Trinity Grammar School

Mathematics Department

2015

TRIAL EXAMINATION

HSC ASSESSMENT TASK 4

Year 12

Mathematics Extension 2

BOSTES Student Number

Class Teacher:

……………………………………………...

Do NOT write solutions to be marked on this question paper. Any working on this question paper will NOT be marked.

Date of Examination:
Thursday, 13 August 2015

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- BOSTES approved calculators for this course may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Write your BOSTES Student Number on this question paper and on any answer sheets or writing booklets used to write your responses
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating N/A and your BOSTES Student Number
- HSC Assessment Weighting: 40%

Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about **15** minutes for this section

Section II

90 marks

- Attempt Questions 11–16
- Allow about **2** hours **45** minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Shade the correct response on the multiple-choice answer sheet for Questions 1–10

Each question is worth 1 mark

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1 If $z = 1 + 2i$ and $w = i - 3$ then $z - \overline{w}$ is?

(A) $3i - 2$

(B) $4 + 3i$

(C) $i - 2$

(D) $4 + i$

2 The function defined by $f(x) = \sqrt{3}\cos x + 3\sin x$ has an amplitude of

(A) $3 - \sqrt{3}$

(B) $\sqrt{3}$

(C) $2\sqrt{3}$

(D) $3 + \sqrt{3}$

3 The equations of the directrices of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ are

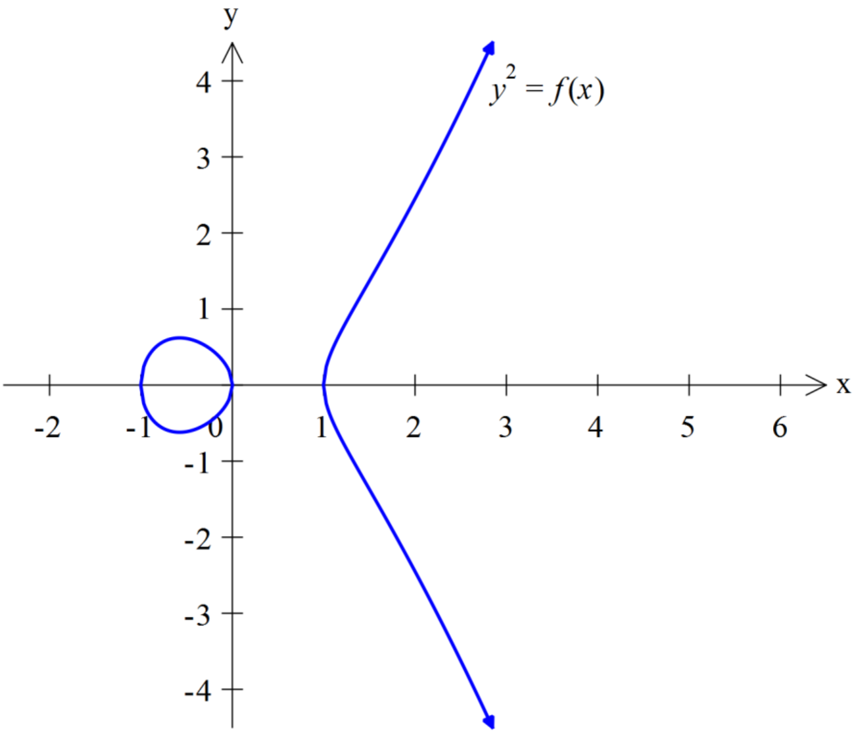
(A) $x = \pm\dfrac{16\sqrt{7}}{7}$

(B) $x = \pm\dfrac{\sqrt{7}}{16}$

(C) $x = \pm\dfrac{16}{5}$

(D) $x = \pm\dfrac{5}{16}$

4 The diagram below shows the graph of $y^2 = f(x)$.



Which expression best represents the function $f(x)$?

(A) $x^2(x-1)$

(B) $x^2(1+x)$

(C) $x(x^2-1)$

(D) $x(1+x^2)$

5 The linear factorisation of x^4+7x^2-18 over the complex number field is:

(A) $(x-i\sqrt{2})(x+i\sqrt{2})(x-3)(x+3)$

(B) $(x-i\sqrt{2})(x-i\sqrt{2})(x+3)(x+3)$

(C) $(x-3i)(x-3i)(x+\sqrt{2})(x+\sqrt{2})$

(D) $(x-3i)(x+3i)(x-\sqrt{2})(x+\sqrt{2})$

6 If α, β, γ are the roots of $x^3 + x - 1 = 0$, then the cubic equation with roots $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ is

(A) $-8x^3 - 2x - 1 = 0$

(B) $8x^3 + 2x - 1 = 0$

(C) $\frac{8}{x^3} + \frac{2}{x} - 1 = 0$

(D) $\frac{x^3}{8} + \frac{x}{2} - 1 = 0$

7 The circle $x^2 + y^2 = 4$ is rotated about the line $x = 3$ through 360°. The volume of the solid formed is given by

(A) $V = 2\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2} \ dx$

(B) $V = 4\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2} \ dx$

(C) $V = 2\pi \int_{0}^{2} (3-x)\sqrt{4-x^2} \ dx$

(D) $V = 4\pi \int_{0}^{2} (4-x)\sqrt{3-x^2} \ dx$

8 The equation $x^2 - y^2 - 2xy - 2 = 0$ defines y implicitly as a function of x.

The value of $\frac{dy}{dx}$ at the point $(-\sqrt{2}, 0)$ is?

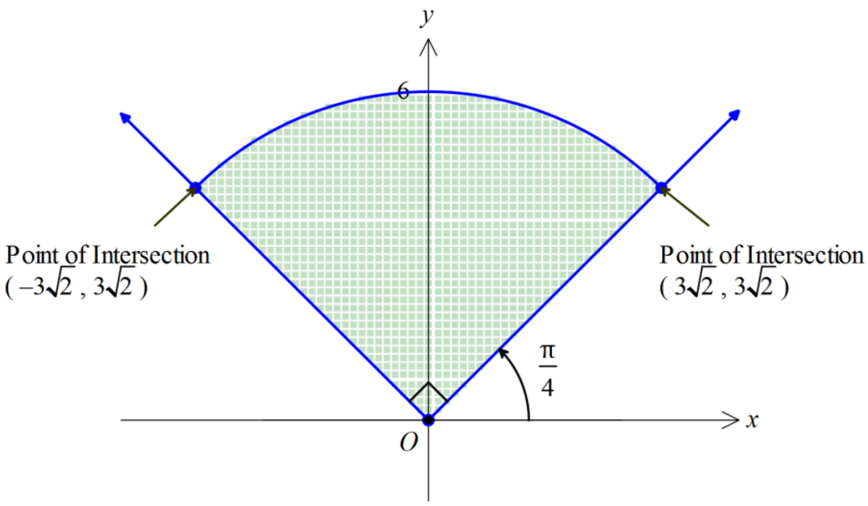
(A) $-\sqrt{2}$

(B) -1

(C) 0

(D) 1

9



Which pair of inequalities represents the shaded region of the Argand plane shown above?

(A) $|z| \le 6$ and $\frac{\pi}{4} \le \arg(z) \le \frac{3\pi}{4}$

(B) $|z| \ge 6$ and $\frac{\pi}{4} \le \arg(z) \le \frac{3\pi}{4}$

(C) $|z| \ge 36$ and $-\frac{\pi}{4} \le \arg(z) \le \frac{\pi}{4}$

(D) $|z| \ge 36$ and $\frac{\pi}{4} \le \arg(z) \le \frac{3\pi}{4}$

10 What is the natural domain of the function $f(x) = \frac{1}{2}\left(x\sqrt{x^2-1} - \log_e\left(x + \sqrt{x^2-1}\right)\right)$?

(A) $x \le -1$

(B) $-1 \le x \le 1$

(C) $x \le -1$ or $x \ge 1$

(D) $x \ge 1$

End of Section I
Section II commences on the next page

Section II

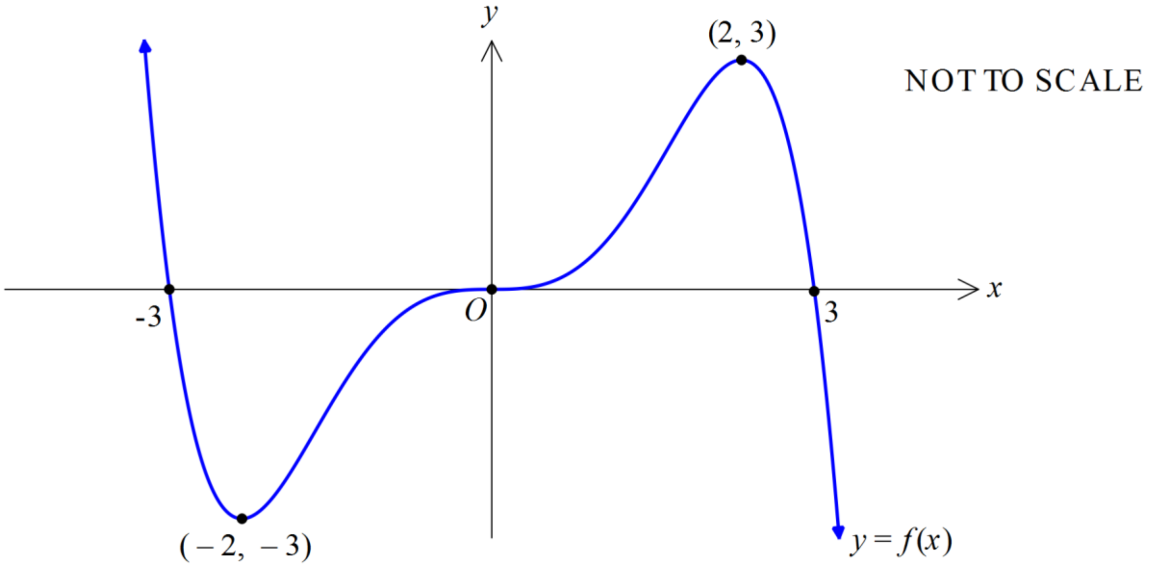
90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Consider the curve $y = f(x)$, with its stationary points and intercepts shown in the diagram below.



(i) ***On separate one-third page number planes***, sketch the graphs of:

(α) $y = \dfrac{1}{f(x)}$ **2**

(β) $y = \sqrt{f(x)}$ **2**

(γ) $|y| = \sqrt{f(x)}$ **2**

(δ) $y = e^{f(x)}$ **2**

(ii) State the number of solutions to the equation $\sqrt{f(x)} = 1$. **1**

Question 11 continues on the following page…

Question 11 continued

(b) Sketch the locus of the complex number z such that:

(i) $|z-2i| \le |z-2|$ **2**

(ii) $0 \le \arg(z-1-i) \le \dfrac{\pi}{6}$ **2**

(iii) $\mathrm{Im}\left(z-\dfrac{1}{z}\right)=0,\ z \ne 0.$ **2**

End of Question 11

Please turn over

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) (i) Show that $2+i$ is a root of $x^3-11x+20=0$. **2**

(ii) Find all the roots of $x^3-11x+20=0$. **2**

(b) Given that α, β and γ are the roots of $x^3+2x^2-x-3=0,$ find the value of $\alpha^2+\beta^2+\gamma^2$. **2**

(c) Given the polynomial $P(x)=2x^4+9x^3+6x^2-20x-24=0$ has a root of multiplicity 3, find all the roots of the equation $P(x)=0$. **3**

(d) (i) Write down all the fifth roots of -1 in the form $\cos\theta+i\sin\theta\ (-\pi<\theta\le\pi)$, **and** plot the five roots on an Argand diagram. **3**

(ii) Hence, express x^5+1 as the product of real factors in the form $(x+1)\left(x^2+(a_1\cos b_1\pi)x+c_1\right)\left(x^2+(a_2\cos b_2\pi)x+c_2\right)$ where a_1, a_2, c_1, c_2 are integers and b_1, b_2 are rational numbers. **3**

End of Question 12

Please turn over

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\int_{1}^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$ using the substitution $x = \tan\theta$. **4**

(b) Find:

(i) $\int_{-1}^{1} \frac{3}{5-2x+x^2}\,dx$ **3**

(ii) $\int xe^{-2x}\ dx$ **2**

(c) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are points on the rectangular hyperbola $xy = c^2$. Tangents to the rectangular hyperbola at P and Q intersect at the point R.

(i) Prove that the coordinates of R in terms of c, p and q are $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **3**

(ii) If P and Q are variable points on the rectangular hyperbola which move such that $p^2 + q^2 = 2$, show that the equation of the locus of R is $y^2 = 2c^2 - xy$, stating any restriction(s) that may apply. **3**

End of Question 13

Please turn over

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) A fly is standing on a small turntable at a distance of 8 cm from the centre. If the turntable is rotating at 45 revolutions per minute, find

(i) the speed of the fly in metres per second (correct to two decimal places); **2**

(ii) the acceleration of the fly (correct to two decimal places). **1**

(b) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the ellipse meets the x-axis at the points $A(a,0)$ and $A'(-a,0)$.

(i) Prove that the equation of the tangent at P is given by $bx\cos\theta + ay\sin\theta = ab$. **2**

(ii) The tangent at P meets the tangents from A and A' at points Q and Q' respectively. Find the coordinates of Q and Q'. **2**

(iii) The points A, A', Q' and Q form a trapezium. Prove that the product of the lengths of the parallel sides is independent of the position of P. **2**

(c) (i) Write down the expansion of $(\cos\theta + i\sin\theta)^3$ in the form a + ib, where a and b are in terms of sin θ and cos θ. **2**

(ii) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(iii) **Hence** solve the equation $\cos 3\theta + \cos\theta = 0$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. **2**

End of Question 14

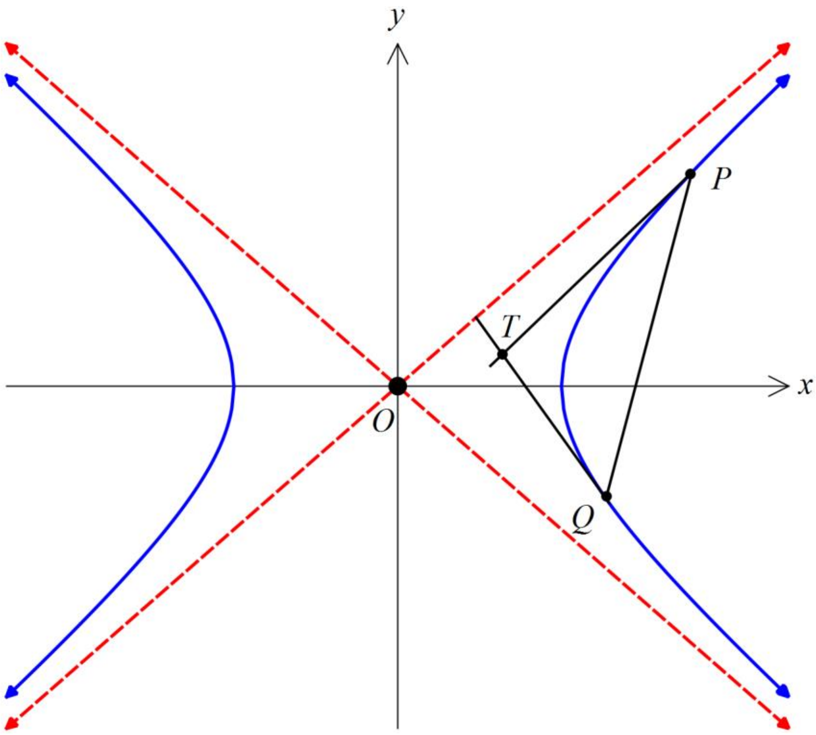
Please turn over

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) Suppose ω is one of the complex roots of $z^3 - 1 = 0$.

(i) Show that $1 + \omega + \omega^2 = 0$. **1**

(ii) Show that $\left(\dfrac{\omega}{1+\omega}\right)^k + \left(\dfrac{\omega^2}{1+\omega^2}\right)^k = (-1)^k 2\cos\dfrac{2k\pi}{3}$, where k is a positive integer. **3**



(b) In the diagram above, the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same branch of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. The tangents at P and Q meet at $T(x_0, y_0)$.

(i) Write down, in Cartesian form, the equation of the tangent at P. **1**

(ii) Hence show that the chord of contact, PQ, has equation $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$. **2**

(iii) The chord PQ passes through the right-hand focus $S(ae, 0)$, where e is the eccentricity of the hyperbola. **2**

Prove that T lies on the directrix of the hyperbola.

Question 15 continues on the next page…

Question 15 continued

(c) (i) Find $\int_0^{\frac{\pi}{2}} \cos x \quad dx$. **1**

(ii) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \quad dx$. **3**

Show, using integration by parts, $I_n = \frac{n-1}{n} I_{n-2}$ (n is an integer ≥ 2).

(iii) Hence, evaluate $I_5 = \int_0^{\frac{\pi}{2}} \cos^5 x \quad dx$. **2**

End of Question 15

Please turn over

Question 16 **(15 marks)** Use a SEPARATE writing booklet

(a) (i) Find real numbers p, q and r such that **3**

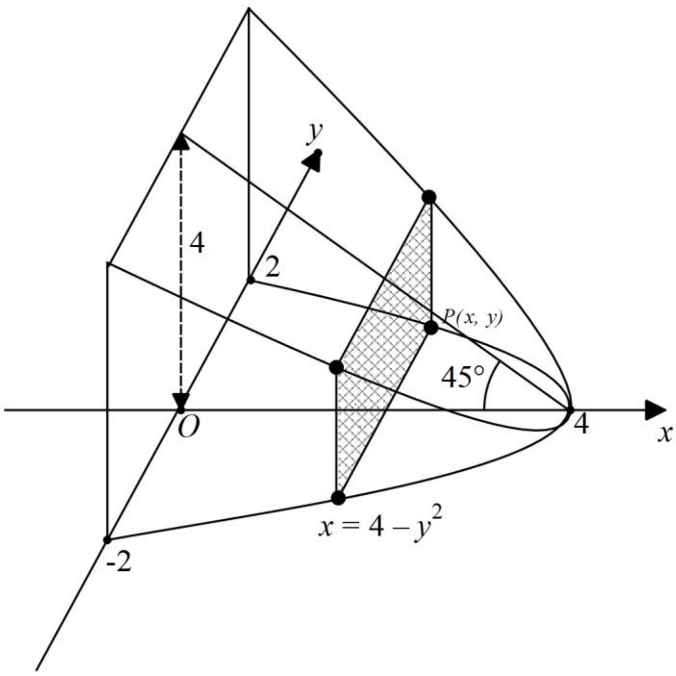
$$\frac{300x}{x^3+1000}=\frac{p}{x+10}+\frac{qx+r}{x^2-10x+100}$$

(ii) A particle of mass m kg is projected vertically upwards in a highly resistive medium at an initial velocity of 5 m/s. The particle is subjected to the force of gravity and to a resistance due to the medium of magnitude $\frac{1}{100}mv^3$. Given the acceleration due to gravity is 10 m/s^2, x is the displacement of the particle from its starting position, and regarding upwards motion as positive

(α) Explain why $\ddot{x}=-\frac{1}{100}(1000+v^3)$. **1**

(β) Hence, find the maximum height reached by the particle, giving your answer correct to one decimal place. **4**

(b) The base of a solid is the region enclosed by the parabola $x=4-y^2$ and the y-axis.

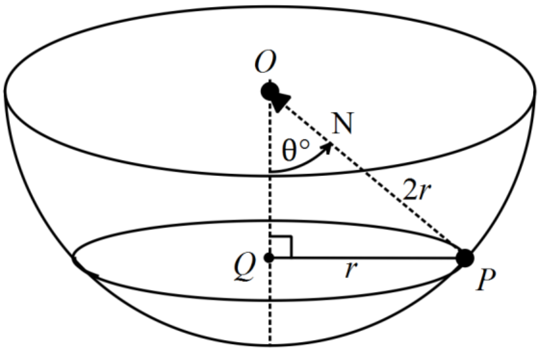


The top of the solid is formed by a plane inclined at 45° to the x-y plane. Each vertical cross-section of the solid taken parallel to the y-axis is a rectangle as shown in the diagram above. The point $P(x,y)$ lies on $x=4-y^2$. **4**

Find the volume of the solid.

Question 16 is continued on the next page

Question 16 continued



(c) The diagram above shows a particle P, of mass m kg moving with constant angular velocity ω in a horizontal circle about Q, of radius r metres on the inside of a fixed smooth hemispherical bowl of internal radius $2r$ metres, centre at O. There is a normal reaction force N acting inward toward the centre of the hemisphere. **3**

By resolving forces vertically and radially at P, show that $\omega^2 = \dfrac{g}{r\sqrt{3}}$.

End of Question 16

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \,, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

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Trinity Grammar School
Mathematics Department
2015
Year 12 Mathematics Extension 2

HSC Assessment Task 4
Section I
Objective Response Answer Sheet

BOSTES Student Number

Class Teacher: EH

- Write your answers for **Section I** on this answer sheet.
- ***Shade completely only ONE answer for each question.***
- To change an answer, erase your previous mark completely and then record your new answer.

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

A table of Standard Integrals is printed on the reverse side of this answer sheet

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Shade the correct response on the multiple-choice answer sheet for Questions 1–10

Each question is worth 1 mark

---

1 If $z=1+2i$ and $w=i-3$ then $z-\overline{w}$ is?

$\overline{w}=-3-i$

$\therefore z-\overline{w}=1+2i-(-3-i)$

$=4+3i$

(A) $3i-2$

~~(B)~~ $4+3i$

(C) $i-2$

(D) $4+i$

2 The function defined by $f(x)=\sqrt{3}\cos x+3\sin x$ has an amplitude of

(A) $3-\sqrt{3}$

(B) $\sqrt{3}$

~~(C)~~ $2\sqrt{3}$

(D) $3+\sqrt{3}$

$f_{max}=\sqrt{(\sqrt{3})^2+3^2}=2\sqrt{3}$

$f_{min}=-\sqrt{(\sqrt{3})^2+3^2}=-2\sqrt{3}$

$\therefore Amp=\dfrac{2\sqrt{3}+|-2\sqrt{3}|}{2}$

$=2\sqrt{3}$

3 The equations of the directrices of the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ are

~~(A)~~ $x=\pm\dfrac{16\sqrt{7}}{7}$

(B) $x=\pm\dfrac{\sqrt{7}}{16}$

(C) $x=\pm\dfrac{16}{5}$

(D) $x=\pm\dfrac{5}{16}$

$a=4,\ b=3$

$b^2=a^2(1-e^2)$

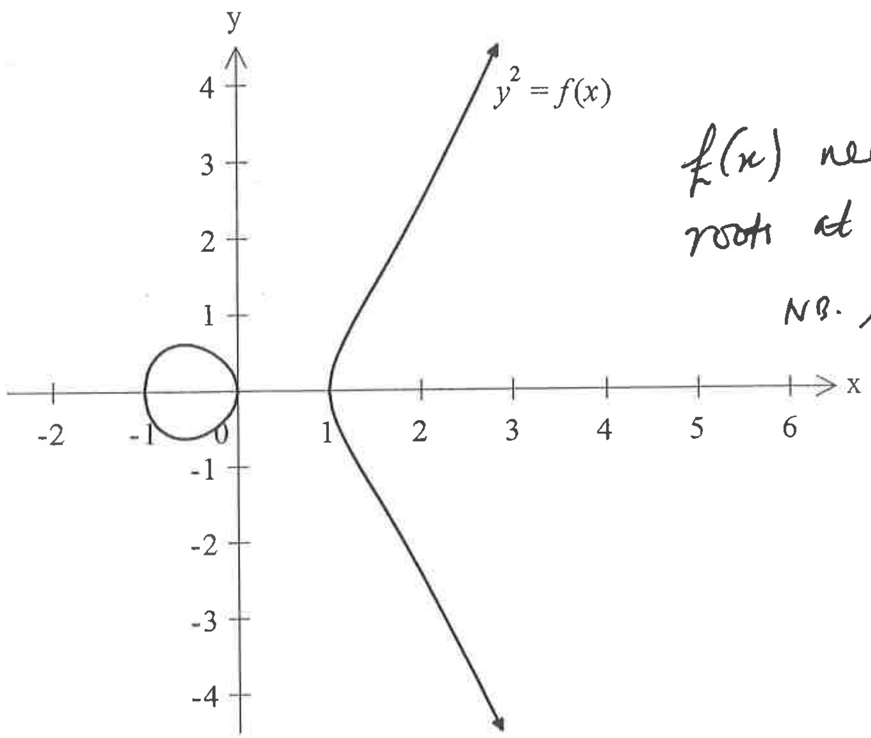
$9=16(1-e^2)$

$\therefore e^2=\frac{7}{16}$

$\therefore x=\pm\frac{a}{e}=\pm\dfrac{4}{\sqrt{7}/4}$

$=\pm\dfrac{16}{\sqrt{7}}$.

4 The diagram below shows the graph of $y^2 = f(x)$.



$f(x)$ needs to have roots at $-1, 0, 1$.

NB. $\lim_{x \to +\infty} y \to \pm\infty$.

Which expression best represents the function $f(x)$?

(A) $x^2(x-1)$

(B) $x^2(1+x)$

~~(C)~~ $x(x^2-1)$ $\Rightarrow x = 0,\ x = \pm 1$.

(D) $x(1+x^2)$

5 The linear factorisation of $x^4 + 7x^2 - 18$ over the complex number field is:

$= (x^2+9)(x^2-2)$

$= (x^2-(3i)^2)(x^2-2)$

$= (x-3i)(x+3i)(x-\sqrt{2})(x+\sqrt{2})$

(A) $(x-i\sqrt{2})(x+i\sqrt{2})(x-3)(x+3)$

(B) $(x-i\sqrt{2})(x-i\sqrt{2})(x+3)(x+3)$

(C) $(x-3i)(x-3i)(x+\sqrt{2})(x+\sqrt{2})$

~~(D)~~ $(x-3i)(x+3i)(x-\sqrt{2})(x+\sqrt{2})$

6 If α, β, γ are the roots of $x^3 + x - 1 = 0$, then the cubic equation with roots $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ is

(A) $-8x^3 - 2x - 1 = 0$

(B) $8x^3 + 2x - 1 = 0$

(C) $\frac{8}{x^3} + \frac{2}{x} - 1 = 0$

(D) $\frac{x^3}{8} + \frac{x}{2} - 1 = 0$

let $y = x/2$ ie $x = 2y$

Subst: $(2y)^3 + 2y - 1 = 0$

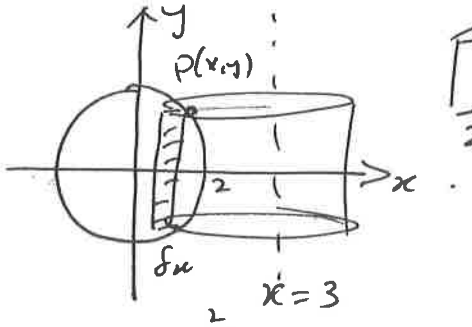
7 The circle $x^2 + y^2 = 4$ is rotated about the line $x = 3$ through 360°. The volume of the solid formed is given by

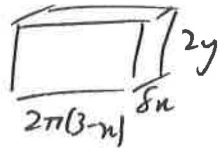
(A) $V = 2\pi\int_{-2}^{2}(3-x)\sqrt{4-x^2}\ dx$

(B) $V = 4\pi\int_{-2}^{2}(3-x)\sqrt{4-x^2}\ dx$

(C) $V = 2\pi\int_{0}^{2}(3-x)\sqrt{4-x^2}\ dx$

(D) $V = 4\pi\int_{0}^{2}(4-x)\sqrt{3-x^2}\ dx$





$V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} 2\pi(3-x)2y\,\delta x$

$= 4\pi\int_{-2}^{2}(3-x)\sqrt{4-x^2}\,dx.$

8 The equation $x^2 - y^2 - 2xy - 2 = 0$ defines y implicitly as a function of x.

The value of $\frac{dy}{dx}$ at the point $(-\sqrt{2}, 0)$ is?

(A) $-\sqrt{2}$

(B) -1

(C) 0

(D) 1

$2x - 2y\frac{dy}{dx} - 2\left[y + x\frac{dy}{dx}\right] = 0$

at $(-\sqrt{2}, 0)$

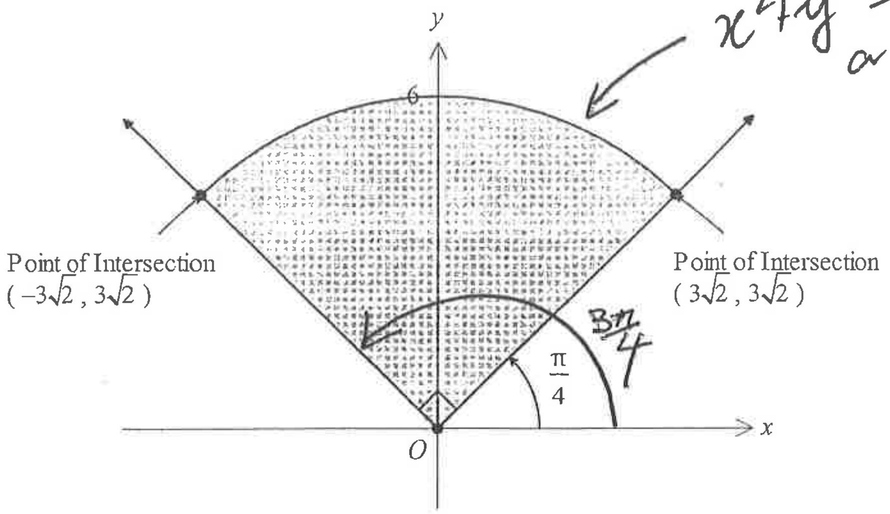
$-2\sqrt{2} - 2\left[0 - \sqrt{2}\frac{dy}{dx}\right] = 0$

$\therefore \frac{dy}{dx} = \frac{2\sqrt{2}}{2\sqrt{2}}$

$= 1.$

9

$x^2 + y^2 = 36$. or $|z| = 6$.



Which pair of inequalities represents the shaded region of the Argand plane shown above?

(A) $|z| \le 6$ and $\frac{\pi}{4} \le \arg(z) \le \frac{3\pi}{4}$

(B) $|z| \ge 6$ and $\frac{\pi}{4} \le \arg(z) \le \frac{3\pi}{4}$

(C) $|z| \ge 36$ and $-\frac{\pi}{4} \le \arg(z) \le \frac{\pi}{4}$

(D) $|z| \ge 36$ and $\frac{\pi}{4} \le \arg(z) \le \frac{3\pi}{4}$

10 What is the natural domain of the function $f(x) = \frac{1}{2}\left(x\sqrt{x^2 - 1} - \log_e\left(x + \sqrt{x^2 - 1}\right)\right)$?

want $x^2 - 1 \ge 0$ and $x \ge 0$

$x \le -1 \cup x \ge 1$ and $x \ge 0$

$\therefore x \ge 1$ only

(A) $x \le -1$

(B) $-1 \le x \le 1$

(C) $x \le -1$ or $x \ge 1$

(D) $x \ge 1$

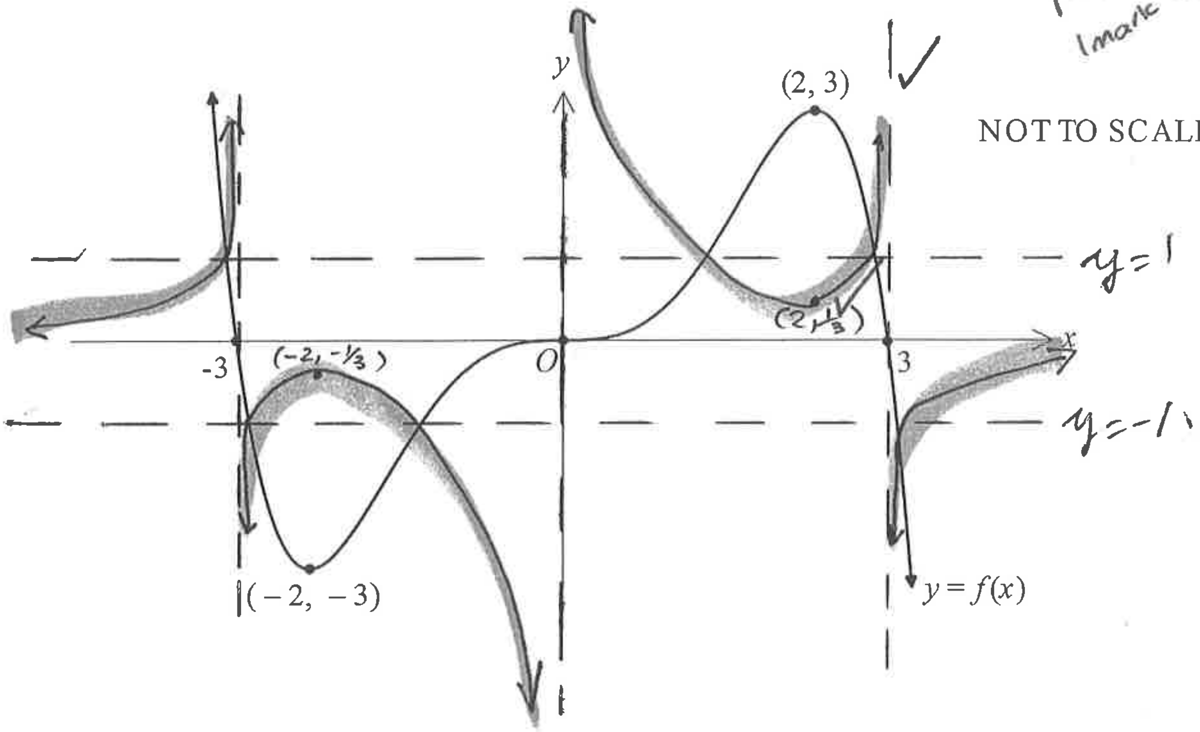
End of Section I

Section II commences on the next page

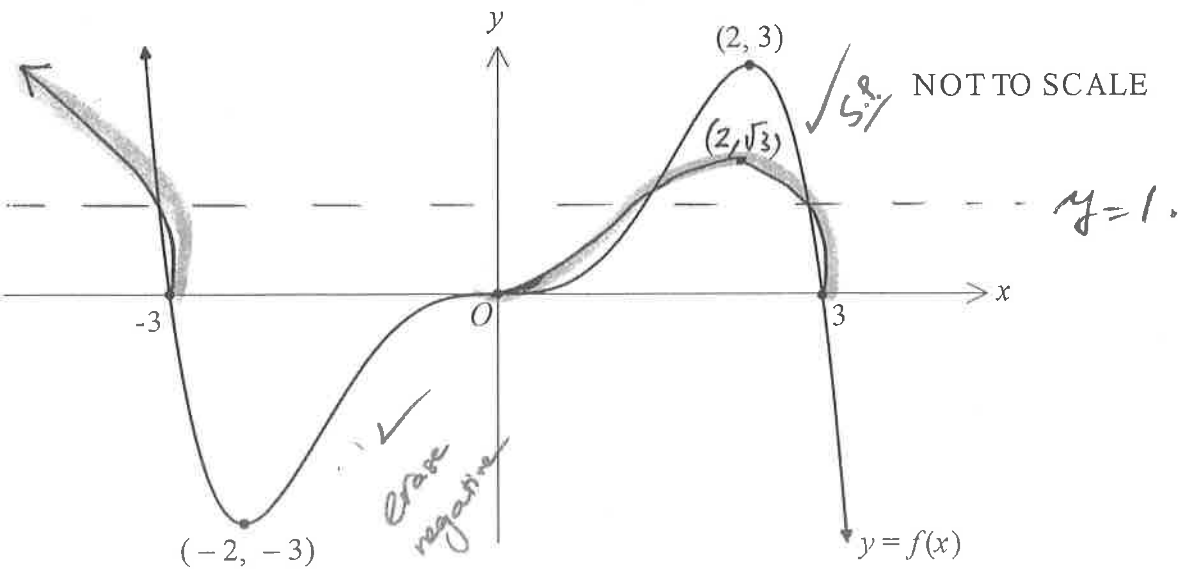
2015 HSC MAXX Task 4 Question 11 (a)

(i) (α) $y = \dfrac{1}{f(x)}$

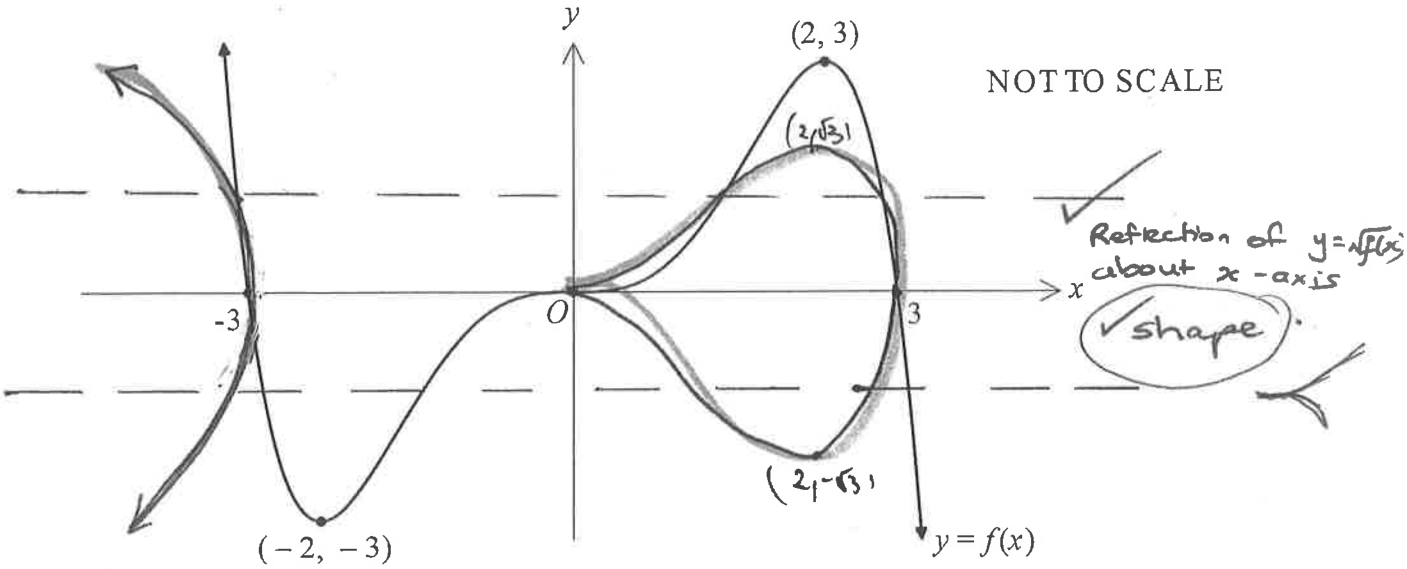
1 = Asymptotes
1 mark - S.P



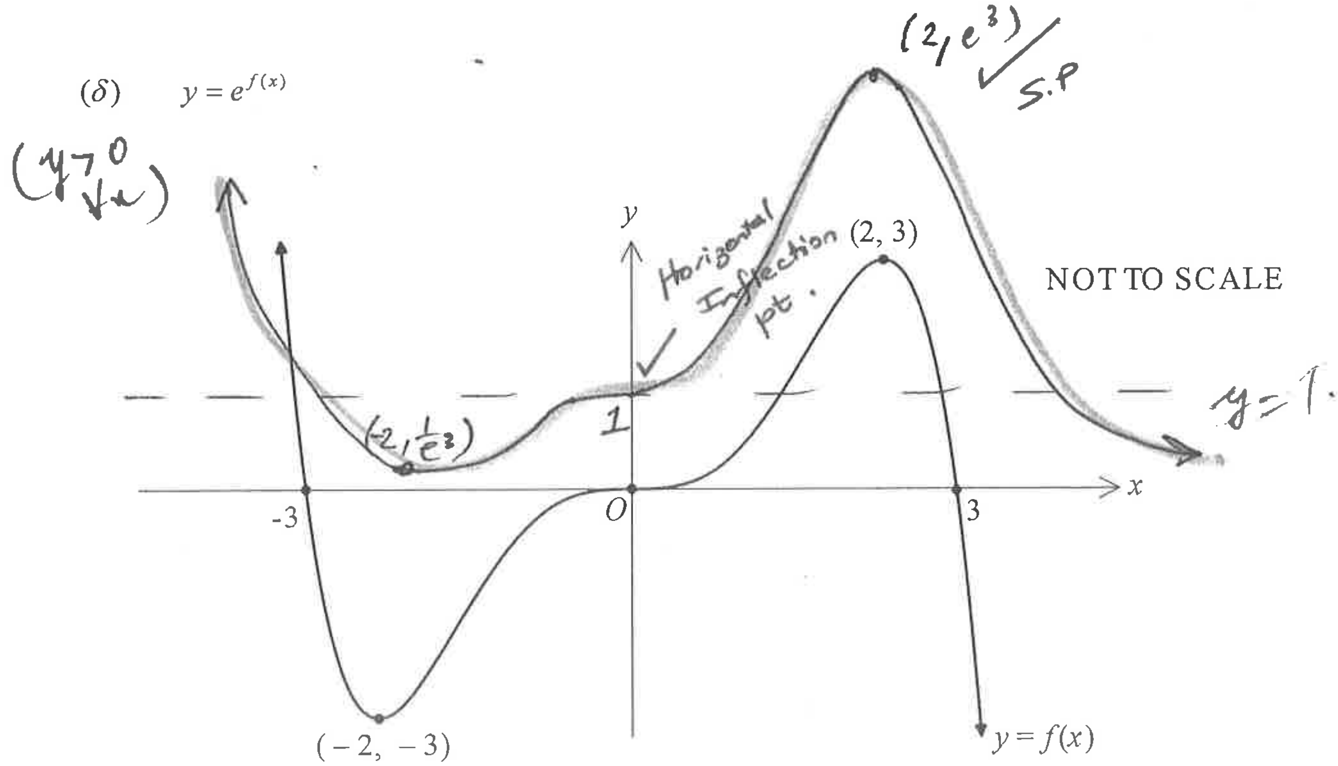
(β) $y = \sqrt{f(x)}$



(γ) $|y| = \sqrt{f(x)}$

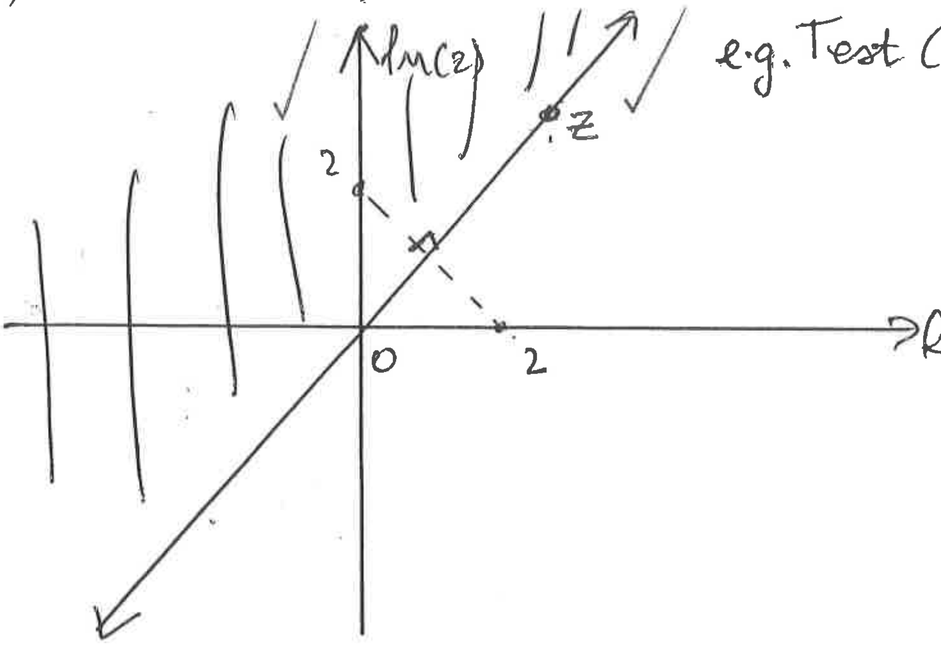


(δ) $y = e^{f(x)}$



iii) $\boxed{3}$ first draw $y=1$ on $y=\sqrt{f(x)}$.

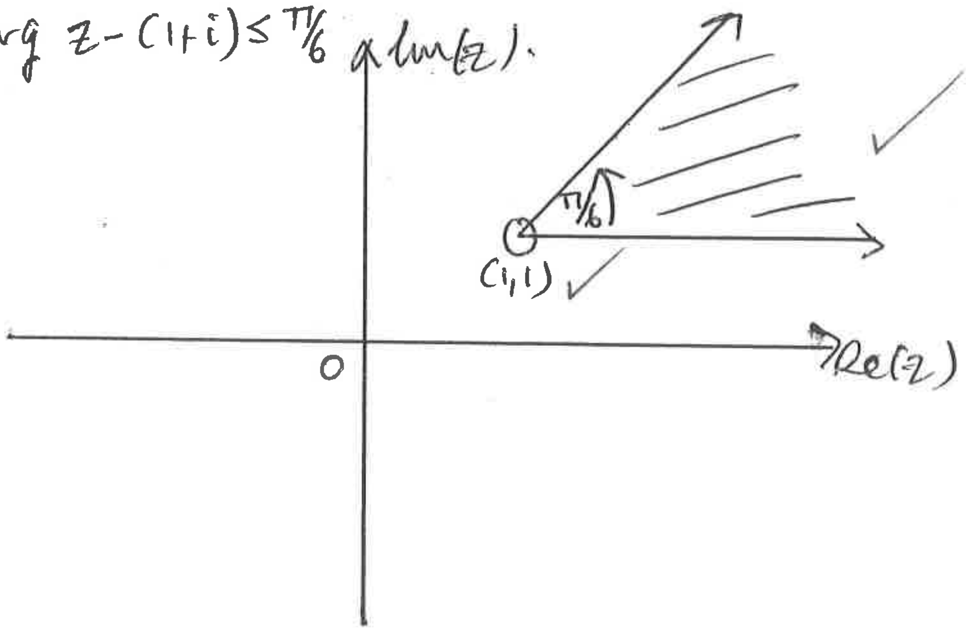
11b) i) $|z-2i| \leq |z-2|$



e.g. Test $(1,0)$ ✓

(2)

ii) $0 < \arg z-(1+i) \leq \pi/6$



open circle
No angle at a point.

(2)

iii) $\operatorname{Im}\left(z-\frac{1}{z}\right)=0,\ z\neq 0$

$\Rightarrow \arg\left(z-\frac{1}{z}\right)=n\pi$

$n\in\mathbb{Z}$ $\therefore$ must lie on real axis $(z\neq 0)$.

NB: $(x+yi)-\left(\frac{1}{x+yi}\right)$

$=\left(x-\frac{x}{x^2+y^2}\right)+i\left(\frac{y}{x^2+y^2}+y\right)$

want $\frac{y}{x^2+y^2}+y=0$

$\Rightarrow y=0$ or $x^2+y^2=-1$ ← impossible in $\mathbb{R}$

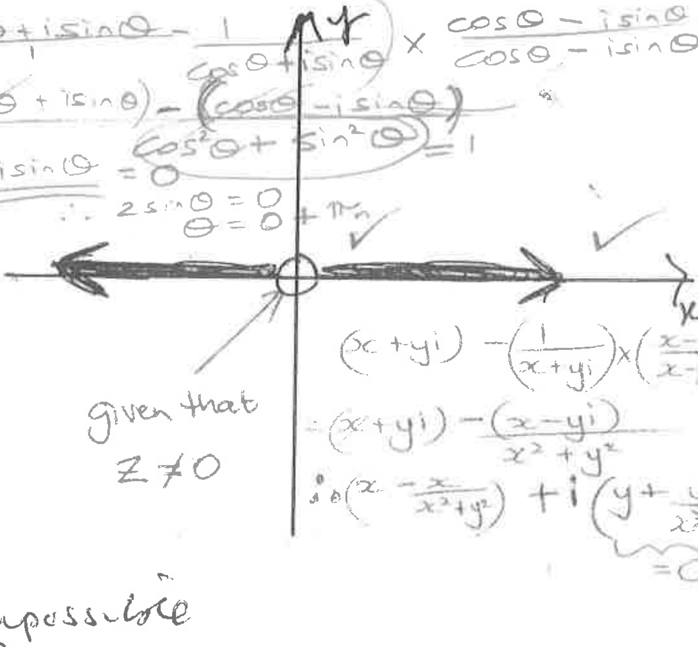
$\left(\cos\theta+i\sin\theta-\frac{1}{\cos\theta+i\sin\theta}\right)\times\frac{\cos\theta-i\sin\theta}{\cos\theta-i\sin\theta}$

$=(\cos\theta+i\sin\theta)-\frac{(\cos\theta-i\sin\theta)}{\cos^2\theta+\sin^2\theta}$ $=1$

$=2i\sin\theta=0$

$\therefore 2\sin\theta=0$

$\theta=0+\pi n$



given that $z\neq 0$

$(x+yi)-\left(\frac{1}{x+yi}\right)\times\left(\frac{x-yi}{x-yi}\right)$

$=(x+yi)-\frac{(x-yi)}{x^2+y^2}$

$\Rightarrow\left(x-\frac{x}{x^2+y^2}\right)+i\underbrace{\left(y+\frac{y}{x^2+y^2}\right)}_{=0}$

12) a) i) let $p(x) = x^3 - 11x + 20$.

$$p(2+i) = (2+i)^3 - 11(2+i) + 20$$
$$= 8 + 12i - 6 - i - 22 - 11i + 20$$
$$= 0 + 0i$$
$$= 0$$

(2)

Since $p(2+i) = 0$ then $x = 2+i$ is a root of $p(x) = 0$.

ii) The coefficients of $p(x)$ are <u>all</u> real and since $x = 2+i$ is a root then $x = 2-i$ (conjugate) is also a root.

OR

Sum of Roots

$2+i+2-i+\alpha = 0$

$4+\alpha = 0$

$\alpha = -4$

Using product of roots

$$(2+i)(2-i)\alpha = -20$$
$$5\alpha = -20$$
$$\therefore \alpha = -4$$

<u>Note;</u>

<u>OR</u> $(x-(2+i))(x-(2-i))$

$= x^2 - 2x + ix - 2x - ix + (4+1)$

$= x^2 - 4x + 5$

Long division;

$$\begin{array}{r} x + 4 \\ x^2-4x+5 \overline{\smash{)} x^3 + 0x^2 - 11x + 20} \\ -\underline{(x^3 - 4x^2 + 5x)} \\ 4x^2 - 16x + 20 \\ \underline{4x^2 - 16x + 20} \end{array}$$

$\therefore$ All the roots of $p(x) = 0$ are $2+i$, $2-i$ and -4.

(2)

[There are alternative methods].

b) The cubic with roots $\alpha^2, \beta^2, \gamma^2$

is $(\sqrt{x})^3 - 2(\sqrt{x})^2 - \sqrt{x} - 3 = 0$

<u>ie</u> $x\sqrt{x} - \sqrt{x} = 2x + 3$

<u>ie</u> $x(x-1)^2 = (2x+3)^2$

<u>ie</u> $x^3 - 6x^2 - 11x - 9 = 0$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = 6 \quad (-b/a)$

To find $\alpha^2 + \beta^2 + \gamma^2$

let $y = x^2$

ie $x = \sqrt{y}$

$P(\sqrt{y}) = y\sqrt{y} + 2y - \sqrt{y} - 3 = 0$

$(y\sqrt{y} - \sqrt{y})^2 = (2y+3)^2$

$y^3 - 2y^2 + y = 4y^2 - 12y + 9$

$y^3 - 6y^2 + 13y - 9 = 0$

Now for $\alpha^2 + \beta^2 + \gamma^2$

find $\left(-\frac{b}{a}\right) = \frac{-(-6)}{1}$

$= \underline{\underline{6}}$

<u>Alternatively:</u>

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= (-2)^2 - 2(-1)$$
$$= 6.$$

(2)

c) Given $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

$P'(x) = 8x^3 + 27x^2 + 12x - 20$

$P''(x) = 24x^2 + 54x + 12$

Solving $24x^2 + 54x + 12 = 0$

ie $4x^2 + 9x + 2 = 0$

$(4x+1)(x+2) = 0$

$\Rightarrow x = -\frac{1}{4},\ x = -2$ ✓

testing $P'(-2) = 8(-2)^3 + 27(-2)^2 + 12(-2) - 20$

$= -64 + 108 - 24 - 20$

$= 0$

$P(-2) = 2(-2)^4 + 9(-2)^3 + 6(-2)^2 - 20(-2) - 24$

$= 0$ ✓

This must be the triple root.

By the product of roots

$(-2)(-2)(-2)(\alpha) = \frac{-24}{2}$

$-8\alpha = -\frac{24}{2}$

$\alpha = \frac{3}{2}$ ✓ (3)

$\therefore$ All the roots are $-2, -2, -2$, and $\frac{3}{2}$.

d) $z^5 = -1$

$(\cos\theta + i\sin\theta)^5 = -1$

$\cos 5\theta + i\sin 5\theta = -1$

$\therefore \cos 5\theta = -1$

$5\theta = \pi + 2k\pi$

$\theta = \frac{\pi + 2k\pi}{5}$

let $k = 0, 1, 2, -1, -2$

$z_1 = \text{cis}\frac{\pi}{5}$ $(k=0)$

$z_2 = \text{cis}\frac{3\pi}{5}$ $(k=1)$

$z_3 = \text{cis}\,\pi$ $(k=2)$

$z_4 = \text{cis}\left(-\frac{\pi}{5}\right)$ $(k=-1)$

$z_5 = \text{cis}\left(-\frac{3\pi}{5}\right)$ $(k=-2)$

d) i) The fifth roots of -1 are equally space on the circle $x^2 + y^2 = 1$ by $\frac{2\pi}{5}^c$ about O. One of the 5th roots of (-1) is -1. $\sqrt[5]{-1} = -1$

Others can be given by $x = \text{cis}\left(\frac{\pi + 2k\pi}{5}\right)$

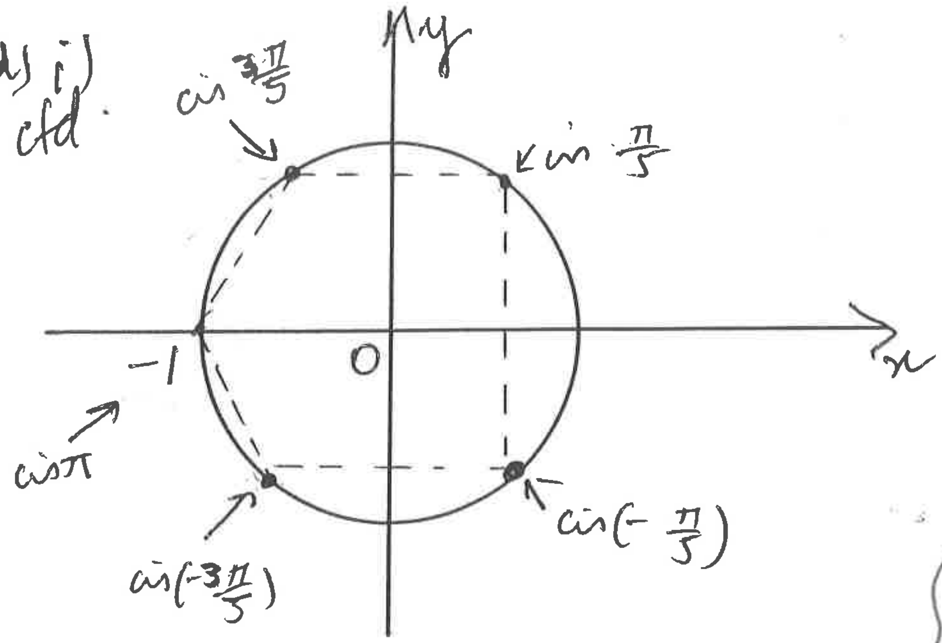
$k = 0, \pm 1, \pm 2$ (general form)

or $x = \text{cis}\,\pi,\ \text{cis}\frac{3\pi}{5},\ \text{cis}\left(-\frac{3\pi}{5}\right),\ \text{cis}\frac{\pi}{5},\ \text{cis}\left(-\frac{\pi}{5}\right)$

$(= -1)$

$\underbrace{\text{cis}\frac{3\pi}{5},\ \text{cis}\left(-\frac{3\pi}{5}\right)}_{\text{conjugates}}$ $\underbrace{\text{cis}\frac{\pi}{5},\ \text{cis}\left(-\frac{\pi}{5}\right)}_{\text{Conjugates!}}$

d) i) ctd.



NB:

$x^5+1=0$

$\Rightarrow (x-\text{cis}\,\pi)(x-\text{cis}\,\frac{3\pi}{5})(x-\text{cis}(-\frac{3\pi}{5}))(x-\text{cis}\,\pi/5)(x-\text{cis}(-\frac{\pi}{5}))$

$=(x-1)(x-\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5})(x-\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5})(x-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5})(x-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5})$

$=(x-1)\left[(x-\cos\frac{3\pi}{5})^2-(i\sin\frac{3\pi}{5})^2\right]\left[(x-\cos\frac{\pi}{5})^2-(i\sin\frac{\pi}{5})^2\right]$

etc ...

ii) let $p(x)=x^5+1$

The zeros of $p(x)$ are the 5^{th} roots of -1 from (i).

let $\alpha=\text{cis}\,\frac{\pi}{5}$, $\beta=\text{cis}(-\frac{\pi}{5})$, $\gamma=\text{cis}(\frac{3\pi}{5})$, $\delta=\text{cis}(-\frac{3\pi}{5})$.

$\therefore p(x)=(x+1)(x^4-x^3+x^2-x+1)$ ✓

$=(x+1)\left[x^2-(\alpha+\beta)x+\alpha\beta\right]\left[x^2-(\gamma+\delta)x+\gamma\delta\right]$

$=(x+1)\left[x^2-2\text{Re}(\alpha)x+1\right]\left[x^2-2\text{Re}(\gamma)x+1\right]$ ✓

$=(x+1)\left[x^2-2\cos\frac{\pi}{5}x+1\right]\left[x^2-2\cos\frac{3\pi}{5}x+1\right]$ ✓ (3)

$b_1=\frac{1}{5}$ $\quad b_2=\frac{3}{5}$

NOTE: $\cos\frac{\pi}{5}=\cos(-\frac{\pi}{5})$

and $\cos\frac{3\pi}{5}=\cos(-\frac{3\pi}{5})$

$(x+1)(x-\alpha)(x-\bar{\alpha})(x-\beta)(x-\bar{\beta})$

$=(x+1)(x^2-(\alpha+\bar{\alpha})x+\alpha\bar{\alpha})(x^2-(\beta+\bar{\beta})x+\beta\bar{\beta})$

$=(x+1)(x^2-\underbrace{2\,\text{Re}\,\alpha}+1)(x^2-2\text{Re}\beta)+1)$

$=(x+1)(x^2-2(\cos\frac{2\pi}{5})x+1)(x^2-2\cos\frac{4\pi}{5}x+1)$

$\alpha+\bar{\alpha}$

$=x+iy+x-iy$

$=2x=2\,\text{Re}(\alpha)$

$(\alpha)(\bar{\alpha})$

$=(x+iy)(x-iy)$ UNIT CIRCLE

$=x^2+y^2=1$

or $x^5+1=(x+1)Q(x)$

$Q(x)=\dfrac{x^5+1}{x+1}$

$=x+1\,\overline{)\,x^5+1}$

by long division

13 a) let $x = \tan\theta$ when $x=1$, $\theta = \pi/4$

$\frac{dx}{d\theta} = \sec^2\theta$ $x=\sqrt{3}$, $\theta = \pi/3$ ✓

$x^2 = \tan^2\theta$.

$\therefore \int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}} \frac{dx}{d\theta} d\theta$

$= \int_{\pi/4}^{\pi/3} \frac{\sec^2\theta \, d\theta}{\tan^2\theta\sqrt{\sec^2\theta}}$ $= dx$ ✓ $= \frac{\sec\theta}{\tan^2\theta}$

$= \int_{\pi/4}^{\pi/3} \sec\theta\cot^2\theta \, d\theta$ $= \int_{\frac{\pi}{4}}^{\pi/3} \frac{1}{\cos\theta} \times \left(\frac{\cos^2\theta}{\sin^2\theta}\right) d\theta$

$= \int_{\pi/4}^{\pi/3} \cot\theta \operatorname{cosec}\theta \, d\theta$ $= \int_{\frac{\pi}{4}}^{\pi/3} \left(\frac{\cos\theta}{\sin\theta}\right) \times \left(\frac{1}{\sin\theta}\right) d\theta$

$= -\operatorname{cosec}\theta \Big|_{\pi/4}^{\pi/3}$ ✓

NB: $\frac{d}{dx}\sec x = \sec x \tan x$

$\therefore \frac{d}{dx}\operatorname{cosec} x = \frac{d}{dx}\sec(\pi/2 - x)$

$= \sec(\pi/2 - x) \cdot \tan(\pi/2 - x)(-1)$

$= -\operatorname{cosec} x \cot x$.

$= -\operatorname{cosec}\pi/3 + \operatorname{cosec}\pi/4$ ✓

$= \sqrt{2} - \frac{2}{\sqrt{3}}$ ✓

b) i) $3\int_{-1}^{1} \frac{1}{4+(x-1)^2} dx$ ✓

$= \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right)\Big|_{-1}^{1}$ ✓

$= \frac{3}{2}[\tan^{-1} 0 - \tan^{-1}(-1)]$

$= \frac{3}{2} \times \pi/4$

$= 3\pi/8$. ✓

(3)

I.B.P

$u = x$ $v' = e^{-2x}$

$u' = 1$ $v = -\frac{1}{2}e^{-2x}$

$x\left(-\frac{1}{2}e^{-2x}\right) + \int \frac{1}{2}e^{-2x} dx$

$= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c$

ii) $\int xe^{-2x} dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$.

$= -\frac{1}{2}e^{-2x}\left(x + \frac{1}{2}\right) + c$

(2)

13(c)

i) Tangent at P:

$$x + p^2 y = 2cp \quad (1)$$

Tangent at Q:

$$x + q^2 y = 2cq \quad (2)$$

tangents ✓

$$(1) - (2) \quad (p^2 - q^2)y = 2cp - 2cq$$

$$\Rightarrow \quad y = \frac{2c(p-q)}{(p-q)(p+q)} \quad (p \neq q)$$

$$y = \frac{2c}{p+q} \quad (3)$$ ✓

Subs (3) into (1)

$$x = 2cp - p^2\left(\frac{2c}{p+q}\right)$$

$$x = \frac{2cp(p+q) - 2cp^2}{p+q}$$ ✓

$$x = \frac{2cpq}{p+q} \quad (4)$$

$$\therefore \quad R = \left[\frac{2cpq}{p+q}, \frac{2c}{p+q}\right].$$ (3)

ii) from (3) and (4)

Square $\quad y = \dfrac{2c}{p+q} \quad (5)$

$$y^2 = \frac{4c^2}{(p+q)^2}$$

$$= \frac{4c^2}{\cancel{p^2+q^2}\,2 + 2pq}$$ ✓

$$= \frac{4c^2}{2+2pq}$$

$$\text{ie} \quad y^2 = \frac{4c^2}{2(1+pq)}$$

now in (4) $\quad (p+q) \times x = \dfrac{2cpq}{p+q}(p+q)$

$$pq = \frac{x(p+q)}{2c}$$

but $\quad p+q = \dfrac{2c}{y}$ from (3)

$$\therefore \quad pq = x/y$$ ✓ both.

hence $\quad y^2 = \dfrac{4c^2}{2(1+pq)} \quad \dfrac{x}{y}$

$$y^2 = \frac{4c^2}{2(1 + x/y)}$$

$$\text{ie} \quad y^2 = \frac{2c^2 y}{x+y}$$

$$y^2(x+y) = 2c^2$$ ✓

$$\text{ie} \quad y^2 = 2c^2 - xy$$

(as required)

Possible Restrictions:

→ $\dfrac{y}{x+y} > 0$; $x \neq -y$; $p \neq -q$

$pq \neq -1$; $y \neq 0$

OR

$$xy = \frac{2cpq}{p+q} \times \frac{2c}{p+q}$$

$$xy = \frac{4c^2pq}{(p+q)^2} \quad \dots (1)$$

$$y^2 = \frac{4c^2}{(p+q)^2} \quad \dots (2)$$

$$xy + y^2 = \frac{4c^2pq + 4c^2}{(p+q)^2} \quad (1)+(2)$$

$$= \frac{4c^2(pq+1)}{p^2+q^2+2pq} \qquad p^2+q^2 = 2$$

$$= \frac{4c^2(pq+1)}{2(1+pq)} = 2c^2$$

Given 3 marks

14) a) i) $45\text{rpm} = \frac{45 \times 2\pi}{60}$ rad/s

$\therefore \omega = \frac{3\pi}{2}$ rad/s.

using $v = r\omega$

$= 0.08 \times \frac{3\pi}{2}$

$= 0.38$ m/s (2dp).

ii) using $a = r\omega^2$

$= 0.08 \times \left(\frac{3\pi}{2}\right)^2$

$= 1.78$ m/s^2 (2dp).

b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2}{a^2}\left(\frac{x}{y}\right)$

at P, $\frac{dy}{dx} = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$

$= -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$

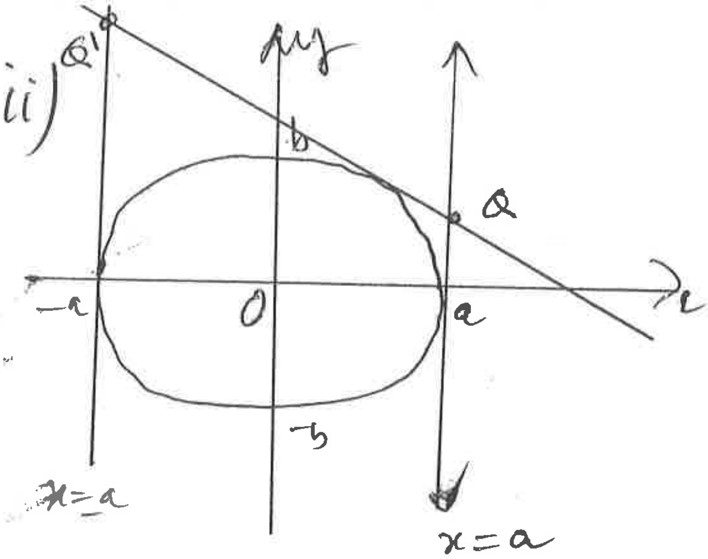
using $y - y_1 = m(x - x_1)$

$y - b\sin\theta = \frac{-b}{a}\frac{\cos\theta}{\sin\theta}(x - a\cos\theta)$

$\Rightarrow ay\sin\theta - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$

$\therefore bx\cos\theta + ay\sin\theta = ab$

ii)



at $x = \pm a$

$\pm ab\cos\theta + ay\sin\theta = ab$

$ay\sin\theta = ab \mp ab\cos\theta$

$y = \frac{b \pm b\cos\theta}{\sin\theta}$ $(a \neq 0)$

$\therefore Q = \left(a, \frac{b + b\cos\theta}{\sin\theta}\right)$

$Q' = \left(-a, \frac{b - b\cos\theta}{\cos\theta}\right)$

iii) $AQ \cdot Q'A'$

$= \left|\frac{b(1+\cos\theta)}{\sin\theta}\right| \left|\frac{b(1-\cos\theta)}{\sin\theta}\right|$

$= \frac{b^2(1-\cos^2\theta)}{\sin^2\theta}$

$= \frac{b^2\sin^2\theta}{\sin^2\theta}$

$= b^2$

$\therefore$ since $|AQ||Q'A'|$ is independent of θ then the product is independent of position P.

c) i)

$(cis\theta)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3$

$= \cos^3\theta - 3\cos^2\theta\sin\theta i - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \underbrace{(\cos^3\theta - 3\cos\theta\sin^2\theta)} + i(3\cos^2\theta\sin\theta - \sin^3\theta)$ ✓ ✓

ii) by de Möivre's Theorem

$(cis\,\theta)^3 = cis\,3\theta$

$\therefore \cos 3\theta = Re[(cis\theta)^3]$ ✓

$= \cos^3\theta - 3\cos\theta\sin^2\theta$ ✓

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$

iii) $\cos 3\theta + \cos\theta = 0$

$4\cos^3\theta - 3\cos\theta + \cos\theta = 0$

$\cos\theta[4\cos^2\theta - 2] = 0$

$\cos\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \pm\frac{\pi}{2}$ ✓ $\theta = \pm\pi/4$ ✓

$[-\pi/2 \leq \theta \leq \pi/2]$.

(15.)

a) $z^3-1=0$

i) $(z-1)(z^2+z+1)=0$

$z=1$ or $z^2+z+1=0$

if ω is a $\mathbb{C}$ root $\left[=\text{cis}\left(\pm\frac{2\pi}{3}\right)\right]$

$\therefore \omega^2+\omega+1=0$

ii) $LHS=\left(\frac{\omega}{1+\omega}\right)^k+\left(\frac{\omega^2}{1+\omega^2}\right)^k$

$=\left(\frac{\omega}{-\omega^2}\right)^k+\left(\frac{\omega^2}{-\omega}\right)^k$ from (i)

$=(-1)^k\omega^{-k}+(-1)^k\omega^k$

$=(-1)^k(\omega^{-k}+\omega^k)$

but $z^n+z^{-n}=2\cos n\theta$

$\therefore LHS=(-1)^k\left[2\cos\frac{2\pi}{3}k\right]$

since $\omega=\text{cis}\frac{2\pi}{3}$ in this requirement.

b) i) $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$

ii) tangent at Q is

$\frac{xx_2}{a^2}-\frac{yy_1}{b^2}=1.$

but T (x_0,y_0) lies on both TP and TQ therefore its coordinates satisfy both tangents at P and Q.

b) $\frac{x_0x_1}{a^2}-\frac{y_0y_1}{b^2}=1\rightarrow(P)$

$\frac{x_0x_2}{a^2}-\frac{y_0y_2}{b^2}=1\rightarrow(Q)$

replacing (x,y) in either (P) & (Q)

$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1.$

iii) Sub $x=ae,\ y=0$

$\frac{x_0ae}{a^2}=1.$

$\therefore x_0=\frac{a}{e}$

which is the equation of the directrix.

c) i) $\int_0^{\pi/2}\cos x\,dx$

$=\sin x\Big|_0^{\pi/2}$

$=\sin\pi/2-\sin 0$

$=1.$

ii) $I_n = \int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \cos x \cdot \cos^{n-1} x$

$u = \cos^{n-1} x \qquad dv = \cos x$

$du = -(n-1)\cos^{n-2} x\,(-\sin x) \qquad v = \sin x$

$I_n = \sin x \cos^{n-1} x \Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^2 x \cos^{n-2} x \, dx$ ✓

$= (n-1)\int_0^{\pi/2} (1-\cos^2 x)\cos^{n-2} x \, dx$ ✓

$I_n = (n-1)\left[I_{n-2} - I_n\right]$

$I_n + (n-1) I_n = (n-1) I_{n-2}$ ✓

$I_n = \left(\frac{n-1}{n}\right) I_{n-2} \qquad n \geq 2$

iii) $I_5 = \int_0^{\pi/2} \cos^5 x \, dx$

$= \frac{4}{5} I_3$ ✓

$= \frac{4}{5} \times \frac{2}{3} \times I_1$

$= \frac{4}{5} \times \frac{2}{3} \times 1$ from (i)

$= \frac{8}{15}$ ✓

only if (i) is correct.
$I_5 > 0$ aswell
irrespective

(16)

a i) Want

$300x \equiv p(x^2-10x+100)+(x+10)(qx+r)$

Sub $x=-10$

$-3000 = p(100+100+100)$

$p=-10$

Sub $x=0$

$0 \equiv 100p + 10r$

$\therefore\ 0 = -1000 + 10r$

$10r = 1000$

$r = 100.$

Sub $x=10$

$3000 = p(100-100+100) + 20(10q+r)$

$3000 = 100p + 20(100+10q)$

$3000 = -1000 + 2000 + 200q$

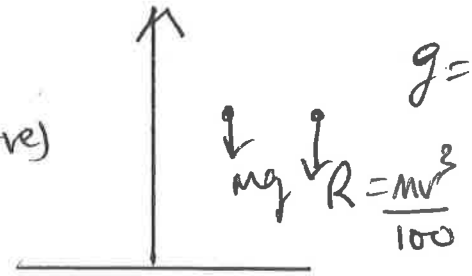
$\dfrac{4000-2000}{200} = q$

$q = \dfrac{2000}{200}$

$= 10.$

$\therefore (p,q,r) = (-10, 10, 100).$

ii) (+ve) $g = 10\,m/s^2.$



$x=0$

$t=0,\ v=5\,m/s$

(α) by Newton's 2nd Law

$m\ddot{x} = \sum F_x$

$m\ddot{x} = -mg - \dfrac{mv^3}{100}$

$\ddot{x} = -g - \dfrac{v^3}{100}$

$\ddot{x} = -\dfrac{1}{100}(100\times 10 + v^3)$

$\ddot{x} = -\dfrac{1}{100}(1000+v^3).$

(β) Let the max. height reached be H metres, $v=0$

Using $v\dfrac{dv}{dx} = -\dfrac{1}{100}(1000+v^3)$

$\Rightarrow \dfrac{dv}{dx} = -\dfrac{(1000+v^3)}{100v}$

$\dfrac{dx}{dv} = -\dfrac{100v}{1000+v^3}$

$\Rightarrow dx = -\dfrac{100v}{1000+v^3}\,dv$

$\int_0^H dx = -\int_5^0 \dfrac{100v}{1000+v^3}\,dv$

$\therefore \int_0^H dx = \int_0^5 \dfrac{100v}{1000+v^3}\,dv$

c) $\int_0^4 dx = \frac{1}{3}\int_0^5 \frac{300v}{1000+v^3}\,dv$

$= \frac{1}{3}\left[\int_0^5 \frac{-10}{v+10} + \frac{10v+100}{v^2-10v+100}\,dv\right]$

from (i)

$\therefore H = \frac{1}{3}\left[-10\ln(v+10)\Big|_0^5 + 10\int_0^5 \frac{v+10}{v^2-10v+100}\,dv\right]$

$= \frac{1}{3}\left(\left[-10\ln 15 + 10\ln 10\right] + 10\int_0^5 \frac{\frac{1}{2}(2v-10)+15}{v^2-10v+100}\,dv\right)$

$= \frac{1}{3}\left[10\ln\frac{10}{15} + 5\ln(v^2-10v+100)\Big|_0^5 + \int_0^5 \frac{150}{75+(v-5)^2}\,dv\right]$

$= \frac{1}{3}\left[10\ln\frac{10}{15} + 5\ln(75) - 5\ln 100 + \frac{150}{5\sqrt{3}}\tan^{-1}\frac{(v-5)}{5\sqrt{3}}\Big|_0^5\right]$

$= \frac{1}{3}\left[10\ln\frac{10}{15} + 5\ln 75 - 5\ln 100 + \frac{150}{5\sqrt{3}}\tan^{-1}0 + \frac{150}{5\sqrt{3}}\tan^{-1}\frac{5}{5\sqrt{3}}\right]$

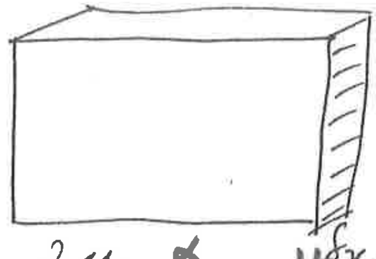
$= \frac{1}{3}\left[10\ln\frac{2}{3} + 5\ln 75 - 5\ln 100 + \frac{150}{5\sqrt{3}}\times\frac{\pi}{6}\right]$

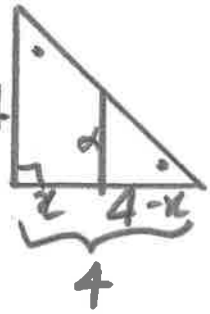
$= 1.19197\ldots$

$= 1.2$ m (1dp)

b) $x = 4-y^2 \Rightarrow y = \sqrt{4-x}$ $(y \geq 0)$.

$4-x = \alpha$





$2y$

$\frac{\alpha}{4-x} = \frac{4}{4}$

$\therefore \alpha = 4-x$.

$\delta A = 2y(4-x)$

$\delta V = 2y(4-x)\,\delta x$.

Total Volume

$V = \lim_{\delta x \to 0} \sum_{x=0}^{4} 2(4-x)y\,\delta x$

$= \lim_{\delta x \to 0} \sum_{x=0}^{4} 2(4-x)\sqrt{4-x}\,\delta x$

$= 2\int_0^4 (4-x)^{3/2}\,dx$

$= 2\left[\frac{2(4-x)^{5/2}}{-5}\right]_0^4$

$= -\frac{4}{5}(4-x)^2\sqrt{4-x}\Big|_0^4$

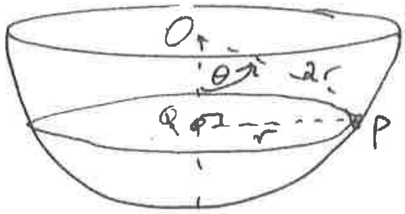
$= \frac{4}{5}(4-x)^2\sqrt{4-x}\Big|_4^0$

$= \frac{4}{5}(4)^2\sqrt{4} - 0$

$= \frac{4\times16\times2}{5}$

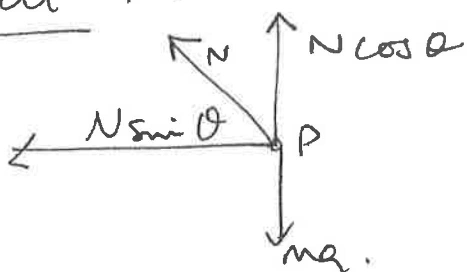
$= \frac{128}{5}\ u^3$.

16(i)



$$OQ^2 = (2r)^2 - r^2$$
$$= 3r^2$$
$$\therefore OQ = r\sqrt{3}$$

Vertically at P.



by Newton's Third Law $\quad N\cos\theta - mg = 0$

$$\Rightarrow N\cos\theta = mg \quad (1)$$

Radially: $\quad m\ddot{x} = \sum F_n$

$$mr\omega^2 = N\sin\theta \quad (2)$$

but in Δ OQP, $\quad \frac{r}{2r} = \sin\theta$

$$\sin\theta = \frac{1}{2}$$

$$mr\omega^2 = \frac{N}{2}$$

$$N = 2mr\omega^2 \quad (3)$$

hence subs (3) in (1)

$$\Rightarrow \quad 2mr\omega^2\cos\theta = mg \quad \text{and } \cos\theta = \frac{r\sqrt{3}}{2r} = \sqrt{3}/2$$

$$\therefore \quad 2mr\omega^2\frac{\sqrt{3}}{2} = mg$$

$$r\omega^2\sqrt{3} = g$$

$$\omega^2 = \frac{g}{r\sqrt{3}} \quad \text{as required.}$$