2013 Bored of Studies Trial Examinations

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 1 – 4

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 5 – 13

60 marks

- Attempt Questions 11 – 13
- Allow about 1 hour 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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1 The point $P(1,4)$ divides the line segment joining $A(-1,8)$ and $B(x,y)$ internally in the ratio $2:3$. What are the coordinates of *B*?

(A) $(4,2)$.

(B) $(4,-2)$.

(C) $(-4,2)$.

(D) $(-4,-2)$.

2 What is $\lim_{x\to 0}\dfrac{1-\cos ax}{x^2}$, where a is a real number?

(A) $\dfrac{a^2}{2}$.

(B) $\dfrac{2}{a^2}$.

(C) $-\dfrac{a^2}{2}$.

(D) $-\dfrac{2}{a^2}$.

3 Which of the following is NOT equivalent to $\dfrac{1+\sin\theta}{\cos\theta}$?

(A) $\sec\theta+\tan\theta$.

(B) $\dfrac{t+1}{t-1}$, where $t=\tan\left(\dfrac{\theta}{2}\right)$.

(C) $\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$.

(D) $\dfrac{1}{\sec\theta-\tan\theta}$.

4 Consider the following statements.

Let C be some curve with an inverse relation C^{-1}.

Which of the following statements is always true for C ?

(A) If C is a function, then C^{-1} is a function.

(B) If C is a function, then C^{-1} is not a function.

(C) If C is not a function, then C^{-1} is a function.

(D) None of the above.

5 Which of the following is the range of $y=\sin^{-1}\left(\dfrac{x-1}{x+1}\right)$?

(A) $-\dfrac{\pi}{2}<y<\dfrac{\pi}{2}$.

(B) $-\dfrac{\pi}{2}\le y<\dfrac{\pi}{2}$.

(C) $-\dfrac{\pi}{2}<y\le\dfrac{\pi}{2}$.

(D) $-\dfrac{\pi}{2}\le y\le\dfrac{\pi}{2}$.

6 Use Newton's Method on $f(x)=x^2-a$ to determine which of the following recursions approaches $x=\sqrt{a}$, given $x_0>0$?

(A) $x_{n+1}=\dfrac{1}{2}\left(x_n+\dfrac{a}{x_n}\right)$.

(B) $x_{n+1}=\dfrac{1}{2}\left(x_n-\dfrac{a}{x_n}\right)$.

(C) $x_{n+1}=x_n\left(1-\dfrac{2}{x_n^2-a}\right)$.

(D) $x_{n+1}=x_n\left(1+\dfrac{2}{x_n^2-a}\right)$.

7 Which of the following is the value of $\int_0^a \frac{x^2}{a^2+x^2}\,dx$?

(A) $a\left(1-\frac{\pi}{4}\right)$.

(B) $a\left(\frac{\pi}{4}-1\right)$.

(C) $a^2\left(1-\frac{\pi}{4}\right)$.

(D) $a^2\left(\frac{\pi}{4}-1\right)$.

8 Cham and Constant are playing a badminton match, where five games must be won in order to win the match. The probability that Cham wins is 0.9 and the probability that Constant wins is 0.1.

Which of the following is the approximate probability of Cham winning after 7 games.

(A) 8.9%.

(B) 9.8%.

(C) 47.8%.

(D) 12.4%.

9 Suppose that $f(x)$ and $g(x)$ are increasing functions, and that $\frac{f(x)}{g(x)}$ is also an increasing function for all real *x*. Which of the following statements is always true?

(A) $\frac{f(x)}{f'(x)} > \frac{g(x)}{g'(x)}$.

(B) $\frac{f(x)}{f'(x)} < \frac{g(x)}{g'(x)}$.

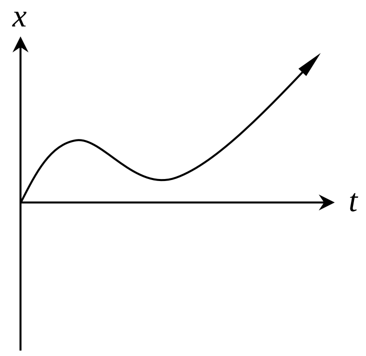
(C) $\frac{f'(x)}{f(x)} > \frac{g'(x)}{g(x)}$.

(D) $\frac{f'(x)}{f(x)} < \frac{g'(x)}{g(x)}$.

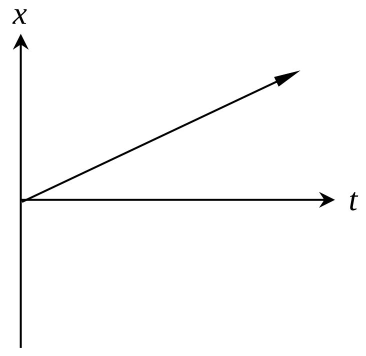
10 A particle's acceleration is given by $\ddot{x} = kx$, where $k > 0$.

Which of the following graphs best describes the behavior of the particle's displacement with respect to time?

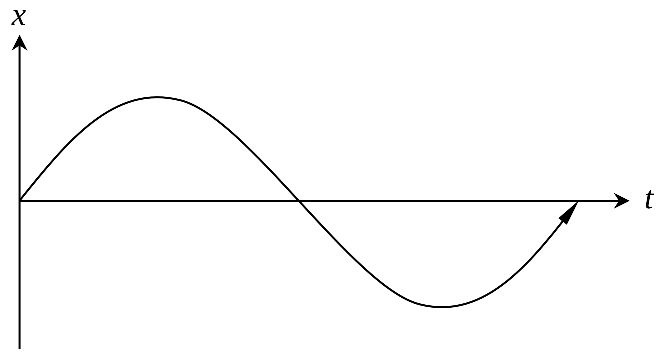
(A)



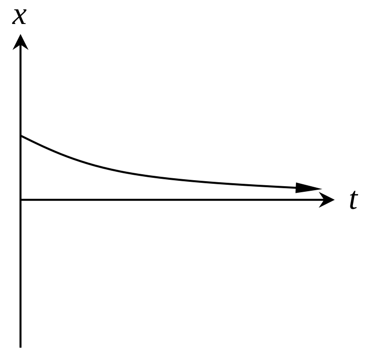
(B)



(C)



(D)



Total marks – 60
Attempt Questions 11 – 14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a)

(i) Find the exact value of the acute angle between the curves $y = \ln x$ and $y = x - 1$. **2**

(ii) Explain the significance of your result. **1**

(b) Use the substitution $u = 1 - \sqrt{x}$ to evaluate $\int_0^1 \sqrt{1-\sqrt{x}}\, dx$. **3**

(c) Let $f(x) = \ln\left(\dfrac{1+\sin x}{1+\cos x}\right)$. **3**

Show that $f(x)$ has no stationary points.

(d) Show that the area between the curve $y = \sin^2(kx)$ and the x axis in the domain $0 \le x \le \dfrac{\pi}{2}$, where k is an integer, is independent of k. **2**

(e) A monic polynomial $P(x)$ has roots α, β, γ. The product of its roots is equal to one and the sum of its roots equal to the sum of its reciprocals. Let the sum of its roots be S.

(i) Prove that $x = 1$ is a root of the polynomial. **2**

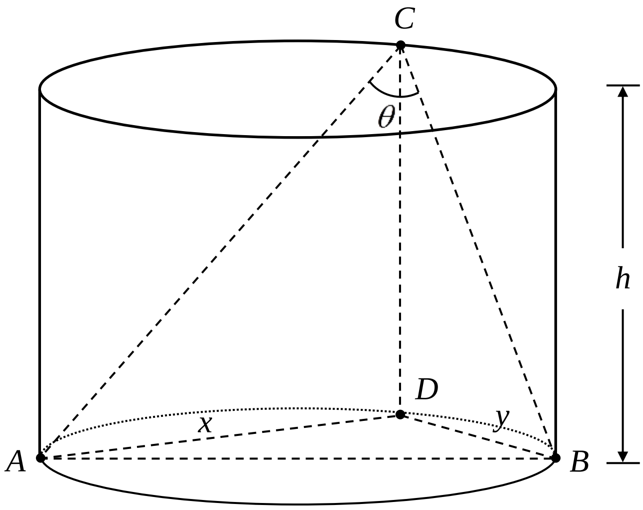
(ii) Hence, find all values of S such that the polynomial has 3 real roots. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider a cylinder of height h and radius r. From the point C on the circumference of the top circle, lines are subtended down to the diameter AB. From C, a perpendicular is dropped down to the circumference of the base circle at D.

Let the angle between AC and BC be θ. Let DA and DB be x and y respectively.



(i) Prove that **2**

$$\cos\theta = \frac{h^2}{\sqrt{h^2+x^2}\sqrt{h^2+y^2}}.$$

(ii) Hence, find the limiting value of θ as $h \to \infty$. **1**

(b)

(i) Prove algebraically that $n(n+1)$ is even for all integers $n \geq 1$. **1**

(ii) Hence use Mathematical Induction to prove that $n^3 + 3n^2 + 2n$ is divisible by 6 for all integers $n \geq 1$. **3**

Question 12 continues on page 7

(c) An island is currently dominated by a breed of pigeons, which has population P_1. A new breed of pigeons with population P_2 is introduced to the island, which competes against P_1 for food.

The rate at which P_1 and P_2 are changing are given respectively by

$$\frac{dP_1}{dt} = -\alpha\left(P_1 - P_2\right),$$

$$\frac{dP_2}{dt} = \beta\left(P_1 - P_2\right),$$

where α and β are positive constants.

Let the original population of P_1 and P_2 be A and B respectively.

(i) By differentiating $\alpha P_2 + \beta P_1$ with respect to t, show that **1**

$$\alpha P_2 + \beta P_1 = k,$$

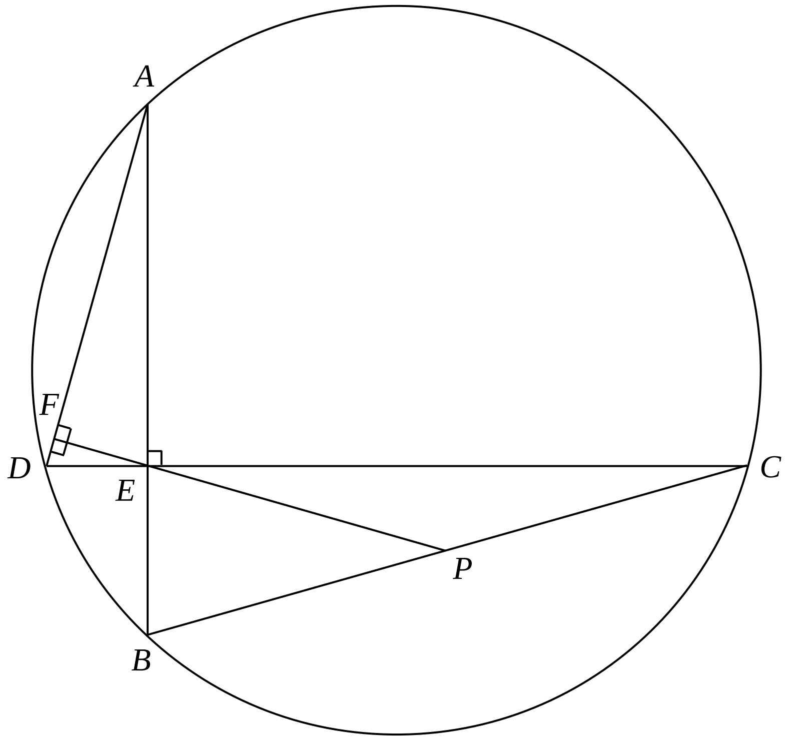
where k is a constant.

(ii) Find the limiting population of both pigeon breeds in terms of α, β and k. **2**

Question 12 continues on page 8

(d) Let AB and CD be perpendicular chords in a circle, that intersect at E. **2**

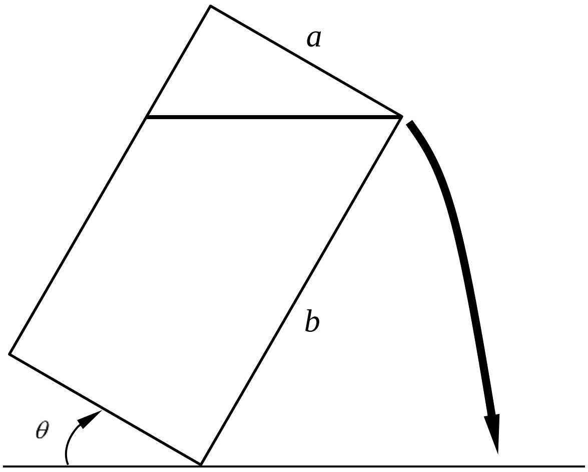
From E, a perpendicular is dropped to AD at F. The line EF is extended to meet BC at P.



Prove that P bisects BC.

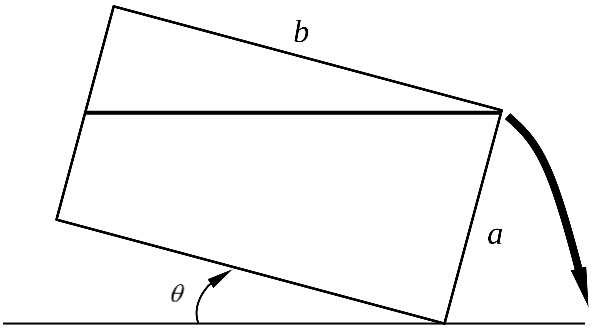
Question 12 continues on page 9

(e) A container in the shape of a rectangular prism has length b, height a and width c, where $0 < b < a$. **3**



It is initially full of water, but the angle it makes with the horizontal slowly increases at a constant rate ω radians per second until the container is empty.

A similar container is also pouring water out in a similar fashion.



Let the rate at which the first and second container loses water, in terms of θ, be R_1 and R_2 respectively.

Prove that $\dfrac{R_1}{R_2} = \dfrac{a^2}{b^2}$.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle moves in simple harmonic motion with amplitude A, period $\dfrac{2\pi}{n}$ and centre of motion at the origin. Initially, the particle starts from the origin and moves towards the positive extremity.

(i) Francis claims that when the particle has displacement $\dfrac{1}{k}$ of the positive extremity, the velocity is also $\dfrac{1}{k}$ of the maximum velocity, where $k > 1$. **3**

Prove that if the claim is true, then $k = \sqrt{2}$.

(ii) Let the period of motion be T. **1**

When do the conditions of (i) first occur, in terms of T?

(b)

(i) By considering $\sin\left(\sin^{-1} x + \cos^{-1} x\right)$, prove that **2**

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

(ii) Sketch the curve **2**

$$f\left(x\right) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x - \frac{\pi}{2},$$

indicating any restrictions.

(iii) Hence, or otherwise, evaluate **1**

$$\int_{-1}^{1} \sin^{-1} x + \cos^{-1} x + \tan^{-1} x \, dx.$$

Question 13 continues on page 11

(c) The diagram shows a point $P\left(2ap, ap^2\right)$ on the parabola $x^2 = 4ay$.

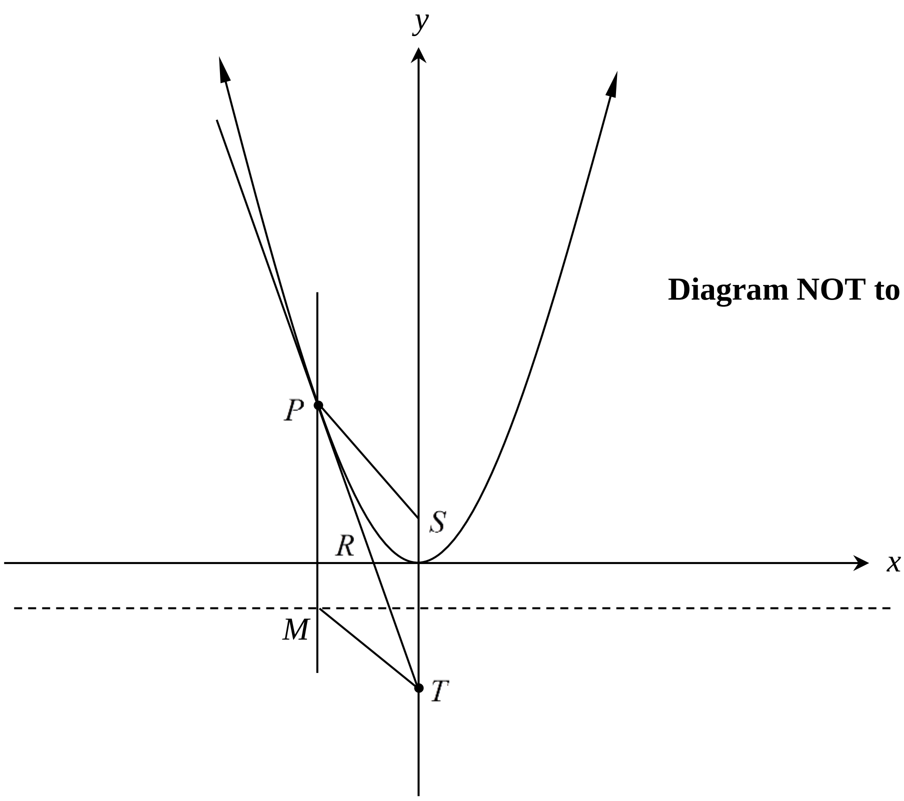


Diagram NOT to scale.

From P, a vertical line is constructed to meet the directrix at M. Also from P, a tangent is constructed that meets the x axis at R and the y axis at T. Let S be the focus.

You may assume that the equation of the tangent is $y = px - ap^2$.

(i) Find the coordinates of R. **1**

(ii) Prove that $PSTM$ is a rhombus. **2**

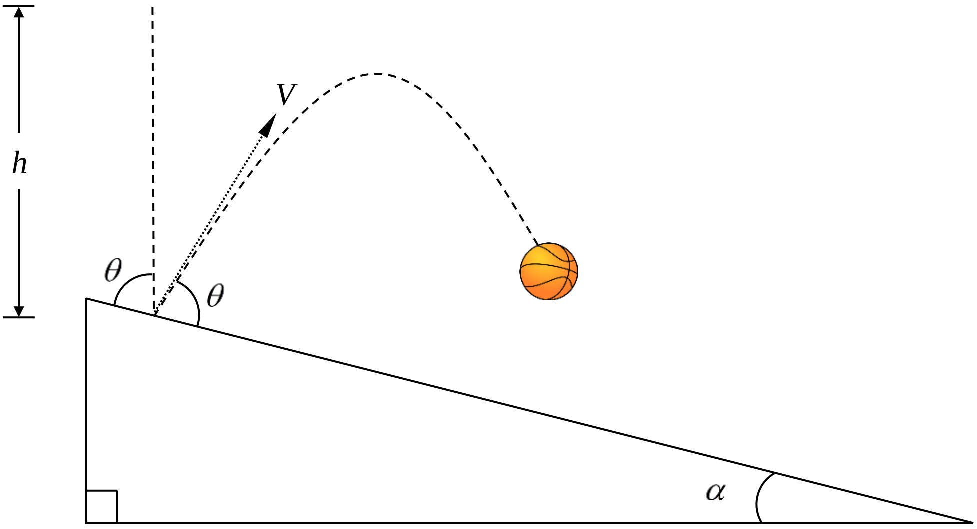
(iii) Prove that $\Delta ORS \,|||\, \Delta RPS$. **2**

(iv) Deduce that $SR^2 = SO \times SP$. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A ball is dropped from rest at height h onto a ramp inclined at an angle α. It rebounds with the same angle θ at which it landed from the ramp, and with the same velocity V at which it landed. Let g be the acceleration due to gravity.



(i) Show that the ball initially lands on the ramp with speed $V = \sqrt{2gh}$. **1**

(ii) Prove the following equations of motion.

(1) $x = Vt \sin 2\alpha$. **1**

(2) $y = -\dfrac{1}{2}gt^2 + Vt\cos 2\alpha$. **1**

(iii) The ball lands on the ramp at some point P. Prove that when $\alpha = \dfrac{\pi}{4}$, the horizontal distance of P from the point where the ball was dropped is maximised. **3**

(iv) Let the ball land on the ramp again with velocity V_0. **2**

Prove that when the horizontal range is maximised, $V_0 = V\sqrt{5}$.

Question 14 continues on page 13

(b) A class has r boys and s girls. A group of k prefects are to be selected.

(i) Explain why the number of ways of choosing k prefects is $\binom{r+s}{k}$. **1**

(ii) By finding another way of choosing k prefects, explain why **1**

$$\sum_{j=0}^{k}\binom{r}{j}\binom{s}{k-j}=\binom{r+s}{k}$$

(c) A basket contain N pieces of fruit, M of which are old.

(i) From the basket, n pieces of fruit are taken. Let $P(k)$ be the probability of k of those pieces being old. Explain why **1**

$$P(k)=\frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

(ii) Prove that $k\binom{M}{k}=M\binom{M-1}{k-1}$. **1**

(iii) Using (b), or otherwise, prove that **3**

$$\frac{1}{n}\sum_{k=0}^{n}k\times P(k)=\frac{M}{N}.$$

End of Exam

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2013 Bored of Studies Trial Examinations

Mathematics Extension 1

SOLUTIONS

Multiple Choice

1. B
2. A
3. B
4. D
5. B
6. A
7. A
8. A
9. B
10. D

Brief Explanations

Question 1 Standard ratio of intervals calculation.

Question 2 Use the double-angle formula $\cos ax = 1 - 2\sin^2\left(ax/2\right)$, then $\lim_{x\to 0} \sin x/x = 1$.

Question 3 Converting the expression using *t* formula does not yield the result in (B).

Question 4 Counter-example to (A) is the upper half of $x^2 + y^2 = 1$. Counter example to (B) is simply $y = x$. Counter example to (C) is $x^2 + y^2 = 1$. Hence, the answer is (D).

Question 5 For domain, solve $-1 \le \dfrac{x-1}{x+1} \le 1$ to get $x \ge 0$. Lower limit occurs when $x = 0$, so $y = -\pi/2$. As $x \to \infty$, $\dfrac{x-1}{x+1} \to 1$, so the upper limit is $\pi/2$, with a weak inequality since $\pi/2$ is the horizontal asymptote.

Question 6 Standard Newton's Method problem, except replace x_0 and x_1 with x_n and x_{n+1} respectively.

Question 7 Express the numerator as $x^2 = \left(a^2 + x^2\right) - a^2$. Split it, then calculate as usual.

Question 8 Cham must win the last game, so the remaining 6 games becomes a normal binomial probability question with Cham winning four games out of six. Multiply this answer by 0.9 to make Cham win the last game.

Question 9 Use the Quotient Rule, but be careful not to make any unjustified manipulations. Although the functions are increasing, they may not necessarily be positive.

Question 10 Note that when $x > 0$, $\ddot{x} > 0$ since $k > 0$. Hence, it must be concave up and the only such curve is (D).

Written Response

Question 11 (a) (i)

Firstly, we need to solve for where they intersect, so we must solve $\ln x = x - 1$. Trivially, $x = 1$ is a solution. We now use the angle between two lines formula.

From $y = \ln x$, $m_1 = \dfrac{1}{x} = 1$, since we substitute in $x = 1$. For $y = x - 1$, $m_2 = 1$ always.

However, notice that the two gradients are the same. This means that the angle between the two tangents is zero. More precisely, the tangents coincide.

Question 11 (a) (ii)

We know that the tangents coincide, but $y = x - 1$ is a line in itself, so it must therefore be a tangent to $y = \ln x$.

Question 11 (b)

$$u = 1 - \sqrt{x} \Rightarrow x = (1-u)^2 = (u-1)^2$$
$$dx = 2(u-1)\,du$$

$$x = 1 \quad \Rightarrow \quad u = 0$$
$$x = 0 \quad \Rightarrow \quad u = 1$$

$$\begin{aligned}
I &= 2\int_1^0 \sqrt{u}\,(u-1)\,du. \\
&= 2\int_0^1 \sqrt{u}\,(1-u)\,du. \\
&= 2\int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du. \\
&= 2\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1 \\
&= 2\left(\frac{2}{3} - \frac{2}{5}\right) \\
&= \frac{8}{15}
\end{aligned}$$

Question 11 (c)

Firstly, we express the function as such

$$\ln\left(\frac{1+\sin x}{1+\cos x}\right)=\ln(1+\sin x)-\ln(1+\cos x)$$

Differentiating the function and letting the derivative be zero, we have

$$\begin{aligned}
&\frac{\cos x}{1+\sin x}+\frac{\sin x}{1+\cos x}=0\\
&\cos x+\cos^2 x+\sin x+\sin^2 x=0\\
&\sin x+\cos x+1=0\\
&\sin x+\cos x=-1
\end{aligned}$$

Using the Auxiliary Angle method, we have

$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=-1$$

We now solve it.

$$\begin{aligned}
\sin\left(x+\frac{\pi}{4}\right)&=-\frac{1}{\sqrt{2}}\\
x+\frac{\pi}{4}&=-\frac{\pi}{4}+2k\pi,\ -\frac{3\pi}{4}+2k\pi, \qquad \text{where } k\in\mathbb{Z}\\
x&=-\frac{\pi}{2}+2k\pi,\ -\pi+2k\pi
\end{aligned}$$

Notice that in either case, we have an undefined y value . Hence, there exist no stationary points. □

Question 11 (d)

We simply compute the integral.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 kx\,dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1-\cos 2kx\,dx \\ &= \frac{1}{2}\left[x-\frac{1}{2k}\sin 2kx\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2}\left[\frac{\pi}{2}-\frac{1}{2k}\sin k\pi\right] \\ &= \frac{\pi}{4}, \text{ since } \sin k\pi = 0, \text{for all integer values of } k.\end{aligned}$$

Hence, the integral is independent of the value of *k*. □

Question 11 (e) (i)

Let the roots be α, β, γ.

We know that $\alpha\beta\gamma = 1$ and that $\alpha+\beta+\gamma = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. The second expression simplifies to become $\alpha+\beta+\gamma = \alpha\beta+\alpha\gamma+\beta\gamma = S$.

From here, we can reconstruct the polynomial.

$$P(x) = x^3 - S\,x^2 + S\,x - 1$$

Since $P(1) = 0$, we have $x = 1$ being a root. □

Question 11 (e) (ii)

Firstly, we factorise our cubic polynomial.

$$\begin{aligned}P(x) &= x^3 - S\,x^2 + S\,x - 1 \\ &= x^3 - 1 - S\,x(x-1) \\ &= (x-1)(x^2+x+1-S\,x) \\ &= (x-1)(x^2+x(1-S)+1)\end{aligned}$$

If the cubic has 3 real roots, then the discriminant of the quadratic must be ≥ 0.

$$\begin{aligned}\Delta &= (1-S)^2 - 4 \\ &\geq 0\end{aligned}$$

$$(1-S)^2 \geq 4$$

$$1-S \geq 2 \qquad \Rightarrow S \leq -1$$

$$1-S \leq -2 \qquad \Rightarrow S \geq 3$$

Question 12 (a) (i)

Using the Cosine Rule, we have $\cos\theta = \dfrac{CA^2 + CB^2 - AB^2}{2 \times CA \times CB}$.

Note that A and B are endpoints of the diameter of the base circle. This means that $\angle ADB = \dfrac{\pi}{2}$ (Thales' Theorem), so ΔADB is right-angled. This means that we can use Pythagoras' Theorem on it.

We now use Pythagoras' Theorem repeatedly to find *CA*, *CB* and *AB*.

$$CA^2 = x^2 + h^2$$

$$CB^2 = y^2 + h^2$$

$$AB^2 = x^2 + y^2$$

Substituting these into our expression for $\cos\theta$, we have

$$\cos\theta = \frac{x^2 + h^2 + y^2 + h^2 - x^2 - y^2}{2\sqrt{x^2+h^2}\sqrt{y^2+h^2}} = \frac{h^2}{\sqrt{h^2+x^2}\sqrt{h^2+y^2}} \qquad \square$$

Question 12 (a) (ii)

We first make a couple of manipulations to prepare us for taking the limit as h gets large.

$$\begin{aligned}\cos\theta &= \frac{h^2}{\sqrt{h^2+x^2}\sqrt{h^2+y^2}} \\ &= \frac{1}{\sqrt{1+\dfrac{x^2}{h^2}}\sqrt{1+\dfrac{y^2}{h^2}}} \qquad \ldots\left(\text{divide top and bottom by } h^2\right)\end{aligned}$$

As $h \to \infty$, $\cos\theta \to 1$, so $\theta \to 0$.

Question 12 (b) (i)

Consider the sum $S_n = 1+2+3+4+5+...+n$.

Using the Sum of AP formula, we have $S_n = \frac{1}{2}n(n+1)$, so

$n(n+1) = 2(1+2+3+4+5+...+n)$ and therefore $n(n+1)$ is even. □

Alternatively

Suppose n is even, so $n = 2k$, where k is an integer.

$n(n+1) = 2k(2k+1)$, which is even since k is an integer.

Suppose n is odd, so $n = 2k+1$, where k is an integer.

$n(n+1) = (2k+1)(2k+2) = 2(2k+1)(k+1)$

In either case, $n(n+1)$ is even. □

Question 12 (b) (ii)

Base Case: $n = 1$

$1+3+2 = 6$, which is obviously divisible by 6.

Induction Hypothesis: $n = k$

$k^3 + 3k^2 + 2k = 6M, \quad M \in \mathbb{Z}$

Inductive Step: $k \Rightarrow k+1$

$$
\begin{aligned}
(k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2 \\
&= (k^3 + 3k^2 + 2k) + 3k^2 + 9k + 6 \\
&= 6M + 3(k^2 + 3k + 2) \qquad \text{...(from the inductive hypothesis)} \\
&= 6M + 3(k+1)(k+2)
\end{aligned}
$$

However, from (i), $k(k+1)$ is even. Let it be expressed as $2N$, where N is an integer.

$$
\begin{aligned}
6M + 3(2N+2) &= 6M + 6(N+1) \\
&= 6(M+N+1)
\end{aligned}
$$

Since M and N are integers, the proof by induction is complete. □

Question 12 (c) (i)

$$\begin{aligned}\frac{d}{dt}\left(\alpha P_2+\beta P_1\right)&=\alpha\left(\beta\left(P_1-P_2\right)\right)+\beta\left(-\alpha\left(P_1-P_2\right)\right)\\&=\alpha\beta\left(P_1-P_2\right)-\alpha\beta\left(P_1-P_2\right)\\&=0\end{aligned}$$

Hence, by integrating both sides with respect to *t*, we have $\alpha P_2+\beta P_1=k$, where *k* is a constant.

Question 12 (c) (ii)

Since the population eventually reaches a limit, the gradient of the population function must gradually decrease or 'flatten out', hence as $t\to\infty$, $\frac{dP_1}{dt}\to 0$.

Thus, to find the limiting value of say P_1, we let $\frac{dP_1}{dt}=0$.

$$\begin{aligned}\frac{dP_1}{dt}&=-\alpha\left(P_1-P_2\right)\\&=0\\P_1&=P_2\end{aligned}$$

This means that both populations eventually approach the same limit. Substituting this into our result from (i), we have

$$\begin{aligned}\alpha P_1+\beta P_1&=k\\P_1&=\frac{k}{\alpha+\beta}\end{aligned}$$

Question 12 (d)

Let $\angle DAB = x$.

$$\begin{aligned}
\angle DCB &= x && \ldots(\text{angles subtended from chord } BD) \\
\angle FEA &= 90^\circ - x && \ldots(\angle \text{ sum of } \Delta FEA) \\
\angle BEP &= 90^\circ - x && \ldots(\text{vertically opposite angles}) \\
\angle PEC &= 90^\circ - \angle BEP && \ldots(\text{since } \angle BEC = 90^\circ) \\
&= x
\end{aligned}$$

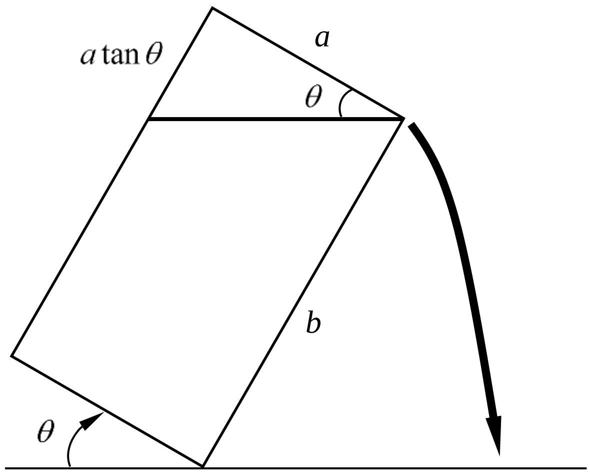
Thus ΔEPC is isosceles with $PE = PC$ since $\angle PEC = \angle DCB = x$.

$$\begin{aligned}
\angle FDE &= 90^\circ - x && \ldots(\angle \text{ sum of } \Delta ADB) \\
\angle ABC &= 90^\circ - x && \ldots(\text{angles subtended from chord } AC)
\end{aligned}$$

Thus ΔEPB is isosceles with $PE = PB$ since $\angle PEB = \angle PBE = 90^\circ - x$.

Hence, since $PB = PC$ and BC is a straight line, P bisects BC. □

Question 12 (e)



We know that the width of the solid is *c*, so to find the volume of it, we must first find the area of the cross section. To do this, we subtract the area of the small triangle from the area of the rectangle.

$$A(\theta) = ab - \frac{a^2}{2}\tan\theta$$

$$V(\theta) = abc - \frac{a^2 c}{2}\tan\theta$$

We want to find the rate at which water flows out, so we want $\frac{dV}{dt}$.

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{d\theta}\times\frac{d\theta}{dt}\\ &= \frac{dV}{d\theta}\times\omega\end{aligned}$$

Since we know what *V* is with respect to θ, we can calculate $\frac{dV}{dt}$.

$$\frac{dV}{dt} = \frac{dV}{d\theta}\times\omega$$

$$R_1 = -\frac{a^2 c\,\omega}{2}\sec^2\theta$$

Since the solid for R_2 will work exactly the same, except *a* and *b* are interchanged, we can simply state that

$$R_2 = -\frac{b^2 c\,\omega}{2}\sec^2\theta$$

And hence, by working out the ratio, we have the result. □

Question 13 (a) (i)

When the particle as at $\frac{1}{k}$ of its positive extremity, $x = \frac{A}{k}$, where A is the amplitude.

We now need to find its maximum velocity.

Using the information provided, we can model the particle's movement by $x = A\sin nt$, so $v = An\cos nt$. Since $\cos nt$ oscillates between ± 1, the maximum velocity must be An, so $\frac{1}{k}$ of that would be $\frac{An}{k}$.

We now find the times where the above conditions occur.

$$A\sin nt = \frac{A}{k} \qquad\qquad An\cos nt = \frac{An}{k}$$

$$\sin nt = \frac{1}{k} \qquad\qquad \cos nt = \frac{1}{k}$$

But $\sin^2 nt + \cos^2 nt = 1$, so

$$\begin{aligned} \frac{1}{k^2} + \frac{1}{k^2} &= 1 \\ k^2 &= 2 \\ k &= \sqrt{2} \qquad \ldots(\text{since } k > 0) \end{aligned}$$

Question 13 (a) (ii)

When $k = \sqrt{2}$, we have

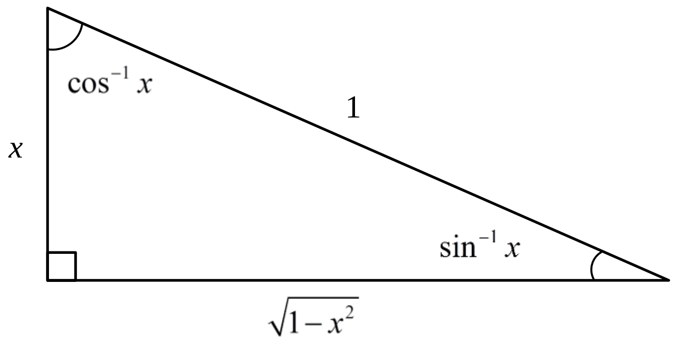
$$\begin{aligned} \sin nt &= \frac{1}{\sqrt{2}} \\ nt &= \frac{\pi}{4} \\ t &= \frac{\pi}{4n} \end{aligned}$$

But the period is $T = \frac{2\pi}{n}$, and we immediately see that $t = \frac{T}{8}$.

Question 13 (b) (i)

$$\begin{aligned}\sin\left(\sin^{-1}x+\cos^{-1}x\right)&=\sin\left(\sin^{-1}x\right)\cos\left(\cos^{-1}x\right)+\sin\left(\cos^{-1}x\right)\cos\left(\sin^{-1}x\right)\\&=x^2+\sqrt{1-x^2}\times\sqrt{1-x^2}\qquad\text{...(by considering the triangles below)}\\&=1\end{aligned}$$

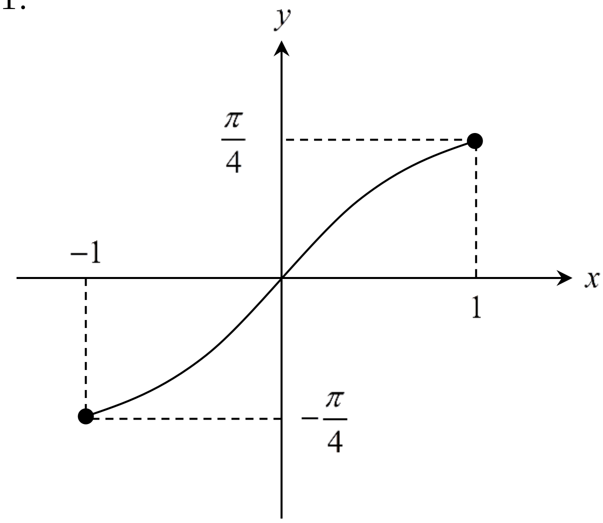
$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\quad\square$$



Note that $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$, so $\cos\left(\sin^{-1}x\right)\ge 0$. Similarly, $0\le\cos^{-1}x\le\pi$, so $\sin\left(\cos^{-1}x\right)\ge 0$.

Question 13 (b) (ii)

Because of the $\sin^{-1} x$ and $\cos^{-1} x$, we have the conditions that $-1 \le x \le 1$. Since $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, we are essentially sketching $y = \tan^{-1} x$, except with the added condition that $-1 \le x \le 1$.



Question 13 (b) (iii)

From the diagram, we see that $f(x)$ is an odd function, so its integral is 0.

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= 0\\ \int_{-1}^{1} \sin^{-1} x + \cos^{-1} x + \tan^{-1} x\,dx &= \int_{-1}^{1} \frac{\pi}{2}\,dx\\ &= \pi\end{aligned}$$

Alternatively

$$\begin{aligned}\int_{-1}^{1} \sin^{-1} x + \cos^{-1} x + \tan^{-1} x\,dx &= \int_{-1}^{1} \frac{\pi}{2} + \tan^{-1} x\,dx\\ &= \int_{-1}^{1} \frac{\pi}{2}\,dx\\ &= \pi \qquad \ldots\left(\text{since } \tan^{-1} x \text{ is an odd function}\right)\end{aligned}$$

Question 13 (c) (i)

Since we know that the equation of the tangent is $y = px - ap^2$, we can immediately find the coordinates of *R*, by letting $y = 0$ and solving for *x*. From this, we have $R(ap, 0)$.

Question 13 (c) (ii)

From the definition of the parabola, we know that $PS = PM$.

$$\text{Midpoint } MS = \left(\frac{2ap+0}{2}, \frac{a-a}{2}\right)$$
$$= (ap, 0) \qquad \ldots(\text{which are the coordinates of } R)$$

Similarly,

$$\text{Midpoint } PT = \left(\frac{2ap+0}{2}, \frac{ap^2-ap^2}{2}\right)$$
$$= (ap, 0) \qquad \ldots(\text{which are the coordinates of } R)$$

Hence, we have two pairs of adjacent sides being equal and the two diagonals bisecting, and thus *PSTM* is a rhombus. □

Question 13 (c) (iii)

$\angle PRS = 90^\circ$ ...(since diagonals intersect perpendicularly)

$\angle SOR = 90^\circ$ ...(angle between coordinate axes)

$\angle PSR = \angle RSO$ ...(since diagonals bisect opposite angles)

Hence, $\Delta ORS \,|||\, \Delta RPS$, as the two triangles are equiangular. □

Question 13 (c) (iv)

Using similar triangle ratios of corresponding sides, we have $\frac{SR}{SO} = \frac{SP}{SR}$ and thus $SR^2 = SO \times SP$. □

Question 14 (a) (i)

We can show from the equations of motion and defining the landing position as the origin:

$$\ddot{y} = -g$$
$$\dot{y} = -gt$$
$$y = -\frac{1}{2}gt^2 + h$$

When $y = 0$, the ball lands on the ramp, which gives the time of flight

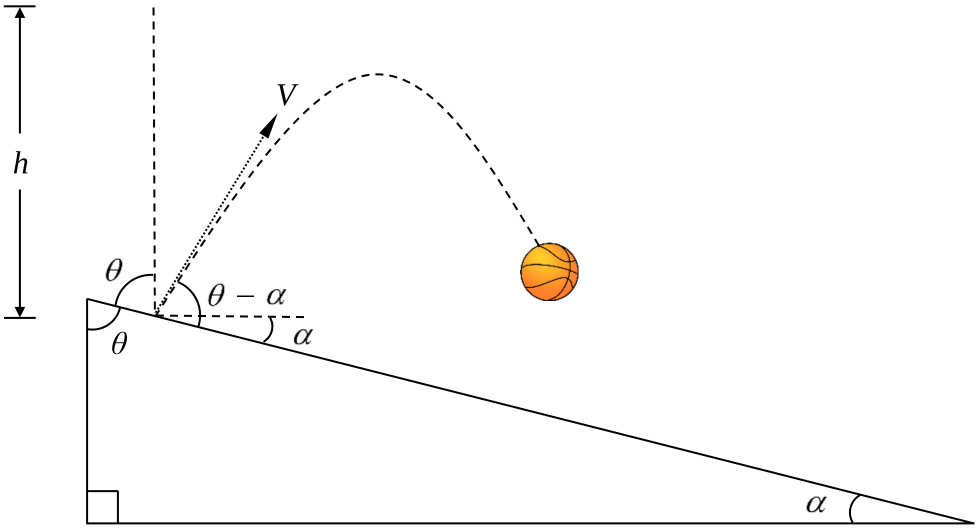
$$t = \sqrt{\frac{2h}{g}}$$
$$\begin{aligned}\dot{y} &= -g \times \sqrt{\frac{2h}{g}} \\ &= -\sqrt{2gh}\end{aligned}$$

But since we defined speed as V, we then have

$$V = \sqrt{2gh} \qquad \square$$

Question 14 (a) (ii)

Note that the angle of inclination to the horizontal is $\theta-\alpha$.



$$\ddot{x}=0$$
$$\dot{x}=V\cos(\theta-\alpha)$$
$$x=Vt\cos(\theta-\alpha)$$
$$\ddot{y}=-g$$
$$\dot{y}=-gt+V\sin(\theta-\alpha)$$
$$y=-\frac{1}{2}gt^2+Vt\sin(\theta-\alpha)$$

But by the angle sum of the right-angled triangle in the ramp we have $\theta+\alpha=\frac{\pi}{2}$, which means that $\theta-\alpha=\frac{\pi}{2}-2\alpha$ so our displacement equations are

$$\begin{aligned}x&=Vt\cos\left(\frac{\pi}{2}-2\alpha\right)\\&=Vt\sin 2\alpha\end{aligned}$$

$$\begin{aligned}y&=-\frac{1}{2}gt^2+Vt\sin\left(\frac{\pi}{2}-2\alpha\right)\\&=-\frac{1}{2}gt^2+Vt\cos 2\alpha\end{aligned} \qquad \square$$

Question 14 (a) (iii)

First find the Cartesian equation of the parabola. We know that

$$t = \frac{x}{V \sin 2\alpha}$$

Substitute this into *y*.

$$y = -\frac{gx^2}{2V^2 \sin^2 2\alpha} + x\frac{\cos 2\alpha}{\sin 2\alpha}$$

To find where the parabola intersects with the ramp, we need the equation of the ramp, where the origin of the number plane is at the point of launch. In this case, it is $y = -x\tan\alpha$. Now solve simultaneously to find the value of *x*, which is the horizontal distance.

$$-\frac{gx^2}{2V^2 \sin^2 2\alpha} + x\frac{\cos 2\alpha}{\sin 2\alpha} = -x\tan\alpha$$

$$x\left(\frac{gx}{2V^2 \sin^2 2\alpha} - \frac{\cos 2\alpha}{\sin 2\alpha} - \tan\alpha\right) = 0$$

$$x = 0 \quad \text{or} \quad x = \frac{2V^2 \sin^2 2\alpha}{g}\left(\tan\alpha + \frac{\cos 2\alpha}{\sin 2\alpha}\right)$$

But $x = 0$ is the launch position so we consider

$$\begin{aligned} x &= \frac{2V^2 \sin^2 2\alpha}{g}\left(\tan\alpha + \frac{\cos 2\alpha}{\sin 2\alpha}\right) \\ &= \frac{2V^2 \sin 2\alpha}{g}(\tan\alpha \sin 2\alpha + \cos 2\alpha) \\ &= \frac{2V^2 \sin 2\alpha}{g}\left(\frac{\sin\alpha}{\cos\alpha} \times 2\sin\alpha\cos\alpha + 1 - 2\sin^2\alpha\right) \\ &= \frac{2V^2 \sin 2\alpha}{g} \end{aligned}$$

Since *V* and *g* are constants the *x* value is maximised when $\sin 2\alpha = 1$ which occurs when $\alpha = \frac{\pi}{4}$.

Question 14 (a) (iv)

When $\alpha = \dfrac{\pi}{4}$, we have

$$\begin{aligned}\dot{x} &= V \sin 2\alpha \\ &= V \\ \dot{y} &= -gt + V \cos 2\alpha \\ &= -gt\end{aligned}$$

But $t = \dfrac{x}{V \sin 2\alpha}$ and $x = \dfrac{2V^2 \sin 2\alpha}{g}$ when the projectile hits the ramp, hence $t = \dfrac{2V}{g}$

Substitute this into $\dot{y}$ and we have $\dot{y} = -2V$.

Now we note that

$$\begin{aligned}V_0 &= \sqrt{(\dot{x})^2 + (\dot{y})^2} \\ &= \sqrt{5V^2} \\ &= V\sqrt{5} \qquad \square\end{aligned}$$

noting that $V > 0$.

Question 14 (a) (i)

We have *r* boys and *s* girls, hence a total of $r+s$ people. We are choosing *k* prefects from the total of $r+s$, which is $\binom{r+s}{k}$.

Question 14 (a) (ii)

Consider the following table, which shows all the different possible combinations of boys and girls from the *k* prefects.

Number of Boys (total of *r*)	**Number of Girls (total of *s*)**	**Number of combinations**
0	k	$\binom{r}{0}\binom{s}{k}$
1	$k-1$	$\binom{r}{1}\binom{s}{k-1}$
2	$k-1$	$\binom{r}{2}\binom{s}{k-2}$
$\dots$	$\dots$	$\dots$
k	0	$\binom{r}{k}\binom{s}{0}$

Adding the total number of combinations, we have $\sum_{j=0}^{k}\binom{r}{j}\binom{s}{k-j}$.

But since we are still essentially choosing *k* from a total of $r+s$, we can conclude the result.

Question 14 (c) (i)

Since k pieces are old, from a total of M, we have $\binom{M}{k}$.

The remaining $n-k$ pieces must not be old, so they must be chosen from the remaining total of $N-M$ pieces of fruit, hence $\binom{N-M}{n-k}$.

We are choosing n pieces from a total of N, so the total number of combinations is $\binom{N}{n}$.

Thus, the probability is $P(k)=\dfrac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$.

Question 14 (c) (ii)

We can explicitly expand the terms.

$$\begin{aligned}
\text{LHS} &= k\times\frac{M!}{k!(M-k)!}\\
&= \frac{M!}{(k-1)!(M-k)!}\\
&= M\times\frac{(M-1)!}{(k-1)!(M-k)!}\\
&= M\times\binom{M-1}{k-1}\\
&= \text{RHS}
\end{aligned}$$

Alternatively

We will use two methods to count the number of ways of choosing a team of *k* from a total of *M*, and one captain from the team of *k*.

Method #1:

Choose *k* team members from a total of *M*, so $\binom{M}{k}$.

Choose one captain from *k*, so $\binom{k}{1}$.

Hence, we have $k\binom{M}{k}$

Method #2:

Choose one captain from the *M*, so $\binom{M}{1}$.

Choose $k-1$ team members from the remaining $M-1$, so $\binom{M-1}{k-1}$.

Hence, we have $M\binom{M-1}{k-1}$.

Since the two expressions count the same thing, we can equate them.

Question 14 (c) (iii)

$$\frac{1}{n}\sum_{k=0}^{n} k\times P(k)=\frac{1}{n}\sum_{k=1}^{n} k\times P(k) \qquad \ldots(\text{since the first term is zero})$$

$$=\frac{1}{n}\sum_{k=1}^{n} k\times\binom{M}{k}\binom{N-M}{n-k}\binom{N}{n}^{-1}$$

$$=\frac{1}{n}\binom{N}{n}^{-1}\sum_{k=1}^{n} M\binom{M-1}{k-1}\binom{N-M}{n-k} \qquad \ldots(\text{from (c) (ii)})$$

$$=\frac{M}{n}\binom{N}{n}^{-1}\sum_{k=1}^{n}\binom{M-1}{k-1}\binom{N-M}{n-k}$$

$$=\frac{M}{n}\binom{N}{n}^{-1}\binom{N-1}{n-1} \qquad \ldots(\text{see below})$$

$$=\frac{M}{n}\times\binom{N}{n}^{-1}\times\frac{n}{N}\binom{N}{n} \qquad \ldots(\text{from (c) (ii) again})$$

$$=\frac{M}{N} \qquad \square$$

Regarding the above manipulation, we used the following:

$$\sum_{k=1}^{n}\binom{M-1}{k-1}\binom{N-M}{n-k}=\binom{M-1}{0}\binom{N-M}{n-1}+\binom{M-1}{1}\binom{N-M}{n-2}+\ldots+\binom{M-1}{n-1}\binom{N-M}{0}$$

Now compare this with

$$\binom{r+s}{k}=\sum_{j=0}^{k}\binom{r}{j}\binom{s}{k-j}=\binom{r}{0}\binom{s}{k}+\binom{r}{1}\binom{s}{k-1}+\ldots+\binom{r}{k}\binom{s}{0}$$

We see that we have to make the substitutions $r=M-1$, $s=N-M$, $k=n-1$ and $j=k$.

Doing so, we have

$$\binom{N-1}{n-1}=\sum_{k=1}^{n}\binom{M-1}{k-1}\binom{N-M}{n-k}$$

Alternatively:

We still proceed the same as above, except we take a different path about halfway through.

$$\frac{1}{n}\sum_{k=0}^{n} k\times P(k) = \frac{M}{n}\sum_{k=1}^{n}\frac{\binom{M-1}{k-1}\binom{N-M}{n-k}}{\binom{N}{n}}$$

$$= \frac{M}{n}\sum_{k=1}^{n}\frac{\binom{M-1}{k-1}\binom{N-M}{n-k}}{\frac{N}{n}\binom{N-1}{n-1}} \qquad \ldots\text{(from (c) (ii))}$$

$$= \frac{M}{N}\sum_{k=1}^{n}\frac{\binom{M-1}{k-1}\binom{N-M}{n-k}}{\binom{N-1}{n-1}}$$

$$= \frac{M}{N}\sum_{k=1}^{n}\frac{\binom{M-1}{k-1}\binom{(N-1)-(M-1)}{(n-1)-(k-1)}}{\binom{N-1}{n-1}}$$

All that remains to prove is that the large summand is equal to 1.

Notice how the summand is the exact same formula as in (c) (i), except with $N-1$ pieces of fruit, $M-1$ of which are old. From the basket, we pick $n-1$ pieces of fruit.

The probability of $k-1$ of those pieces being old is $\dfrac{\binom{M-1}{k-1}\binom{(N-1)-(M-1)}{(n-1)-(k-1)}}{\binom{N-1}{n-1}}$.

Since $k-1$ can vary from 0 to $n-1$, we have *k* varying from 1 to *n*.

This means that $\displaystyle\sum_{k=1}^{n}\frac{\binom{M-1}{k-1}\binom{N-M}{n-k}}{\binom{N-1}{n-1}}$ is the sum across the sample space of the probabilities,

and so it must be equal to 1, where the required result follows immediately. □