*-axis. The distance between the foci is 4 units and the distance between the directrices is 16 units. Find the equation of the ellipse.

(A) $\frac{x^2}{18}+\frac{y^2}{10}=1$

(B) $\frac{x^2}{16}+\frac{y^2}{10}=1$

(C) $\frac{x^2}{16}+\frac{y^2}{12}=1$

(D) $\frac{x^2}{20}+\frac{y^2}{12}=1$

**8.** Solve the equation: $2x^3-13x^2-26x+16=0$, given that the roots are in geometric progression.

(A) $\frac{1}{2}, 2, 9$

(B) $\frac{1}{2}, -2, 8$

(C) $\frac{1}{3}, -2, 6$

(D) $\frac{1}{2}, -3, 2$

**9.** Find $\int \frac{x^3 dx}{x^2+x+1}$

(A) $\frac{x^2}{4} - x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$

(B) $\frac{x^2}{2} + x - \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$

(C) $\frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$

(D) $\frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \sin^{-1} \frac{2x+1}{\sqrt{3}} + c$

**10.** A motorcyclist with a mass of 82 kg is riding a motorcycle which has a mass of 225 kg. If he rides around a circular bend on level ground of radius 63 m, what is the minimum frictional force that must act on the wheels of the motorcycle to keep it moving around the bend at 85 km/h ?

(A) $6.4 \times 10^2\ N$

(B) $2.1 \times 10^3\ N$

(C) $2.7 \times 10^3\ N$

(D) $3.5 \times 10^4\ N$

**Section II starts on the next page**

## Section II

**90 marks**
**Attempt Questions 11 – 16**
**Allow about 2 hours and 45 minutes for this section**

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

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**Question 11 (15 Marks)** Use a separate page/booklet **Marks**

(a) Given $C = 1 + \sqrt{3}i$

(i) Write $C$ in mod-arg form **2**

(ii) Hence, using De Moivre's theorem, find $C^3$ **2**

(b) Using the substitution $x = \sin\theta$, find $\int \frac{\sqrt{1-x^2}\,dx}{x}$ **3**

(c) Evaluate $\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}dx$. **3**

(d) Show that $\int_0^1 x^3(1-x)^6 \; dx \; = \; \frac{1}{840}$ **2**

(e) Find real $x$ and $y$ such that $(x+iy)^2 = 3+4i$. **3**

**Question 12 starts on the next page**

**Question 12 (15 Marks)** Use a separate page/booklet **Marks**

(a) For the hyperbola $x^2 - y^2 = 16$, find

(i) the eccentricity **1**

(ii) the coordinates of the foci **1**

(iii) the equations of the directrices **1**

(iv) the equations of the asymptotes. **1**

(b) If $P(x) = 4x^3 + 15x^2 + 12x - 4$ has a double zero, find all the zeros and factorise $P(x)$ fully over the real numbers. **3**

(c) Find $\int_0^{\pi/2} \sqrt{\cos x} \sin^3 x dx$ **3**

(d) Show that the equation of the circle on the diameter joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ **2**

(e) A non-real number $w$ is such that $|w| = 1$. **3**

If $z = \dfrac{1 + w}{1 - w}$ find the locus of $z$ as $w$ moves on the complex number plane.

**Question 13 starts on the next page**

**Question 13 (15 Marks)** Use a separate page/booklet **Marks**

(a) Find $\int_{\sqrt{2}}^{3} \frac{1}{\sqrt{(x^2-1)}}dx$ . **2**

(b) Given $f(x)=\sin x+|\sin x|$

(i) Sketch the curve $y=f(x)$ in the domain -2*p* £ *x* £ 2*p* **2**

(ii) On separate axes, sketch the following curves, clearly showing intercepts and other key points.

$(\alpha)$ $y = f(-x), \ -2\pi \le x \le 2\pi$ **2**

$(\beta)$ $y = f\left(x+\frac{\pi}{2}\right), \ -2\pi \le x \le 2\pi$ **2**

$(\gamma)$ $y=1-f(x), \ -2\pi \le x \le 2\pi$ **2**

$(\delta)$ $y=\log_e\left(f(x)\right), \ -2\pi \le x \le 2\pi$ **2**

(c) Show that $1+i$ is a root of the equation $z^4+3z^2-6z+10=0$. Hence solve the equation over the complex field. **3**

**Question 14 starts on the next page**

**Question 14 (15 Marks)** Use a separate page/booklet **Marks**

(a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci $S$ and $S'$. The normal at $P$ meets $SS'$ at $G$. Show that $PG^2=(1-e^2)PS\cdot PS'$. **4**

(b) Sketch the graph of $y=\frac{x^2+1}{x^2-1}$. Use this graph to solve the inequality $\frac{x^2+1}{x^2-1}<1$. **3**

(c) Find $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} \ dx$ **3**

(d) Let $I_n=\int_0^{\frac{\pi}{2}} \sin^n x \ dx$ where n is a non-negative number.

(i) Show that $I_n=(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x \ dx$ when $n\geq 2$ **2**

(ii) Deduce that $I_n=\frac{(n-1)}{n}I_{n-2}$ when $n\geq 2$ **2**

(iii) Evaluate $I_4$. **1**

**Question 15 starts on the next page**

**Question 15 (15 Marks)** Use a separate page/booklet **Marks**

(a) Show that the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\pi ab$. **2**

(b) A solid is formed by rotating the region enclosed by the parabola $y^2=4ax$, its vertex (0,0) and the line $x=a$, about the $x$-axis.

What is the volume of this solid using the method of cylindrical shells? **2**

(c) A particle is moving vertically downward under gravity in a medium which exerts a resistance to the motion which is proportional to the speed of the particle. It is released from rest at $O$, and its terminal velocity is $V$. Find the distance it has fallen below $O$ and the time taken when its velocity is one half of its terminal velocity. **4**

(d) Show that $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c)$ **3**

(e) The polynomial equation $x^3+x^2-x-4=0$ has roots $\alpha, \beta$ and $\gamma$. **2**

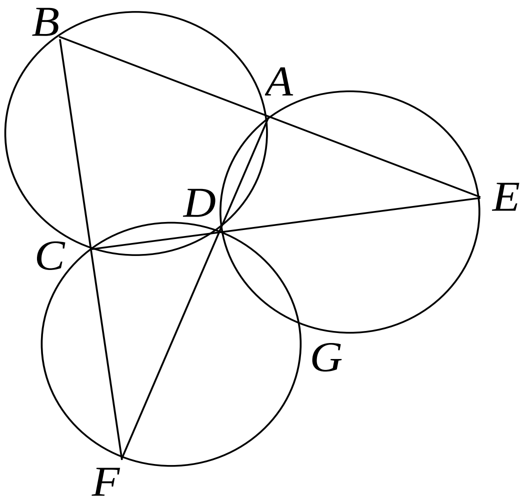
(i) Find the equation with $2\alpha, 2\beta$ and $2\gamma$ as the roots.

(ii) Find the equation with $\alpha-2, \beta-2$ and $\gamma-2$ as the roots. **2**

**Question 16 starts on the next page**

**Question 16 (15 Marks)** Use a separate page/booklet **Marks**

(a) *ABCD* is a cyclic quadrilateral. *BA* and *CD* are *both* produced to intersect at *E*.
*BC* and *AD* produced intersect at *F*. The circles *EAD*, *FCD* intersect at *G* as well as *D*.
Prove that the points *E*, *G* and *F* are collinear.




**3**

(b) (i) Prove by mathematical induction that when $z \neq 1$, where $n \geq 1$

$$1 + 2z + 3z^2 + ..........nz^{n-1} = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}$$ **4**

(ii) Hence, or otherwise, prove that

$$2 + 3(\frac{1}{2}) + 4(\frac{1}{2})^2 + ...... + n(\frac{1}{2})^{n-2} = 6 - (n+2)(\frac{1}{2})^{n-2}$$ **3**

(iii) Show that, when $z \neq 0$, the right hand side of the identity in part (i) may be written

$$\frac{z^{-1} - (n+1)z^{n-1} + nz^n}{z^{-1} - 2 + z}$$ **1**

(iv) Hence, by writing $z = \cos\theta + i\sin\theta$ and using de Moivre's theorem, show that the sum of the series

$$1 + 2\cos\theta + 3\cos 2\theta + ... + n\cos(n-1)\theta = \frac{(n+1)\cos(n-1)\theta - n\cos n\theta - \cos\theta}{2(1-\cos\theta)}$$

when $\theta$ is not a multiple of $2\pi$. **4**

**End of Exam**

## STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, n \neq -1;\ x \neq 0,\ \text{if}\ n < 0$$

$$\int \frac{1}{x} dx = \ln x,\quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax dx, = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE : $\ln x = \log_e x, \qquad x > 0$

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**YEAR 12 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**
**MATHEMATICS EXTENSION 2-MULTIPLE CHOICE ANSWER SHEET**

**Sample** $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

(A) ○ (B) ● (C) ○ (D) ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

(A) ○ (B) ~~●~~ (C) ● (D) ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows ***correct***

(A) ○ (B) ~~●~~ (C) ~~●~~ (D) ○

**1** (A)○ (B)○ (C)○ (D)○

**2** (A)○ (B)○ (C)○ (D)○

**3** (A)○ (B)○ (C)○ (D)○

**4** (A)○ (B)○ (C)○ (D)○

**5** (A)○ (B)○ (C)○ (D)○

**6** (A)○ (B)○ (C)○ (D)○

**7** (A)○ (B)○ (C)○ (D)○

**8** (A)○ (B)○ (C)○ (D)○

**9** (A)○ (B)○ (C)○ (D)○

**10** (A)○ (B)○ (C)○ (D)○

B A D A A C C B C C



1) B $\frac{1}{2-5i} \times \frac{2+5i}{2+5i} = \frac{2+5i}{4+25} = +\left(\frac{2+5i}{29}\right).$

2) $x^3+2x-1=0$

$\alpha+\beta+\gamma=0$

$\alpha\beta+\alpha\gamma+\beta\gamma=2$

$\alpha\beta\gamma=1.$

$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\alpha\gamma+2\beta\gamma$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0-2\times 2 = -4$

A

3) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

$u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}\quad dx.$

$2\,du = \frac{dx}{\sqrt{x}}$

$2\int \cos u\,du.$

$\int \sin u\,du = \sin\sqrt{x}$

$= 2\sin\sqrt{x} + C$

D

4) $\int \frac{dx}{2\sqrt{x^2+9}} = \frac{1}{2}\ln\left(x+\sqrt{x^2+9}\right)+C$

A

5) A

Change.. ② Q6.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{OB} = 2\operatorname{cis}\frac{2\pi}{3}\, z_1$

$\overrightarrow{AB} = (-1+\sqrt{3}i)z_1 - z_1$

$= z_1(-1+\sqrt{3}i-1) = z_1(-2+\sqrt{3}i).$

$2(\cos 120 + i\sin 120)$

$2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right).$

6). $\frac{8-2x}{(1+x)(4+x^2)} = \frac{a}{1+x} + \frac{bx+c}{4+x^2}$

$8-2x = 4a+ax^2+bx+c+bx^2+cx.$

$8 = 4a+c$

$-2 = b+c$

$0 = a+b$

C.

7) $ae=2$ $\quad \frac{a}{e}=8$ $\quad a=8e.$

$\frac{a^2}{8}=2$ $\quad b^2=a^2(1-e^2).$

C $\quad a^2=16$ $\quad a=4$ $\quad e=\frac{1}{2}$ $\quad b^2=16\left(\frac{3}{4}\right)$ $\quad b=2\sqrt{3}$

8) B $2x^3-13x^2-26x+16=0$ $\quad$ B

C or D.

9) C

$x^2+x+1\,\overline{)\,x^3}$ quotient $x-1$

$x^3+x^2+x$

$-x^2-x$

$-x^2-x-1$

$1.$

$\int x-1+\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}.$

$\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right.$ $\quad \frac{2x+1}{\sqrt{3}}$

100% Recycled Paper

10. Change ② Q8.

C// $\quad v = r\omega \qquad a = r\omega^2.$

$85\text{ km/h} = \frac{85000}{60\times 60} = 23.6$

$f = \frac{mv^2}{r} = \frac{(82+225)(23.6)^2}{63} = 2.7\times 10^3 N$

Section II

Q11. a) $C = 1+\sqrt{3}i$

i) $|1+\sqrt{3}i| = \sqrt{1^2+\sqrt{3}^2} = 2. \qquad \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$2\,\text{cis}\frac{\pi}{3}$

ii) $\left(2\,\text{cis}\frac{\pi}{3}\right)^3 = 8\,\text{cis}\,\pi = 8(-1) = -8.$

b) $x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\, d\theta$

$\int \frac{\sqrt{1-x^2}}{x}\,dx.$

$= \int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cdot\cos\theta\, d\theta$

$= \int \frac{\cos^2\theta\, d\theta}{\sin\theta.} = \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta$

$= \int \operatorname{cosec}\theta - \sin\theta\, d\theta.$

$= -\ln(\operatorname{cosec}\theta + \cot\theta) + \cos\theta + c.$

$\theta = \sin^{-1}x$

$= -\ln(\operatorname{cosec}(\sin^{-1}x) + \cot(\sin^{-1}x)) + \cos(\sin^{-1}x) + C$

c) $\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}\,dx = \int \frac{a}{4-x} + \frac{bx+c}{x^2+1}\,dx.$

$1+4x = ax^2 + a + 4bx + 4c - bx^2 + cx$

$1 = a + 4c$

$4 = 4b - c$

$0 = a - b$

$\cancel{(1 = -b+4c)\times 4}$

$\cancel{4 = 4b + c}$

$\cancel{4 = -4b+16c}$

$\cancel{8 = 17c}$

$\cancel{c = \frac{8}{17}}$

$a = b$

$1 = b + 4c$

$4 = 4b - c$

$4 = 4b + 16c$

$c = 0$

$b = 1 = a.$

$\int_0^2 \frac{1}{4-x} + \frac{x}{x^2+1}\,dx = \left[-\ln(4-x) + \frac{1}{2}\ln(x^2+1)\right]_0^2$

$= -\ln 2 + \frac{1}{2}\ln 5 + \ln 4 - 0$

$= \frac{1}{2}\ln 5 + \ln 2.$

Q11d). $\int_0^1 x^3(1-x)^6\,dx = \frac{1}{840}$ few ways.

$\cancel{f(x) = x^3(1-x)^6}$

$\cancel{f(-x) = (-x)^3}$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

$$\int_0^1 x^3(1-x)^6\,dx = \int_0^1 (1-x)^3 x^6\,dx.$$

$$= \int_0^1 (1-3x+3x^2-x^3)x^6\,dx$$

$$= \int_0^1 x^6-3x^7+3x^8-x^9\,dx.$$

$$\left[\frac{x^7}{7} - \frac{3x^8}{8} + \frac{3x^9}{9} - \frac{x^{10}}{10}\right]_0^1 = \frac{1}{840}.$$

e) $(x+iy)^2 = 3+4i$

$x+iy = \sqrt{3+4i}$

$x^2 + 2ixy - y^2 = 3+4i$

$x^2 - y^2 = 3$

$2xy = 4$

$y = \frac{2}{x}.$

$x^2 - \frac{4}{x^2} = 3$

$x^4 - 3x^2 - 4 = 0$

$(x^2-4)(x^2+1) = 0$

$x = \pm 2$ No soln.

$y = \pm 1$

$2+i$ or $-2-i$

Question 12.

a) $x^2 - y^2 = 16$

$a^2 = 16$ $b^2 = 16$

$a = 4$ $b = 4$

$b^2 = a^2(e^2-1)$

$16 = 16(e^2-1)$

i) $e = \sqrt{2}.$

ii) $(\pm ae, 0) = (\pm 4\sqrt{2}, 0).$

iii) $x = \pm\frac{a}{e} = \pm\frac{4}{\sqrt{2}} = \pm 2\sqrt{2}.$

iv) $y = \pm\frac{b}{a}x$ $y = \pm x.$

12b) $P(x) = 4x^3 + 15x^2 + 12x - 4$

$P'(x) = 12x^2 + 30x + 12 = 0 \quad \div 2.$

$6x^2 + 15x + 6 \overset{36}{=} 0$

$\dfrac{(6x+12)(6x+3)}{6} =$

$(x+2)(6x+3) = 0$

$x = -2 \text{ or } -\tfrac{1}{2}.$

check $P(-2) = 4 \times -8 + 15 \times 4 - 24 - 4 = -32 + 60 - 24 - 4 = 0$

$x = -2$ Double root.

$$\begin{array}{r|l} & 4x - 1 \\ \hline x^2 + 4x + 4 & 4x^3 + 15x^2 + 12x - 4 \\ & 4x^3 + 16x^2 + 16x \\ \hline & -x^2 - 4x - 4 \\ & -x^2 - 4x - 4 \\ \hline & 0. \end{array}$$

$(x+2)^2(4x-1).$

$x = -2 \text{ or } \tfrac{1}{4}.$

c). $\int_0^{\pi/2} \sqrt{\cos x} \; \sin^3 x \, dx.$

$\int_0^{\pi/2} \sqrt{\cos x}\,(1 - \cos^2 x) \sin x \, dx$

$\int_0^{\pi/2} \sqrt{\cos x} \sin x \, \cancel{dx} - \sqrt{\cos x} \cos^2 x \sin x \, dx$

let $u = \cos x$

$\dfrac{du}{dx} = -\sin x$

$-du = \sin x \, dx.$

$\int -u^{1/2} + u^{1/2} u^2 \; du = \int -u^{1/2} + u^{5/2} \, du$

$= \left[ -\tfrac{2}{3} u^{3/2} + \tfrac{2}{7} u^{7/2} \right]_1^0$

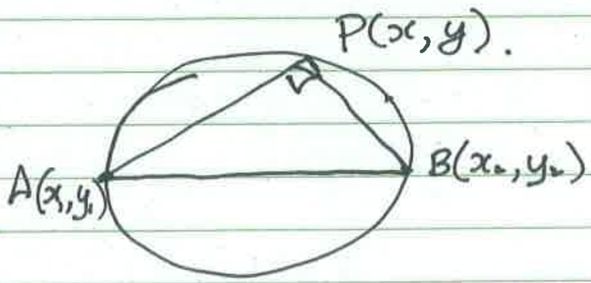
$\cos \pi/2 = 0$

$\cos 0 = 1$

$= 0 - \left( -2/3 + \tfrac{2}{7} \right)$

$= \tfrac{2}{3} - \tfrac{2}{7} = \tfrac{14}{21} - \tfrac{6}{21} = \tfrac{8}{21}$

d).




$\dfrac{y - y_1}{x - x_1} \times \dfrac{y - y_2}{x - x_2} = -1$

$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

$(y - y_1)(y - y_2) + (x - x_1)(x - x_2) = 0$

12e)  $|w| = 1$

$$z = \frac{1+w}{1-w}$$

$$z(1-w) = 1+w$$

$$z - zw = 1+w$$

$$z-1 = w(z+1)$$

$$w = \frac{z-1}{z+1}$$

$$|w| = \frac{|z-1|}{|z+1|} = 1$$

$$|z-1| = |z+1|$$

Equidistant from $1$ & $-1$ | straight line $x=0$ or $y$ axis.

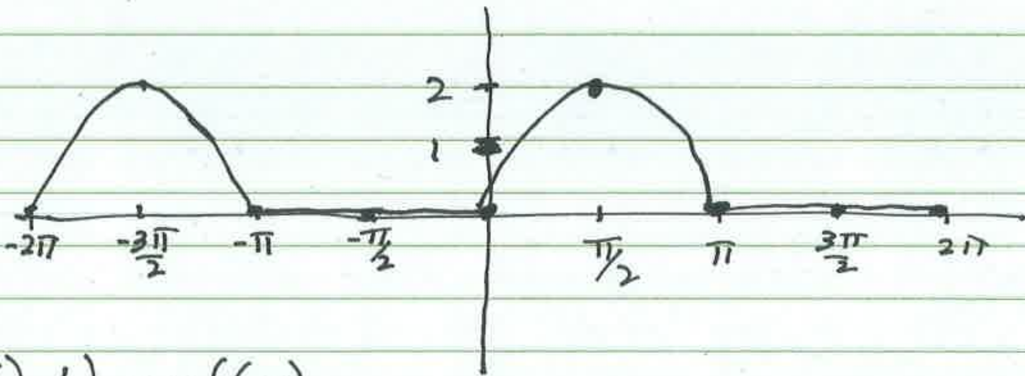
Q13 a) $\displaystyle\int_{\sqrt{2}}^{3} \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x+\sqrt{x^2+a^2}\right|$

$$= \ln\left[x+\sqrt{x^2-1}\right]_{\sqrt{2}}^{3}$$

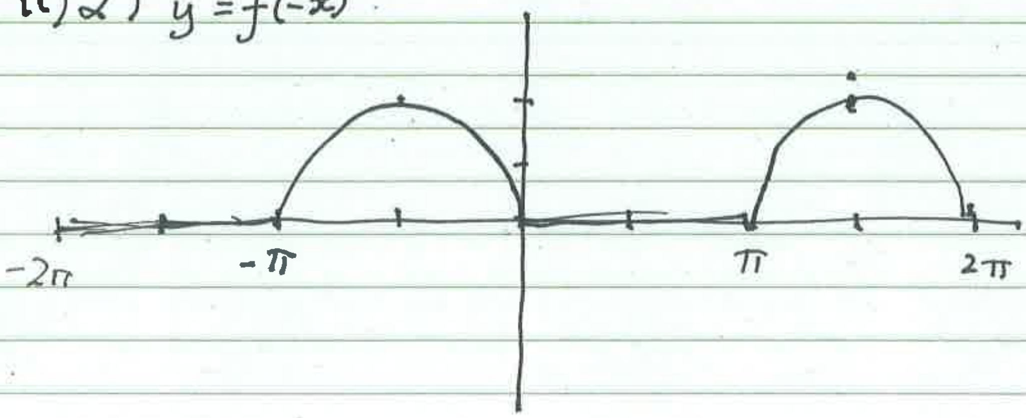
$$= \ln(3+\sqrt{8}) - \ln(\sqrt{2}+1).$$

b). $f(x) = \sin x + |\sin x|$.

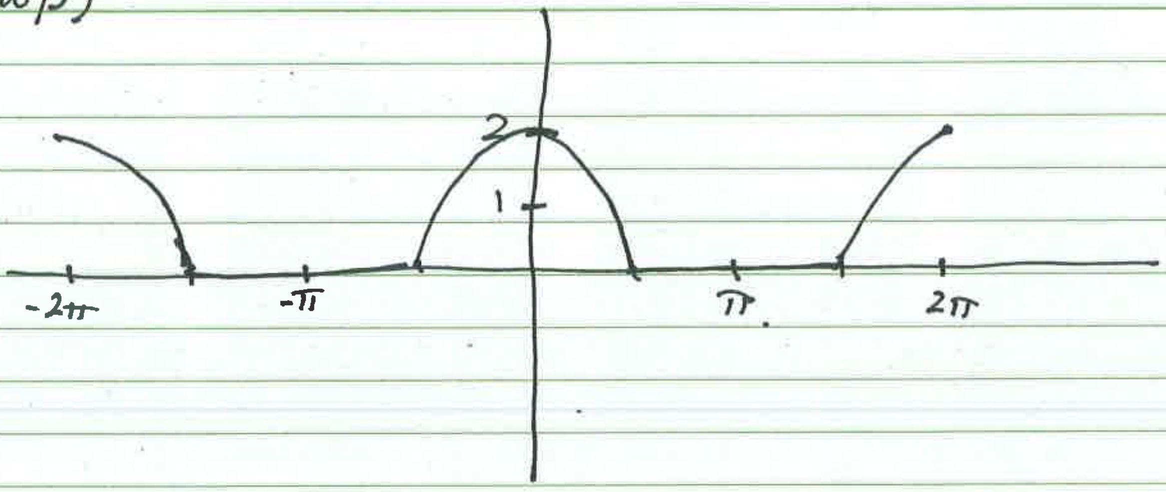
i) $-2\pi \leq x \leq 2\pi$



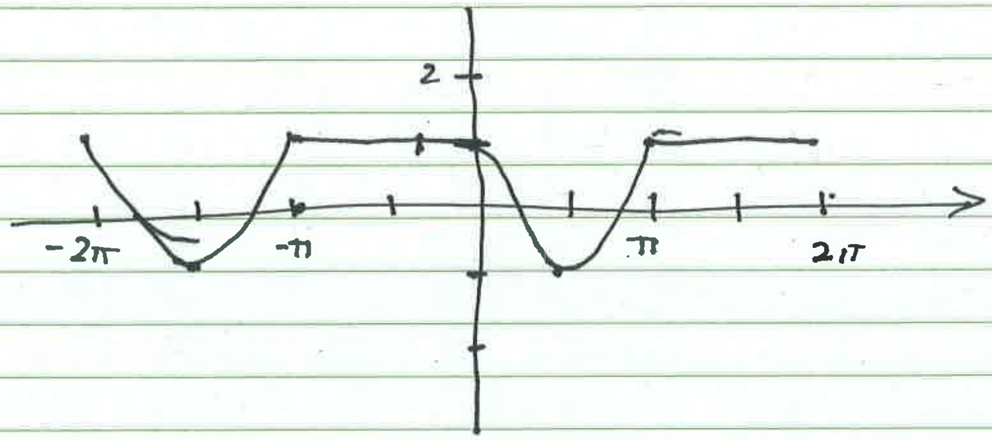

ii) $\alpha$) $y = f(-x)$



Q13 b ii β)

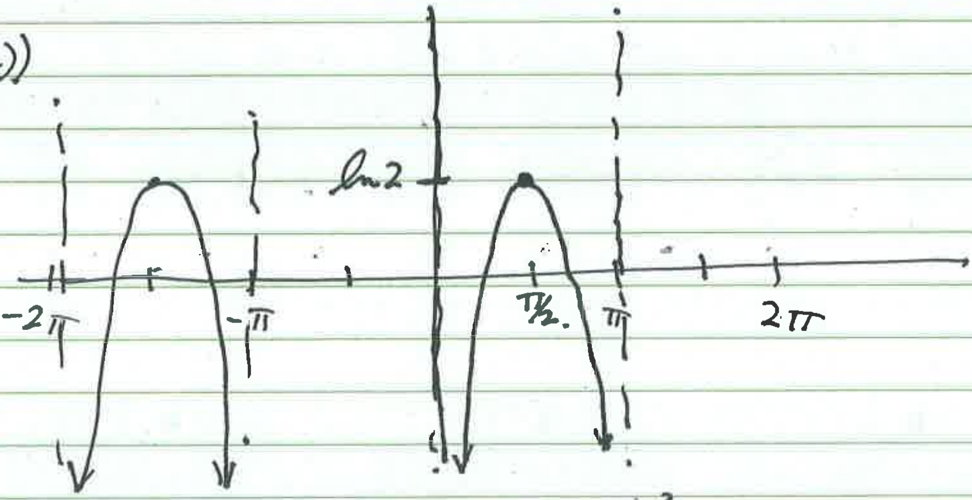



ii γ)




ii δ)

$y = \ln(f(x))$




Q13 c) $1+i$ $\quad (1+i)^4 + 3(1+i)^2 - 6(1+i) + 10$

$(1+i)^2 = 1 + 2i - 1 = 2i$

$(2i)^2 = -4$

$-4 + 3(2i) - 6 - 6i + 10$

$-10 + 6i - 6i + 10 = 0.$

if $1+i$ then conjugate also root. $1-i$

$(z-(1+i))(z-(1-i)) = z^2 - 2z + 2.$

$a^2(1-e^2)(1-e\cos\theta)(1+e\cos\theta)$
$a^2(1-e^2)(1-e^2\cos^2\theta).$

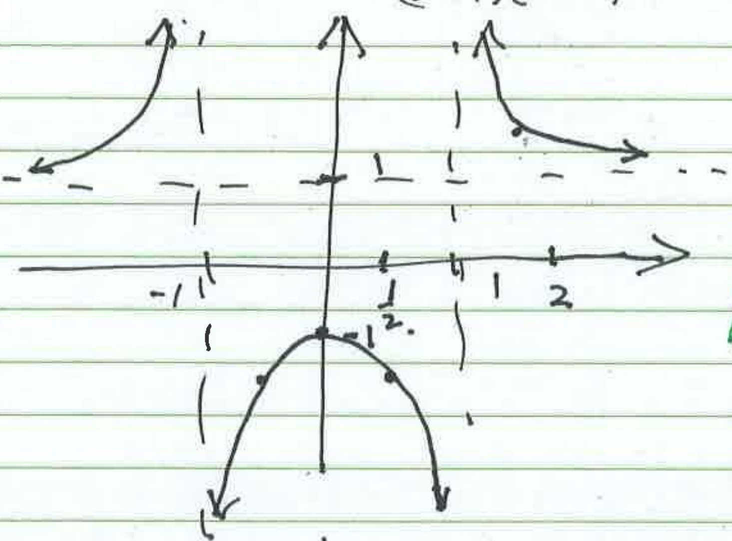
$PG^2 = (1-e^2)(b^2\cos^2\theta + a^2\sin^2\theta)$
$(1-e^2)(a^2(1-e^2)\cos^2\theta + a^2\sin^2\theta.$
$= a^2(1-e^2)(\cos^2\theta - e^2\cos^2\theta + \sin^2\theta)$
$= a^2(1-e^2)(1-e^2\cos^2\theta).$

b) $y = \dfrac{x^2+1}{x^2-1}$
$= \dfrac{x^2+1}{(x-1)(x+1)}$

Solve $\dfrac{x^2+1}{x^2-1} < 1$

$x \neq 1$ or $-1$.

as $x \to \pm\infty$
$y \to 1^+$




$x = \frac{1}{2}$ $\quad y = \dfrac{\frac{5}{4}}{-\frac{3}{4}} = -\frac{5}{3}.$

$x = -\frac{1}{2}$ $\quad y = \dfrac{5/4}{-3/4} = -5/3.$

soln $-1 < x < 1$

c) $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x}$

let $t = \tan\frac{x}{2}.$ $\quad x = 1$
$\dfrac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}.$ $\quad x = 0$
$\dfrac{2dt}{1+t^2} = dx.$

$\displaystyle\int \frac{1}{2+\frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2}$

$\displaystyle 2\int \frac{dt}{2+2t^2+1-t^2}$

$\displaystyle 2\int \frac{dt}{3+t^2} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)\right]_0^1$
$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1} 0\right)$
$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$

Q13c)

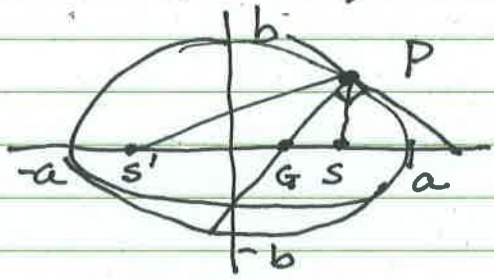
$$
\begin{array}{r}
z^2+2z+5 \\
z^2-2z+2 \overline{\big) z^4 \qquad +3z^2-6z+10} \\
\underline{z^4-2z^3+2z^2} \\
2z^3+z^2-6z \\
\underline{2z^3-4z^2+4z} \\
5z^2-10z+10 \\
\underline{5z^2-10z+10} \\
0.
\end{array}
$$

$z^2+2z+5=0$

$z = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

$z = -1 \pm 2i, \ 1 \pm i$

Q14 a). $P(a\cos\theta, b\sin\theta)$ $\qquad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $\qquad b^2=a^2(1-e^2)$




Show $PG^2=(1-e^2)\,PS\,.\,PS'$

Normal at P is

$\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2-b^2$

To find G. Now $y=0$ $\qquad \dfrac{ax}{\cos\theta} = a^2-b^2$

$x = \dfrac{\cos\theta}{a}(a^2-b^2).$ $\quad y=0$

$PG^2 = \left(a\cos\theta - \cos\theta\left(\dfrac{a^2-b^2}{a}\right)\right)^2 + (b\sin\theta - 0)^2$

$= \cos^2\theta\left(a - \dfrac{(a^2-b^2)}{a}\right)^2 + b^2\sin^2\theta.$

$= \cos^2\theta\left(\dfrac{a^2-a^2+b^2}{a}\right)^2 + b^2\sin^2\theta.$

$= \cos^2\theta\left(\dfrac{b^4}{a^2}\right) + b^2\sin^2\theta.$

$= \dfrac{b^2}{a^2}(b^2\cos^2\theta) + b^2\sin^2\theta. = \underline{\dfrac{b^2}{a^2}(b^2\cos^2\theta + a^2\sin^2\theta)}$

$= (1-e^2)(b^2\cos^2\theta) + b^2\sin^2\theta = b^2\cos^2\theta + b^2\sin^2\theta - e^2\cos^2\theta$

$= \underline{b^2 - e^2 b^2\cos^2\theta}$

$PS = \sqrt{(a\cos\theta - ae)^2 + (b\sin\theta - 0)^2} = \sqrt{a^2(\cos^2\theta - 2e\cos\theta + e^2) + b^2}$

$PS' = \sqrt{(a\cos\theta + ae)^2 + b^2\sin^2\theta}.$

$= \sqrt{a^2(\cos^2\theta + 2e\cos\theta + e^2) + b^2\sin^2\theta}$

$b^2 = a^2(1-e^2).$

$= \sqrt{a^2\cos^2\theta + 2a^2e\cos\theta + a^2e^2 + a^2\sin^2\theta - a^2e^2\sin^2\theta}$

$= \sqrt{a^2(1+2e\cos\theta + e^2\cos^2\theta)}. = \sqrt{a^2(1+e\cos\theta)^2}$

$= a(1+e\cos\theta)$

$PS = a(1-e\cos\theta).$

14 d) i) $I_n = \int_0^{\pi/2} \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$

$u = \sin^{n-1} x \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = (n-1)\sin^{n-2} x.\cos x \qquad v = -\cos x$

$I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\pi/2} + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$

ii)

$= 0 \qquad + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$

$+ (n-1)\int \sin^{n-2} x - \sin^n x\,dx.$

$n I_n = (n-1) I_{n-2}.$

$I_n = \frac{n-1}{n} I_{n-2}.$

iii) $I_4 = \frac{3}{4} I_2$

$I_2 = \int \sin^2 x\,dx = \frac{1}{2}\int 1 - \cos 2x\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$

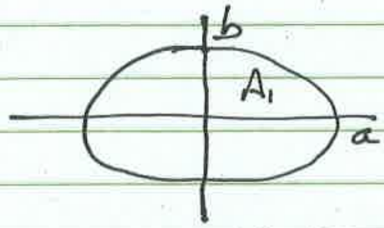
$= \frac{1}{2}\left[\pi/2 - 0 - (0-0)\right].$

$= \frac{\pi}{4}$

$I_4 = \frac{3}{4} \times \frac{\pi}{4} = \frac{3\pi}{16}.$

Question 15

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$




$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$y = b\sqrt{1 - \frac{x^2}{a^2}}$

$A_1 = b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx.$

$x = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$dx = a\cos\theta\,d\theta$

$A_1 = b\int \sqrt{1 - \frac{a^2\sin^2\theta}{a^2}}\,.\,a\cos\theta\,d\theta$

$= ab\int \cos^2\theta\,d\theta$

$= \frac{1}{2}ab\int 1 + \cos 2\theta\,d\theta$

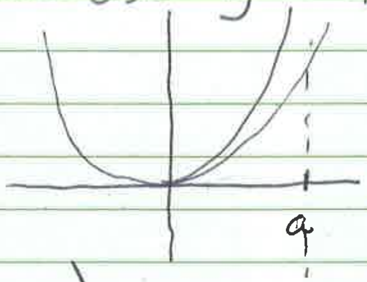
$= \frac{1}{2}ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}ab\left[\pi/2\right] = \frac{\pi ab}{4}$

four quarters

Area $= \pi ab.$

Q15 b) $y^2 = 4ax$




$V(0,0)$

radius

$$a - x = a - \frac{y^2}{4a}.$$

$$V = \lim_{\delta y \to 0} \sum 2\pi y \left(a - \frac{y^2}{4a}\right) \delta y$$

$$V = 2\pi \int_0^{2a} y\left(a - \frac{y^2}{4a}\right) dy$$

$$= 2\pi \int_0^{2a} ay - \frac{y^3}{4a} dy$$

$$= 2\pi \left[\frac{ay^2}{2} - \frac{y^4}{16a}\right]_0^{2a} = 2\pi a^3$$

15 c) $\uparrow R \downarrow mg$

$\ddot{x} \to 0$

$v \to \frac{g}{k}$

$k = \frac{g}{V}$

$$F = mg - mkv$$

$$a = g - kv.$$

$$\frac{v\,dv}{dx} = g - kv.$$

$$\int \frac{v\,dv}{g - kv} = \int dx.$$

$$-\frac{1}{k}\int \left(\frac{g - kv - g}{g - kv}\right) = \int dx$$

$$-\frac{1}{k} v + \frac{g}{k^2} \ln\left(\frac{1}{g - kv}\right) + c = x.$$

$x = 0 \quad v = 0 \qquad c = \frac{g}{k^2} \ln g.$

$$x = \frac{-v}{k} + \frac{g}{k^2} \ln\left(\frac{g}{g - kv}\right). \quad \rightarrow x = -\frac{V^2}{2g} + \frac{V^2}{g} \ln 2.$$

$$\frac{dv}{dt} = g - kv.$$

$$\int \frac{dv}{g - kv} = \int dt$$

$$-\frac{1}{k}\ln(g - kv) + c_1 = t \qquad t = 0 \quad v = 0$$

$$A = \frac{1}{k} \ln g$$

$$t = \frac{1}{k} \ln\left(\frac{g}{g - kv}\right). \qquad \text{but } k = \frac{g}{V}$$

$$t = \frac{V}{g} \ln 2.$$

15d) Show $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c)$

$(a-b)^2 \geq 0$

$a^2+b^2 \geq 2ab$

$\frac{a^2+b^2}{2} \geq ab.$ $\quad ac \leq \frac{a^2+c^2}{2} \quad bc \leq \frac{b^2+c^2}{2}.$ ---(1)

$ab+ac+bc \leq \frac{a^2+b^2}{2} + \frac{a^2+c^2}{2} + \frac{b^2+c^2}{2}$

$\leq a^2+b^2+c^2$ ---(2)

replace $a$ by $a^2$ etc

$a^2b^2+a^2c^2+b^2c^2 \leq a^4+b^4+c^4.$

using (1)

Now. $abc(a+b+c) = a^2bc+ab^2c+abc^2 \leq \cancel{bc}\left(\frac{a^2+b^2}{2}\right)+ \cancel{ac}$

$\leq a^2\left(\frac{a^2+b^2}{2}\right)+b^2\left(\frac{a^2+c^2}{2}\right)+c^2\left(\frac{b^2+c^2}{2}\right)$

$= a^2b^2+b^2c^2+a^2c^2$

Hence Answer.

15e) 12a $x^3+x^2-x-4=0$ roots $\alpha, \beta$ & $\gamma$.

i) roots $2\alpha, 2\beta, 2\gamma$

$x = 2\alpha \quad \alpha = \frac{x}{2}.$

$\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^2-\left(\frac{x}{2}\right)-4=0$

$\left(\frac{x^3}{8}+\frac{x^2}{4}-\frac{x}{2}-4=0\right)\times 8$

$x^3+2x^2-4x-32=0$

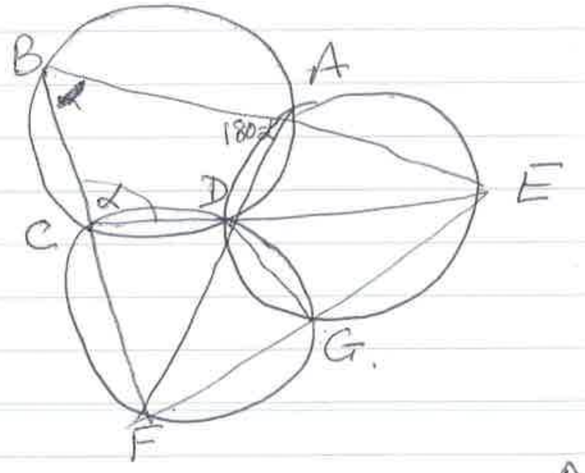
ii) $x = \alpha - 2 \quad x+2 = \alpha$

$(x+2)^3+(x+2)^2-(x+2)-4=0$

$x^3+6x^2+12x+8+(x^2+4x+4)-x-2-4=0$

$x^3+7x^2+15x+6=0.$

Question 16.a)




Join FG, DG, EG.

Prove E, G, F are collinear.

FCDG cyclic quad.

Let $\angle BCD = \alpha$.

$\angle BCD = \angle DGF$ $\left(\begin{array}{l}\angle DCF = 180-\alpha \\ \therefore \angle DGF = \alpha \quad \text{cyclic quad.}\end{array}\right)$

$\angle BAD = 180-\alpha$

$\therefore \angle DAE = \alpha$ (st line)

$\therefore \angle DGE = 180-\alpha$. cyclic

Now $\angle FGD + \angle DGE = \alpha + 180 - \alpha = 180$

st line.

b). $1 + 2z + 3z^2 + \cdots n z^{n-1} = \dfrac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}$

Prove true for $n=1$

L.HS. $= 1$

$RHS = \dfrac{1-2z+z^2}{(1-z)^2}$

$= \dfrac{(1-z)^2}{(1-z)^2} = 1$

$\therefore$ true for $n=1$.

Assume true for $n=k$

$1 + 2z + \cdots + kz^{k-1} = \dfrac{1-(k+1)z^k + kz^{k+1}}{(1-z)^2}$

Hence prove true for $n=k+1$.

$1 + 2z + \cdots + kz^{k-1} + (k+1)z^k = \dfrac{1-(k+2)z^{k+1} + (k+1)z^{k+2}}{(1-z)^2}$

$\dfrac{1-(k+1)z^k + kz^{k+1}}{(1-z)^2} + (k+1)z^k$

$+ \dfrac{(k+1)z^k \overbrace{(1-z)^2}^{1-2z+z^2}}{(1-z)^2}$

$\dfrac{1-(k+1)z^k + kz^{k+1}}{(1-z)^2} + \dfrac{(k+1)z^k - 2(k+1)z^{k+1} + (k+1)z^{k+2}}{(1-z)^2}$

$\dfrac{1-(k+2)z^{k+1} + (k+1)z^{k+2}}{(1-z)^2}$

$\therefore$ true for $n = 1, 2 \ldots$

$\therefore$ true for all positive numbers $n \geq 1$.

16 b) ii) $z = \frac{1}{2}$

$$\left[1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + \dots + n\left(\tfrac{1}{2}\right)^{n-1} = \frac{1-(n+1)\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^2}\right] \div \tfrac{1}{2}$$

$$2 + 2 + 3\left(\tfrac{1}{2}\right) + \dots + n\left(\tfrac{1}{2}\right)^{n-2} = 8\left[1-(n+1)\tfrac{1}{2}^n + n\tfrac{1}{2}^{n+1}\right]$$

$$= 8\left[1 - n.\tfrac{1}{2}^n - \tfrac{1}{2}^n + \tfrac{1}{2}n\,\tfrac{1}{2}^n\right]$$

$$= 8\left[1 - \left(\tfrac{1}{2}\right)^n\left(n + 1 - \tfrac{1}{2}n\right)\right]$$

$$= 8 - 8.\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}n + 1\right)$$

$$= 8 - 4\left(\tfrac{1}{2}\right)^n(n+2)$$

$$= 8 - \left(\tfrac{1}{2}\right)^{n-2}(n+2)$$

subtract $2 + 3\left(\tfrac{1}{2}\right) + \dots n\left(\tfrac{1}{2}\right)^{n-2} = 6 - (n+2)\left(\tfrac{1}{2}\right)^{n-2}$.

iii) $\dfrac{1-(n+1)z^n + nz^{n+1}}{1 - 2z + z^2}$ Divide top & bottom by $z$

$$\frac{z^{-1} - (n+1)z^{n-1} + nz^n}{z^{-1} - 2 + z}.$$

iv) $z = \cos\theta + i\sin\theta$

$$1 + 2\cos\theta + 3\cos 2\theta + \dots + n\cos(n-1)\theta = \frac{\phantom{x}}{2(\cos\theta - 1)}.$$

Now $z^{-1} = \dfrac{1}{\cos\theta + i\sin\theta} \times \dfrac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$$= \cos\theta - i\sin\theta.$$

Denominator

$\cos\theta - i\sin\theta - 2 + \cos\theta + i\sin\theta$

$2\cos\theta - 2.$

Numerator

$\cos\theta - i\sin\theta - (n+1)(\cos(n-1)\theta - i\sin(n-1)\theta)$

$+ n(\cos n\theta + i\sin n\theta)$

Real Part

$$1 + 2\cos\theta + 3\cos 2\theta + \dots n\cos(n-1)\theta = \frac{\cos\theta - (n+1)\cos(n-1)\theta + n\cos n\theta}{2(\cos\theta - 1)}$$

$\div$ by $-1$ to get answer.