Name:	
Class:	

YEAR 12
ASSESSMENT TEST 1
TERM 4, 2016

MATHEMATICS EXTENSION 2

Time Allowed – 90 Minutes
(Plus 5 minutes Reading Time)

- *All* questions may be attempted
- *All* questions are of equal value
- Department of Education approved calculators and templates are permitted
- In every Question, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged work
- A reference sheet will be given to each student

The answers to all questions are to be returned in separate bundles clearly labeled Question 1, Question 2, etc.
Each question must show (in the top right hand corner) your Candidate Number.

QUESTION 1 (15 Marks) **Marks**

(a) If $z = 2 - i$ and $w = -3 - 2i$ find:

(i) $3z - 2w$. **2**

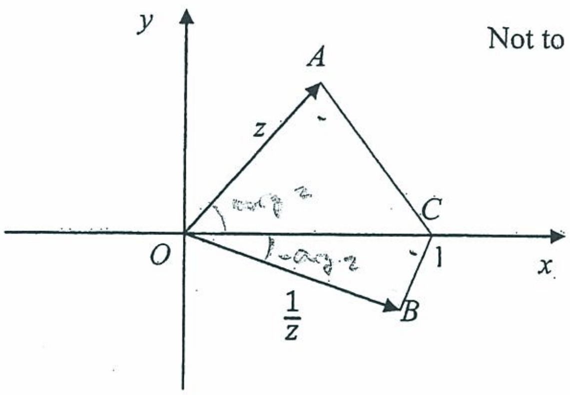
(ii) $\dfrac{\bar{z}}{w}$ in the form $a + ib$. **2**

(b) Points A and B represent the complex numbers z_1 and z_2 on an Argand diagram such that OA and OB are two sides of a right angled isosceles triangle, where O is the origin and $\angle AOB = 90°$.

Prove that $z_1{}^2 + z_2{}^2 = 0$, giving reasons. **3**

(c) Solve the equation $(x + iy)^2 = x - iy$ for real values of x and y. **4**

(d) Points A, B and C represent the complex numbers z, $\frac{1}{z}$ and 1 respectively as shown on the diagram below such that $0 < \arg z < \frac{\pi}{2}$ and O is the origin.



Not to scale

Prove that $\angle OAC = \angle BCO$, giving reasons. **4**

QUESTION 2 (15 Marks) START A NEW PAGE

(a) If $z = x + iy$ and $P(x, y)$ moves in the complex plane, find the Cartesian equation of the locus of P if:

(i) $|z + i| = |z - 2 - i|$ **2**

(ii) $\bar{z} - iz = 0$ **2**

(b) (i) Find the four solutions of $z^4 + 16i = 0$ **4**

(ii) Display your answers on an Argand diagram. **2**

Question 2 continued **Marks**

(c) (i) Given that ω is a non real root of $x^3 - 1 = 0$. **2**
Show that ω is also a root of $x^2 + x + 1 = 0$.

(ii) Hence or otherwise show that $(x+1)^{2n} + x^{2n} + 1$ is NOT divisible by $x^2 + x + 1$ when n is a multiple of 3. **3**

QUESTION 3 (15 Marks) START A NEW PAGE

(a) Given that $2\sqrt{3} + i$ is a cube root of the complex number w, find the other two cube roots of w. **3**

(b) Sketch the locus on the Argand diagram for $\arg(z+2) = \arg(z-2i)$. **2**

(c) (i) Given that $w = 1 - (\cos 2\theta + i\sin 2\theta)$ show that $|w| = 2|\sin\theta|$ **3**

(ii) Express $z^n - 1$ as a product of linear factors for $n = 2, 3, 4 \ldots\ldots\ldots\ldots$ **2**

(iii) Hence show that: **2**

$$\left(1 - \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)\left(1 - \cos\frac{4\pi}{n} - i\sin\frac{4\pi}{n}\right)\ldots\ldots\ldots\left(1 - \cos\frac{2(n-1)\pi}{n} - \sin\frac{2(n-1)\pi}{n}\right) = n$$

for $n = 2, 3, 4 \ldots\ldots$

(iv) Hence prove that:

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$ for $n = 2, 3, 4 \ldots\ldots\ldots$ **3**

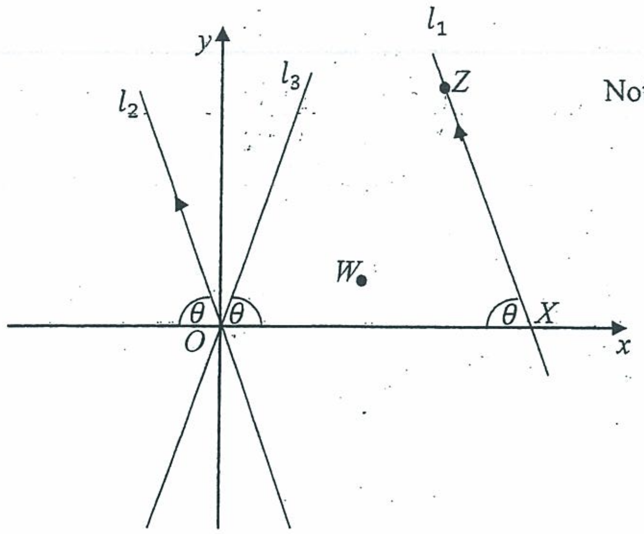
QUESTION 4 (15 Marks) START A NEW PAGE

(a) Express $1 - i$ in modulus-argument form. **2**

(b) Sketch the region defined by the intersection of $1 \le |z + i| \le 3$ and $-\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$. **4**

(c) On the Argand diagram below, points W and Z represent the complex numbers w and z respectively.

A line l_1 is drawn through point Z so that $\angle ZXO = \theta,\ 0 < \theta < \frac{\pi}{2}$. Line l_2 is parallel to line l_1 and passes through the origin. Line l_3 is the reflection of line l_2 about the x-axis.



Not to scale

(i) On the diagram provided on the separate sheet, mark the position of point:

(α) P which represents the complex number $z - w$. **1**

(β) P' which represents the complex number $\frac{1}{z-w}$. **2**

(γ) P'' which represents the complex number $\frac{\frac{1}{z-w}}{\cos\theta + i\sin\theta}$. **1**

(ii) If point P'' represents the complex number $a + ib$ explain why $b < 0$ for all positions of W on the complex plane which are to the left of l_1. **2**

(iii) Given that the point Z represents the complex number z and c_n are the complex numbers represented by C_n, the vertices of an n-sided polygon, prove if **3**

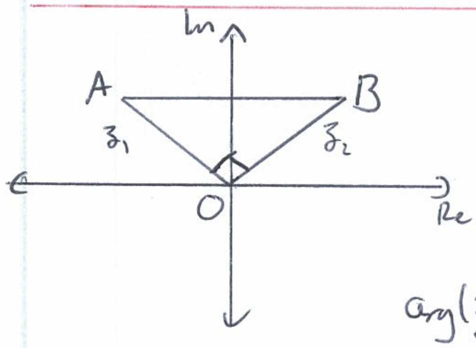
$$\frac{1}{z-c_1} + \frac{1}{z-c_2} + \frac{1}{z-c_3} \ldots\ldots\ldots\ldots\ldots\ldots. \frac{1}{z-c_n} = 0$$

then the point Z cannot lie outside the n-sided polygon.

End of Examination

2016 T4 Test

Ext 2 **MATHEMATICS: Question** 1

Suggested Solutions	Marks Awarded	Marker's Comments
(a) (i) $3z - 2w = 3(2-i) - 2(-3-2i)$	1	
$= 6 - 3i + 6 + 4i$		
$= 12 + i$	1	
(ii) $\dfrac{\bar{z}}{w} = \dfrac{2+i}{-3-2i} \times \dfrac{-3+2i}{-3+2i}$	1	
$= \dfrac{(-6-2) - 3i + 4i}{9+4}$		
$= \dfrac{-8+i}{13}$	1	
(b)  $z_1 = \overrightarrow{OA}$ $z_2 = \overrightarrow{OB}$ $\arg(z_1) > \arg(z_2)$		
$\angle AOB = 90^\circ \Rightarrow z_1 = iz_2$ (multiplication by i $\Leftrightarrow$ rotation of vector by $\frac{\pi}{2}$)	1	

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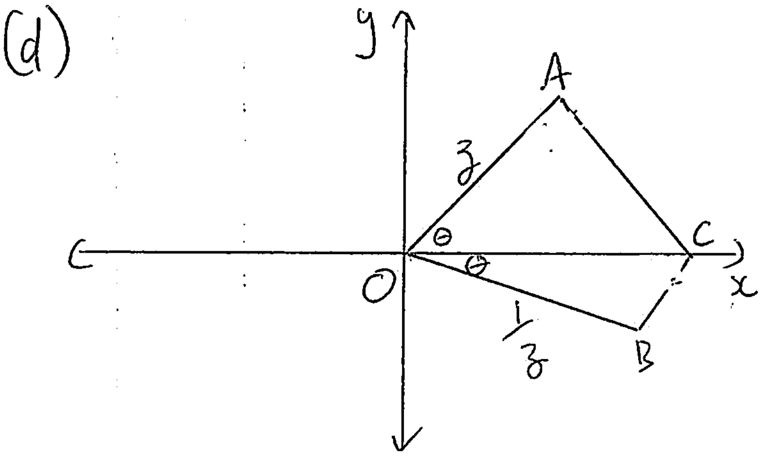
Ext **MATHEMATICS: Question** 1

Suggested Solutions	Marks Awarded	Marker's Comments
$z_1^2 + z_2^2 = (iz_2)^2 + z_2^2$ LHS:	1	
$= -1z_2^2 + z_2^2$		
$= 0$	1	
$= RHS$ as required		
(c) $(x+iy)^2 = x-iy$		
$x^2+2xyi-y^2 = x-iy$		
Equate real and imaginary:		
$x^2-y^2-x=0$ ①		
$2xy+y=0$ ②	1	
②: $y(2x+1)=0 \Rightarrow y=0,\ x=-\frac{1}{2}$	1	
Case $y=0 \rightarrow$ ①		
$x^2-x=0$		
$\Rightarrow x=0,\ x=1$	1	
Case $x=-\frac{1}{2} \rightarrow$ ①		
$\frac{1}{4}-y^2+\frac{1}{2}=0$		
$y^2=\frac{3}{4}$		
$\Rightarrow y = \pm\frac{\sqrt{3}}{2}$	1	

Ext 2 **MATHEMATICS: Question** 1

Suggested Solutions	Marks Awarded	Marker's Comments

(d)



$$\angle AOB = \arg(z) - \arg\left(\tfrac{1}{z}\right)$$

$$= \arg(z^2)$$

$$= 2\arg(z) \quad \left(\text{note } \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)\right)$$

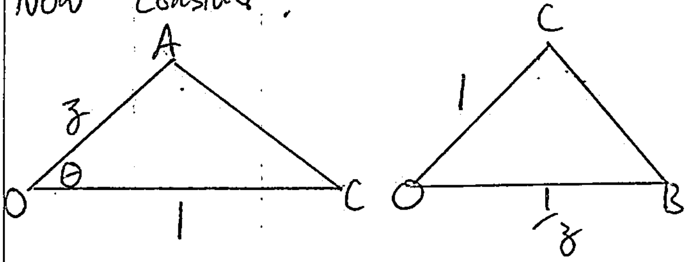
$\therefore\ \angle AOC = \angle COB$

$= \arg(z)$

$= \theta$

1

Now consider:



$\dfrac{OA}{OC} = z \quad \boxed{1}$, $\quad \dfrac{OC}{OB} = \dfrac{1}{\frac{1}{z}} = z \quad \boxed{1}$

$= 2$

Ext 2

MATHEMATICS: Question....1...

Suggested Solutions	Marks Awarded	Marker's Comments
$\therefore$ In ΔOAC and ΔOBC $\angle AOC = \angle COB$ (proven above) $\frac{OA}{OC} = \frac{OC}{OB}$ (proven above) $\therefore \Delta OAC \parallel\!\mid \Delta OBC$ (two pairs of matching sides in the same ratio, included angles equal) $\therefore \angle OAC = \angle BCO$ (corresponding angles in similar triangles $\Delta OAC \parallel\!\mid \Delta OBC$ are equal) As required. ↓ For alternative	1	

Ext 2 MATHEMATICS Extension 1 : Question 1		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
Alternative Method $\angle OAC + \angle AOC = \angle ACX$ (exterior angle of ΔOAC equal sum of interior opposite angles) $\therefore \angle OAC = \angle ACX - \angle AOC$ Now, $\angle ACX = \arg(\overrightarrow{CA})$ and $\overrightarrow{CA} = \overrightarrow{CO} - \overrightarrow{OA}$ $= z - 1$ And $\overrightarrow{OA} = z$ $\therefore \angle OAC = \arg(z-1) - \arg(z)$ $= \arg\left(\frac{z-1}{z}\right)$	1 1	

MATHEMATICS Extension 1 : Question……..		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
Also, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$ $= 1 - \frac{1}{z}$ $= \frac{z-1}{z}$ $\angle OCB = \arg(\overrightarrow{BC})$ $= \arg\left(\frac{z-1}{z}\right)$ $= \angle OAC$ - as required	1 1	

TERM 4 2016 - ASSESSMENT TASK 1

MATHEMATICS Extension 2: Question ..2... Page 1

Suggested Solutions	Marks Awarded	Marker's Comments
a) i) Let $z = x+iy$		
$\therefore \lvert x+iy+i \rvert = \lvert x+iy-2-i \rvert$		
$\lvert x+i(y+1) \rvert = \lvert x-2+i(y-1) \rvert$		
$\therefore x^2+(y+1)^2 = (x-2)^2+(y-1)^2$	1	Correct expression
$\cancel{x^2}+\cancel{y^2}+2y+\cancel{1} = \cancel{x^2}-4x+4+\cancel{y^2}-2y+\cancel{1}$		
$2y = -4x-2y+4$		
$\therefore 4x+4y-4=0$		
$x+y-1=0$	1	
Comment: Generally very well done. Some careless errors.		
OR		
$\lvert z+i \rvert = \lvert z-2-i \rvert$		
Perpendicular bisector between $(0,-1)$ and $(2,1)$	1	
Midpoint $(1,0)$		
Perpendicular gradient $=-1$		
$\therefore y-0=-1(x-1)$		
$y=-x+1$	1	
Comment: Generally well done. Some careless errors.		

TERM 4 ASSESSMENT TASK 1 - 2016

MATHEMATICS Extension 2: Question 2 Page 2

Suggested Solutions	Marks Awarded	Marker's Comments
a) ii) $\bar{z} - iz = 0$		
Let $z = x+iy$, $\bar{z} = x-iy$		
$\therefore x-iy - i(x+iy) = 0+0i$	1	
$x-iy-ix-i^2y = 0+0i$		
$x-iy-ix+y = 0+0i$		
$\therefore x+y+i(-x-y) = 0+0i$ (1)		
Either Equate real and imaginary parts		
$\therefore x+y=0$ or $-x-y=0$		
$\therefore x+y=0$ or $y=-x$	1	
	OR	
or from (1)		
$(x+y)(1-i)=0$		
As $1-i \neq 0$, $x+y=0$		
$y=-x$	1	

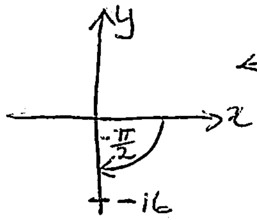
Comment

Generally well done. Some made careless errors, possibly from trying to save time by leaving out steps and doing work in their head.

A number of students didn't realise they had to equate real and imaginary parts.

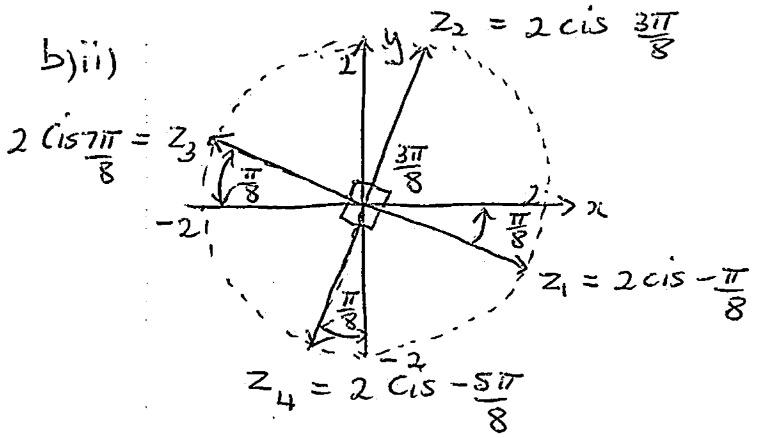
TERM 4 - ASSESSMENT TASK 1 - 2016

MATHEMATICS Extension 2: Question...2... Page 3

Suggested Solutions	Marks Awarded	Marker's Comments
b) i) $z^4 + 16i = 0$ $z^4 = -16i$ 		It can be useful to always draw this.
$= 16\left(\cos -\frac{\pi}{2} + i \sin -\frac{\pi}{2}\right)$	1	
OR $z^4 = 16 \operatorname{cis}\left(-\frac{\pi}{2}\right)$ $= 16 \operatorname{cis}\left(-\frac{\pi}{2} + 2k\pi\right)$ $z = 16^{\frac{1}{4}} \operatorname{cis}\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right)$ $= 2 \operatorname{cis}\left(\frac{\frac{-\pi + 4k\pi}{2}}{4}\right)$		
$= 2 \operatorname{cis} \frac{\pi(4k-1)}{8}$	1	
$k=0$, $z_1 = 2\left(\cos\left(-\frac{\pi}{8}\right) + i \sin\left(-\frac{\pi}{8}\right)\right)$ $k=1$, $z_2 = 2\left(\cos\left(\frac{3\pi}{8}\right) + i \sin\left(\frac{3\pi}{8}\right)\right)$ $k=2$, $z_3 = 2\left(\cos\left(\frac{7\pi}{8}\right) + i \sin\left(\frac{7\pi}{8}\right)\right)$ $k=3$, $z_4 = 2\left(\cos\left(\frac{11\pi}{8}\right) + i \sin\left(\frac{11\pi}{8}\right)\right)$ $= 2\left(\cos\left(-\frac{5\pi}{8}\right) + i \sin\left(-\frac{5\pi}{8}\right)\right)$	2	
Note $z_1 = 2\left(\cos\left(\frac{15\pi}{8}\right) + i \sin\left(\frac{15\pi}{8}\right)\right)$ and $z_4 = 2\left(\cos\left(\frac{11\pi}{8}\right) + i \sin\left(\frac{11\pi}{8}\right)\right)$ were accepted.		See note

Term 4 2016 Assessment Task 1

MATHEMATICS Extension 2: Question 2 Page 4

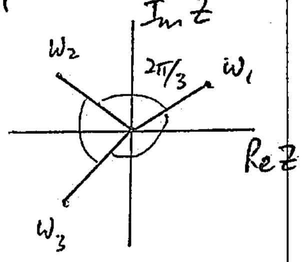
Suggested Solutions	Marks Awarded	Marker's Comments
Comments Common errors included: i) arguments $\frac{\pi}{2}$, $-\frac{3\pi}{4}$, $-\pi$ ii) mod $z^4 = -16$ iii) leaving answers in cis form. - Marks not deducted for this.		
b)ii) 	① ①	For showing roots and making it clear about the angles Making it clear that $\|z_n\| = 2$ Better responses drew circle $x^2+y^2=4$ as shown.
Comment Generally well done. Some needed to pay attention to detail.		
c)i) $x^3-1=0$ If ω is a root then $\omega^3-1=0$ $\therefore (\omega-1)(\omega^2+\omega+1)=0$ As $\omega \neq 1$, $\omega^2+\omega+1=0$ $\therefore \omega$ is a root of $x^2+x+1=0$	 ① ①	Comment. Most were able to factorize ω^3-1. Some didn't recognise that ω was not real.

TERM 4 2016 ASSESSMENT TASK 1

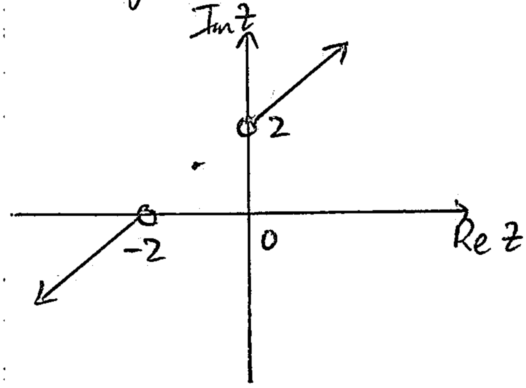
MATHEMATICS Extension 2: Question 2 Page 5

Suggested Solutions	Marks Awarded	Marker's Comments
c) ii) Let $n = 3m$ $\quad m \in \mathbb{Z}$	①	
Let $P(x) = (x+1)^{6m} + x^{6m} + 1$ ---- 1		Allow 1 mark for substituting $\omega^3 = 1$ or $x^2+x+1=0$ appropriately if no other marks are given.
Substitute $x = \omega$		
$P(\omega) = (\omega+1)^{6m} + \omega^{6m} + 1$		
Now $\omega + 1 = -\omega^2$		
$\therefore \quad = (-\omega^2)^{6m} + (\omega^3)^{2m} + 1$	①	
$= (-\omega^3)^{4m} + 1^{2m} + 1$		
$= (-1)^{4m} + 1 + 1$		
$= 1 + 1 + 1$		
$= 3$		
$\neq 0$	①	
$\therefore \omega$ is not a zero		
$\therefore x^2 + x + 1$ is not a divisor		
Comments • Students who let $n = 3$ tended to oversimplify the proof and generally were awarded a maximum of 1 mark. • Some students incorrectly used the remainder theorem by substituting $\omega^2 + \omega + 1$ for x expression $(x+1)^{2n} + x^{2n} + 1$		

JRAHS Term 4 2016 MATHEMATICS Extension 2: Question 3 Task 1

Suggested Solutions	Marks	Marker's Comments
Question 3 a) Since the cube roots of a complex number are equally spaced $\frac{2\pi}{3}$ radians apart on an Argand diagram, if	1	For $\arg z_1 - \arg z_2 = \frac{2\pi}{3}$
 $w_1 = 2\sqrt{3} + i$, then $w_2 = (2\sqrt{3}+i)(\text{cis}\,\frac{2\pi}{3})$ $= (2\sqrt{3}+i)(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ $= (2\sqrt{3}+i)(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ $= -\frac{3\sqrt{3}}{2} + \frac{5}{2}i$ ----- $= -2.6 + 2.5i$	1	For correct simplified form
and $w_3 = (2\sqrt{3}+i)(\text{cis}\,(-2\pi/3))$ $= (2\sqrt{3}+i)(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$ $= -\frac{\sqrt{3}}{2} - \frac{7}{2}i$ ...... $= -0.87 - 3\frac{1}{2}i$	1	For correct simplified form
If $w_2 = \sqrt{13}\left[\text{cis}\left(\tan^{-1}\frac{1}{2\sqrt{3}} + \frac{2\pi}{3}\right)\right]$ $w_3 = \sqrt{13}\left[\text{cis}\left(\tan^{-1}\frac{1}{2\sqrt{3}} - \frac{2\pi}{3}\right)\right]$, max 2 marks $w_2 = \sqrt{13}\,\text{cis}\,(2.375)$ $w_3 = \sqrt{13}\,\text{cis}\,(-1.813)$, max 2 marks		

JRAHS Term 4 2016 **MATHEMATICS Extension 2: Question** 3 Task 1

Suggested Solutions	Marks	Marker's Comments
b) $\arg(z+2) = \arg(z-2i)$		
	1	For both arms (st. line)
	1	For arrows + open circles

Note: The equation for the locus (which was not asked) can be obtained as follows:

If $z = x + iy$, then

$$\arg(x+2+iy) = \arg(x+i(y-2))$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x+2}\right) = \tan^{-1}\left(\frac{y-2}{x}\right)$$

$$\Rightarrow \quad yx = (x+2)(y-2)$$

$$\Rightarrow \quad y = x+2$$

For $x < -2$ and $x > 0$

JRAHS Term 4 2016 **MATHEMATICS Extension 2: Question** 3 task 1

Suggested Solutions	Marks	Marker's Comments
c) (i) $w = 1-\cos 2\theta - i\sin 2\theta$		
$\|w\| = \sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2}$	1	For correct real and imaginary parts
$= \sqrt{1 - 2\cos 2\theta + \underbrace{\cos^2 2\theta + \sin^2 2\theta}}$		
$= \sqrt{2-2\cos 2\theta}$	1	For simplification
$= \sqrt{2}\sqrt{1-\cos 2\theta}$, $1-\cos 2\theta = 2\sin^2\theta$		
$= \sqrt{2}.\sqrt{2}.\sqrt{\sin^2\theta}$	1	For double angle substitution and $\|x\| = \sqrt{x^2}$
$= 2\|\sin\theta\|$ as $\|x\| = \sqrt{x^2}$		
Alternate Solution:		
$w = 1-\cos 2\theta + i\sin 2\theta$		
$= 1-(1-2\sin^2\theta) + 2i\sin\theta\cos\theta$	1	For simplification ↕
$= 2\sin^2\theta + 2i\sin\theta\cos\theta$	1	For double angle substitution
$= 2\sin\theta(\sin\theta + i\cos\theta)$		
$= 2i\sin\theta(\cos\theta - i\sin\theta)$		
$\therefore \|w\| = \|2\|\cdot\|i\|\cdot\|\sin\theta\|\cdot\|\cos\theta - i\sin\theta\|$ -----	1	For correctly finding moduli of all quantities
$= 2\cdot 1\cdot\|\sin\theta\|\cdot 1$		
$= 2\|\sin\theta\|$		

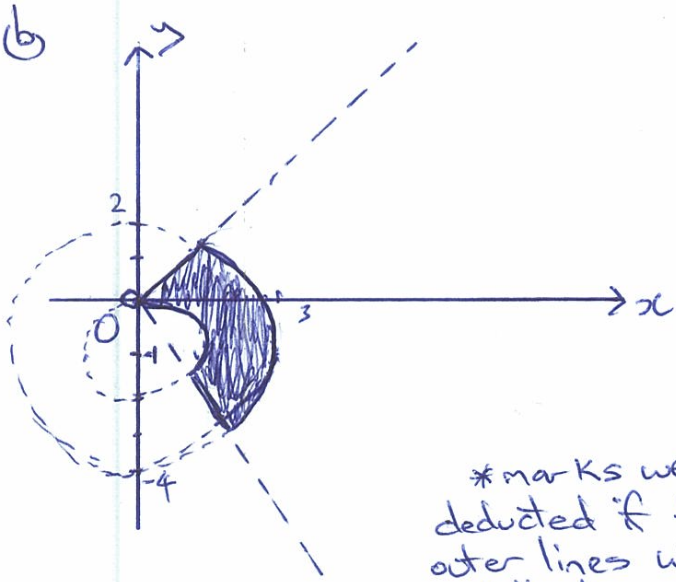
Suggested Solutions	Marks	Marker's Comments
c) (ii) $z^n - 1 = (z-1)(z-z_1)(z-z_2)\cdots(z-z_{n-1})$ where $z_k = cis\left(\frac{2\pi k}{n}\right) \quad 1 \le k \le n-1$ $= \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$	1 1	For correct linear factors including $z_0 = z-1$ Define $z_k = cis\left(\frac{2\pi k}{n}\right)$
(iii) $z^n - 1 = (z-1)(1+z+z^2+\cdots+z^{n-1})$ $\therefore 1+z+z^2+\cdots+z^{n-1} = (z-z_1)(z-z_2)\cdots(z_{n-1})$ If $z=1$, then LHS $= 1+1+1^2+\cdots+1$ (n times) $= 1\times n$ $= n$ RHS $= (1-z_1)(1-z_2)\cdots(1-z_{n-1})$ But $z_k = cis\left(\frac{2\pi k}{n}\right)$ $\therefore z_1 = cis(2\pi/n)$ $z_2 = cis(4\pi/n)$ $\vdots$ $z_{n-1} = cis\left[\frac{2(n-1)}{n}\pi\right]$ $\therefore n = \left(1-\cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}\right)\left(1-\cos\frac{4\pi}{n} - i\sin\frac{4\pi}{n}\right)\cdots\left(1-\cos\frac{2(n-1)\pi}{n} - \sin\frac{2(n-1)\pi}{n}\right)$	1 1	For expressing z^n-1 as a product of linear and "polynomial" factors For showing LHS $= n$ AND RHS $= (\cdots)(\cdots)\cdots(\cdots)$

JRAHS Term 4 2016 **MATHEMATICS Extension 2: Question** 3 Task 1

Suggested Solutions	Marks	Marker's Comments
c) (iv)		
$n = \left(1-\text{cis}\frac{2\pi}{n}\right)\left(1-\text{cis}\frac{4\pi}{n}\right)\cdots\left(1-\text{cis}\frac{2(n-1)\pi}{n}\right)$		... from (iii)
Take moduli of LHS and RHS	1	
$\lvert n\rvert = \left\lvert\left(1-\text{cis}\frac{2\pi}{n}\right)\left(1-\text{cis}\frac{4\pi}{n}\right)\cdots\left(1-\text{cis}\,2\frac{(n-1)\pi}{n}\right)\right\rvert$		
$\Rightarrow n = \left\lvert 1-\text{cis}\frac{2\pi}{n}\right\rvert\left\lvert 1-\text{cis}\frac{4\pi}{n}\right\rvert\cdots\left\lvert 1-\text{cis}\frac{2(n-1)}{n}\pi\right\rvert$ as $n \in \mathbb{R}$ and $\lvert z_1 z_2\cdots\rvert = \lvert z_1\rvert\cdot\lvert z_2\rvert\cdot\lvert z_3\rvert\cdots$		
$\Rightarrow n = 2\left\lvert\sin\frac{\pi}{n}\right\rvert\cdot 2\left\lvert\sin\frac{2\pi}{n}\right\rvert\cdot 2\left\lvert\sin\frac{4\pi}{n}\right\rvert\cdots\cdot 2\left\lvert\sin\frac{(n-1)}{n}\pi\right\rvert$ Using result of (i)		1
But $0 < \frac{(n-1)\pi}{n} \leq \pi$ for $n = 2, 3, 4\ldots$		
and $\sin\frac{(n-1)}{n}\pi > 0$	1	
$\therefore n = 2^{n-1}\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\left(\sin\frac{4\pi}{n}\right)\cdots\left(\sin\frac{(n-1)}{n}\pi\right)$		
$\Rightarrow \sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdot\sin\frac{4\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$		1 moduli of RHS & RHS 1 removal of moduli 1 2^{n-1} factors and using $\lvert w\rvert = 2\lvert\sin\theta\rvert$

1/4

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks Awarded	Marker's Comments
(a) $1-i=\sqrt{2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right)$	① for $\sqrt{2}$ and $-\pi/4$ or $\frac{7\pi}{4}$ ① for expanded form	
(b)  *marks weren't deducted if the outer lines weren't dotted.	① circle boundaries ① angle $\pm\pi/4$ ① open circle at centre ① shaded area	* If the open circle wasn't obvious, the mark wasn't given *A lot of students drew the smaller circle in the wrong place
(c) (i) see template *A lot of students drew the free vector $z-w$ *Some students didn't reflect $\frac{1}{z-w}$ about the x-axis. *Angles weren't measured, just placed rather than measured accurately ① Having $OP \parallel WZ$ and OP is 8cm in length ① OP' is reflection of OP in the x-axis ① $OP' = 2$cm in length ① angle between OP' and $OP'' = \theta = 70°$ and both $OP' = OP''$ in length		

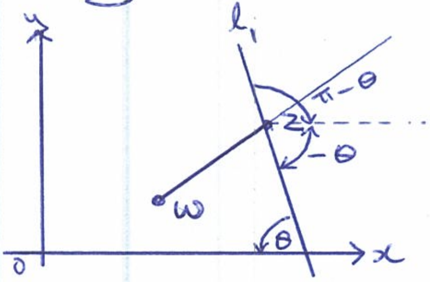
MATHEMATICS Extension 2: Question 4 continued

Suggested Solutions	Marks Awarded	Marker's Comments

(ii) * Badly done, a lot of students just reworded the question, or assumed "ω" was in the first quadrant.

* A lot of students mixed the point P up with a complex number.

* There were three cases, but most students only considered the case that was given and so scored zero marks.



If ω is to the left of l_1 then

$$-\theta < Arg(z-\omega) < \pi-\theta$$

now, $\theta > Arg\left(\frac{1}{z-\omega}\right) > \theta-\pi$ or/ $-\theta < -arg\left(\frac{1}{z-\omega}\right) < \pi-\theta$ 1

If a complex number is divided by $cis\theta$, then its a rotation by $-\theta$.
(ie subtract θ from each term using De Moivre's theorem)

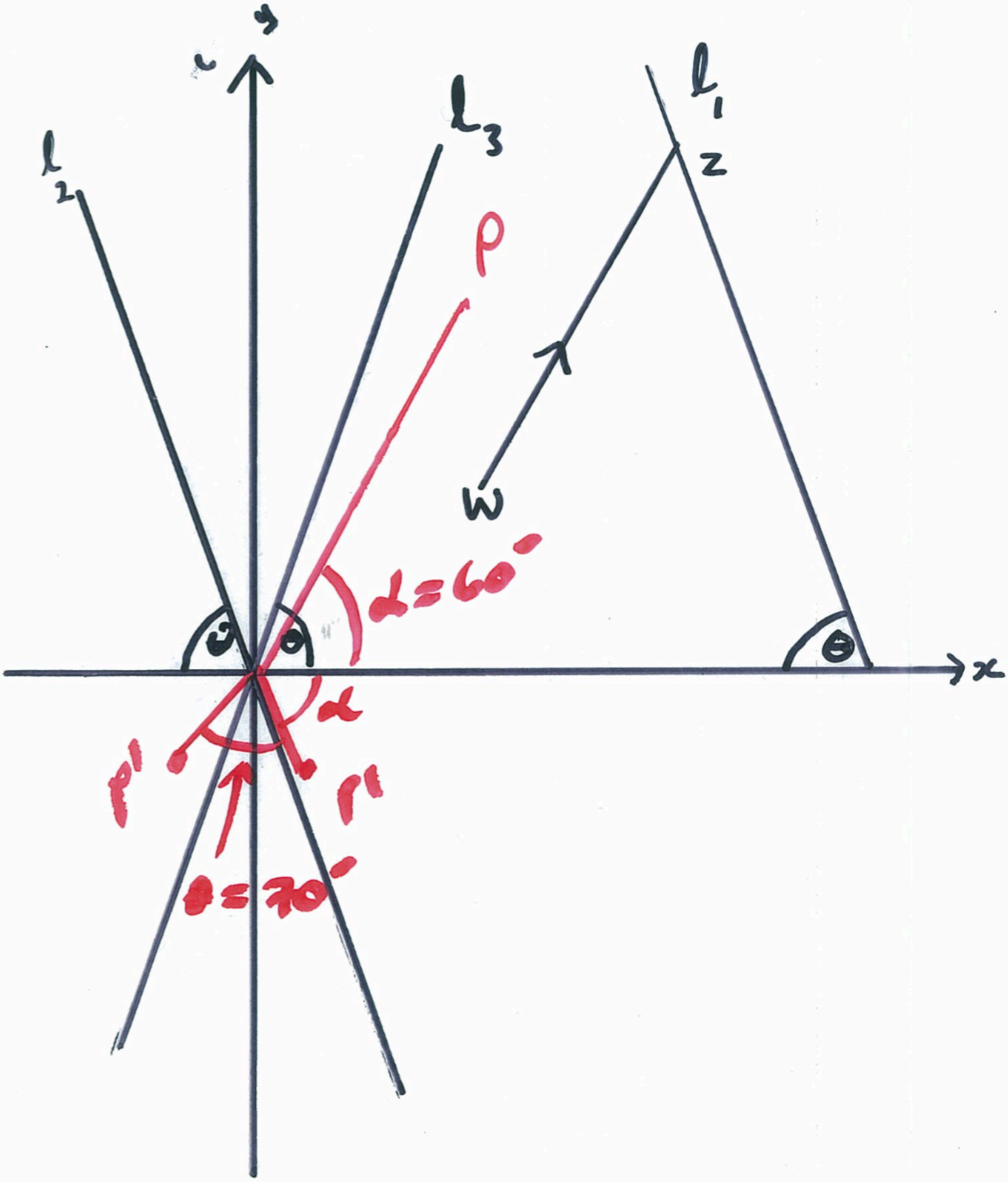
$$\therefore 0 > Arg\left(\frac{\frac{1}{z-\omega}}{cis\theta}\right) > -\pi$$

$$ie \quad -\pi < Arg\left(\frac{\frac{1}{z-\omega}}{cis\theta}\right) < 0$$ 1

Thus P'' lies below the x-axis, the imaginary part is negative.

* If P'' is represented by $a+bi$ then $b<0$.

* If this method was used then both endpoints needed to be discussed.

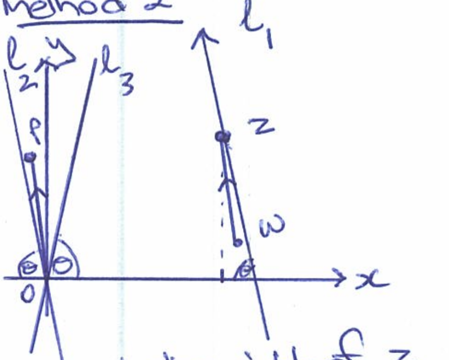


y
x
l1
l2
l3
z
w
p
p'
p'
α = 60°
α
θ
θ
θ = 70°

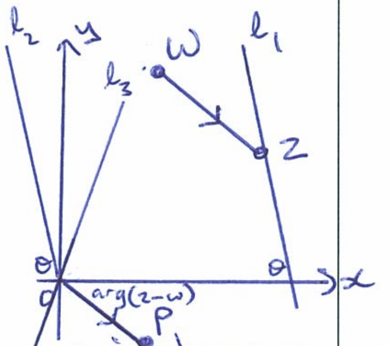
MATHEMATICS Extension 2: Question 4 continued

Suggested Solutions	Marks Awarded	Marker's Comments

Method 2

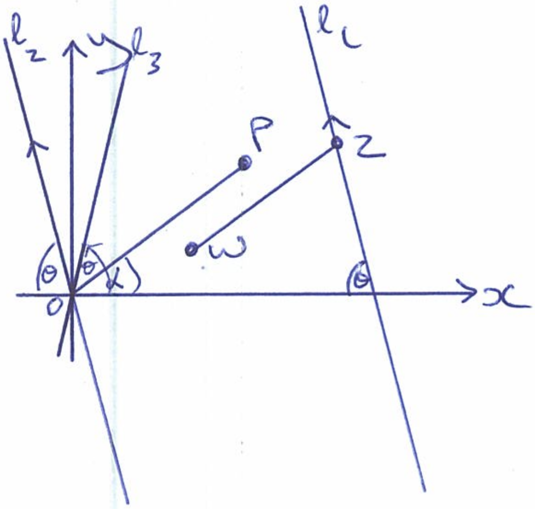


w is to the right of z but still to the left of l_1



w is above z but to the left of l_1

*needed to show all 3 cases and discuss them clearly.



*Some students introduced new variables without defining them.

let $\alpha = \arg(z-w)$

As w is to the left of l_1

$(z-w)$ is to the right of l_2

P', $\left(\frac{1}{z-w}\right)$, is to the right of l_3

P', $\left(\frac{1}{z-w}\right)$, is the reflection about the x-axis — (1)

$\therefore$ Rotate $\left(\frac{1}{z-w}\right)$ by $\text{cis}(-\theta)$

P'' will be below the x-axis

$\therefore \text{cis}(-\theta) \times \frac{1}{z-w} = a + bi$

$\therefore \text{Im}\left[\text{cis}(-\theta) \times \frac{1}{z-w}\right] < 0$ — (1)

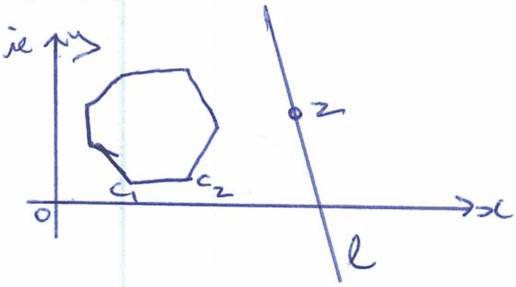
$\therefore b < 0$ for all points w to the left of l_1

MATHEMATICS Extension 2: Question ...4... cont.

Suggested Solutions	Marks Awarded	Marker's Comments

(iii) Assume z is outside a given convex polygon. Then its always possible to draw a line, l, through z which doesn't intersect the polygon.

ie



*needed to use (ii) and hence talk about a line, and the points being to the left or right of it.

If $\frac{1}{z-c_1}+\frac{1}{z-c_2}+\frac{1}{z-c_3}+\dots+\frac{1}{z-c_n}=0$

then $\left[\frac{1}{z-c_1}+\frac{1}{z-c_2}+\dots+\frac{1}{z-c_n}\right]\times \text{cis}(-\theta)=0$ for $\text{cis}(-\theta)\neq 0$

$$\frac{\text{cis}(-\theta)}{z-c_1}+\frac{\text{cis}(-\theta)}{z-c_2}+\dots+\frac{\text{cis}(-\theta)}{z-c_n}=0$$

$$\text{Re}\left[\frac{\text{cis}(-\theta)}{z-c_1}+\frac{\text{cis}(-\theta)}{z-c_2}+\dots+\frac{\text{cis}(-\theta)}{z-c_n}\right]+\text{Im}\left[\frac{\text{cis}(-\theta)}{z-c_1}+\dots+\frac{\text{cis}(-\theta)}{z-c_n}\right]=0$$

$$\therefore \text{Im}\left[\frac{\text{cis}(-\theta)}{z-c_1}+\frac{\text{cis}(-\theta)}{z-c_2}+\dots+\frac{\text{cis}(-\theta)}{z-c_n}\right]=0 \quad (1)$$

From (ii) for all points c_n to the left of l,

$$\text{Im}\left[\frac{\text{cis}(-\theta)}{z-c_n}\right]<0 \quad (1)$$

$$\therefore \text{Im}\left[\frac{\text{cis}(-\theta)}{z-c_1}\right]+\text{Im}\left[\frac{\text{cis}(-\theta)}{z-c_2}\right]+\dots+\text{Im}\left[\frac{\text{cis}(-\theta)}{z-c_n}\right]<0$$

$$\therefore \text{Im}\left[\frac{\text{cis}(-\theta)}{z-c_1}+\dots+\frac{\text{cis}(-\theta)}{z-c_n}\right]\neq 0 \quad (1)$$

$\therefore$ contradiction

*similarly if all c_n lies to the right of l

$\therefore$ z can't be outside the polygon