Name: ______________________

Class: ______________________

YEAR 12

ASSESSMENT TASK 3

TERM 2, 2018

Mathematics Extension 2

General Instructions

- Working time – 90 minutes + 5 minutes reading
- Board approved calculators may be used
- Write using black or blue pen
- All necessary working should be shown in all questions
- One double sided sheet of A4 paper may be brought in and placed on the desk
- Write your student number at the top of every page of your answer sheets.

5 Long Response Questions–12 marks each

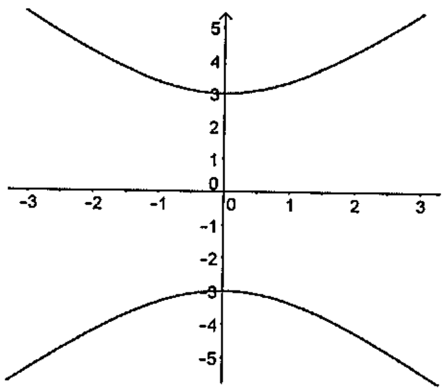
Total marks – 60

Attempt all questions

Start each question on a new sheet of paper.

Question 1 (12 marks)

a) The region bounded by the lines $x = \pm 2$ and the hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1$ is rotated about the $y-$axis.



i) Using the method of cylindrical shells, show that $V = 4\pi \int_0^2 x\sqrt{9 + \frac{9}{4}x^2}\, dx$. **2**

ii) Hence, evaluate the volume of the solid. **2**

b) Points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the hyperbola $xy = c^2$.

i) Show that the equation of the tangent to the hyperbola at P is $x + p^2y = 2cp$. **2**

ii) If the tangents at the points P and Q meet at the point $R(x_0, y_0)$, prove that

$pq = \frac{x_0}{y_0}$ and $p + q = \frac{2c}{y_0}$. **3**

iii) If the length of chord PQ is d units, show that $d^2 = c^2(p-q)^2(1 + \frac{1}{p^2q^2})$. **1**

iv) If d remains fixed, deduce that the locus of R has equation

$4c^2(x^2 + y^2)(c^2 - xy) = x^2y^2d^2$. **2**

Question 2 (12 marks) Start a NEW sheet of paper

a) A particle is projected vertically upwards in a resisting medium where the resistance varies as the square of the velocity and k is the constant of variation. If the velocity of projection is $v_0 \tan\alpha$,

i) Show that the equation of motion is given by $\ddot{x} = -kv^2 - g$ where $\ddot{x}$ is the acceleration of the particle and g is the acceleration due to gravity **1**

ii) Show that the maximum height, H, reached is given by: $H = \dfrac{1}{2k}\ln(\dfrac{g + k{v_0}^2\tan^2\alpha}{g})$ **3**

iii) Show that the particle returns to the point of projection with velocity $v_0 \sin\alpha$ given that v_0 is the terminal velocity **4**

iv) Show that the time of ascent is $\dfrac{v_0\alpha}{g}$ **4**

Question 3 (12 marks) Start a NEW sheet of paper

a) A particle of mass, m kg, moves in a straight line with velocity v metres/second, under a constant force, P newtons, and a resistance R newtons. Initially, the particle has a speed v_0 m/s. It is given that $R = 5 + 3v$ and $P = 10$.

i) Show that $\ddot{x} = \dfrac{1}{m}(5 - 3v)$. **1**

ii) Show that the velocity, $v = \dfrac{5}{3}(1 - e^{-\frac{3t}{m}}) + v_0 e^{-\frac{3t}{m}}$. **3**

iii) Find the terminal velocity of the particle. **1**

iv) When the particle accelerates from v_0 to v_1, show that the distance travelled, x metres, is given by: $x = \dfrac{m}{9}[3(v_0 - v_1) + 5\ln(\dfrac{5 - 3v_0}{5 - 3v_1})]$. **3**

b) The area bounded by the lines $y = x + 1, x = 2, x = 4$ and the x−axis, is rotated about the line $x = 5$. By taking slices perpendicular to the y−axis, find the volume of the solid formed. **4**

Question 4 (12 marks) Start a NEW sheet of paper

a) On a rectangular hyperbola $xy = c^2$, point P has coordinates $(cp, \frac{c}{p})$. It is given that the equations of the tangent and the normal at this point are $x + p^2y - 2cp = 0$ and $p^3x - py = c(p^4 - 1)$ respectively. (Do NOT prove these).

i) Find the coordinates of R where the normal crosses the other branch of the hyperbola. **2**

ii) The coordinates of Q where the tangent meets the y−axis is $(0, \frac{2c}{p})$. Hence show that the area of ΔPQR is $\frac{c^2}{2p^4}(1 + p^4)^2$. **4**

iii) Hence, find the value(s) of p such that the area is a minimum. **2**

b) Using the letters of the word CALCULUS, how many 'words' can be made if the 'words' consist of:

i) All eight letters? **1**

ii) Only five letters? **3**

Question 5 (12 marks) Start a NEW sheet of paper

a) The base of a solid is the region bounded by the line $y = 2x$, the parabola $y = 4x - x^2$ and the $x-$axis. Cross-sections parallel to the $x-$axis are right-angled isosceles triangles with the hypotenuse in the base of the solid.

i) Show that the volume of the solid is given by: $V = \dfrac{1}{16}\displaystyle\int_0^4 (4 + 2\sqrt{4-y} - y)^2 dy$ **3**

ii) Hence, find the exact volume of the solid **2**

b) Two players A and B play a game of chance comprising several turns in which each player throws a fair 6-sided die. The possible outcomes are a draw (A, B throw the same score), A wins (A's score is higher than B's) or B wins. The game is over when either player first records two wins. Let p_n be the probability the game ends on the n^{th} turn. Let q_n be the probability the game does not end in n or fewer turns.

i) Explain why, on each turn, the probability that A wins is $\dfrac{5}{12}$ **1**

ii) Explain why $p_2 + q_2 = 1$ **1**

iii) Explain why $p_n + q_n = q_{n-1}$, for $n = 3, 4, 5...$ and deduce that $\displaystyle\sum_{n=2}^{n} p_k = 1 - q_n$, $n \geq 2$ **2**

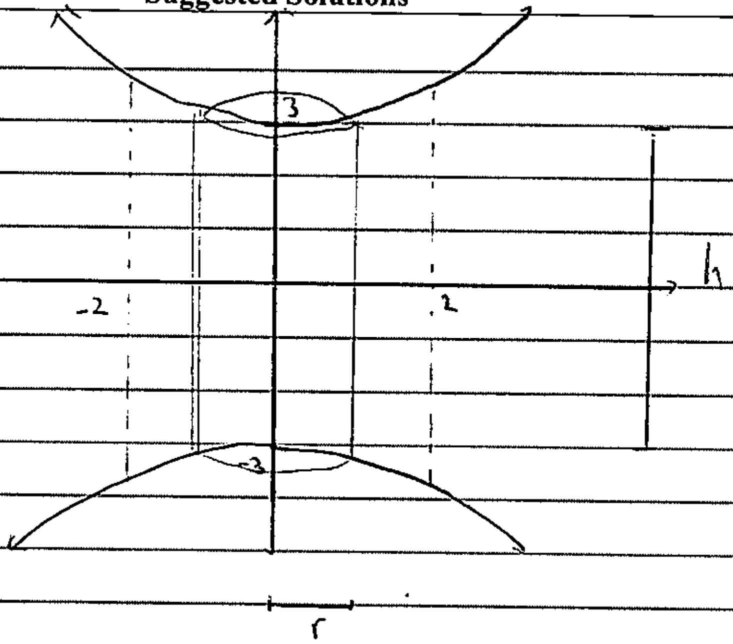
iv) Show that $q_n = \dfrac{25n^2 - 5n + 4}{4 \times 6^n}$ and $p_n = \dfrac{25(n-1)(5n-8)}{4 \times 6^n}, n \geq 2$ **3**

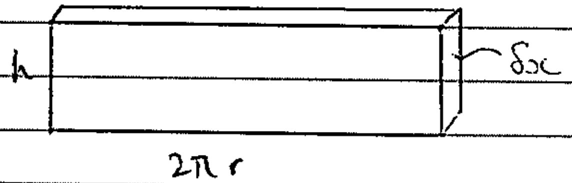
End of examination

MATHEMATICS Extension 2: Question Q1

Suggested Solutions	Marks	Marker's Comments

a) i)





$h = 2|y| \quad r = |x|$

$\therefore V_{\text{Shell}} = 2\pi r h \delta x$

$= 2\pi |x| . 2|y| \delta x$

$= 4\pi |x| \sqrt{9 + \frac{9x^2}{4}} \delta x$

$\frac{y^2}{9} - \frac{x^2}{4} = 1$

$\frac{y^2}{9} = 1 + \frac{x^2}{4}$

$y^2 = 9\left(1 + \frac{x^2}{4}\right)$

$= 9 + \frac{9x^2}{4}$

$\therefore |y| = \sqrt{9 + \frac{9x^2}{4}}$ — 1

Taking limits from $x = 2$ to $x = 0$.

$\therefore V_{\text{solid}} = \lim_{\delta x \to 0} \sum_{0}^{2} 4\pi x \sqrt{9 + \frac{9x^2}{4}} \delta x$

$= 4\pi \int_0^2 x \sqrt{9 + \frac{9}{4}x^2}\, dx$ — 1

Must have limits!!!

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
ii) $V = 4\pi\int_0^2 x\left(9+\frac{9}{4}x^2\right)^{\frac{1}{2}}dx$		
$= \frac{8\pi}{9}\int_0^2 \frac{9x}{2}\left(9+\frac{9}{4}x^2\right)^{\frac{1}{2}}dx$		
$= \frac{8\pi}{9}\left[\frac{2\left(9+\frac{9}{4}x^2\right)^{3/2}}{3}\right]_0^2$	1	
$= \frac{16\pi}{27}\left[18^{3/2} - 9^{3/2}\right]$		
$= \frac{16\pi}{27}\left[18\sqrt{18} - 9\sqrt{9}\right]$		
$= \frac{16\pi}{27}\left(54\sqrt{2} - 27\right)$		
$= 16\pi\left(2\sqrt{2} - 1\right)$	1	
b) i) $y = \frac{c^2}{x}$		
$\frac{dy}{dx} = -c^2x^{-2}$		
When $x = cp$, $\frac{dy}{dx} = \frac{-c^2}{c^2p^2} = \frac{-1}{p^2}$	1	
$\therefore$ tangent at P is given by:		
$y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$		
$p^2y - cp = -x + cp$		
$\therefore x + p^2y = 2cp$	1	

MATHEMATICS Extension 1 : Question........

Suggested Solutions	Marks	Marker's Comments
ii) Similarly, tangent at $Q\left(cq, \frac{c}{q}\right)$ is given by		
$x + q^2 y = 2cq$ - Eqn 2		
Eqn 1 - Eqn 2		
$x + p^2 y = 2cp$		
$-\ \ x + q^2 y = 2cq$		
$y(p^2 - q^2) = 2c(p-q)$		
$\therefore y(p+q) = 2c$		
$y = \dfrac{2c}{p+q} \quad \therefore y_0 = \dfrac{2c}{p+q}$		1
Sub $y_0 = \dfrac{2c}{p+q}$ into Eqn 1.		
$x_0 + p^2\left(\dfrac{2c}{p+q}\right) = 2cp$		
$(p+q)x_0 + 2cp^2 = 2cp(p+q)$		
$\therefore px_0 + qx_0 + 2cp^2 = 2cp^2 + 2cpq$		
$\therefore x_0(p+q) = 2cpq$		1
$\therefore pq = \dfrac{x_0(p+q)}{2c}$		
$= \dfrac{x_0}{2c/p+q}$		
$= \dfrac{x_0}{y_0}$		1
and		
$p+q = \dfrac{2c}{y_0}$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments

iii) $d^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$

$= c^2(p-q)^2 + c^2\left(\frac{1}{p} - \frac{1}{q}\right)^2$

$= c^2(p-q)^2 + c^2\left(\frac{q-p}{pq}\right)^2$ — must see this line!!!

$= c^2(p-q)^2 + c^2\left(\frac{p-q}{pq}\right)^2$ (Since $(p-q)^2 = (q-p)^2$)

$= c^2(p-q)^2\left(1 + \frac{1}{p^2q^2}\right)$

iv) $x_0 = \frac{2cpq}{p+q}$ $\quad y_0 = \frac{2c}{p+q}$

$(p-q)^2 = (p+q)^2 - 4pq$

$\therefore \quad = \frac{4c^2}{y_0^2} - \frac{4x_0}{y_0}$ — 1

$\frac{1}{p^2q^2} = \frac{1}{(pq)^2} = \frac{y_0^2}{x_0^2}$

$\therefore d^2 = c^2\left(\frac{4c^2}{y_0^2} - \frac{4x_0}{y_0}\right)\left(1 + \frac{y_0^2}{x_0^2}\right)$

$= c^2\left(\frac{4c^2 - 4x_0y_0}{y_0^2}\right)\left(\frac{x_0^2 + y_0^2}{x_0^2}\right)$

$\therefore x_0^2y_0^2d^2 = c^2(4c^2 - 4x_0y_0)(x_0^2 + y_0^2)$

$\therefore 4c^2(x_0^2 + y_0^2)(c^2 - x_0y_0) = x_0^2y_0^2d^2$

$\therefore R$ has locus $4c^2(x^2+y^2)(c^2 - xy) = x^2y^2d^2$ — 1

X2 MATHEMATICS: Question 2		
Suggested Solutions	**Marks**	**Marker's Comments**
$\uparrow$ +ve $\quad \downarrow kv^2 \downarrow mg$		students failed to indicate positive direction
(i) $m\ddot{x} = -kv^2 - mg$ since $m=1$ $\ddot{x} = -kv^2 - g$	1	or did not have $m\ddot{x} =$ expression.
(ii) $\ddot{x} = v\frac{dv}{dx}$ $v\frac{dv}{dx} = -kv^2 - g$ $\int_{v_0 \tan\alpha}^{0} \frac{v\,dv}{kv^2+g} = \int_0^H -dx$	1	including correct limits
$\therefore -H = \frac{1}{2k}\left[\ln(kv^2+g)\right]_{v_0\tan\alpha}^{0}$	1	Some students had issue with negative or limits around wrong way.
$\therefore H = \frac{1}{2k}\left[\ln(kv^2+g)\right]_0^{v_0\tan\alpha}$ $= \frac{1}{2k}\left[-\ln g + \ln(kv_0^2\tan^2\alpha + g)\right]$		A show question this step must appear.
$kv^2+g>0 \quad = \frac{1}{2k}\left[\ln\left(\frac{kv_0^2\tan^2\alpha+g}{g}\right)\right]$	1	
could also use $+C$		

X2 MATHEMATICS: Question 2		
Suggested Solutions	**Marks**	**Marker's Comments**
$\uparrow kv^2$ $\downarrow mg$ $\quad \downarrow +ve$ OR $\uparrow kv^2 \quad \uparrow +ve$ $\downarrow mg$ $\ddot{x} = kv^2 - g$ but limits will be different $m\ddot{x} = mg - kv^2 \quad m = 1$ $\ddot{x} = g - kv^2$ $\int_0^V \frac{v\,dv}{g - kv^2} = \int_0^H dx$	1	Students didn't find positive direction.
$H = -\frac{1}{2k}\left[\ln(g - kv^2)\right]_0^V \quad g - kv^2 > 0.$ $= -\frac{1}{2k}\left[\ln(g - kv^2) - \ln g\right]$ $= \ln\left(\frac{g}{g - kv^2}\right)$	1	
Vo is terminal velocity $\ddot{x} = 0 \quad g - kv_0^2 = 0$ $\therefore v_0^2 = \frac{g}{k}$	1	
equate H from part (i) $\frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right) = \frac{1}{2k}\ln\left(\frac{g + kv_0^2\tan^2\alpha}{g}\right)$ $\therefore \frac{g}{g - kv^2} = \frac{g + kv_0^2\tan^2\alpha}{g}$		

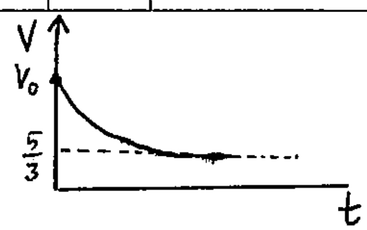
MATHEMATICS: Question

Suggested Solutions	Marks	Marker's Comments
$g - kv^2 = \dfrac{g^2}{g + kv_0^2\tan^2\alpha}$	1	
but $v_0^2 = \dfrac{g}{k}$.		Students failed to read question carefully and ~~failed~~ did not use this fact
$\therefore \cancel{g}\, kv^2 = g - \dfrac{g^2}{g+g\tan^2\alpha}$.		
$kv^2 = \dfrac{g^2+g^2\tan^2\alpha-g^2}{g+g\tan^2\alpha}$.		
$v^2 = \dfrac{g(\cancel{}\ \tan^2\alpha)}{k(1+\tan^2\alpha)}$		Careless algebraic errors losing g's and k's.
Now $1+\tan^2\alpha = \sec^2\alpha$.		
$= \dfrac{g}{k} \times \cos^2\alpha \times \dfrac{\sin^2\alpha}{\cos^2\alpha}$.		
$v = v_0^2\sin^2\alpha$	1	
$\therefore v = v_0\sin\alpha$. as $v > 0$.		

MATHEMATICS: Question		
Suggested Solutions	**Marks**	**Marker's Comments**
$\ddot{x} = -kv^2 - g$		
$\frac{dv}{dt} = -(kv^2+g)$.		
$-\int_{v_0\tan\alpha}^{0} \frac{1}{kv^2+g} = \int_0^T t\,dt$.	1	
$-T = \frac{1}{k}\left[\frac{\sqrt{k}}{\sqrt{g}} \tan^{-1} \frac{V}{\sqrt{g/k}}\right]_{v_0\tan\alpha}^{0}$	1	
$= \frac{1}{\sqrt{kg}} \tan^{-1} \frac{\sqrt{k}}{\sqrt{g}} v_0 \tan\alpha - \tan^{-1} 0$	1	
but $v_0 = \sqrt{\frac{g}{k}}$.		
$= \frac{1}{\sqrt{kg}} \tan^{-1} \frac{\sqrt{k}}{\sqrt{g}} \times \frac{\sqrt{g}}{k} \tan\alpha$.		
$= \frac{1}{\sqrt{kg}} \tan^{-1} (\tan\alpha)$		NB $\tan^{-1}(2\tan\alpha) \neq 2$. ∴ had to clearly indicate $\tan^{-1}\sqrt{\frac{k}{g}} v_0 \tan\alpha = \tan^{-1}(\tan\alpha)$ not $\sqrt{\frac{kv_0}{g}}\,\alpha$.
$= \frac{\sqrt{g}}{\sqrt{k}} \times \frac{1}{g}\alpha$		
$= \frac{v_0}{g}\alpha$		

Task 3 : Term 2 : 2018 **MATHEMATICS Extension 2: Question** 3 JRAHS

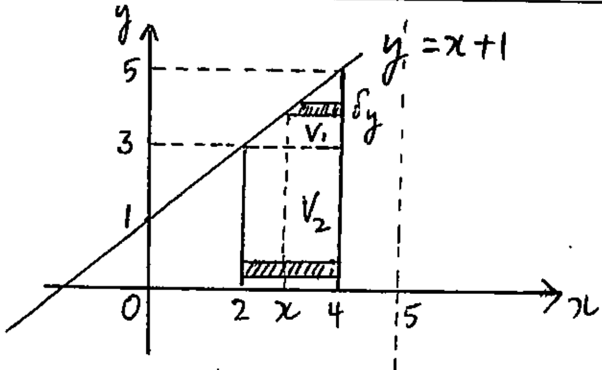
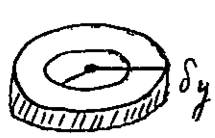
Suggested Solutions	Marks	Marker's Comments
a) (i) $R = 5+3v \longleftarrow m \longrightarrow P = 10 \quad + \rightarrow$ $m\ddot{x} = F_{net} \cdots$ (Newton's 2nd law) $= P - R$ $= 10 - 5 - 3v$ $\therefore \ddot{x} = \frac{1}{m}(5-3v)$	1	For correct derivation Note: when $v=0,\ R=5<P$ $v=1,\ R=8<P$ $v=2,\ R=11>P$ $v=\frac{5}{3},\ R=10=P$ $v=v_0,\ R=5+V_0\ ?\ P$
(ii) $\frac{dv}{dt} = \frac{1}{m}(5-3v)$ $\int_0^t dt = \frac{1}{m}\int_{v_0}^{v}\frac{dv}{5-3v}$	1	For splitting differential Eq. and integrating
$\Rightarrow \quad t = -\frac{m}{3}\ln\lvert 5-3v\rvert \Big]_{v_0}^{v}$ $\Rightarrow \ln(5-3v) - \ln(5-3v_0) = -\frac{3t}{m}$	1	For integrating
$\Rightarrow \ln\left(\frac{5-3v}{5-3v_0}\right) = -\frac{3t}{m}$ i.e. $\frac{5-3v}{5-3v_0} = e^{-3t/m}$ $5-3v = 5e^{-3t/m} + 3V_0 e^{-3t/m}$ $3v = 5\left(1-e^{-3t/m}\right) + 3V_0 e^{-3t/m}$ $\therefore v = \frac{5}{3}\left(1-e^{-3t/m}\right) + V_0 e^{-3t/m}$ as reqd.	1	Algebraic manipulation Note: If $5-3v<0$, then same v is obtained!!!



Task 3 Term 2 : 2018 **MATHEMATICS Extension 2: Question** 3 JRAHS

Suggested Solutions	Marks	Marker's Comments
a) (iii) Terminal velocity reached when $\ddot{x}=0 \Rightarrow v_T = \frac{5}{3}\ m.s^{-1}$	1	Note $v \to v_T$ when $t \to \infty$ $\Rightarrow e^{-3t/m} \to 0$
(iv) $\frac{v\,dv}{dx} = \frac{1}{m}(5-3v)$ $\Rightarrow \int_0^x dx = \int_{V_0}^{V_1} \frac{vm\,dv}{5-3v}$ $\Rightarrow \frac{x}{m} = -\frac{1}{3}\int \frac{5-3v-5}{5-3v}\,dv$		Note: $\frac{v}{5-3v} \equiv A + B$ $\Rightarrow v = A(5-3v) + B(5-3v)$ $\Rightarrow A = -\frac{1}{3}$ and $B = +\frac{5}{3}$
$\therefore x = -\frac{m}{3}\int_{V_0}^{V_1}\left[1 - \frac{5}{5-3v}\right]dv$	1	Splitting integral
$= -\frac{m}{3}\left[v + \frac{5}{3}\ln(5-3v)\right]\Big\|_{V_0}^{V_1}$	1	For actual integration
$= -\frac{m}{9}\left[3(V_1 - V_0) + 5\ln\left(\frac{5-3V_1}{5-3V_0}\right)\right]$	1	For algebraic manipulation
$= \frac{m}{9}\left[3(V_0 - V_1) + 5\ln\left(\frac{5-3V_0}{5-3V_1}\right)\right]$ as reqd.		

Task 3 : Term 2 : 2018 **MATHEMATICS Extension 2: Question** 3 JRAHS

Suggested Solutions	Marks	Marker's Comments
b) 		 $r_{outer} = 5-x = 6-y$ $r_{inner} = 5-4 = 1$
Consider region V_1: $\delta V_1 = \pi\left[(5-x)^2 - 1^2\right]\delta y$ $= \pi\left[(6-y)^2 - 1\right]\delta y$ $\therefore V_1 = \lim_{\delta y \to 0} \sum_{y=3}^{5} \pi\left[(6-y)^2 - 1\right]\delta y$	1	For correctly setting up Volume of slice $\perp$ axis (zero if // to axis)
$= \pi\int_3^5 (35 - 12y + y^2)\,dy$ $= \pi\left(35y - 6y^2 + y^3/3\right)\Big\vert_3^5$ $= \frac{20\pi}{3}\ u^3$	1	For correct computation
Consider region V_2: $V_2 = \int_0^3 \pi\left[(5-2)^2 - (5-4)^2\right]dy$ $= 24\pi\ u^3$	1	For volume of cylinder with hole
$\therefore$ Total Volume $= \frac{20\pi}{3} + 24\pi$ $= 30\frac{2}{3}\pi\ u^3$	1	For combined volume

Note: if slice // axis of rotation
max 2 of 4
if $V_T = \frac{92}{3}\pi\ u^3$

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments
a) i) normal: $p^3x - py = c(p^4-1) \quad -①$		
$xy = c^2$		
$y = \dfrac{c^2}{x} \quad -②$		
sub. ② into ①		
$p^3x - p\left(\dfrac{c^2}{x}\right) = c(p^4-1)$		
$p^3x^2 - pc^2 = xc(p^4-1)$		
$p^3x^2 - c(p^4-1)x - pc^2 = 0$		
sum of roots: (let roots be x_1, x_2)		
$x_1 + x_2 = \dfrac{c(p^4-1)}{p^3}$		
one root is cp.		
$\therefore x_1 + cp = \dfrac{cp^4 - c}{p^3}$		
$x_1 + cp = cp - \dfrac{c}{p^3}$		
$\therefore x_1 = -\dfrac{c}{p^3}$	1	
$\therefore y = \dfrac{c^2}{-\frac{c}{p^3}}$		
$= -cp^3$		
$\therefore R$ is $\left(-\dfrac{c}{p^3}, -cp^3\right)$	1	
ii) $P\left(cp, \dfrac{c}{p}\right) \quad Q\left(0, \dfrac{2c}{p}\right)$		
$d_{PQ} = \sqrt{(cp)^2 + \left(\dfrac{c}{p} - \dfrac{2c}{p}\right)^2}$		
$= \sqrt{c^2p^2 + \dfrac{c^2}{p^2}}$		
$= \sqrt{c^2\left(p^2 + \dfrac{1}{p^2}\right)}$		
$= c\sqrt{\dfrac{p^4+1}{p^2}}$	1	must show that c is outside of square root.
$= \dfrac{c}{p}\sqrt{p^4+1}$		

MATHEMATICS Extension 2: Question…4…

Suggested Solutions	Marks	Marker's Comments
$d_{PR} = \sqrt{\left(cp + \frac{c}{p^3}\right)^2 + \left(\frac{c}{p} + cp^3\right)^2}$		
$= \sqrt{c^2\left(\frac{p^4+1}{p^3}\right)^2 + c^2\left(\frac{1+p^4}{p}\right)^2}$	1	some factorising or expanding needed
$= c\sqrt{\frac{(p^4+1)^2}{p^6} + \frac{(p^4+1)^2}{p^2}}$		
$= c(p^4+1)\sqrt{\frac{1}{p^6} + \frac{1}{p^2}}$		
$= c(p^4+1)\sqrt{\frac{1}{p^2}\left(\frac{1}{p^4}+1\right)}$		
$= \frac{c(p^4+1)}{p}\sqrt{\frac{1+p^4}{p^4}}$		
$= \frac{c(p^4+1)}{p^3}\sqrt{p^4+1}$	1	
$\therefore$ Area $= \frac{1}{2} \times d_{PQ} \times d_{PR}$		
$= \frac{1}{2} \times \frac{c}{p}\sqrt{p^4+1} \times \frac{c(p^4+1)}{p^3}\sqrt{p^4+1}$	1	correct formula and substitution.
$= \frac{c^2}{2p^4}(p^4+1)(p^4+1)$		
$= \frac{c^2}{2p^4}(p^4+1)^2$		
NOTE: Perpendicular distance:		
$x + p^2y - 2cp = 0 \quad R\left(-\frac{c}{p^3}, -cp^3\right)$		
$d = \frac{\left\lvert -\frac{c}{p^3} + p^2(-cp^3) - 2cp \right\rvert}{\sqrt{1^2 + p^4}}$		maximum 3 marks.
$= \frac{\left\lvert -\frac{c}{p^3} - cp^5 - 2cp \right\rvert}{\sqrt{1+p^4}}$		Could not explain why a negative sign was needed, since p can be positive or negative.
$= \frac{\left\lvert \frac{-c(1+p^8+2p^4)}{p^3} \right\rvert}{\sqrt{1+p^4}}$		

MATHEMATICS Extension 2: Question 4		
Suggested Solutions	Marks	Marker's Comments
iii) $A = \frac{c^2}{2p^4}(1+p^4)^2$		
$\frac{dA}{dp} = \frac{-4c^2}{2p^5}(1+p^4)^2$		
$+\frac{c^2}{2p^4} \times 2(1+p^4) \times (4p^3)$		
$= \frac{-2c^2}{p^5}(1+p^4)^2 + \frac{4c^2}{p}(1+p^4)$		
$= \frac{2c^2}{p}(1+p^4)\left(-\frac{(1+p^4)}{p^4}+2\right)$		
for stationary points, $\frac{dA}{dp} = 0$.		
$\therefore \frac{2c^2}{p}(1+p^4)\left(\frac{-1-p^4+2p^4}{p^4}\right) = 0$		
$\therefore 1+p^4 = 0$, $\frac{p^4-1}{p^4} = 0$		
no solution $\therefore p^4-1 = 0$		
$p^4 = 1$		
$p = \pm 1$	1	
$\frac{dA}{dp} = \frac{2c^2}{p^5}(1+p^4)(p^4-1)$		
$= \frac{2c^2}{p^5}(p^8-1)$		
$\therefore \frac{d^2A}{dp^2} = \frac{-10c^2}{p^6}(p^8-1) + \frac{2c^2}{p^5}(8p^7)$		
$= -10c^2p^2 + \frac{10c^2}{p^6} + 16c^2p^2$		
$= 6c^2p^2 + \frac{10c^2}{p^6}$		
when $p=1$, $\frac{d^2A}{dp^2} = 6c^2 + 10c^2$		
$= 16c^2 > 0$		
$\therefore p=1$ is a minimum as it concaves up.		

MATHEMATICS Extension 2: Question 4

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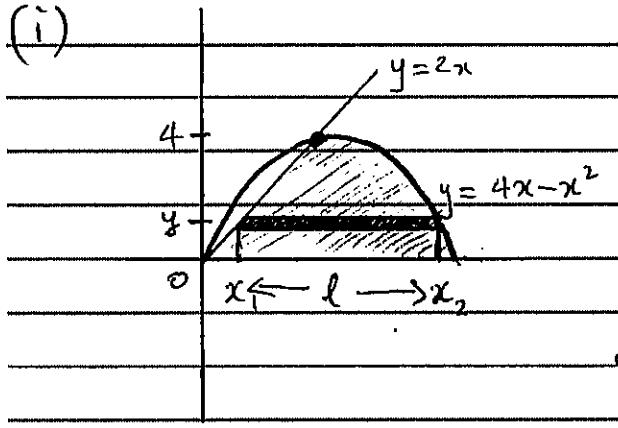
when $p=-1$, $\frac{d^2A}{dp^2} = 6c^2 + 10c^2$		
$= 16c^2 > 0$		
$\therefore$ concave up		
$\therefore p=-1$ is a minimum	1	$\Rightarrow$ testing $p = \pm 1$
$\therefore$ minimum at $p = \pm 1$		
b) i) 2C 2L 2U AS		
$\frac{8!}{2!2!2!} = 5040$	1	
ii) 1: 5 different letters: $5! = 120$	1	$\Rightarrow$ one case correct
2: one double, 3 different		
$\frac{5!}{2!} \times {}^3C_1 \times {}^4C_3 = 720$	1	$\Rightarrow$ another case correct
${}^3C_1$: 3 doubles to choose from; ${}^4C_3$: 4 remaining letters to choose		
3: two doubles, 1 different		
$\frac{5!}{2!2!} \times {}^3C_2 \times {}^3C_1 = 270$	1	$\Rightarrow$ third case correct and total correct.
${}^3C_2$: 3 double, choose 2; ${}^3C_1$: 3 letters remaining, choose 1.		
$\therefore$ total $= 120 + 720 + 270$		
$= 1110$		

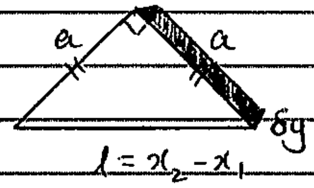
MATHEMATICS Extension 2: Question 5

Suggested Solutions	Marks	Marker's Comments

(a)

(i)





$$\delta V = \tfrac{1}{2}a^2\,\delta y$$

i.e. $\delta V = \dfrac{l^2}{4}\,\delta y$

since $2a^2 = l^2$ by Pythag. theorem.

At $y = 4x - x^2 \Rightarrow x = 2 \pm \sqrt{4-y}$

Hence $x_2 = 2 + \sqrt{4-y}$ (want $x_2 > x_1$).

Also, $y = 2x_1 \Rightarrow x_1 = y/2$.

$\therefore\ l = x_2 - x_1$

$= 2 + \sqrt{4-y} - y/2$ — 1

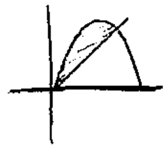
$= \frac{1}{2}\left(4 + 2\sqrt{4-y} - y\right)$

So

$\delta V = \dfrac{l^2}{4} = \dfrac{1}{16}\left(4 + 2\sqrt{4-y} - y\right)^2 \delta y$ — 1

Marker's Comments:

Many identified incorrect region



and then attempted to 'fit' their integrand to that required.

1

MATHEMATICS Extension 2: Question 5

Suggested Solutions	Marks	Marker's Comments
Now, Limits: $y = 0$ to y at point of intersection:		
$2x = x(4-x)$		
$\Rightarrow \quad x(x-2) = 0$		Limits were not calculated explicitly by many candidates.
So $x = 0, \; x = 2$		You must establish all components of the integral.
So $y = 0, \; y = 4$.		
$\therefore$ limits: $y = 0$ to 4.		
So $V = \lim\limits_{\delta y \to 0} \sum\limits_{y=0}^{4} \frac{1}{16}\left(4 + 2\sqrt{4-y} - y\right)^2 \delta y$	1	
$= \frac{1}{16}\int_0^4 \left(4 + 2\sqrt{4-y} - y\right)^2 dy$		
(ii) $V = \frac{1}{16}\int_0^4 \left((4-y) + 2(4-y)^{1/2}\right)^2 dy$		
$= \frac{1}{16}\int_0^4 (4-y)^2 + 4(4-y)^{3/2} + 4(4-y) \, dy$		Majority of candidature did not obtain correct integral or had trouble integrating their expansion.
$= \frac{1}{16}\left[-\frac{(4-y)^3}{3} - 4 \cdot \frac{2}{5}(4-y)^{5/2} - \frac{4(4-y)^2}{2}\right]_0^4$	1	
$= \frac{1}{16}\left[\frac{4^3}{3} + \frac{8}{5} \cdot 4^{5/2} + 2 \cdot 4^2\right]$		
$= \frac{98}{15}$ units^3.	1	

MATHEMATICS Extension 2: Question 5		
Suggested Solutions	**Marks**	**Marker's Comments**
(b) (i) Let $D \equiv$ draw $A \equiv$ A wins turn $B \equiv$ B wins turn In one turn, either A or B or D all mutually exclusive. So, $1 = \mathbb{P}(A \text{ or } B \text{ or } D)$ $= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(D)$ $= 2\mathbb{P}(A) + \frac{1}{6}$ $\quad \left(\mathbb{P}(D) = \frac{1}{6}\right)$ $\left(\mathbb{P}(A) = \mathbb{P}(B), \text{ symmetry}\right)$ so $\mathbb{P}(A) = \frac{5}{12}$ //	1	Diagrammatic reasoning may be used - also accepted.
(ii) $q_2 = \mathbb{P}(\text{game does not end by 2nd turn})$ $= \mathbb{P}(\text{game does not end at 1st turn}$ & $\text{game does not end at 2nd turn})$ $= 1 - \mathbb{P}(\text{game ends on 1st turn}$ OR $\text{game ends on 2nd turn})$ $= 1 - [\mathbb{P}(\text{game ends on 1st turn})$ $+ \mathbb{P}(\text{game ends on 2nd turn})$ $- \mathbb{P}(\text{game ends on 1st turn}$ &		

$\text{game ends on 2nd turn})]$

$= 1 - [0 + p_2 - 0]$ 1

$= 1 - p_2 \longrightarrow p_2 + q_2 = 1.$

MATHEMATICS Extension 2: Question……..		
Suggested Solutions	**Marks**	**Marker's Comments**
(iii) Let $G(n)$ be the event that the game ends <u>on</u> the n^{th} turn, and $E(n)$ be the event that the game does not end <u>by</u> the n^{th} turn. Now, for $n \geq 2$: The game ends on the n^{th} turn or does not end by the n^{th} turn if and only if the game does not end by the $(n-1)^{th}$ turn; i.e. $G(n) \cup E(n) = E(n-1)$ So $\mathbb{P}(G(n) \cup E(n)) = \mathbb{P}(E(n-1))$ i.e. $\mathbb{P}(G(n)) + \mathbb{P}(E(n)) - \mathbb{P}(G(n) \cap E(n)) = \mathbb{P}(E(n-1))$ i.e. $p_n + q_n - \mathbb{P}(\emptyset) = q_{n-1}$ $\therefore\ p_n + q_n = q_{n-1}$	1	

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
Now, since $p_k = q_{k-1} - q_k$		
for $k \geq 2$,		
$\sum_{k=2}^{n} p_k = \sum_{k=2}^{n} (q_{k-1} - q_k)$		
$= (q_1 - q_2) + (q_2 - q_3) + \cdots$		
$\cdots + (q_{n-2} - q_{n-1}) + (q_{n-1} - q_n)$		
$= q_1 - q_n$ (telescoping series)		
$= 1 - q_n$		
since $q_1 = \mathbb{P}$(game does not end by 1st turn)		
$= 1$	1	
as game must comprise at least two turns.		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
(iv) Consider q_n = $\mathbb{P}$(game does not end by n^{th} turn).		
This is the case iff, upon conclusion of n^{th} turn, either		
① all n turns are draws;		
② A wins once, all other $(n-1)$ turns are draws;	1	
③ B wins once, all other $(n-1)$ turns are draws;		
④ A wins once, B wins once, all other $(n-2)$ turns are draws.		
Scenario — No. ways can happen		
① $\{\underbrace{D, D, \ldots, D}_{n}\}$: $n!/n! = 1$		
$\mathbb{P}(\underbrace{D, D, \ldots, D}_{n}) = 1 \times \left(\frac{1}{6}\right)^n = \frac{1}{6^n}$		
② $\{A, \underbrace{D, D, \ldots, D}_{n-1}\}$: $\frac{n!}{(n-1)!} = n$ ways,		
so $\mathbb{P}(A, \underbrace{D, \ldots, D}_{n-1}) = n \cdot \frac{5}{12} \cdot \frac{1}{6^{n-1}}$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
③ Similarly, $\mathbb{P}(B,\underbrace{D,\ldots,D}_{n-1}) = n\cdot\frac{5}{12}\cdot\frac{1}{6^{n-1}}$		
④ $\{A,B,\underbrace{D,\ldots,D}_{n-2}\}$: $\frac{n!}{(n-2)!} = n(n-1)$ ways,		
so $\mathbb{P}(A,B,\underbrace{D,\ldots,D}_{n}) = n(n-1)\left(\frac{5}{12}\right)^2\frac{1}{6^{n-2}}$		
Since all scenarios are mutually exclusive,		
$q_n = \frac{1}{6^n} + 2\cdot n\cdot\frac{5}{12}\cdot\frac{1}{6^{n-1}} + n(n-1)\left(\frac{5}{12}\right)^2\frac{1}{6^{n-2}}$		
$= \frac{1}{6^n}\left[1 + \frac{6\cdot 2\cdot n\cdot 5}{12} + n(n-1)\cdot\frac{25}{144}\cdot 6^2\right]$		
$= \frac{1}{6^n}\left[1 + 5n + \frac{25n(n-1)}{4}\right]$		
$= \frac{1}{4\times 6^n}\left[4 + 20n + 25n^2 - 25n\right]$		
$= \frac{25n^2 - 5n + 4}{4\times 6^n}$	1	

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Given $p_n = q_{n-1} - q_n$,		
then		
$p_n = \dfrac{25(n-1)^2 - 5(n-1) + 4}{4\times 6^{n-1}} - \dfrac{25n^2 - 5n + 4}{4\times 6^n}$		
$= \dfrac{1}{4\times 6^n}\left[6(25n^2 - 50n + 25 - 5n + 5 + 4 - 25n^2 + 5n - 4\right]$		
$= \dfrac{1}{4\times 6^n}\left(125n^2 - 325n + 200\right)$		
$= \dfrac{25}{4\times 6^n}\left(5n^2 - 13n + 8\right)$		
$= \dfrac{25}{4\times 6^n}(5n-8)(n-1)$ //	1	