Trinity Grammar School

Mathematics Department

2016

TRIAL EXAMINATION

HSC ASSESSMENT TASK 4

Year 12

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen.
- BOSTES approved calculators for this course may be used.
- A Reference Sheet for this course is supplied.
- In Questions **11-16**, show all relevant mathematical reasoning and/or calculations.
- Write your BOSTES Student Number (Year 12 HSC) and your Class teacher on the question paper and on any answer sheets or writing booklets used to write your responses to the questions submitted.
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 40%

BOSTES Student Number
(Year 12 only)

Class Teacher:

………………………………………………………………

Name:
(Year 10 or 11 only)

………………………………………………………………

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

Date of examination:

Monday, 22 August 2016

Total marks – 100

Section I

10 marks

- Attempt all Questions.
- Allow about **15** minutes for this section.

Section II

90 marks

- Attempt all Questions.
- Allow about **2** hours **45** minutes for this section.

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Section I – 10 marks

- **Attempt Questions 1 – 10.**
- **Allow about 15 minutes for this section.**
- **Shade the correct response on the multiple-choice answer sheet for Questions 1-10.**
- **Each question is worth one mark.**

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1 Which of the following is $(-1+2i)^3$?

(A) $-11-2i$

(B) $-11+2i$

(C) $11-2i$

(D) $-1-8i$

2 The cartesian equation of the ellipse is $x^2+4y^2=4$.
What are the parametric equations of this ellipse?

(A) $x=2\cos\theta$ and $y=\sin\theta$

(B) $x=2\sin\theta$ and $y=\cos\theta$

(C) $x=4\cos\theta$ and $y=\sin\theta$

(D) $x=4\sin\theta$ and $y=\cos\theta$

3 Let $z=2-3i$. What is the value of z^{-1}?

(A) $-\frac{1}{5}(2+3i)$

(B) $\frac{1}{13}(2+3i)$

(C) $\frac{1}{5}(2-3i)$

(D) $\frac{1}{13}(2-3i)$

4 A disc of radius 4 *cm* is spinning such that a point on the circumference is moving at a speed of $80\ cm/\min$. The angular speed of the disc (in revolutions per minute) is?

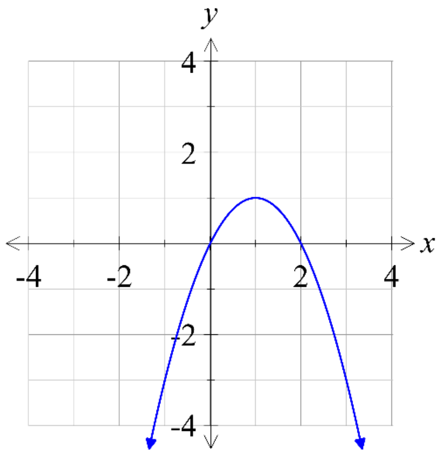
(A) 320 rpm

(B) 20 rpm

(C) $\frac{10}{\pi}$ rpm

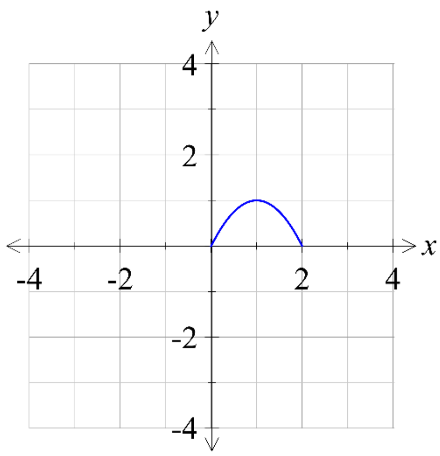
(D) $\frac{160}{\pi}$ rpm

5 The diagram below shows the graph of the function $y = f(x)$.

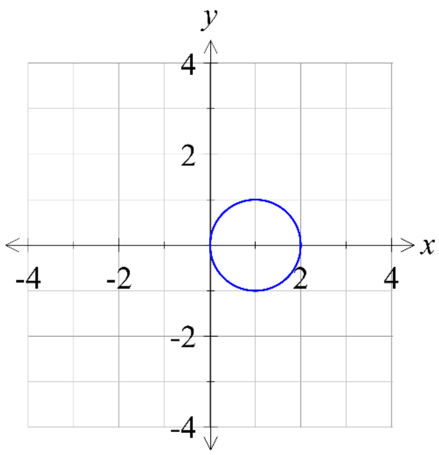


Which of the following is the graph of $y = \sqrt{f(x)}$?

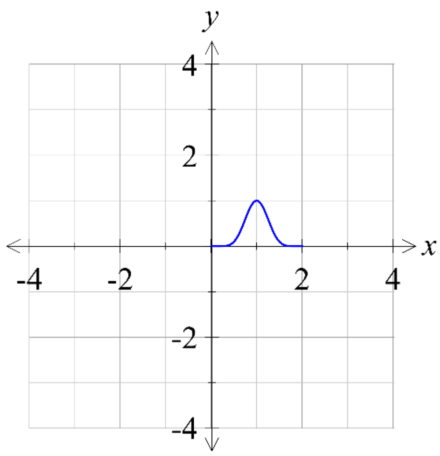
(A)



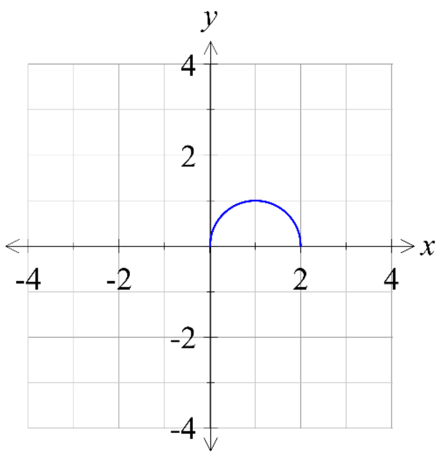
(B)



(C)



(D)



6 Which of the following is the correct expression for $\int \frac{1}{\sqrt{4x^2+36}}dx$?

(A) $\sin^{-1}\frac{x}{3}+C$

(B) $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$

(C) $\log_e\left(x+\sqrt{x^2+9}\right)+C$

(D) $\frac{1}{2}\log_e\left(x+\sqrt{x^2+9}\right)+C$

7 For the ellipse with the equation $\frac{(x-2)^2}{5}+\frac{(y+3)^2}{3}=1$. What is the eccentricity?

(A) $\sqrt{\frac{2}{5}}$

(B) $\sqrt{\frac{2}{3}}$

(C) $\sqrt{\frac{5}{3}}$

(D) $\sqrt{\frac{5}{2}}$

8 A solid is formed by rotating the region enclosed by the parabola $y^2=4ax$, its vertex (0,0) and the line $x=a$ about the x-axis. What is the volume of this solid using the method of cylindrical shells?

(A) $\frac{7\pi a^3}{4}$ units^3

(B) $\frac{7\pi a^3}{8}$ units^3

(C) $\frac{7\pi a^3}{16}$ units^3

(D) $2\pi a^3$ units^3

9 What is the value of $\int_0^2 \sqrt{\frac{x}{4-x}}dx$ using the substitution $x=4\sin^2\theta$?

(A) 0.75π

(B) $\pi-2$

(C) $\pi+6$

(D) $3\pi-8$

10 What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3+pz+q=0$?

(A) $p=-2$ and $q=-4$

(B) $p=-2$ and $q=4$

(C) $p=2$ and $q=4$

(D) $p=2$ and $q=4$

End of Section I

Section II commences on the next page

Section II - 90 marks

- **Attempt Questions 11 – 16.**
- **Allow about 2 hours and 45 minutes for this section.**
- **Answer each question in a SEPARATE writing booklet or answer sheet. Extra writing booklets are available.**
- **In Questions 11 – 16 your responses should include all relevant mathematical reasoning and/or calculations.**

Question 11 (15 marks) **Marks**

(a) Find $\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\,dx$. **2**

(b) (i) Show that $(1-2i)^2 = -3-4i$. **1**

(ii) Hence solve the equation $z^2 - 5z + (7+i) = 0$. **2**

(c) Use integration by parts to evaluate $\int xe^{-x}\,dx$. **2**

(d) Let $z = 1 - i\sqrt{3}$.

(i) What is the exact value of $|z|$ and arg z? **2**

(ii) Find the exact value of z^6. **2**

(e) Consider the function $y = \cos^{-1}(e^x)$.

(i) Find the domain and the range. **2**

(ii) Sketch the graph of $y = \cos^{-1}(e^x)$? **1**

(iii) Hence, or otherwise, sketch the graph of $y = \left[\cos^{-1}(e^x)\right]^2$. **1**

Question 12 (15 marks) **Marks**

(a) Let two complex numbers be $z_1 = 2(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ and $z_2 = 2i$.

(i) On an Argand diagram sketch the vectors OA and OB to represent z_1 and z_2 respectively. **1**

(ii) Draw the vectors $z_1 + z_2$ and $z_1 - z_2$ on the same Argand diagram. **1**

(iii) What are the exact values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$? **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x + 2\sin x} dx$. **4**

(c) Let $f(x) = (2 - x)(x - 4)$. Draw separate one-third page sketches of the following functions indicating clearly any asymptotes and intercepts with the axes.

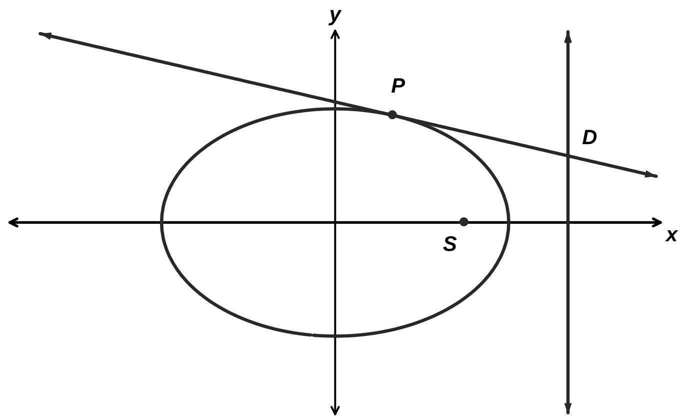
(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = e^{f(x)}$ **2**

(d) The function $y = x^3$ is rotated about the y axis $\{x : 0 \le x \le 2\}$ to form a solid. Calculate the volume of this solid using the method of slicing. **3**

Question 13 (15 marks) **Marks**

(a)



NOT TO SCALE

$P\left(a\cos\theta, b\sin\theta\right)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P meets a directrix of the ellipse at D. S is the corresponding focus.

(i) Show that the tangent at P has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **2**

(ii) Show that $\angle PSD = 90^\circ$. **3**

(b) The polynomial equation $x^3+2x+1=0$ has roots α, β and γ.

(i) Find the equation with α^{-1}, β^{-1} and γ^{-1} as the roots. **2**

(ii) Find the equation with α^{-2}, β^{-2} and γ^{-2} as the roots. **2**

(c) (i) Find real numbers a, b, c and d such that: **2**

$$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{cx+d}{x^2+1}$$

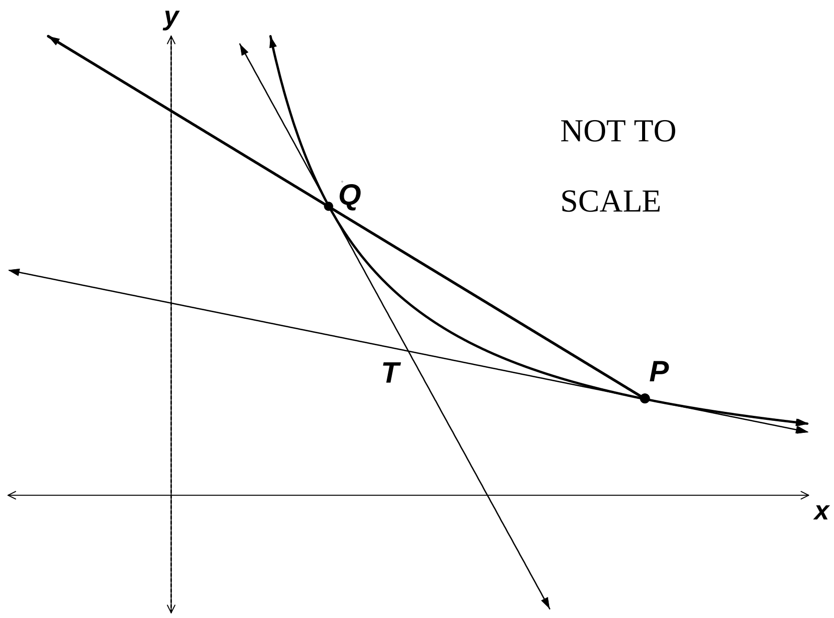
(ii) Hence, evaluate in simplest form: **2**

$$\int\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}\,dx$$

(d) The polynomial $P(x)=x^4+ax^2+bx+28$ has a double root at $x=2$. **2**
What are the values of a and b?

Question 14 (15 marks) **Marks**

(a) The variable points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the hyperbola $xy = c^2$ as shown in the diagram below.



The tangents to the hyperbola at P and Q intersect at the point T. M is the midpoint of PQ. The equation of the tangent at P is $x + p^2 y = 2cp$.

(i) Show that the coordinates of T are: $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **2**

(ii) If O is the origin, show that O, T and M are collinear. **1**

(iii) Find an expression for q in terms of p if T, M and S are collinear, where S is the focus of the hyperbola. **2**

(b) A solid is formed by rotating about the y-axis the region bounded by the curve $y = \sin x$ and the x-axis between $0 \le x \le \pi$. Find the volume of this solid using the method of cylindrical shells. **4**

(c) (i) Find the three cubic roots of -1 by solving the equation $z^3 + 1 = 0$ **2**

(ii) Let α be a cubic root of –1, where α is not real. **1**
Show that $\alpha^2 = \alpha - 1$.

(iii) Hence simplify $(1-\alpha)^6$. **1**

(d) Find the equation of the tangent to the curve $x^2 y + 2x - 2xy = 0$ at (1, 2). **2**

Question 15 (15 marks) **Marks**

(a) A train of mass 2000 kg is travelling around a curve of radius 1 km on a track banked so that the outer rail is 3 cm higher than the inner rail, where the rails are 1.2 m apart. The banked track makes an angle θ with the horizontal. When the train has a speed of v ms^{-1}, the normal reaction force exerted by the track on the train is N Newtons and the lateral thrust exerted by one of the rails on the train is F Newtons. Take $g = 10$ ms^{-2}.

(i) By considering the forces acting on the train with F directed up the slope, show that $F = 2\cos\theta(10{,}000\tan\theta - v^2)$. **3**

(ii) Find the magnitude of the lateral thrust when v is 80% of v_0 if the lateral thrust F is zero when $v = v_0$. **2**

(b) Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. The complex number w is a root of $P(z) = 0$.

(i) Show that iw and $\frac{1}{w}$ are also roots of $P(z) = 0$. **2**

(ii) Find all the roots of $P(z) = 0$ in exact form. **2**

(c) (i) Show that $\sin x + \sin 3x = 2\sin 2x\cos x$. **1**

(ii) Hence, or otherwise, solve $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **2**

(d) A particle of mass m is moving in a straight line under the action of a force. **3**

$$F = \frac{m}{x^3}(6 - 10x)$$

What is the velocity in any position, if the particle starts from rest at $x = 1$?

Question 16 (15 marks) **Marks**

(a) The point P on an Argand diagram represents the complex number $z = x + iy$. If P moves such that $\text{Im}(z+i) = |z+1|$ find the minimum value of $|z+1|$. **3**

(b) (i) Let $I_n = \int_0^1 (1-x^r)^n \, dx$, where $r > 0$, for $n = 0,1,2,3,\ldots$

Show that $I_n = \dfrac{nr}{nr+1} I_{n-1}$. **3**

(ii) Hence or otherwise, find the value of $I_n = \int_0^1 (1-x^{\frac{3}{2}})^3 \, dx$. **2**

(c) From a point O, a particle of mass m is projected upwards with an initial velocity of u metres per second. It moves under a constant gravitational force mg and an air resistance of kmg, where $k < 1$.

Show that:

(i) the equation of the upward motion is $\ddot{x} = -g(1+k)$. **1**

(ii) the greatest height reached by the particle is $\dfrac{u^2}{2g(1+k)}$. **3**

(iii) the total time T that elapses before the particle returns to its point of projection is: $\dfrac{u}{g}\left\{\dfrac{1}{1+k} + \dfrac{1}{\sqrt{1-k^2}}\right\}$ **3**

End of Section II

End of Examination

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180°$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

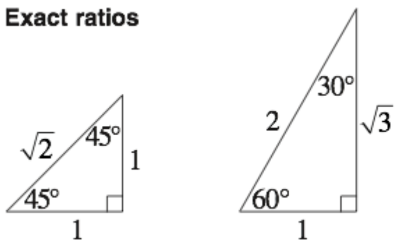
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$\alpha + \beta = -\frac{b}{a}$ $\qquad$ $\alpha\beta = \frac{c}{a}$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$\text{Area} = \frac{1}{2}r^2\theta$

Mathematics Extension 1

Angle sum identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \quad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \quad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \quad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $(2at, at^2)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At (x_1, y_1),

tangent: $xx_1 = 2a(y + y_1)$

normal: $y - y_1 = -\frac{2a}{x_1}(x - x_1)$

Chord of contact from (x_0, y_0): $xx_0 = 2a(y + y_0)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$x = b + a\cos(nt + \alpha)$

$\ddot{x} = -n^2(x - b)$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

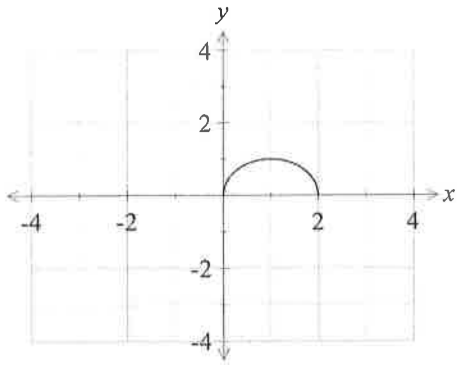
Binomial theorem

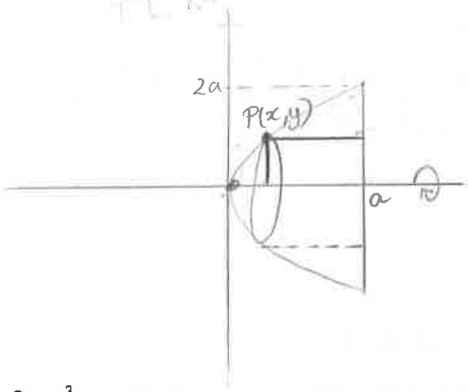
$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

ACE Examination 2016

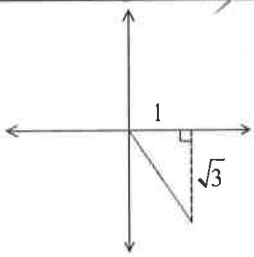
HSC Mathematics Extension 2 Yearly Examination

Worked solutions and marking guidelines

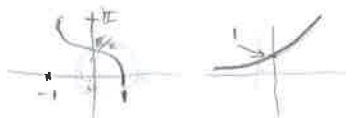
Section I		
	Solution	**Criteria**
1	$(-1+2i)^3$ $=(-1)^3+3(-1)^2(2i)+3(-1)(2i)^2+(2i)^3$ $=-1+6i+12-8i$ $=11-2i$	1 Mark: C
2	$x^2+4y^2=4$ $\frac{x^2}{4}+\frac{y^2}{1}=1$ Hence $a=2$ and $b=1$. Parametric equations are $x=2\cos\theta$ and $y=\sin\theta$	1 Mark: A
3	$z^{-1}=\frac{1}{2-3i}\times\frac{2+3i}{2+3i}$ $=\frac{2+3i}{4+9}$ $=\frac{1}{13}(2+3i)$	1 Mark: B
4	$v=r\omega$ $80=4\omega$ $\omega=20$ rad/min $=20\times\frac{1}{2\pi}$ rev/min $=\frac{10}{\pi}$ rpm	1 Mark: C
5		1 Mark: D
6	$\int\frac{1}{\sqrt{4x^2+36}}\,dx=\frac{1}{2}\int\frac{1}{\sqrt{x^2+9}}\,dx$ $=\frac{1}{2}\log_e\left(x+\sqrt{x^2+9}\right)+C$	1 Mark: D

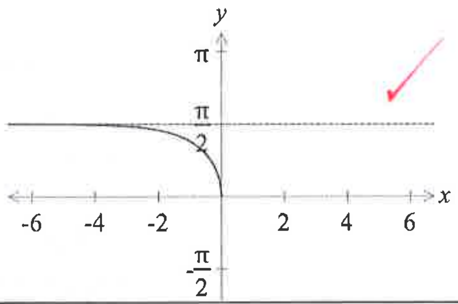
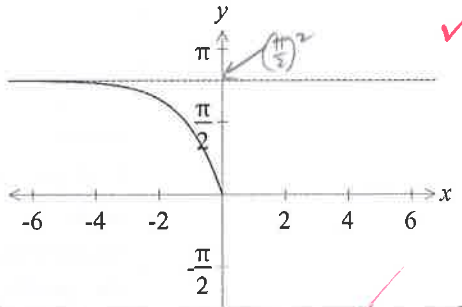
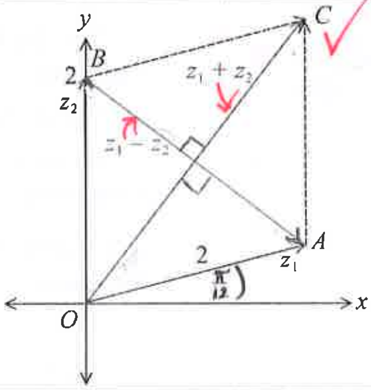
7	$b^2 = a^2(1-e^2)$ $3 = 5(1-e^2)$ $(1-e^2) = \frac{3}{5}$ or $e^2 = \frac{2}{5}$ or $e = \sqrt{\frac{2}{5}}$	1 Mark: A
8	Cylindrical shells radius is y and height $a - x = a - \frac{y^2}{4a}$ $V = \lim_{\delta y \to 0} \sum_{y=0}^{2a} 2\pi y(a - \frac{y^2}{4a})\delta y$ $= 2\pi \int_0^{2a} y(a - \frac{y^2}{4a})dy$ $= 2\pi \int_0^{2a} (ya - \frac{y^3}{4a})dy$ $= 2\pi \left[\frac{ay^2}{2} - \frac{y^4}{16a} \right]_0^{2a}$ $= 2\pi \left[(\frac{4a^3}{2} - \frac{16a^4}{16a}) - 0 \right] = 2\pi a^3$  	1 Mark: D
9	$x = 4\sin^2\theta$ and $dx = 8\sin\theta\cos\theta d\theta$ When $x = 0$ then $\theta = 0$ and when $x = 2$ then $\theta = \frac{\pi}{4}$ $\int_0^2 \sqrt{\frac{x}{4-x}} dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{4\sin^2\theta}{4 - 4\sin^2\theta}} \times 8\sin\theta\cos\theta d\theta$ $= \int_0^{\frac{\pi}{4}} \tan\theta \times 8\sin\theta\cos\theta d\theta$ $= \int_0^{\frac{\pi}{4}} 8\sin^2\theta d\theta$ $= 8\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 2\theta) d\theta$ $= 4\left[x - \frac{1}{2}\sin 2\theta \right]_0^{\frac{\pi}{4}} = \pi - 2$	1 Mark: B
10	Using the conjugate root theorem $1+i$ and $1-i$ are both roots of the equation $z^3 + pz + q = 0$. $(1+i) + (1-i) + \alpha = 0$ (sum of the roots) $\alpha = -2$ $(1+i) \times (1-i) \times -2 = -q$ (product of the roots) $(1+1) \times -2 = -q$ $q = 4$ $(1+i)(1-i) + (1-i) \times -2 + (1+i) \times -2 = p$ $p = -2$ Therefore $p = -2$ and $q = 4$.	1 Mark: B

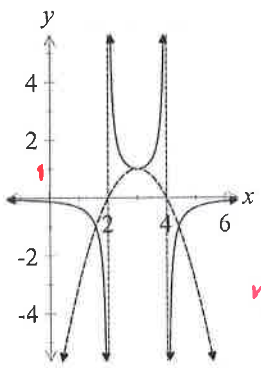
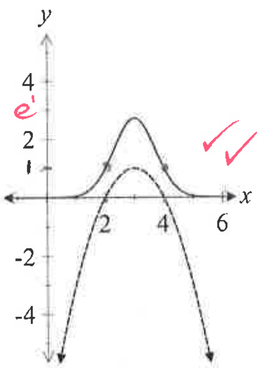
Section II

	Solution	Criteria
11(a)	$\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}dx = \int \frac{\sqrt{1-x}\times\sqrt{1+x}}{\sqrt{1-x}}dx$ $= \int \sqrt{1+x}dx$ $= \frac{2}{3}(1+x)^{\frac{3}{2}} + C$ (2)	2 Marks: Correct answer. 1 Mark: Simplifies the integrand.
11(b) (i)	$(1-2i)^2 = 1-4i-4$ $= -3-4i$ (1)	1 Mark: Correct answer.
11(b) (ii)	$z = \frac{-(-5)\pm\sqrt{(-5)^2-4\times1\times(7+i)}}{2\times1}$ $= \frac{5\pm\sqrt{-3-4i}}{2} = \frac{5\pm(1-2i)}{2}$ $\therefore z = 3-i$ or $z = 2+i$ (2)	2 Marks: Correct answer. 1 Mark: Shows some understanding of the problem.
11(c)	$\int xe^{-x}dx = \int x\frac{d}{dx}(e^{-x})dx$ $= -xe^{-x} + \int e^{-x}dx$ $= -xe^{-x} - e^{-x} + C$ $u = x$ | $v' = e^{-x}$ $u' = 1$ | $v = -e^{-x}$ (2) OR $-e^{-x}(x+1)$	2 Marks: Correct answer. 1 Mark: Sets up the integration by parts.
11(d) (i)	$\lvert z\rvert = \sqrt{1^2+(\sqrt{3})^2}$ $= \sqrt{4}$ $= 2$ $\arg z = -\frac{\pi}{3}$ OR $\frac{5\pi}{3}$ (2) 	2 Marks: Correct answer. 1 Mark: Determines $\lvert z\rvert$ or arg z.
11(d) (ii)	$z = 1-i\sqrt{3}$ $= 2\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]$ $z^6 = 2^6\left[\cos\left(-\frac{6\pi}{3}\right)+i\sin\left(-\frac{6\pi}{3}\right)\right]$ De Moivre's Theorem $= 2^6[\cos(-2\pi)+i\sin(-2\pi)]$ $= 2^6(1-i\times0)$ $= 64$ (2)	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
11(e) (i)	$\cos^{-1}(e^x): -1\le e^x\le 1$ but $e^x>0$ for all x hence $0<e^x\le1$ Domain: $x\le0$ Range: $0\le y<\frac{\pi}{2}$ (2)	2 Marks: Correct answer. 1 Mark: Finds the domain or range.

For $y = \cos^{-1}x$
$D: -1 \le x \le 1$
$R: 0 \le y \le \pi$
For $y = \cos^{-1}(e^x)$
$D: -1 \le e^x \le 1$ ($\log_e$ all)
$x \le 0$

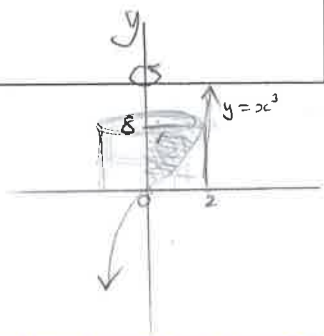


11(e) (ii)	 (1)	1 Mark: Correct answer.
11(e) (iii)	Horizontal asymptote at $y=\dfrac{\pi^2}{4}$  (1)	1 Mark: Correct answer.
12(a) (i)	 (2)	1 Mark: Correct answer.
12(a) (ii)	See Argand diagram above.	1 Mark: Correct answer.
12(a) (iii)	Vectors OC and AB form a parallelogram. However $OA = OB = 2$. Hence $OBCA$ is a rhombus. $\angle AOC=\frac{1}{2}\times\left(\frac{\pi}{2}-\frac{\pi}{12}\right)=\frac{5\pi}{24}$ (diagonals of a rhombus bisect the angles through which they pass) $\arg(z_1+z_2)=\frac{\pi}{12}+\frac{5\pi}{24}$ $=\frac{7\pi}{24}$ $AB\perp OC$ (diagonals of a rhombus intersect at right angles) $\arg(z_1-z_2)=\frac{\pi}{2}+\frac{7\pi}{24}$ (z_1+z_2 can be rotated in an anticlockwise direction hence add $\frac{\pi}{2}$) $=\frac{19\pi}{24}$ (2)	2 Marks: Correct answer. 1 Mark: Finds the value of $\arg(z_1+z_2)$ or $\arg(z_1-z_2)$ or shows some understanding.

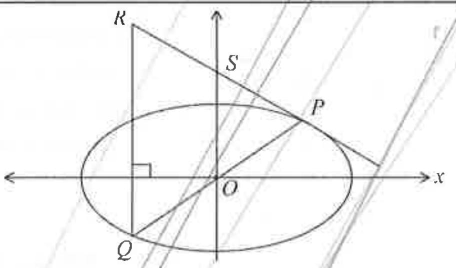
12(b)	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}dx$ or $dt=\frac{1}{2}(1+t^2)dx$ $dx=\frac{2}{1+t^2}dt$ When $x=0$ then $t=0$ and when $x=\frac{\pi}{2}$ then $t=1$ $2-\cos x+2\sin x=\frac{2(1+t^2)-(1-t^2)+4t}{1+t^2}$ $\quad 2-\left(\frac{1-t^2}{1+t^2}\right)+2\left(\frac{2t}{1+t^2}\right)$ $=\frac{3t^2+4t+1}{1+t^2}=\frac{(3t+1)(t+1)}{1+t^2}$ $\int_0^{\frac{\pi}{2}}\frac{1}{2-\cos x+2\sin x}dx=\int_0^1\frac{1+t^2}{(3t+1)(t+1)}\times\frac{2}{1+t^2}dt$ $\quad =2\int_0^1\frac{1}{(3t+1)(t+1)}dt$ $\frac{1-t^2}{1+t^2}$ $\quad \frac{2(2t)}{1+t^2}$ $=\int_0^1\frac{3}{(3t+1)}-\frac{1}{(t+1)}dt$ $\quad \frac{A}{(3t+1)}+\frac{B}{(t+1)}$ $\quad =2\int\frac{3/2}{(3t+1)}-\frac{\frac{1}{2}}{(t+1)}$ $=\left[\log_e(3t+1)-\log_e(t+1)\right]_0^1$ $=(\log_e 4-\log_e 2)-(\log_e 1-\log_e 1)$ $=\log_e 2.$ (4)	4 Marks: Correct answer 3 Marks: Correctly determines the primitive function. 2 Marks: Correctly expresses the integral in terms of t. Partial fractions 1 Mark: Correctly finds $d\theta$ in terms of dt and determines the new limits.
12(c) (i)	$y=\frac{1}{f(x)}=\frac{1}{(2-x)(x-4)}$ Vertical asymptote at $x=2$ & $x=4$ $\lim\limits_{x\to\infty}\frac{1}{(2-x)(x-4)}=0^-$ Horizontal asymptote $y=0$ 	2 Marks: Correct answer. 1 Mark: Determines the asymptotes or shows some understanding.
12(c) (ii)	$\lim\limits_{x\to\infty}e^{(2-x)(x-4)}=0$ Horizontal asymptote $y=0$ When $x=2$ then $y=e^0=1$ When $x=4$ then $y=e^0=1$ When $x=3$ then $y=e^1=e$ 	2 Marks: Correct answer. 1 Mark: Determines the asymptotes or shows some understanding.
12(d)	Area of the slice is a circle radius is x and height y $A=\pi x^2=\pi(y^{\frac{1}{3}})^2$ $\quad y=x^3$ $\quad y^{\frac{1}{3}}=x$ $=\pi y^{\frac{2}{3}}$	3 Marks: Correct answer. OR $V=\pi\int_0^8 4-y^{2/3}$

$dV=\pi(R^2-r^2)dy$
$=\pi(4-x^2)dy$
$=\pi(4-(y^{\frac{1}{3}})^2)dy$

$V=\lim\limits_{\delta y\to 0}\sum\limits_{y=0}^{8}\pi y^{2/3}\delta y$
$=\int_0^8\pi y^{2/3}dy$
$=\pi\frac{3}{5}\left[y^{5/3}\right]_0^8$
$=\pi\left[\frac{3}{5}8^{5/3}\right]=\frac{96\pi}{5}$ units^3



$=\pi\left[4y-\frac{3y^{5/3}}{5}\right]_0^8$
$=\pi\left[32-\frac{96}{5}\right)$
$=\frac{64\pi}{5}$ / 12.8π

12(d)	$\delta V = \delta A.\delta y \qquad V = \lim_{\delta y \to 0} \sum_{y=0}^{8} \pi y^{\frac{2}{3}} \delta y$ $= \int_0^8 \pi y^{\frac{2}{3}} dy$ $= \pi \left[\frac{3}{5} y^{\frac{5}{3}} \right]_0^8 = \frac{3\pi}{5} \times 8^{\frac{5}{3}} = \frac{96\pi}{5}$ cubic units	2 Marks: Correct integral for the volume of the solid. 1Mark: Sets up the area of the slice.
13(a) (i)	To find the equation of tangent through P $x = a\cos\theta \qquad y = b\sin\theta$ $\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$ $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = b\cos\theta \times \frac{1}{-a\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$ Equation of the tangent $y - y_1 = m(x - x_1)$ $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ $ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$ $bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$ $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$	2 Marks: Correct answer 1 Mark: Correctly calculates the gradient.
13(a) (ii) 13b	 $OS \parallel QR$ and O is the midpoint of PQ (diameter). Therefore $\Delta PQR \parallel\!\mid \Delta POS$ Area $\Delta PQR = 4 \times$ Area ΔPOS (enlargement factor of 2) $S\left(0, \frac{b}{\sin\theta}\right)$ and $P(a\cos\theta, b\sin\theta)$ Area $\Delta POS = \frac{1}{2} \times \frac{b}{\lvert\sin\theta\rvert} \times a\lvert\cos\theta\rvert = \frac{1}{2} \times \frac{ab}{\lvert\tan\theta\rvert}$ Area $\Delta PQR = 4 \times \frac{1}{2} \times \frac{ab}{\lvert\tan\theta\rvert} = \frac{2ab}{\lvert\tan\theta\rvert}$ Area of the ellipse is πab $\frac{\text{Area } \Delta PQR}{\text{Area of ellipse}} = \frac{\frac{2ab}{\lvert\tan\theta\rvert}}{\pi ab} = \frac{2}{\pi\lvert\tan\theta\rvert} = 2 : \pi\lvert\tan\theta\rvert$	3 Marks: Correct answer. 2 Marks: Finds the area of ΔPOS. 1 Mark: Recognises similar triangles and the enlargement factor.

a) (ii) (3 marks)

Outcomes assessed: E2, E3

Targeted Performance Bands: E4

Criteria	Marks
• Correctly establishes the result.	3
• Finds gradient of SD (or equivalent progress).	2
• Finds the coordinates of D (or equivalent progress).	1

Sample answer:

$$\text{Tangent is } \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

$$\text{For } D,\ x=\frac{a}{e}$$

$$\frac{a\cos\theta}{ae}+\frac{y\sin\theta}{b}=1$$

$$\frac{y\sin\theta}{b}=1-\frac{\cos\theta}{e}$$

$$y=\frac{b(e-\cos\theta)}{e\sin\theta}$$

$$\therefore D \text{ is } \left(\frac{a}{e},\ \frac{b(e-\cos\theta)}{e\sin\theta}\right)$$

let $x=\frac{a}{e}$ to find y value of D.

$$m_{SD}=\frac{\frac{b(e-\cos\theta)}{e\sin\theta}-0}{\frac{a}{e}-ae}$$

$$=\frac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}$$

finding gradient

$$m_{PS}=\frac{b\sin\theta-0}{a\cos\theta-ae}$$

$$=\frac{b\sin\theta}{a(\cos\theta-e)}$$

$$m_{SD}\times m_{PS}=\frac{b(e-\cos\theta)}{a\sin\theta(1-e^2)}\times\frac{b\sin\theta}{a(\cos\theta-e)}$$

$$=\frac{-b^2}{a^2(1-e^2)}$$

$$=\frac{-b^2}{b^2}$$

$$=-1$$

$$\therefore \angle PSD=90^\circ$$

(3)

7

Question 13:

Outcomes assessed: E3

Targeted Performance Bands: E2-E3

Criteria	Marks
• Correctly derives the equation of the tangent	**2**
• Correctly finds gradient of tangent	**1**

a) Sample answer:

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

differentitate w.r.t x

$\frac{2x}{a^2}+\frac{2y}{b^2}\times\frac{dy}{dx}=0$

$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$

At $P(a\cos\theta, b\sin\theta)$

$\frac{dy}{dx}=-\frac{b\cos\theta}{a\sin\theta}$

$m_{\text{tangent}}=-\frac{b\cos\theta}{a\sin\theta}$ ✓

equation of tangent

$y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$

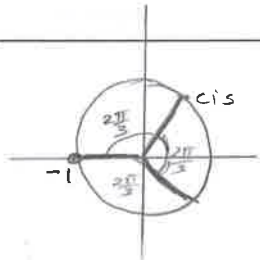
$bx\cos\theta+ay\sin\theta=ab(\sin^2\theta+\cos^2\theta)$

$bx\cos\theta+ay\sin\theta=ab$

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ ✓ (2)

13(b) (i)	The equation $x^3+2x+1=0$ has roots α,β,γ. $x=\alpha^{-1},\beta^{-1},\gamma^{-1}$ satisfies $\left(\frac{1}{x}\right)^3+2\left(\frac{1}{x}\right)+1=0$ $1+2x^2+x^3=0$ $\therefore x^3+2x^2+1=0$ has roots α^{-1}, β^{-1} and γ^{-1}	2 Marks: Correct answer. 1 Mark: Writes an equation in $\frac{1}{x}$ with roots α^{-1}, β^{-1} and γ^{-1}
13(b) (ii)	$x=\alpha^{-2},\beta^{-2},\gamma^{-2}$, $\alpha=\frac{1}{\sqrt{x}}$ satisfies $\left(\frac{1}{\sqrt{x}}\right)^3+2\left(\frac{1}{\sqrt{x}}\right)+1=0$ $x^0+2x^1+x^{\frac{3}{2}}=0$ $x^{\frac{3}{2}}=-2x-1$ $x^3=(-2x-1)^2$ $x^3-4x^2-4x-1=0$ $x^3-4x^2-4x-1=0$ has roots α^{-2}, β^{-2} and γ^{-2}.	2 Marks: Correct answer. 1 Mark: Writes an equation in $\frac{1}{\sqrt{x}}$ with roots α^{-2}, β^{-2} and γ^{-2}
13(c) (i)	$\dfrac{5x^3-3x^2+2x-1}{x^2(x^2+1)}=\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{cx+d}{x^2+1}$ $5x^3-3x^2+2x-1=a(x)(x^2+1)+b(x^2+1)+(cx+d)(x^2)$ $=ax^3+ax+bx^2+b+cx^3+dx^2$ $a+c=5$ (coefficients x^3) $b+d=-3$ (coefficients x^2) $a=2$ (coefficients x) $b=-1$ (constant) Therefore $c=3$ and $d=-2$ $\therefore a=2,\ b=-1,\ c=3$ and $d=-2$	2 Marks: Correct answer. 1 Mark: Makes some progress in finding a,b,c or d.
13(c) (ii)	$\displaystyle\int\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}dx=\int\frac{2}{x}-\frac{1}{x^2}+\frac{3x-2}{x^2+1}dx$ $\displaystyle=\int\frac{2}{x}-\frac{1}{x^2}+\frac{3x}{x^2+1}-\frac{2}{x^2+1}dx$ $\displaystyle=2\log_e x+\frac{1}{x}+\frac{3}{2}\log_e(x^2+1)-2\tan^{-1}x+C$	2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.
13(d)	$P(x)=x^4+ax^2+bx+28$ $P'(x)=4x^3+2ax+b$ Root at $x=2$ $P(2)=2^4+a\times2^2+b\times2+28=0$ $44+4a+2b=0 \quad (1)$ $P'(2)=4\times2^3+2a\times2+b=0$ $32+4a+b=0 \quad (2)$ P.T.O	2 Marks: Correct answer. 1 Mark: Finds either a or b. Alternatively uses the double root by differentiating the polynomial.

P.T.O for Question 14a & 14b

13(d)	Equation (1) – (2) $12+b=0$ $b=-12$ Substitute $b=-12$ into Equation (2) $32+4a-12=0$ $a=-5$ Therefore $a=-5$ and $b=-12$.	
14(c) (i)	$z^3+1=0$ $(z+1)(z^2-z+1)=0$ $\therefore z=-1$ or $z=\dfrac{1\pm\sqrt{(-1)^2-4\times1\times1}}{2}$ $=\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}$  OR $z=-1$ OR $z=\mathrm{cis}\,\frac{\pi}{3}$ OR $\mathrm{cis}\left(\frac{-\pi}{3}\right)$ (2)	2 Marks: Correct answer. 1 Mark: Finds one root.
14(c) (ii)	The two non-real cubic roots of -1 are $\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$. These roots satisfy the equation $(z^2-z+1)=0$. However since α is not real it also satisfies the equation. $\therefore \alpha^2-\alpha+1=0$ show $\alpha^2=\alpha-1$ (1)	1 Mark: Correct answer.
14(c) (iii)	$(1-\alpha)^6=(-\alpha^2)^6$ $=\alpha^{12}$ $=(\alpha^3)^4$ $=(-1)^4=1$ (1)	1 Mark: Correct answer.
14(d)	$x^2y+2x-2xy=0$ $2xy+x^2\dfrac{dy}{dx}+2-2y-2x\dfrac{dy}{dx}=0$ (Implicit differentiation) $\dfrac{dy}{dx}(x^2-2x)=2y-2xy-2$ $\dfrac{dy}{dx}=\dfrac{2y-2xy-2}{x^2-2x}$ for differentiating and getting $\frac{dy}{dx}$ on one side At (1, 2) $\dfrac{dy}{dx}=\dfrac{2\times2-2\times1\times2-2}{1^2-2\times1}=\dfrac{-2}{-1}=2$ $y-y_1=m(x-x_1)$ $y-2=2(x-1)$ $2x-y=0$ $y=2x$ (2)	2 Marks: Correct answer. 1 Mark: Finds the derivative using implicit differentiation.

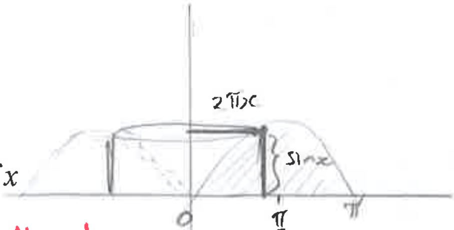
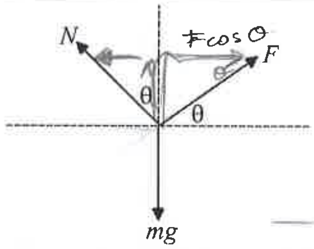
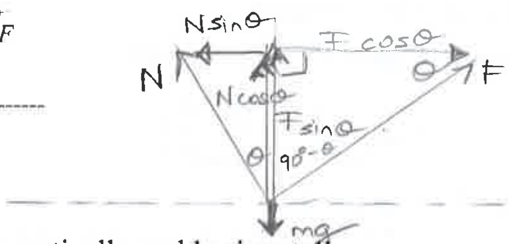
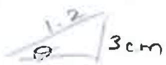
14(a) (i)	Sample answer: Tangent at $P: x+p^2y=2cp$ Tangent at $Q: x+q^2y=2cq$ subtract $(p^2-q^2)y=2c(p-q)$ $y=\dfrac{2c}{p+q}$ ✓ show $x+p^2\left(\dfrac{2c}{p+q}\right)=2cp$ $(p+q)x+2cp^2=2cp^2+2cpq$ $x=\dfrac{2cpq}{p+q}$ ✓ show $\therefore T\ is\ \left(\dfrac{2cpq}{p+q},\dfrac{2c}{p+q}\right)$ (2)	2 Marks: Correct answer. 1 Mark shows some understanding.
14(a) (ii)	M is $\left(\dfrac{c(p+q)}{2},\dfrac{c\left(\frac{1}{p}+\frac{1}{q}\right)}{2}\right)$ M is $\left(\dfrac{c(p+q)}{2},\dfrac{c(p+q)}{2pq}\right)$ ✓ finding the midpt $m_{OM}=\dfrac{\frac{c(p+q)}{2pq}-0}{\frac{c(p+q)}{2}-0}$ $=\dfrac{1}{pq}$ $m_{OT}=\dfrac{\frac{2c}{p+q}-0}{\frac{2cpq}{p+q}-0}$ $=\dfrac{1}{pq}$ (1) $m_{OM}=m_{OT}$ $\therefore O,T,M$ are collinear	1 Mark: Correct answer.

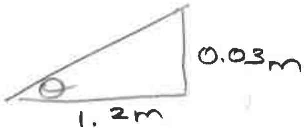
14aiii) If T, M and S are collinear then from part (ii) O, T, M and S are collinear ✓
O and S lie on $y=x$
$\therefore M$ lies on $y=x$ M $\left(\frac{c(p+q)}{2},\frac{c(p+q)}{2pq}\right)$ $x=y$

$\therefore \dfrac{c(p+q)}{2}=\dfrac{c(p+q)}{2pq}$ 2 Marks

$\therefore pq=1\Rightarrow q=\dfrac{1}{p}$ ✓

(2)

14(b)	Cylindrical shell – inner radius x, outer radius $x+\delta x$, height y. $\delta V = \pi\left[(x+\delta x)^2 - x^2\right]y$ $= \pi\left[2x\delta x + \delta x^2\right]y$ $= \pi(2x+\delta x)(\sin x)\delta x$  $V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} \pi(2x+\cancel{\delta x})\sin x\,\delta x$ $= 2\pi\int_0^{\pi} x\sin x\,dx$ Integration by Parts. $u = x$, $u' = 1$, $v' = \sin x$, $v = -\cos x$ $= 2\pi\left(\left[-x\cos x\right]_0^{\pi}\right) - 2\pi\int_0^{\pi} -\cos x\,dx$ $= 2\pi(\pi - 0) + 2\pi\left[\sin x\right]_0^{\pi}$ $= 2\pi^2$ 4	4 Marks: Correct answer. 3 Marks: Correct integral for the volume of the solid. 2 Marks: Correct expression for δV. 1 Mark: Determines the radius or height of the cylindrical shell.
15(a) (i)	  Resolving the forces vertically and horizontally $N\cos\theta + F\sin\theta - mg = 0 \quad (1)$ $N\sin\theta - F\cos\theta = \dfrac{mv^2}{r} \quad (2)$ Equation (1) $\times\sin\theta$ – Equation (2) $\times\cos\theta$ $F(\underbrace{\sin^2\theta + \cos^2\theta}_{=1}) = mg\sin\theta - \dfrac{mv^2\cos\theta}{r}$ eliminating N $F = \dfrac{m\cos\theta}{r}\left(gr\tan\theta - v^2\right)$ $F = \dfrac{2000\times\cos\theta}{1,000}\left(10\times 1,000\tan\theta - v^2\right)$ $= 2\cos\theta(10,000\tan\theta - v^2)$ 3	3 Marks: Correct answer. 2 Marks: Eliminates N to give an expression for F. 1 Mark: Resolves the forces vertically and horizontally.
15(a) (ii)	Now $F = 0$ when $v = v_0$. $F = 2\cos\theta(10,000\tan\theta - v^2)$ $0 = 2\cos\theta(10,000\tan\theta - v_0^{\ 2})$ $v_0^{\ 2} = 10,000\tan\theta$  Now $v = 0.80\times v_0$ $v^2 = 0.64\times v_0^{\ 2}$ $= 0.64\times(10,000\tan\theta) = 6,400\tan\theta$	2 Marks: Correct answer. 1 Mark: Finds the correct expression for v^2 or shows some understanding.

15(a) (ii)	$F = 2\cos\theta(10,000\tan\theta - v^2)$ $= 2\cos\theta(10,000\tan\theta - 6,400\tan\theta)$ $= 2\cos\theta \times 3,600\tan\theta$ $= 7,200\sin\theta$ Also $\sin\theta = \frac{0.03}{1.2}$ $= 0.025$  $F = 7,200 \times 0.025$ $= 180\ N$ Therefore the lateral thrust is 180 N (2)	
15(b) (i)	The complex number w is a root of $P(z) = 0$ Therefore $P(w) = w^8 - \frac{5}{2}w^4 + 1 = 0$ Now $P(iw) = (iw)^8 - \frac{5}{2}(iw)^4 + 1$ $= i^8w^8 - \frac{5}{2}i^4w^4 + 1$ $= w^8 - \frac{5}{2}w^4 + 1 = 0$ show Also $P(\frac{1}{w}) = (\frac{1}{w})^8 - \frac{5}{2}(\frac{1}{w})^4 + 1$ $= \frac{1}{w^8}\left(1 - \frac{5}{2}w^4 + w^8\right)$ show $= \frac{1}{w^8} \times 0 = 0$ Therefore iw and $\frac{1}{w}$ are roots of $P(z) = 0$. (2)	2 Marks: Correct answer. 1 Mark: Shows iw or $\frac{1}{w}$ are roots of the polynomial or alternatively shows some understanding.
15(b) (ii)	$z^8 - \frac{5}{2}z^4 + 1 = 0$ $2z^8 - 5z^4 + 2 = 0$ $(2z^4 - 1)(z^4 - 2) = 0$ $\therefore z^4 = \frac{1}{2}$ or $z^4 = 2$ Roots are $z = \pm\sqrt[4]{2}$ or $z = \pm\frac{1}{\sqrt[4]{2}}$. (2)	2 Marks: Correct answer. 1 Mark: Factorises the polynomial.

15(c) (i)	$\text{LHS} = \sin x + \sin 3x$ $= \sin(2x - x) + \sin(x + 2x)$ show $= \sin 2x \cos x - \cos 2x \sin x + \sin x \cos 2x + \cos x \sin 2x$ $= 2\sin 2x \cos x$ $= \text{RHS}$ (1)	1 Mark: Correct answer.
15(c) (ii)	$\sin x + \sin 2x + \sin 3x = 0$ $\sin 2x + 2\sin 2x \cos x = 0$ from part (i) $\sin 2x(2\cos x + 1) = 0$ $\sin 2x = 0$ or $2\cos x + 1 = 0$ $x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ (2)	2 Marks: Correct answer. 1 Mark: Uses the result in part (i) and factorises.
15(d)	$F = \frac{m}{x^3}(6 - 10x)$ $ma = \frac{m}{x^3}(6 - 10x)$ $v\frac{dv}{dx} = \frac{6}{x^3} - \frac{10}{x^2}$ $\int v\,dv = \int (\frac{6}{x^3} - \frac{10}{x^2})dx$ $\frac{1}{2}v^2 = (\frac{6x^{-2}}{-2} - \frac{10x^{-1}}{-1}) + C$ $\frac{1}{2}v^2 = (\frac{-3}{x^2} + \frac{10}{x}) + C$ When $v = 0$ and $x = 1$ $\frac{1}{2}0^2 = (\frac{-3}{1^2} + \frac{10}{1}) + C$ $C = -7$ $\frac{1}{2}v^2 = (\frac{-3}{x^2} + \frac{10}{x}) - 7$ $v^2 = (\frac{-6}{x^2} + \frac{20}{x}) - 14$ $= \frac{-6 + 20x - 14x^2}{x^2}$ $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$ (3)	3 Marks: Correct answer. 2 Marks: Integrates and uses the initial conditions to find an expression for $\frac{1}{2}v^2$. 1 Mark: Uses $v\frac{dv}{dx}$ for acceleration.

16(a)	$z = x + iy$ $\operatorname{Im}(z+i) = \lvert z+1\rvert \Rightarrow y+1 = \sqrt{(x+1)^2 + y^2}$ $\operatorname{Im}(x + i(y+1))$ // $x+iy+1$ $(y+1)^2 = (x+1)^2 + y^2$ square both sides $2y = x^2 + 2x$ $y = \frac{1}{2}\left[(x+1)^2 - 1\right]$ minimum value of $y = -\frac{1}{2}$ minimum value of $\operatorname{Im}(z+i) = \frac{1}{2}$ $\therefore$ minimum value of $\lvert z+1\rvert = \frac{1}{2}$	3 Marks: Correct answer. 2 Marks: Makes significant progress. 1 Mark: Uses the sum of binomial square to obtain the required expression.
16(b) (i)	$I_n = \int_0^1 (1-x^r)^n \, dx$ $u = (1-x^r)^n$, $u' = n(1-x^r)^{n-1}$ $v' = 1$, $v = x$ $= \left[x(1-x^r)^n\right]_0^1 - n\int_0^1 x(1-x^r)^{n-1}(-rx^{r-1})dx$ IBP $\rightarrow rn\int_0^1 x^r(1-x^r)^{n-1}$ $= 0 - nr\int_0^1 \underbrace{[(1-x^r)-1]}_{\text{from } -x^r}(1-x^r)^{n-1}dx$ $= 0 - nr\int_0^1 (1-x^r)^n - (1-x^r)^{n-1}dx$ $= nr(-I_n + I_{n-1})$ $(nr+1)I_n = nrI_{n-1}$ $I_n = \frac{nr}{nr+1}I_{n-1}$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up the integration and shows some understanding.
16(b) (ii)	For $r = \frac{3}{2}$ and $n = 3$ $I_3 = \frac{3\times\frac{3}{2}}{3\times\frac{3}{2}+1}\times I_2$, $I_2 = \frac{2\times\frac{3}{2}}{2\times\frac{3}{2}+1}\times I_1$, $I_1 = \frac{1\times\frac{3}{2}}{1\times\frac{3}{2}+1}\times I_0$ But $I_0 = \int_0^1 (1-x^r)^0 dx = \int_0^1 1dx = 1$ $I_3 = \frac{3\times\frac{3}{2}}{3\times\frac{3}{2}+1}\times\frac{2\times\frac{3}{2}}{2\times\frac{3}{2}+1}\times\frac{1\times\frac{3}{2}}{1\times\frac{3}{2}+1}\times 1$ $= \frac{81}{220}$	2 Marks: Correct answer.

(c) (iii) To find the time it takes to reach maximum height,

$$\frac{dv}{dt} = -g(1+k)$$

Integrating with respect to t:

$$v = -g(1+k)t + c$$

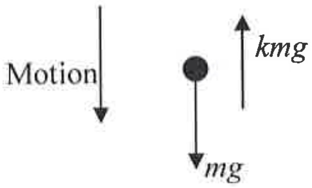
When $t = 0, v = u \Rightarrow c = u$

At the maximum height, $v = 0, t = t_1$

$$\therefore\ 0 = -g(1+k)t_1 + u$$

$$\therefore\ t_1 = \frac{u}{g(1+k)}$$

On the downward motion:



The equation of motion is:

$$m\ddot{x} = mg - kmg$$

$$\therefore\ \ddot{x} = \frac{dv}{dt} = g(1-k)$$

Integrating with respect to t:

$$v = g(1-k)t + c$$

Initially, $v = 0, t = 0 \Rightarrow c = 0$

$$\therefore\ v = g(1-k)t$$

$$\therefore \int \frac{dx}{dt}\,dt = g(1-k)\int t\,dt$$

$$x = g(1-k)\frac{t^2}{2} + c$$

when $t=0, x=0 \therefore c=0$

$$\therefore\ x = g(1-k)\frac{t^2}{2}$$

when $x = H, t = t_2$

$$\therefore\ H = g(1-k)\frac{t_2^2}{2} \quad --- (2)$$

subst (1) into (2)

$$\frac{u^2}{2g(1+k)} = g(1-k)\frac{t_2^2}{2}$$

$$\therefore\ t_2^2 = \frac{u^2}{2g(1+k)} \times \frac{2}{g(1-k)}$$

$$t_2^2 = \frac{u^2}{g^2(1-k^2)}$$

$$t_2 = \frac{u}{g\sqrt{1-k^2}}$$

$\therefore$ Total time $= t_1 + t_2$

$$= \frac{u}{g(1+k)} + \frac{u}{g\sqrt{1-k^2}}$$

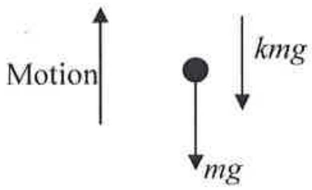
$$= \frac{u}{g}\left\{\frac{1}{1+k} + \frac{1}{\sqrt{1-k^2}}\right\}$$

as required.

(3)

Question 16c :

(c) (i)



Resolving forces in the direction of the motion:

$$m\ddot{x} = -(mg + kmg)$$

$$\ddot{x} = -g(1+k) \text{ as required.}$$

(1)

(c) (ii) Using $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -g(1+k)$

Integrating both sides with respect to x:

$$\frac{1}{2}v^2 = -g(1+k)x + c$$

Initially, $x = 0, v = u \Rightarrow c = \frac{1}{2}u^2$

$$\therefore \frac{1}{2}v^2 = -g(1+k)x + \frac{1}{2}u^2 \quad \ldots. (1)$$

At the maximum height, $v = 0, x = H$,
Substituting into (1)

OR

$$v \cdot \frac{dv}{dx} = -g(1+k)$$

$$\frac{dv}{dx} = \frac{-g(1+k)}{v}$$

$$\frac{dx}{dv} = \frac{v}{-g(1+k)}$$

$$-g(1+k)\int \frac{dx}{dv}\,dv = \int v\,dv$$

$$-g(1+k)x = \frac{v^2}{2} + c$$

$x = 0, v = u$

$$0 = \frac{u^2}{2} + c$$

$$\therefore c = -\frac{1}{2}u^2$$

$$\therefore \frac{1}{2}v^2 - \frac{1}{2}u^2 = -g(1+k)x$$

$$\frac{1}{2}v^2 = -g(1+k)x + \frac{1}{2}u^2$$

At max height, $v = 0$ and $x = H$

$$0 = -g(1+k)H + \frac{1}{2}u^2$$

$$\therefore g(1+k)H = \frac{1}{2}u^2$$

$$\therefore H = \frac{u^2}{2g(1+k)} \quad --- 1$$

(3)