TRINITY GRAMMAR SCHOOL

Mathematics Faculty

(NESA Student Number)

2025

Year 12

Mathematics Extension 2

TRIAL EXAMINATION

ASSESSMENT TASK 4

Date of Assessment Task: Monday, 18 August 2025

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Erasable pens and all electronic devices (or timekeepers), unless otherwise advised, are not permitted in this Assessment
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- *Write your NESA Student Number at the top of this page*

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

- **Year 12 HSC Assessment Weighting:** 30%

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 If $w = a - 2ai$, which of the following is equivalent to i^3w?

A. $-2a - ai$

B. $-a - 2ai$

C. $2a - ai$

D. $-a + 2ai$

2 The sign on a door reads as follows: **"If you are not permitted, you do not have access."**

Which of the following statements is logically equivalent to the one above?

A. If you do not have access, then you are not permitted.

B. If you have access, then you are permitted.

C. If you are not permitted, then you have access.

D. If you are permitted, then you have access.

3 Which of the following is equivalent to $\left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^3$?

A. $2 + 2i$

B. $2 - 2i$

C. $-2 + 2i$

D. $-2 - 2i$

4 The acceleration of a body is given by $a = 3v$, where v is the velocity of the body. Which of the following equations is correct given the velocity is $4\ \text{ms}^{-1}$ when the body is 2 m to the right of the origin?

A. $v = 2x$

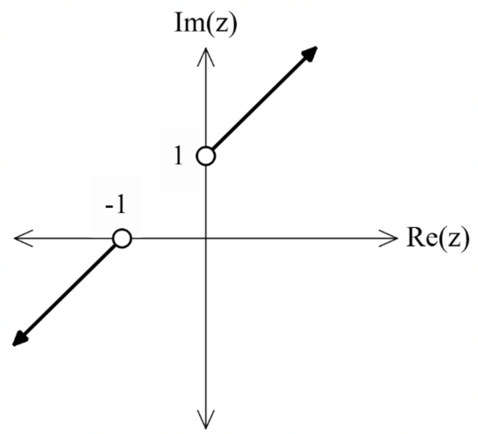
B. $v = 4x - 4$

C. $v = 3x - 2$

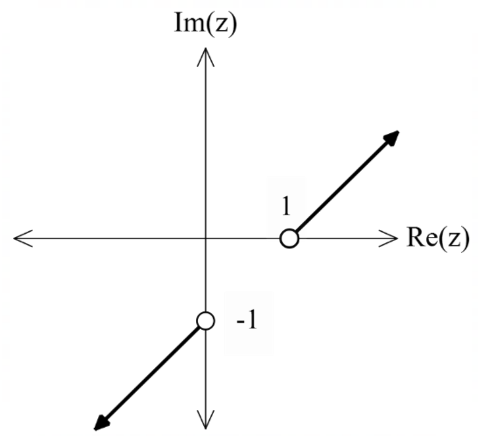
D. $v = -3x + 10$

5 Which of the following Argand diagrams represents the solution to the equation $\text{Arg}(z+i) = \text{Arg}(z-1)$?

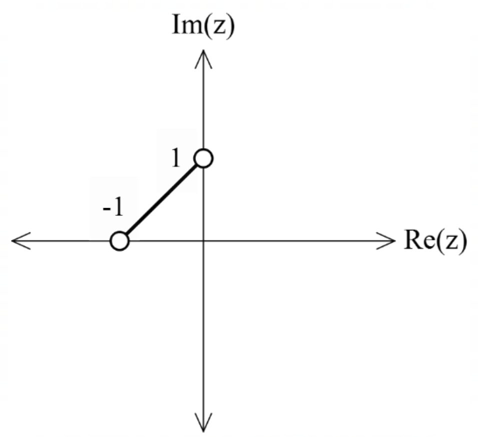
A.



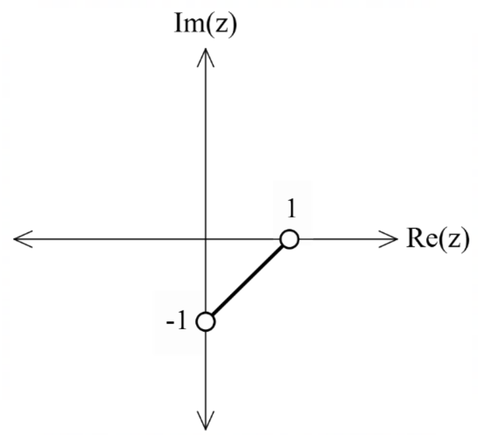
B.



C.



D.



6 Which expression is equal to $\int x\sqrt{1+x}\,dx$?

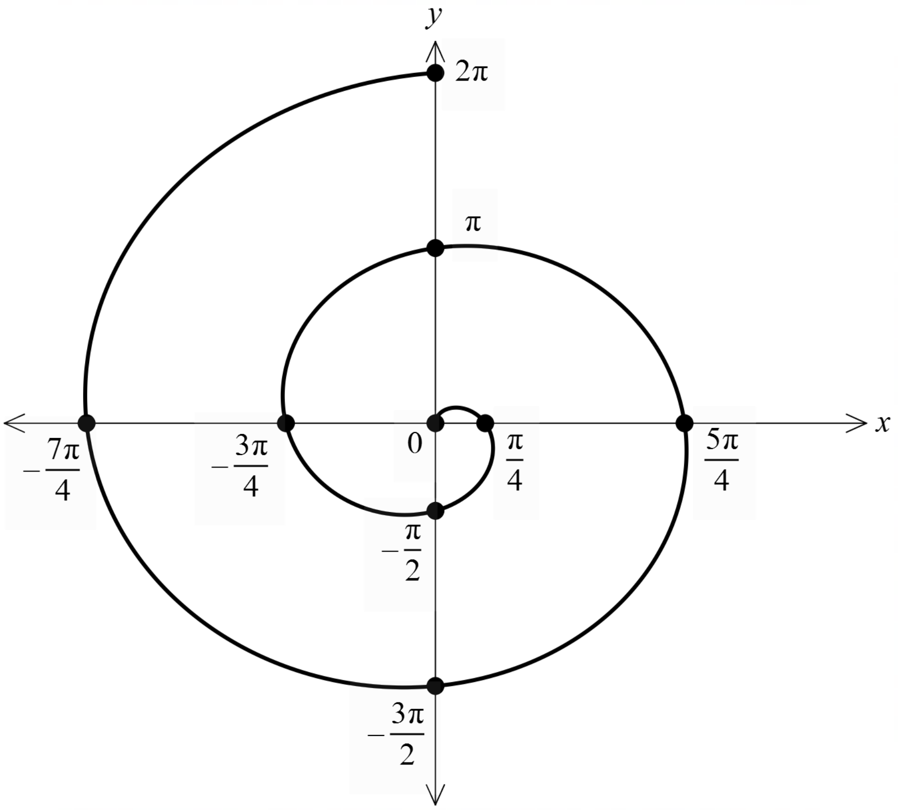
A. $\frac{2}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

B. $-\frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

C. $-\frac{2}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

D. $\frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

7 The spiral below describes the position (x, y) of a particle for time $0 \le t \le T$.



Which of the following describes the spiral?

A. $x = \frac{t}{2}\sin t,\ y = \frac{t}{2}\cos t,\ T = 4\pi$

B. $x = t\sin(2t),\ y = t\cos(2t),\ T = 4\pi$

C. $x = t\sin t,\ y = t\cos t,\ T = 2\pi$

D. $x = \frac{t}{2}\sin t,\ y = -\frac{t}{2}\cos t,\ T = 4\pi$

8 What is the perimeter of a regular n-gon formed by the solutions of $z^n = 1$, for $n \ge 3$?

A. $2n\cos\left(\frac{2\pi}{n}\right)$

B. $2n\cos\left(\frac{\pi}{n}\right)$

C. $2n\sin\left(\frac{2\pi}{n}\right)$

D. $2n\sin\left(\frac{\pi}{n}\right)$

9 Which of the following is the negation of $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ : xy = 1$?

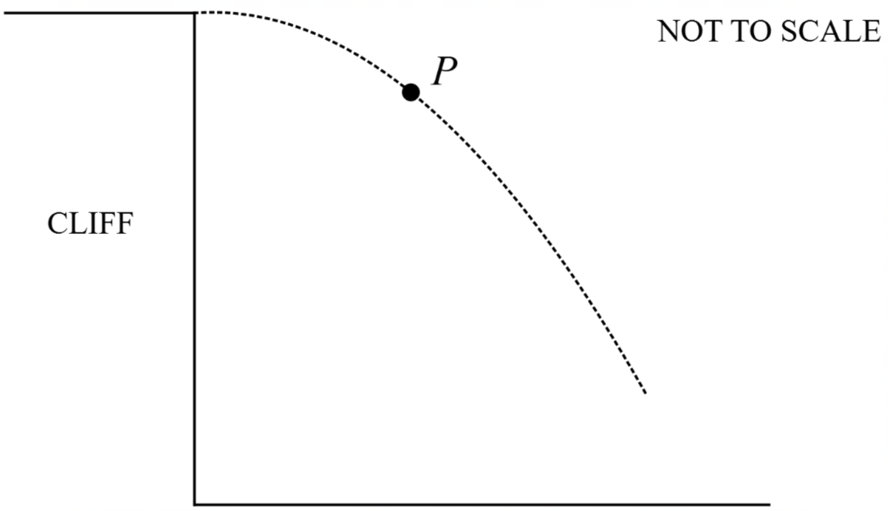
A. $\exists x, y \in \mathbb{R}^+ : xy \neq 1$

B. $\forall x, y \in \mathbb{R}^+, xy \neq 1$

C. $\exists x \in \mathbb{R}^+ : \forall y \in \mathbb{R}^+, xy \neq 1$

D. $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ : xy \neq 1$

10 On the diagram below, the point P represents the position of a unit mass that has been projected horizontally from the top of a cliff. The air resistance is $\vec{R}$, the gravitational force is $\vec{G}$, the position of the mass is $\vec{r}$, and the velocity is $\vec{v}$.



Which of the following is consistent with the motion of the particle at P?

A. $\dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|} < \dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|} < \dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|}$

B. $\dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|} < \dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|} < \dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|}$

C. $\dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|} < \dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|} < \dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|}$

D. $\dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|} < \dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|} < \dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|}$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet

(a) Let $z_1 = -3 - 2i$, $z_2 = 2 - i$, and $z_3 = i\left(z_1 - \overline{z_2}\right)$. Calculate $|z_3|$. **3**

(b) Sketch the region of the Argand diagram described by $iz - i\overline{z} > 4$. **2**

(c) Consider the line $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

(i) Show that the point $(0, 7)$ lies on $\underset{\sim}{r}$. **1**

(ii) Show that the line through the points $A(3, 2)$ and $B(8, 4)$ is perpendicular to $\underset{\sim}{r}$. **2**

(d) It is given that $2 + 3i$ is a zero of $P(z) = z^3 - 2z^2 + 5z + 26$. Solve $P(z) = 0$. **2**

(e) Consider the following statement: for $x \in \mathbb{R}$, $x^2 - x > 0 \Rightarrow x < 0$.

(i) Show that the statement is false. **1**

(ii) State the converse of the statement. **1**

(iii) Determine whether the converse is true. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Consider the complex number $z = 1 + \sqrt{3}i$.

(i) Write z in mod-arg form. **1**

(ii) Evaluate z^6. **1**

(iii) Find all values of n for which z^n is purely real. **1**

(b) Consider the points $A(1, 0, 1)$ and $B(0, 2, 2)$. Find the projection of $\overrightarrow{AB}$ onto the xy-plane. **2**

(c) Use partial fractions to show that $\displaystyle\int_0^1 \frac{x+7}{(x-2)(x+1)}\,dx = -5\ln 2$. **3**

(d) Consider the line $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$.

(i) Find the Cartesian equation of the line l that passes through the point $(-2, 1, 4)$ and is parallel to $\underset{\sim}{r}$. **2**

(ii) Determine where the line l meets the xz-plane. **2**

(e) The motion of a section of a particular roller coaster track can be modelled by simple harmonic motion. The force acting on the roller coaster is given by $F = -kx$, where $k \in \mathbb{R}$, and $x(t)$ is the displacement of the roller coaster from the centre of motion at time t.

(i) Show that $x = A\cos\left(\sqrt{\frac{k}{m}}\,t\right)$ satisfies $F = -kx$, where m is the mass of the roller coaster and A is the amplitude of the motion. **1**

(ii) The mechanical energy of the roller coaster is given by $M(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, where v is the velocity of the rollercoaster at time t. Show that $M(t)$ is proportional to A^2. **1**

(iii) Show that the maximum velocity of the roller coaster is $v_{max} = -A\sqrt{\frac{k}{m}}$. **1**

End of Question 12

Question 13 (16 marks) Use a SEPARATE writing booklet

(a) Find the value of k such that the points $(-2, k, 3)$, $(1, -1, 3)$, and $(3, 5, 3)$ are collinear. **3**

(b) Let $z = x + iy$, $z_1 = 6 + 2i$, and $z_2 = 2 + 2i$, where $x, y \in \mathbb{R}$.

Find $\dfrac{z - z_1}{z - z_2}$ with a real denominator. **2**

(c) Sketch the following on separate Argand diagrams.

(i) $|z - 1| = 4$ **2**

(ii) $-\dfrac{\pi}{3} < \arg(z + i) \leq \dfrac{\pi}{4}$ **3**

(d) Consider the spheres S_1 and S_2. **3**

$$S_1 : x^2 - 2x + y^2 + 4y + z^2 - 4z = 0$$
$$S_2 : x^2 + 10x + y^2 + z^2 + 2z + 10 = 0$$

Show that S_1 and S_2 intersect at a point.

(e) Let $z_2 = 1 + i$ and, for $n > 2$, let $z_n = z_{n-1}\left(1 + \dfrac{i}{|z_{n-1}|}\right)$. **3**

Use mathematical induction to prove that $|z_n| = \sqrt{n}$ for all integers $n \geq 2$.

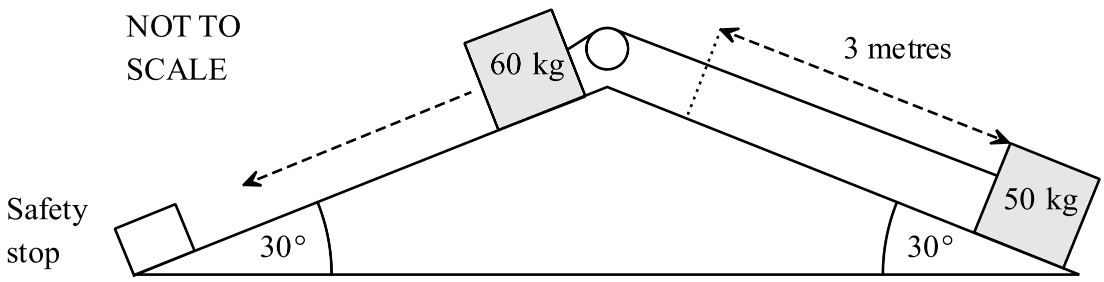
End of Question 13

Question 14 (16 marks) Use a SEPARATE writing booklet

(a) Prove $a^3 + b^3 > ab(a+b)$ for $a, b \in \mathbb{R}^+$, $a \neq b$. **2**

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{4x-3}{4x^2-4x+5}\,dx$. **4**

(c) A pulley system has been set up using a 60 kg counterweight to move a 50 kg box of tiles up on a roof. The roof is sloped at 30º to the horizontal on both sides and the safety stop ensures the box is pulled exactly 3 metres up the roof.



Both masses are initially at rest and the acceleration due to gravity is $10\ \text{ms}^{-2}$. Air resistance is negligible but the frictional force opposing the direction of each mass is equal to half the normal force for each box, respectively.

(i) Determine whether the pulley system will work as intended. **3**

(ii) Find the maximum weight of the box that can be pulled using this system. **2**

(d) The complex number z lies on the unit circle with $\text{Arg}(z) = \theta$.

(i) Show that $\dfrac{1+z}{1-z} = i\cot\left(\dfrac{\theta}{2}\right)$. **2**

(ii) Hence, or otherwise, solve $\left(\dfrac{w-1}{w+1}\right)^8 = 1$, where w is a complex number. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) For $p, q \in \mathbb{R}^+$, explain why $\dfrac{1}{p-2q} \neq \dfrac{1}{p} - \dfrac{1}{2q}$, where $p \neq 2q$. **3**

(b) Consider the complex number $z = \cos\theta + i\sin\theta$.

(i) Expand and simplify $(z^3 - 1)(z^6 + z^3 + 1)$. **1**

(ii) Show that **3**

$$z^6 + z^3 + 1 = \left(z^2 - 2z\cos\left(\frac{2\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{8\pi}{9}\right) + 1\right).$$

(iii) Hence, show that **2**

$$2\cos(3\theta) + 1 = 8\left(\cos\theta - \cos\left(\frac{2\pi}{9}\right)\right)\left(\cos\theta - \cos\left(\frac{4\pi}{9}\right)\right)\left(\cos\theta - \cos\left(\frac{8\pi}{9}\right)\right).$$

(c) Double factorials are defined as follows:

$n!! = n(n-2)(n-4)...6 \times 4 \times 2$, where n is an even positive integer

e.g. $10!! = 10 \times 8 \times 6 \times 4 \times 2$

$n!! = n(n-2)(n-4)...5 \times 3 \times 1$, where n is an odd positive integer

e.g. $11!! = 11 \times 9 \times 7 \times 5 \times 3 \times 1$

(i) Show that $(2k)!! = k!2^k$ for even integers $n = 2k$. **1**

(ii) Hence, prove that $(2k + 1)!! = \dfrac{(2k+1)!}{k!2^k}$ for odd integers $n = 2k + 1$. **2**

(iii) Hence, or otherwise, prove that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n}(x)\, dx = \frac{(2n)!\pi}{2^{2n+1}(n!)^2}$. **3**

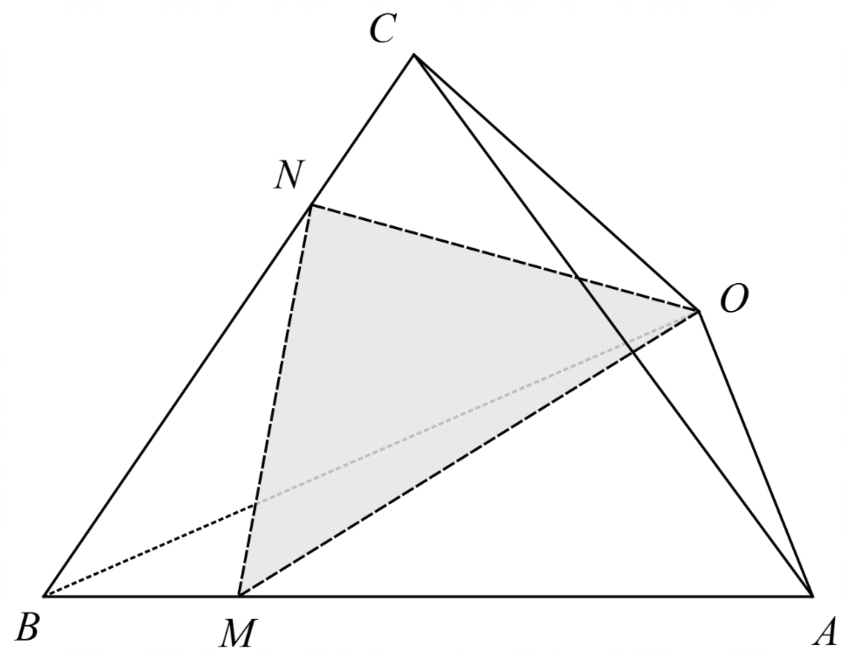
End of Question 15

Question 16 (14 marks) Use a SEPARATE writing booklet

(a) A unit mass is projected vertically upwards with an initial speed of S ms^{-1} in a resisting medium that exerts a force of kv opposing the direction of motion, where k is a positive constant and the acceleration due to gravity is g ms^{-2}. **5**

Find an expression for the velocity, $v(t)$, independent of S, given that the mass reaches maximum height after T seconds.

(b) A regular triangular pyramid of side length 1, $OABC$, is shown below. M and N divide the sides AB and BC in the ratio $x : 1 - x$, respectively. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$.



(i) Let $f(x) = \dfrac{1 + x - x^2}{1 - x + x^2}$, where $0 < x < 1$. Prove that $1 < f(x) \leq \dfrac{5}{3}$. **3**

(ii) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{c} \cdot \underset{\sim}{a} = \dfrac{1}{2}$. **1**

(iii) Show that $\left|\overrightarrow{ON}\right| = \sqrt{x^2 - x + 1}$. **2**

(iv) Determine the range of values of $\angle MON$. **3**

End of Question 16

End of Paper

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TRINITY GRAMMAR SCHOOL

Mathematics Faculty

S	O	L	U	T	I	O	N	S

(NESA Student Number)

2025

Year 12

Mathematics Extension 2

TRIAL EXAMINATION

ASSESSMENT TASK 4

Date of Assessment Task: Monday, 18 August 2025

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Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
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- **Year 12 HSC Assessment Weighting:** 30%

Section I

10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

A	B	C	C	B	D	A	D	C	B

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1 If $w = a - 2ai$, which of the following is equivalent to i^3w?

A. $-2a - ai$

B. $-a - 2ai$

C. $2a - ai$

D. $-a + 2ai$

$w = a - 2ai$

$i^3 w = -iw$

$= -ai - 2a$

$\therefore$ A

2 The sign on a door reads as follows: **"If you are not permitted, you do not have access."**

Which of the following statements is logically equivalent to the one above?

A. If you do not have access, then you are not permitted.

B. If you have access, then you are permitted.

C. If you are not permitted, then you have access.

D. If you are permitted, then you have access.

B, by contrapositive.

3 Which of the following is equivalent to $\left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^3$?

A. $2 + 2i$

B. $2 - 2i$

C. $-2 + 2i$

D. $-2 - 2i$

$\left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^3 = 2\sqrt{2}\, e^{\frac{3\pi}{4}i}$

$= 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= -2 + 2i$

$\therefore$ C

4 The acceleration of a body is given by $a = 3v$, where v is the velocity of the body. Which of the following equations is correct given the velocity is 4 ms^{-1} when the body is 2 m to the right of the origin?

A. $v = 2x$

B. $v = 4x - 4$

C. $v = 3x - 2$

D. $v = -3x + 10$

$a = 3v$

$v\dfrac{dv}{dx} = 3v$

$\int 1\, dv = \int 3\, dx$

$v = 3x + C$

$4 = 3 \times 2 + C$

$C = -2$

$\therefore v = 3x - 2$

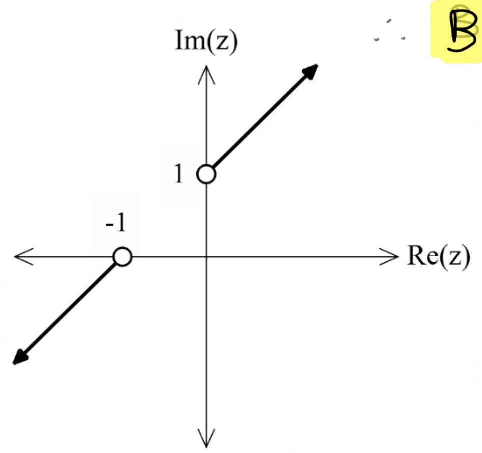
$\therefore$ C

5 Which of the following Argand diagrams represents the solution to the equation $\text{Arg}(z + i) = \text{Arg}(z - 1)$?

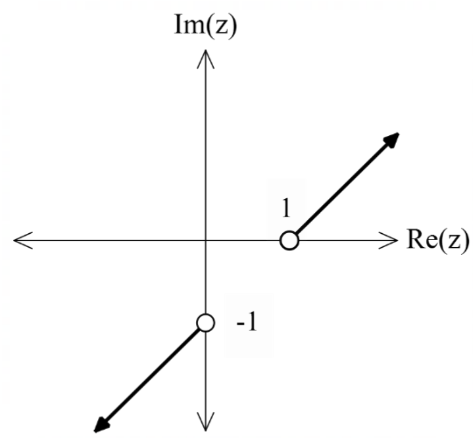
angle to z from (0,-1) and from (1,0) is the same

$\therefore$ B

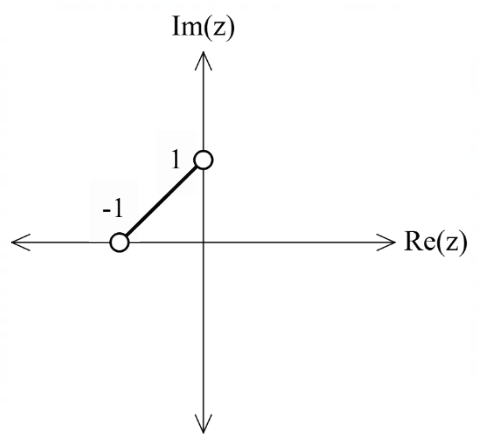
A.



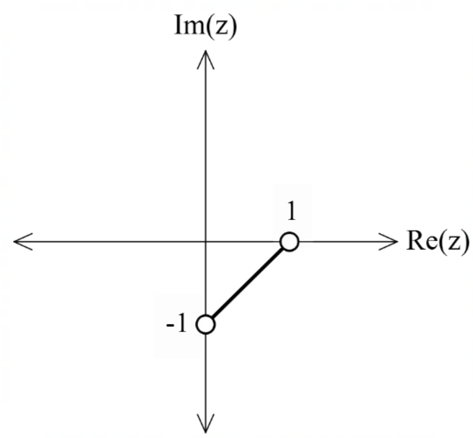
B.



C.



D.



6 Which expression is equal to $\int x\sqrt{1+x}\,dx$?

A. $\frac{2}{5}(1+x)^{\frac{5}{2}}+\frac{2}{3}(1+x)^{\frac{3}{2}}+C$

B. $-\frac{2}{5}(1+x)^{\frac{5}{2}}-\frac{2}{3}(1+x)^{\frac{3}{2}}+C$

C. $-\frac{2}{5}(1+x)^{\frac{5}{2}}+\frac{2}{3}(1+x)^{\frac{3}{2}}+C$

D. $\frac{2}{5}(1+x)^{\frac{5}{2}}-\frac{2}{3}(1+x)^{\frac{3}{2}}+C$

$\int x\sqrt{1+x}\,dx$

$u = 1+x$

$x = u-1$

$\frac{dx}{du} = 1$

$= \int (u-1)\sqrt{u}\,du$

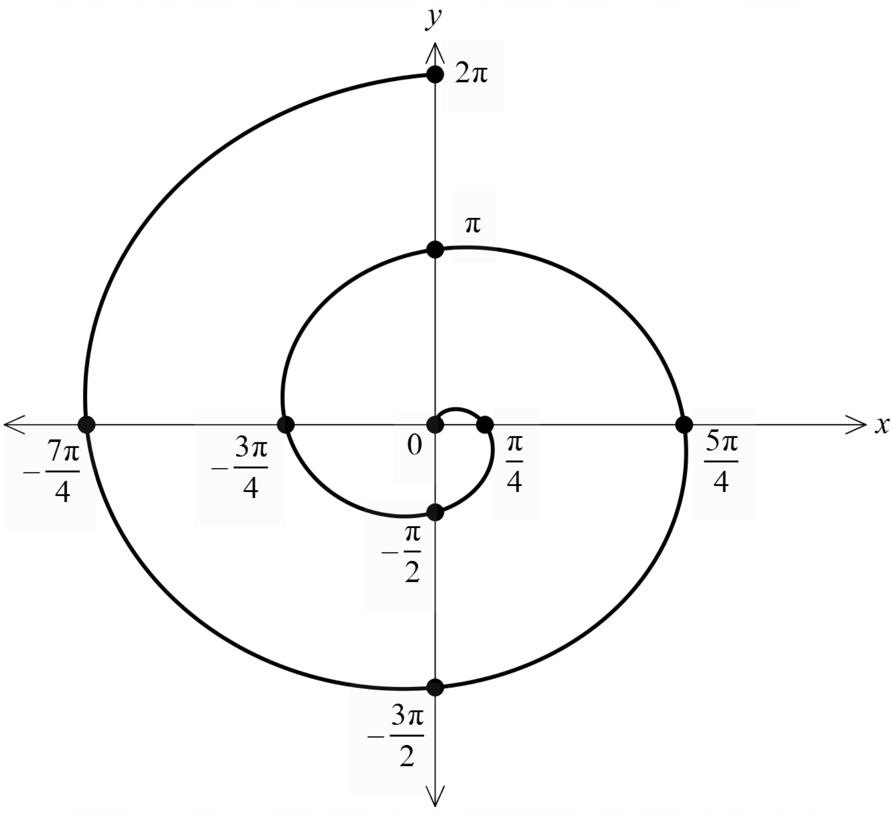
$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

$\therefore$ D

7 The spiral below describes the position (x, y) of a particle for time $0 \le t \le T$.



Which of the following describes the spiral?

A. $x = \frac{t}{2}\sin t,\ y = \frac{t}{2}\cos t,\ T = 4\pi$

B. $x = t\sin(2t),\ y = t\cos(2t),\ T = 4\pi$

C. $x = t\sin t,\ y = t\cos t,\ T = 2\pi$

D. $x = \frac{t}{2}\sin t,\ y = -\frac{t}{2}\cos t,\ T = 4\pi$

Test points:

$(\frac{\pi}{4}, 0)$ only works for A

$\therefore$ A

8 What is the perimeter of a regular n-gon formed by the solutions of $z^n = 1$, for $n \geq 3$?

A. $2n\cos\left(\frac{2\pi}{n}\right)$

B. $2n\cos\left(\frac{\pi}{n}\right)$

C. $2n\sin\left(\frac{2\pi}{n}\right)$

D. $2n\sin\left(\frac{\pi}{n}\right)$

angle between roots is $\frac{2\pi}{n}$, $r=1$

edge length $= |z_1 - z_0|$

$= |e^{i\frac{2\pi}{n}} - 1|$

$= |e^{i\frac{\pi}{n}}||e^{i\frac{\pi}{n}} - e^{-i\frac{\pi}{n}}|$

$= |1 \times |2i\sin\frac{\pi}{n}||$

$= 2\sin\frac{\pi}{n}$

n-gon so multiply by n $\therefore$ D

9 Which of the following is the negation of $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ : xy = 1$?

A. $\exists x, y \in \mathbb{R}^+ : xy \neq 1$

B. $\forall x, y \in \mathbb{R}^+, xy \neq 1$

C. $\exists x \in \mathbb{R}^+ : \forall y \in \mathbb{R}^+, xy \neq 1$

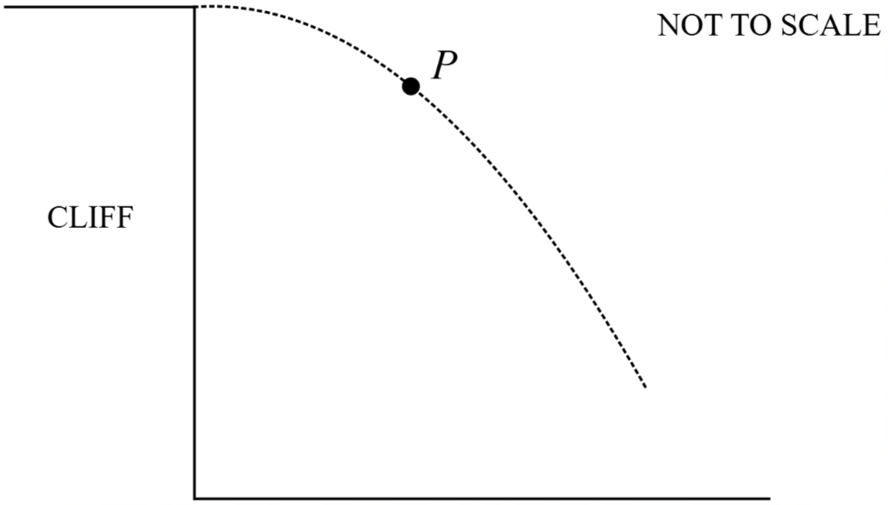
D. $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ : xy \neq 1$

negation swaps $\forall$ and $\exists$

i.e. $\exists x \in \mathbb{R}^+$ s.t. $\forall y \in \mathbb{R}^+$, $xy \neq 1$

$\therefore$ C

10 On the diagram below, the point P represents the position of a unit mass that has been projected horizontally from the top of a cliff. The air resistance is $\vec{R}$, the gravitational force is $\vec{G}$, the position of the mass is $\vec{r}$, and the velocity is $\vec{v}$.



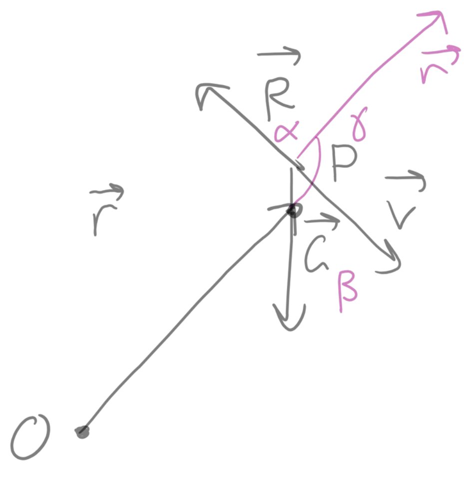
Which of the following is consistent with the motion of the particle at P?

A. $\dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|} < \dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|} < \dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|}$

B. $\dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|} < \dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|} < \dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|}$

C. $\dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|} < \dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|} < \dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|}$

D. $\dfrac{\vec{R}\cdot\vec{r}}{|\vec{R}||\vec{r}|} < \dfrac{\vec{G}\cdot\vec{v}}{|\vec{G}||\vec{v}|} < \dfrac{\vec{G}\cdot\vec{r}}{|\vec{G}||\vec{r}|} < \dfrac{\vec{R}\cdot\vec{v}}{|\vec{R}||\vec{v}|}$



Label the cosines of each angle

$\cos\alpha$

- $\vec{R}$ and $\vec{r}$ $= \cos\pi = -1$
- $\vec{R}$ and $\vec{v}$ $= \cos\beta$
- $\vec{G}$ and $\vec{r}$ $= \cos\gamma$

by inspection

$\beta < \alpha < \gamma < \pi$

$\therefore \cos\pi < \cos\gamma < \cos\alpha < \cos\beta$

$\therefore$ B

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet

(a) Let $z_1 = -3 - 2i$, $z_2 = 2 - i$, and $z_3 = i\left(z_1 - \overline{z_2}\right)$. Calculate $|z_3|$. **3**

$z_3 = i(-3-2i-(2+i))$

$= -5i+3$

$|z_3| = \sqrt{25+9}$

$= \sqrt{34}$

or $|z_3| = |i(z_1 - \bar{z}_2)|$

$= |z_1 - \bar{z}_2|$

1 for sub

1 for recognising this

1 for $|z_3|$

1 mark for substitution

1 mark for z_3

1 mark for $|z_3|$

(b) Sketch the region of the Argand diagram described by $iz - i\bar{z} > 4$. **2**

$iz - i\bar{z} > 4$ let $z = x+iy$, $x, y \in \mathbb{R}$

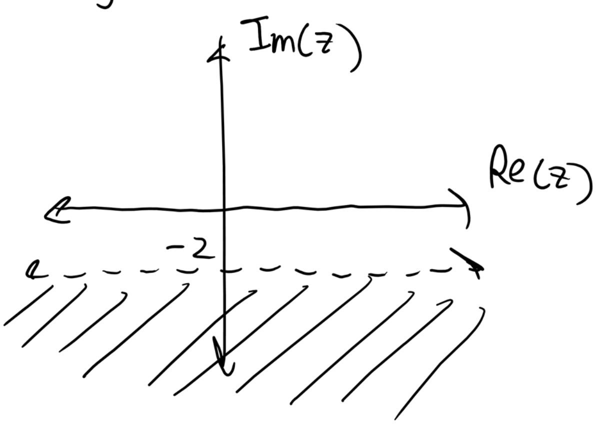
$i(x+iy-x+iy) > 4$

$i(2iy) > 4$

$-2y > 4$

$y < -2$

1 mark for correct algebra



1 mark for correct graph, inc. dotted line

(c) Consider the line $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

(i) Show that the point $(0, 7)$ lies on $\underset{\sim}{r}$. **1**

$\begin{pmatrix} 0 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

$0 = 4 - 2\lambda$

$\lambda = 2$

$7 = -3 + 5\lambda$

$10 = 5\lambda$

$\lambda = 2$ same λ $\therefore$ point is on line

1 mark for showing $\lambda = 2$ twice

(ii) Show that the line through the points $A(3, 2)$ and $B(8, 4)$ is perpendicular to $\underset{\sim}{r}$. **2**

direction through AB:

$\begin{pmatrix} 8-3 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

1 mark for AB direction

perpendicular if dot product of direction vectors is zero

$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \end{pmatrix} = 5 \times -2 + 2 \times 5$

$= 0$

1 mark for showing dot product is zero

(d) It is given that $2 + 3i$ is a zero of $P(z) = z^3 - 2z^2 + 5z + 26$. Solve $P(z) = 0$. **2**

real coefficients $\therefore$ $2 - 3i$ is a zero 1 mark

cubic, $\therefore$ third root is real, let it be α

sum of roots: $(2+3i) + (2-3i) + \alpha = 2$

$4 + \alpha = 2$

$\alpha = -2$

1 mark for getting third zero

$\therefore z = 2+3i,\ 2-3i,\ -2$

(e) Consider the following statement: for $x \in \mathbb{R}$, $x^2 - x > 0 \Rightarrow x < 0$.

(i) Show that the statement is false. **1**

counterexample: $x = 2$

1 mark for suitable show

(ii) State the converse of the statement. **1**

for $x \in \mathbb{R}$, $x < 0 \Rightarrow x^2 - x > 0$

1 mark (don't need $x \in \mathbb{R}$)

(iii) Determine whether the converse is true. **2**

$x^2 - x = x(x-1)$

$\left.\begin{array}{l}\text{both of these factors are} \\ \text{negative when } x < 0\end{array}\right\}$ 1 mark for using factors and/or inequality

$\therefore\ x^2 - x > 0$

$\therefore$ converse is true

1 mark for conclusion

End of Question 11

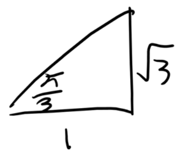
Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Consider the complex number $z = 1 + \sqrt{3}i$.

(i) Write z in mod-arg form. **1**

$r = \sqrt{1^2 + \sqrt{3}^2}$

$= 2$



$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

1 mark (or equivalent result)

(ii) Evaluate z^6. **1**

$z^6 = 2^6\left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right)$

$= 64$

1 mark for result

(iii) Find all values of n for which z^n is purely real. **1**

$z^n = 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right)$

purely real when $\sin\frac{n\pi}{3} = 0$

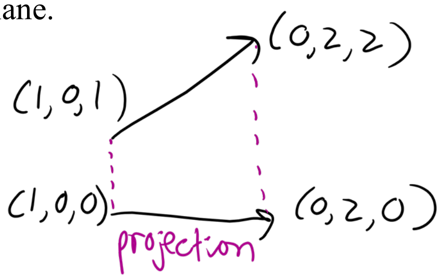
1 mark (or equivalent result)

i.e. $\frac{n\pi}{3} = k\pi, \ k \in \mathbb{Z}$

$n = 3k$

i.e. n is any integer multiple of 3.

(b) Consider the points $A(1, 0, 1)$ and $B(0, 2, 2)$. Find the projection of $\overrightarrow{AB}$ onto the xy-plane. **2**



1 mark for $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

or clearly showing projection e.g. diagram

vector is $\begin{pmatrix} 0-1 \\ 2-0 \\ 0-0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

1 mark for $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

(full marks for vector with suitable explanation)

(c) Use partial fractions to show that $\displaystyle\int_0^1 \frac{x+7}{(x-2)(x+1)}\,dx = -5\ln 2$. **3**

$\displaystyle\int_0^1 \frac{3}{x-2} - \frac{2}{x+1}\,dx$

$= \Big[3\ln|x-2| - 2\ln|x+1|\Big]_0^1$

$= 3\ln 1 - 2\ln 2 - 3\ln 2 + 2\ln 1$

$= -5\ln 2$ as required

1 mark for decomposition

1 mark for integration

1 mark for showing result

(d) Consider the line $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$.

(i) Find the Cartesian equation of the line l that passes through the point $(-2, 1, 4)$ and is parallel to $\underset{\sim}{r}$. **2**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

1 mark (or equivalent progress)

$$\therefore \frac{x+2}{2} = \frac{y-1}{5} = 4 - z$$

1 mark (accept $\frac{z-4}{-1}$ for RHS or equivalent)

(ii) Determine where the line l meets the xz-plane. **2**

xz-plane where $y = 0$

$\therefore 0 = 1 + 5\lambda$

$\lambda = -\frac{1}{5}$

1 mark for appropriate λ

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 - \frac{2}{5} \\ 1 - 1 \\ 4 + \frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{5} \\ 0 \\ \frac{21}{5} \end{pmatrix}$$

meets xz-plane at $\left(-\frac{12}{5}, 0, \frac{21}{5}\right)$

1 mark for coordinate

(e) The motion of a section of a particular roller coaster track can be modelled by simple harmonic motion. The force acting on the roller coaster is given by $F = -kx$, where $k \in \mathbb{R}$, and $x(t)$ is the displacement of the roller coaster from the centre of motion at time t.

(i) Show that $x = A\cos\left(\sqrt{\frac{k}{m}}\, t\right)$ satisfies $F = -kx$, where m is the mass of the roller coaster and A is the amplitude of the motion. **1**

$$\dot{x} = -\sqrt{\frac{k}{m}}\, A \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$\ddot{x} = -\frac{k}{m}\, A \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

1 mark for showing derivatives and $F = m\ddot{x}$

$$F = m\ddot{x}$$
$$= -kA\cos\left(\sqrt{\frac{k}{m}}\, t\right)$$
$$= -kx$$

(ii) The mechanical energy of the roller coaster is given by $M(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$, where v is the velocity of the rollercoaster at time t. Show that $M(t)$ is proportional to A^2. **1**

$$M(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\left[\frac{k}{m}A^2\sin^2\left(\sqrt{\frac{k}{m}}\, t\right)\right] + \frac{1}{2}k\left[A^2\cos^2\left(\sqrt{\frac{k}{m}}\, t\right)\right]$$

$$= \frac{1}{2}kA^2\left[\sin^2\sqrt{\frac{k}{m}}\, t + \cos^2\sqrt{\frac{k}{m}}\, t\right]$$

$$= \frac{1}{2}kA^2$$

which is proportional to A^2

1 mark for clearly showing

(iii) Show that the maximum velocity of the roller coaster is $v_{max} = -A\sqrt{\frac{k}{m}}$. **1**

V_{max} when sine at maximum i.e. 1

$$V_{max} = -\sqrt{\frac{k}{m}}\, A \times 1$$

$$= -A\sqrt{\frac{k}{m}}$$

1 mark for showing result

End of Question 12

Question 13 (16 marks) Use a SEPARATE writing booklet

(a) Find the value of k such that the points $(-2, k, 3)$, $(1, -1, 3)$, and $(3, 5, 3)$ are collinear. **3**

Let $\underset{\sim}{a} = \begin{pmatrix} -2 \\ k \\ 3 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\underset{\sim}{c} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$

collinear when $\underset{\sim}{b} - \underset{\sim}{a} = p\,(\underset{\sim}{c} - \underset{\sim}{a})$ for some $p \in \mathbb{R}$

$$\begin{pmatrix} 1+2 \\ -1-k \\ 3-3 \end{pmatrix} = p\begin{pmatrix} 3+2 \\ 5-k \\ 3-3 \end{pmatrix}$$

1 mark for a correct relationship

$$\begin{pmatrix} 3 \\ -1-k \\ 0 \end{pmatrix} = p\begin{pmatrix} 5 \\ 5-k \\ 0 \end{pmatrix}$$

1 mark for finding parameter or equivalent

$\therefore p = \frac{3}{5}$ $\quad \therefore -1-k = \frac{3}{5}(5-k)$

$$-5-5k = 15-3k$$

$$-20 = 2k$$

$$k = -10$$

1 mark for result

(b) Let $z = x + iy$, $z_1 = 6 + 2i$, and $z_2 = 2 + 2i$, where $x, y \in \mathbb{R}$.

Find $\dfrac{z - z_1}{z - z_2}$ with a real denominator. **2**

$$\frac{z - z_1}{z - z_2} = \frac{x+iy-6-2i}{x+iy-2-2i}$$

$$= \frac{(x-6)+(y-2)i}{(x-2)+(y-2)i} \times \frac{(x-2)-(y-2)i}{(x-2)-(y-2)i}$$

$$= \frac{(x-6)(x-2)+(y-2)^2 + i(y-2)(x-2-x+6)}{(x-2)^2+(y-2)^2}$$

1 mark for attempt at realising

$$= \frac{(x-6)(x-2)+(y-2)^2 + 4(y-2)i}{(x-2)^2+(y-2)^2}$$

1 mark for simplified result

(c) Sketch the following on separate Argand diagrams.

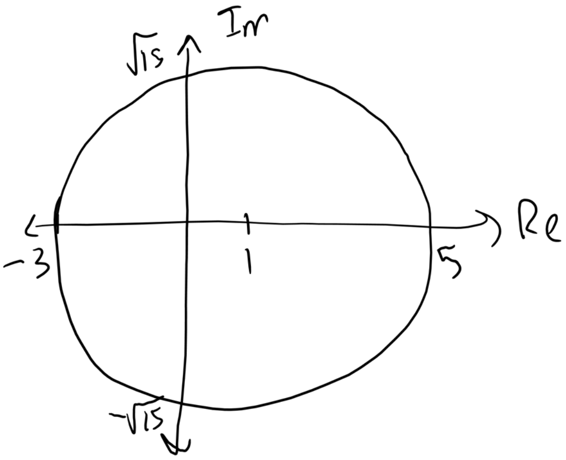
(i) $|z-1| = 4$ **2**

circle centred at $(1,0)$, radius 4

$(x-1)^2 + y^2 = 16$

intercepts: $y = \pm\sqrt{15}$

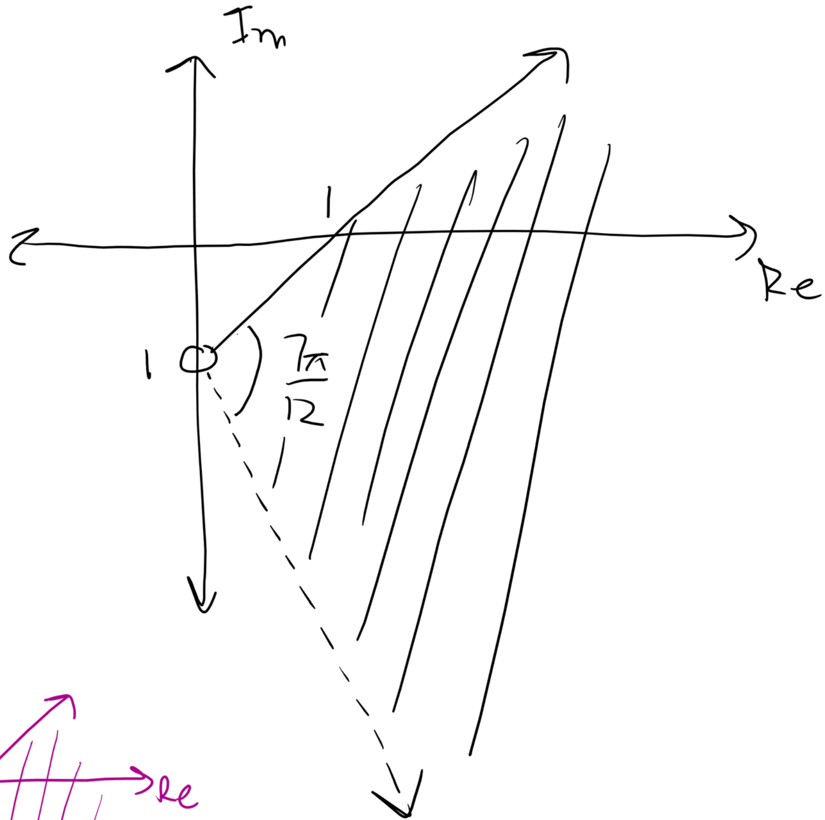
$x = -3, 5$



1 mark with attempt at sketch with either correct radius or centre

1 mark for fully correct sketch (y-intercepts not required, must have centre and x-intercepts or max/min identified)

(ii) $-\dfrac{\pi}{3} < \arg(z+i) \leq \dfrac{\pi}{4}$ **3**

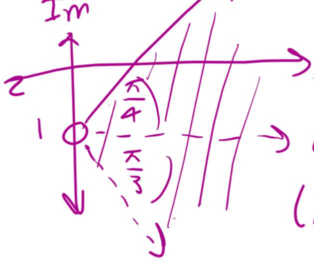


1 mark for correct region

1 mark for open circle and at least one correct boundary

1 mark for other boundary and angle labeled

Accept



→ as generous (horizontal line may imply not included)

(d) Consider the spheres S_1 and S_2. **3**

$$S_1 : x^2 - 2x + y^2 + 4y + z^2 - 4z = 0$$
$$S_2 : x^2 + 10x + y^2 + z^2 + 2z + 10 = 0$$

Show that S_1 and S_2 intersect at a point.

$S_1: x^2-2x+1+y^2+4y+4+z^2-4z+4=1+4+4$

$(x-1)^2+(y+2)^2+(z-2)^2=9 \quad C:(1,-2,2), r=3$

$S_2: x^2+10x+25+y^2+z^2+2z+1=25+1-10$

$(x+5)^2+y^2+(z+1)^2=16 \quad C:(-5,0,-1), r=4$

1 mark for finding centre and radii

distance between centres:

$\sqrt{(1+5)^2+(-2+0)^2+(2+1)^2} = \sqrt{49}$

$= 7$

$=$ sum of the two radii

1 mark for progress with distance

$\therefore$ spheres must intersect at one point

1 mark for conclusion

(e) Let $z_2 = 1 + i$ and, for $n > 2$, let $z_n = z_{n-1}\left(1 + \dfrac{i}{|z_{n-1}|}\right)$. **3**

Use mathematical induction to prove that $|z_n| = \sqrt{n}$ for all integers $n \geq 2$.

① Prove first case $n=2$

$LHS = |z_2|$

$= |1+i|$

$= \sqrt{2}$

$= RHS \quad \therefore$ true for first case

1 mark

② Assume true for $n=k$, $k \in \mathbb{Z}^+$, $k>2$

i.e. $|z_k| = \sqrt{k}$ and $z_k = z_{k-1}\left(1 + \dfrac{i}{|z_{k-1}|}\right)$

Prove true for $n=k+1$

i.e. prove $|z_{k+1}| = \sqrt{k+1}$

$LHS = |z_{k+1}|$

$= \left| z_k \left(1 + \frac{i}{|z_k|}\right)\right|$ as defined initially

$= |z_k| \left|1 + \frac{i}{|z_k|}\right|$

$= \sqrt{k}\left|1 + \frac{i}{\sqrt{k}}\right|$ by assumption

$= |\sqrt{k} + i|$

$= \sqrt{\sqrt{k}^2 + 1^2}$

$= \sqrt{k+1}$, as required

1 mark for using definition and/or assumption with progress

(3) $\therefore |z_n| = \sqrt{n}$ for integers $n \geq 2$ by mathematical induction

1 mark for fully complete proof

End of Question 13

Question 14 (16 marks) Use a SEPARATE writing booklet

(a) Prove $a^3 + b^3 > ab(a+b)$ for $a, b \in \mathbb{R}^+$, $a \neq b$. **2**

$a^3+b^3 > ab(a+b)$, $a,b \in \mathbb{R}^+$

Prove $a^3+b^3-a^2b-ab^2 > 0$

i.e. $a^2(a-b)-b^2(a-b) > 0$

i.e. $(a-b)(a^2-b^2) > 0$

i.e. $(a-b)^2(a+b) > 0$

which is always true as $a,b \in \mathbb{R}^+, a \neq b$

$\Rightarrow (a-b)^2 > 0$ and $a+b > 0$

$\therefore a^3+b^3 > ab(a+b)$

1 mark for progress (factorising cubes or squares and comparing with 0 or using cases)

1 mark for complete proof using conditions

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{4x-3}{4x^2-4x+5}\,dx$. **4**

$$\int_0^{\frac{1}{2}} \frac{4x-3}{4x^2-4x+5}\,dx$$

$$= \frac{1}{4}\int_0^{\frac{1}{2}} \frac{4x-3}{x^2-x+\frac{5}{4}}\,dx$$

$$= \frac{1}{4}\int_0^{\frac{1}{2}} \frac{4x-3}{x^2-x+\frac{1}{4}+1}\,dx$$

1 mark for appropriate method (complete square)

$$= \frac{1}{4}\int_0^{\frac{1}{2}} \frac{4x-3}{(x-\frac{1}{2})^2+1}\,dx \qquad \frac{d}{dx}\left(x-\tfrac{1}{2}\right)^2 = 2\left(x-\tfrac{1}{2}\right) = 2x-1$$

$$= \frac{1}{4}\int_0^{\frac{1}{2}} \frac{4x-2}{(x-\frac{1}{2})^2+1} - \frac{1}{(x-\frac{1}{2})^2+1}\,dx$$

1 mark for correct split

$$= \frac{1}{4}\left[2\ln\left(\left(x-\tfrac{1}{2}\right)^2+1\right) - \tan^{-1}\left(x-\tfrac{1}{2}\right)\right]_0^{\frac{1}{2}}$$

1 mark for correct progress with integration

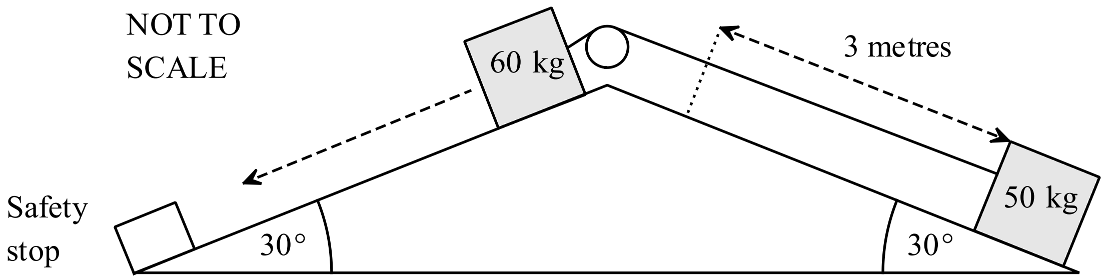
$$= \frac{1}{4}\left[0 - 0 - 2\ln\left(\tfrac{1}{4}+1\right) + \tan^{-1}\left(-\tfrac{1}{2}\right)\right]$$

$$= -\frac{1}{2}\ln\frac{5}{4} + \frac{1}{4}\tan^{-1}\left(-\tfrac{1}{2}\right)$$

1 mark for answer following

[accept $-\ln\frac{5}{4} = \ln\frac{4}{5}$ and $\tan^{-1}\left(-\frac{1}{2}\right) = -\tan^{-1}\left(\frac{1}{2}\right)$]

(c) A pulley system has been set up using a 60 kg counterweight to move a 50 kg box of tiles up on a roof. The roof is sloped at 30º to the horizontal on both sides and the safety stop ensures the box is pulled exactly 3 metres up the roof.

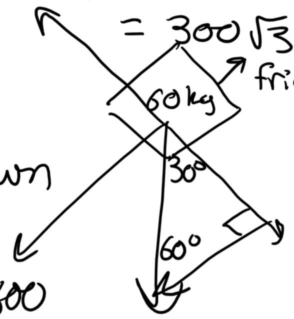


Both masses are initially at rest and the acceleration due to gravity is $10\ \mathrm{ms}^{-2}$. Air resistance is negligible but the frictional force opposing the direction of each mass is equal to half the normal force for each box, respectively.

(i) Determine whether the pulley system will work as intended. **3**

Counterweight:

$N = \cos 30° \times 600$
$= 300\sqrt{3}$



force down roof
$= \cos 60° \times 600$
$= 300$

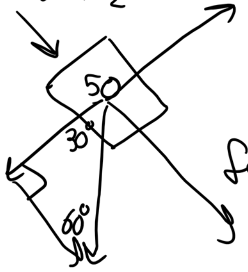
$W = 60 \times 10$
$= 600$

$\therefore$ resultant force down the roof
$= 300 - 150\sqrt{3}$

1 mark for finding either
- both frictional forces, or
- both weight forces, or
- one resultant force

Box:

$N = \cos 30° \times 500$
$= 250\sqrt{3}$



force down roof
$= \cos 60° \times 500$
$= 250$

$W = 50 \times 10$
$= 500$

force up the roof
$= -250 - 125\sqrt{3}$

1 mark for both resultant forces found

total force in system $= 300 - 150\sqrt{3} - 250 - 125\sqrt{3}$
$= -426\ldots$ i.e. not working

1 mark for determining result

(ii) Find the maximum weight of the box that can be pulled using this system. **2**

force > 0, let the mass of the box be x kg

$$300 - 150\sqrt{3} - 5x - \frac{5\sqrt{3}}{2}x > 0$$

1 mark for correct inequality

$$300 - 150\sqrt{3} > \left(5 + \frac{5\sqrt{3}}{2}\right)x$$

$$120 - 60\sqrt{3} > (2 + \sqrt{3})x$$

$$\therefore x < \frac{120 - 60\sqrt{3}}{2 + \sqrt{3}}$$

$$= 420 - 240\sqrt{3}$$

$$= 4.3078\ldots \text{ kg}$$

$\longleftarrow$ 1 mark for result with evidence (accept this line)

(d) The complex number z lies on the unit circle with $\operatorname{Arg}(z) = \theta$.

(i) Show that $\dfrac{1+z}{1-z} = i\cot\left(\dfrac{\theta}{2}\right)$. **2**

$$\frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}} \times \frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}}$$

1 mark for using an appropriate substitution for z

$$= \frac{e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}}$$

$$= \frac{2\cos\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}} \times \frac{\frac{i}{2}}{\frac{i}{2}}$$

$$= \frac{i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= i\cot\frac{\theta}{2}$$

1 mark for showing result

(ii) Hence, or otherwise, solve $\left(\frac{w-1}{w+1}\right)^8 = 1$, where w is a complex number. **3**

$$\left(\frac{w-1}{w+1}\right)^8 = 1$$

$$= \text{cis}\, 2k\pi$$

$$\therefore \frac{w-1}{w+1} = \text{cis}\left(\frac{k\pi}{4}\right), \quad k = -3, -2, \ldots, 2, 3, 4$$

1 mark for de Moivre

$$w - 1 = w\,\text{cis}\left(\frac{k\pi}{4}\right) + \text{cis}\left(\frac{k\pi}{4}\right)$$

$$w\left(1 - \text{cis}\frac{k\pi}{4}\right) = 1 + \text{cis}\left(\frac{k\pi}{4}\right)$$

$$w = \frac{1 + \text{cis}\frac{k\pi}{4}}{1 - \text{cis}\frac{k\pi}{4}}$$

$$= i\cot\frac{k\pi}{8}, \text{ from (i)}$$

$$k = -3, -2, -1, 1, 2, 3$$

test $k = 0, 4$ separately, as undefined in cotangent

1 mark for using (i) (even if incorrectly assumed $z = w$)

$k = 0$ $\quad w - 1 = w + 1$

no solutions

$k = 4$ $\quad w - 1 = -w - 1$

$w = 0$

$$\therefore w = -i\cot\frac{3\pi}{8}, -i\cot\frac{\pi}{4} = -i,$$

$$-i\cot\frac{\pi}{8}, \; i\cot\frac{\pi}{8}, \; i\cot\frac{\pi}{4} = i,$$

$$i\cot\frac{3\pi}{8}, \; 0$$

1 mark for full results with check

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) For $p, q \in \mathbb{R}^+$, explain why $\dfrac{1}{p-2q} \neq \dfrac{1}{p} - \dfrac{1}{2q}$, where $p \neq 2q$. **3**

Assume $\dfrac{1}{p-2q} = \dfrac{1}{p} - \dfrac{1}{2q}$

1 mark for proof by contradiction or equivalent appropriate method

$\therefore \dfrac{1}{p-2q} = \dfrac{2q-p}{2pq}$

$2pq = (2q-p)(p-2q)$

$= -4q^2 + 4pq - p^2$

i.e. $4q^2 - 4pq + p^2 = -2pq$

$(2q-p)^2 = -2pq$

1 mark for reaching part of a contradiction

which is impossible as $p, q > 0$

$\therefore$ RHS < 0 and LHS > 0

also, LHS $\neq 0$ as $2q \neq p$

$\therefore$ we have a contradiction

$\therefore$ assumption false

1 mark for fully complete proof

$\therefore \dfrac{1}{p-2q} \neq \dfrac{1}{p} - \dfrac{1}{2q}$ $\square$

(b) Consider the complex number $z = \cos\theta + i\sin\theta$.

(i) Expand and simplify $(z^3 - 1)(z^6 + z^3 + 1)$. **1**

$z^9 + z^6 + z^3 - z^6 - z^3 - 1 = z^9 - 1$

1 mark for result

i.e. $(z^3-1)(z^6+z^3+1) = z^9 - 1$

(ii) Show that

$$z^6 + z^3 + 1 = \left(z^2 - 2z\cos\left(\frac{2\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{4\pi}{9}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{8\pi}{9}\right) + 1\right).$$ **3**

Let $z = e^{i\theta}$

$\therefore$ Let $z^9 = e^{9i\theta} = 1 = e^{2\pi i}$

1 mark for finding roots

$\therefore \theta = \frac{2\pi}{9}$

$\therefore z = 1, \underbrace{e^{\frac{2\pi}{9}i}, e^{-\frac{2\pi}{9}i}}_{\text{conj.}}, \underbrace{e^{\frac{4\pi}{9}i}, e^{-\frac{4\pi}{9}i}}_{\text{conj.}}, \underbrace{e^{\frac{6\pi}{9}i}, e^{-\frac{6\pi}{9}i}}_{\text{conj.}}, \underbrace{e^{\frac{8\pi}{9}i}, e^{-\frac{8\pi}{9}i}}_{\text{conj.}}$

$(z - w)(z - \overline{w}) = z^2 - zw - z\overline{w} + w\overline{w}$

$= z^2 - z(w + \overline{w}) + 1$

$= z^2 - 2z\,\mathrm{Re}(w) + 1$

1 mark for using conjugate

$\therefore z^9 - 1 = (z-1)(z^2 - 2z\cos\frac{2\pi}{9} + 1)$
$(z^2 - 2z\cos\frac{4\pi}{9} + 1) \ldots (z^2 - 2z\cos\frac{8\pi}{9} + 1)$

$z^3 - 1 = (z-1)(z^2 - 2z\cos\frac{6\pi}{9} + 1)$

1 mark for $z^3 - 1$ in this form and showing

$\frac{z^9 - 1}{z^3 - 1} = (z^2 - 2z\cos\frac{2\pi}{9} + 1)(z^2 - 2z\cos\frac{4\pi}{9} + 1)$
$(z^2 - 2\cos\frac{8\pi}{9} + 1)$ $\square$

(iii) Hence, show that

$$2\cos(3\theta) + 1 = 8\left(\cos\theta - \cos\left(\frac{2\pi}{9}\right)\right)\left(\cos\theta - \cos\left(\frac{4\pi}{9}\right)\right)\left(\cos\theta - \cos\left(\frac{8\pi}{9}\right)\right).$$ **2**

$z^6 + z^3 + 1 = z^3(z - 2\cos\frac{2\pi}{9} + \frac{1}{z})(z - 2\cos\frac{4\pi}{9} + \frac{1}{z})(z - 2\cos\frac{8\pi}{9} + \frac{1}{z})$

$z^n + z^{-n} = 2\cos n\theta$

1 mark for factoring z

$\therefore z^3 + 1 + z^{-3} = (z + z^{-1} - 2\cos\frac{2\pi}{9})(z + z^{-1} - 2\cos\frac{4\pi}{9})(z + z^{-1} - 2\cos\frac{8\pi}{9})$

$2\cos(3\theta) + 1 = 2^3(\cos\theta - \cos\frac{2\pi}{9})(\cos\theta - \cos\frac{4\pi}{9})(\cos\theta - \cos\frac{8\pi}{9})$

$\square$

1 mark for this line or equivalent showing $z^n + z^{-n} = 2\cos n\theta$

(c) Double factorials are defined as follows:

$n!! = n(n-2)(n-4)...6 \times 4 \times 2$, where n is an even positive integer

e.g. $10!! = 10 \times 8 \times 6 \times 4 \times 2$

$n!! = n(n-2)(n-4)...5 \times 3 \times 1$, where n is an odd positive integer

e.g. $11!! = 11 \times 9 \times 7 \times 5 \times 3 \times 1$

(i) Show that $(2k)!! = k!2^k$ for even integers $n = 2k$. **1**

$$\left.\begin{aligned}(2k)!! &= (2k)(2k-2)(2k-4)\ldots 2\\ &= 2\times k\times 2(k-1)\times 2(k-2)\times\ldots\times 2\times 1\\ &= 2^k(k!) \text{ as there are } k \text{ terms}\end{aligned}\right\} \text{1 mark}$$

(ii) Hence, prove that $(2k+1)!! = \dfrac{(2k+1)!}{k!2^k}$ for odd integers $n = 2k+1$. **2**

$$(2k+1)!! = (2k+1)(2k-1)(2k-3)\ldots 5\times 3\times 1$$

$\times (2k)!!$ $\quad (2k+1)!! \times k!2^k = (2k+1)(2k)(2k-1)\ldots 5\times 4\times 3\times 2\times 1$

1 mark for using (i)

$$\left.\begin{aligned}(2k+1)!! \times k!2^k &= (2k+1)!\\ \therefore (2k+1)!! &= \frac{(2k+1)!}{k!\,2^k}\end{aligned}\right\} \text{1 mark}$$

(iii) Hence, or otherwise, prove that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n}(x)\,dx = \frac{(2n)!\pi}{2^{2n+1}(n!)^2}$. **3**

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{2n-1} x \cos x \, dx$$

$u = \cos^{2n-1} x \qquad v' = \cos x$

$u' = (2n-1)(-\sin x)\cos^{2n-2} x \qquad v = \sin x$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \left[\cos^{2n-1} x \sin x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (2n-1)\sin^2 x \cos^{2n-2} x\,dx$$

1 mark for integration by parts

$$= (2n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{2n-2} x\,dx$$

$$= (2n-1)\int_0^{\frac{\pi}{2}} \cos^{2n-2} x - \cos^{2n} x\,dx$$

Let $\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = I_n$

$$\therefore I_{2n} = (2n-1)\left[I_{2n-2} - I_{2n}\right]$$

$$= 2n I_{2n-2} - 2n I_{2n} - I_{2n-2} + I_{2n}$$

$$\therefore 2n I_{2n} = (2n-1) I_{2n-2}$$

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times I_{2n-4}$$

1 mark for applying recursion to achieve double facterials

$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \times I_0$$

$$= \frac{(2n-1)!!}{(2n)!!} \int_0^{\frac{\pi}{2}} 1\,dx$$

Also, from (ii) $(2n-1)!! = \dfrac{(2n)!}{n!\,2^n}$ as there is one less term

from (i) $(2n)!! = n!\,2^n$

$$\therefore I_{2n} = \frac{(2n)!}{(n!)^2 2^{2n}} \times \frac{\pi}{2}$$

$$= \frac{(2n)!\,\pi}{(n!)^2 2^{2n+1}} \quad \square$$

1 mark for substituting and proving result

End of Question 15

Question 16 (14 marks) Use a SEPARATE writing booklet

(a) A unit mass is projected vertically upwards with an initial speed of S ms^{-1} in a resisting medium that exerts a force of kv opposing the direction of motion, where k is a positive constant and the acceleration due to gravity is g ms^{-2}. **5**

Find an expression for the velocity, $v(t)$, independent of S, given that the mass reaches maximum height after T seconds.

Solution using S.

$v_0 = S$ motion upwards and $m = 1$ $\therefore \ddot{y} = -g - kv$

$m\ddot{y} = mg + kv$ down

$$\frac{dv}{dt} = -(g+kv)$$

1 mark for ~~establishing~~ initial equation of motion

$$-\int_S^v \frac{1}{g+kv}\,dv = \int_0^t 1\,dt$$

$$-\left[\frac{1}{k}\ln(g+kv)\right]_S^v = t$$

1 mark for integrating for v

$$\ln\left(\frac{g+kS}{g+kv}\right) = kt$$

$$\frac{g+kS}{g+kv} = e^{kt}$$

$$ge^{-kt} + kSe^{-kt} = g+kv$$

$$\therefore v = \frac{1}{k}\left(ge^{-kt} + kSe^{-kt} - g\right)$$

$$= \frac{g}{k}\left(e^{-kt}-1\right) + Se^{-kt}$$

1 mark for a correct v(t)

when $v=0$, $t=T$

$$\therefore 0 = \frac{g}{k}\left(e^{-kT}-1\right) + Se^{-kT}$$

$$\frac{g}{k}\left(1-e^{-kT}\right) = Se^{-kT}$$

$$S = \frac{g}{k}\left(e^{kT}-1\right)$$

1 mark for progress in eliminating S using initial conditions

$$\therefore v = \frac{g}{k}\left(e^{-kt}-1\right) + \frac{g}{k}\left(e^{kT}-1\right)e^{-kt}$$

$$= \frac{g}{k}\left[e^{-kt} - 1 + e^{kT-kt} - e^{-kt}\right]$$

$$= \frac{g}{k}\left[e^{k(T-t)} - 1\right]$$

1 mark for result independent of S (accept results with m)

Alternative solution:

$$\frac{dv}{dt} = -g - kv$$ (1)

$$\int \frac{1}{kv+g}\, dv = \int -1\, dt$$

$$\frac{1}{k}\ln(kv+g) = -t + C_1$$ (1)

$$\ln(kv+g) = -kt + C_2$$

$$kv+g = C_3 e^{-kt}$$ (1)

At maximum height, $v=0$, $t=T$

$$g = C_3 e^{-kT}$$

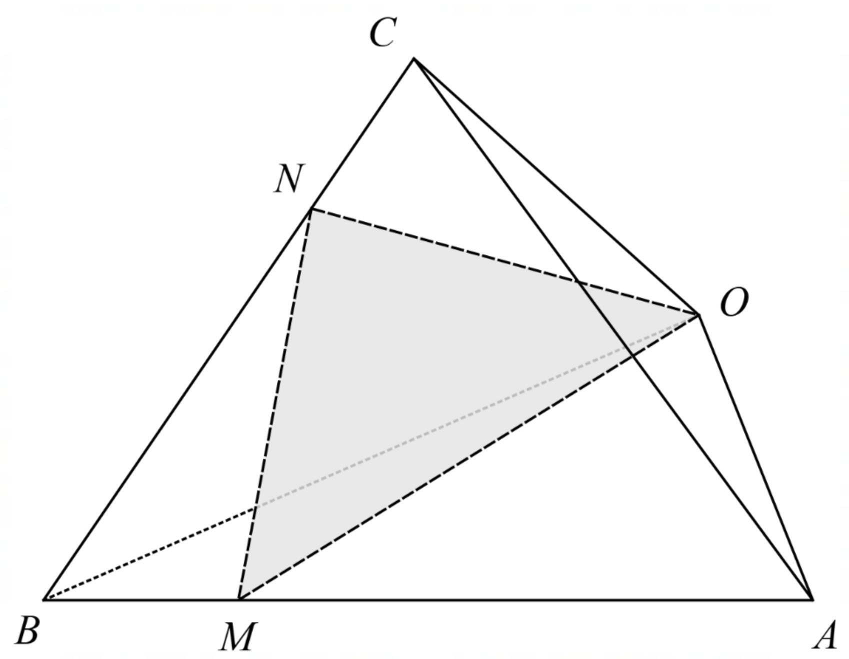
$$C_3 = ge^{kT}$$ (1)

$$\therefore kv+g = ge^{kT}e^{-kt}$$

$$v = \frac{ge^{k(T-t)} - g}{k}$$ (1) for $v(t)$

$$= \frac{g}{k}\left(e^{k(T-t)} - 1\right)$$

(b) A regular triangular pyramid of side length 1, $OABC$, is shown below. M and N divide the sides AB and BC in the ratio $x:1-x$, respectively. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$.



(i) Let $f(x) = \dfrac{1+x-x^2}{1-x+x^2}$, where $0 < x < 1$. Prove that $1 < f(x) \le \dfrac{5}{3}$. **3**

Non-calculus:

Consider $0 < x < 1$

$$-\frac{1}{2} < x - \frac{1}{2} < \frac{1}{2}$$

squaring gives $0 \le \left(x - \frac{1}{2}\right)^2 < \frac{1}{4}$

$$0 \le x^2 - x + \frac{1}{4} < \frac{1}{4}$$

$$\frac{3}{4} \le x^2 - x + 1 < 1$$

check discriminant for $x^2 - x + 1$

$\Delta = 1^2 - 4$

< 0 $\therefore$ no zeroes

$\therefore$ $\frac{4}{3} \geq \frac{1}{x^2-x+1} > 1$

i.e. $1 < \frac{1}{x^2-x+1} \le \frac{4}{3}$

$$2 < \frac{2}{x^2-x+1} \le \frac{8}{3}$$

$$1 < \frac{2}{x^2-x+1} - \frac{x^2-x+1}{x^2-x+1} \le \frac{5}{3}$$

$$1 < \frac{1+x-x^2}{x^2-x+1} \le \frac{5}{3}$$

as required.

multiple ways to prove:

① for correct progress/logic in proof e.g. using conditions or calculus

① left inequality e.g. proving $f(x) - 1 > 0$

① right inequality e.g. proving $\frac{5}{3} - f(x) \geq 0$

Calculus: $f(x) = \frac{1+x-x^2}{1-x+x^2}$

$= \frac{2-1+x-x^2}{1-x+x^2}$

$= \frac{2}{1-x+x^2} - 1$

$f'(x) = \frac{-2(-1+2x)}{(1-x+x^2)}$

$= 0$ when $x = \frac{1}{2}$

1 for $x = \frac{1}{2}$

for $1-x+x^2$, $\Delta = 1^2 - 4$

< 0 $\therefore$ no zeroes

$\therefore f(x)$ continuous

$f(\frac{1}{2}) = \frac{5}{3}$ either max or min

endpoints: $f(0) = 1$, $f(1) = 1$

$\therefore f(\frac{1}{2})$ is max and $f(0) = f(1)$ min

1 for endpoints

$\therefore 1 < f(x) \leq \frac{5}{3}$

1 for inequality with clear argument (continuity, equality)

(ii) Show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{c} \cdot \underset{\sim}{a} = \frac{1}{2}$. **1**

all sides lengths 1

$\therefore$ all faces equilateral Δ's

$\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = 1 \times 1 \times \cos 60^\circ$

$= \frac{1}{2}$

Similarly, $\hat{b} \cdot \underset{\sim}{c} = \frac{1}{2}$ and $\underset{\sim}{c} \cdot \hat{a} = \frac{1}{2}$

1 mark for clear show

(iii) Show that $\left|\overrightarrow{ON}\right| = \sqrt{x^2 - x + 1}$. **2**

$\overrightarrow{ON} = \underset{\sim}{b} + x(\underset{\sim}{c} - \underset{\sim}{b})$

accept equivalent use of ratios

accept cosine rule

$\therefore |\overrightarrow{ON}|^2 = |\underset{\sim}{b} + x(\underset{\sim}{c} - \underset{\sim}{b})|^2$

$= ((1-x)\underset{\sim}{b} + x\underset{\sim}{c})((1-x)\underset{\sim}{b} + x\underset{\sim}{c})$

$= (1-x)^2\,\underset{\sim}{b}\cdot\underset{\sim}{b} + 2x(1-x)\,\underset{\sim}{b}\cdot\underset{\sim}{c} + x^2\,\underset{\sim}{c}\cdot\underset{\sim}{c}$

1 mark for using magnitude and dot product

$\underset{\sim}{b}\cdot\underset{\sim}{b} = \underset{\sim}{c}\cdot\underset{\sim}{c} = 1$

$\underset{\sim}{b}\cdot\underset{\sim}{c} = \frac{1}{2}$ $\quad\therefore |\overrightarrow{ON}|^2 = (1-x)^2 + x(1-x) + x^2$

$= 1 - 2x + x^2 + x - x^2 + x^2$

$= x^2 - x + 1$

$\therefore |\overrightarrow{ON}| = \sqrt{x^2 - x + 1}$

1 mark for using (ii) to show result

(iv) Determine the range of values of $\angle MON$. **3**

Also, $\overrightarrow{OM} = \underset{\sim}{a} + x(\underset{\sim}{b} - \underset{\sim}{a})$

will give $|\overrightarrow{OM}| = \sqrt{x^2 - x + 1}$

1 mark for deduction

$\therefore \cos\angle MON = \dfrac{\overrightarrow{OM}\cdot\overrightarrow{ON}}{|\overrightarrow{OM}||\overrightarrow{ON}|}$

$= \dfrac{((1-x)\underset{\sim}{a} + x\underset{\sim}{b})\cdot((1-x)\underset{\sim}{b} + x\underset{\sim}{c})}{x^2 - x + 1}$

$= \dfrac{(1-x)^2\,\underset{\sim}{a}\cdot\underset{\sim}{b} + x(1-x)\,\underset{\sim}{b}\cdot\underset{\sim}{b} + x^2\,\underset{\sim}{b}\cdot\underset{\sim}{c} + x(1-x)\,\underset{\sim}{a}\cdot\underset{\sim}{c}}{x^2 - x + 1}$

1 mark for applying dot product with (i) and (ii)

$$= \frac{\frac{1}{2}(1-x)^2 + x(1-x) + \frac{1}{2}x^2 + \frac{1}{2}x(1-x)}{x^2 - x + 1}$$

$$= \frac{1 - 2x + x^2 + 2x - 2x^2 + x^2 + x - x^2}{2(x^2 - x + 1)}$$

$$= \frac{1 + x - x^2}{2(x^2 - x + 1)}$$

$$= \frac{f(x)}{2} \text{ from (i)}$$

$$\therefore \quad \frac{1}{2} < \cos(\angle MON) \leq \frac{5}{6}$$

$$\therefore \cos^{-1}\left(\frac{5}{6}\right) \leq \angle MON < 60^\circ$$

1 mark for result

End of Question 16

End of Paper