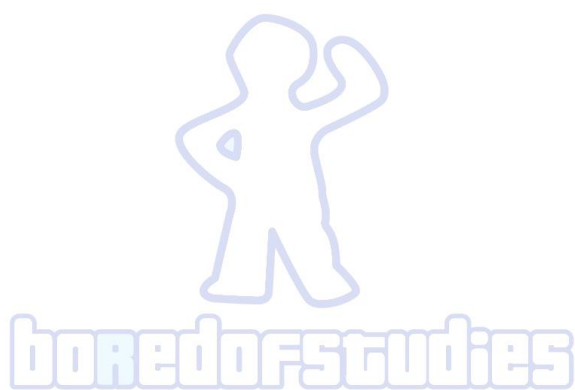




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2020

BORED OF STUDIES TRIAL EXAMINATION

9th October

Mathematics Extension 1

**General
instructions**

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

**Total marks:
70**

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–11)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

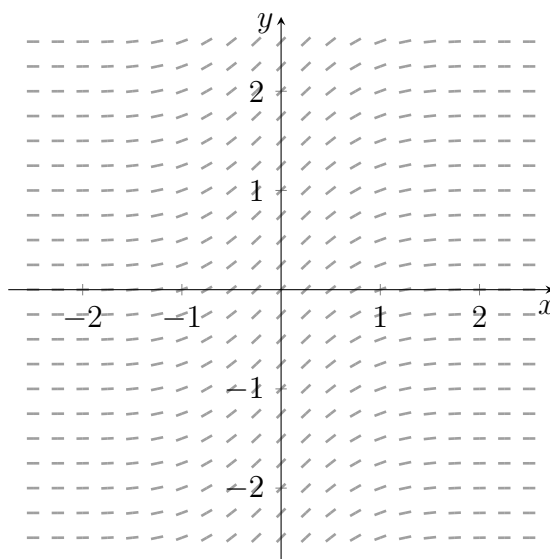
10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

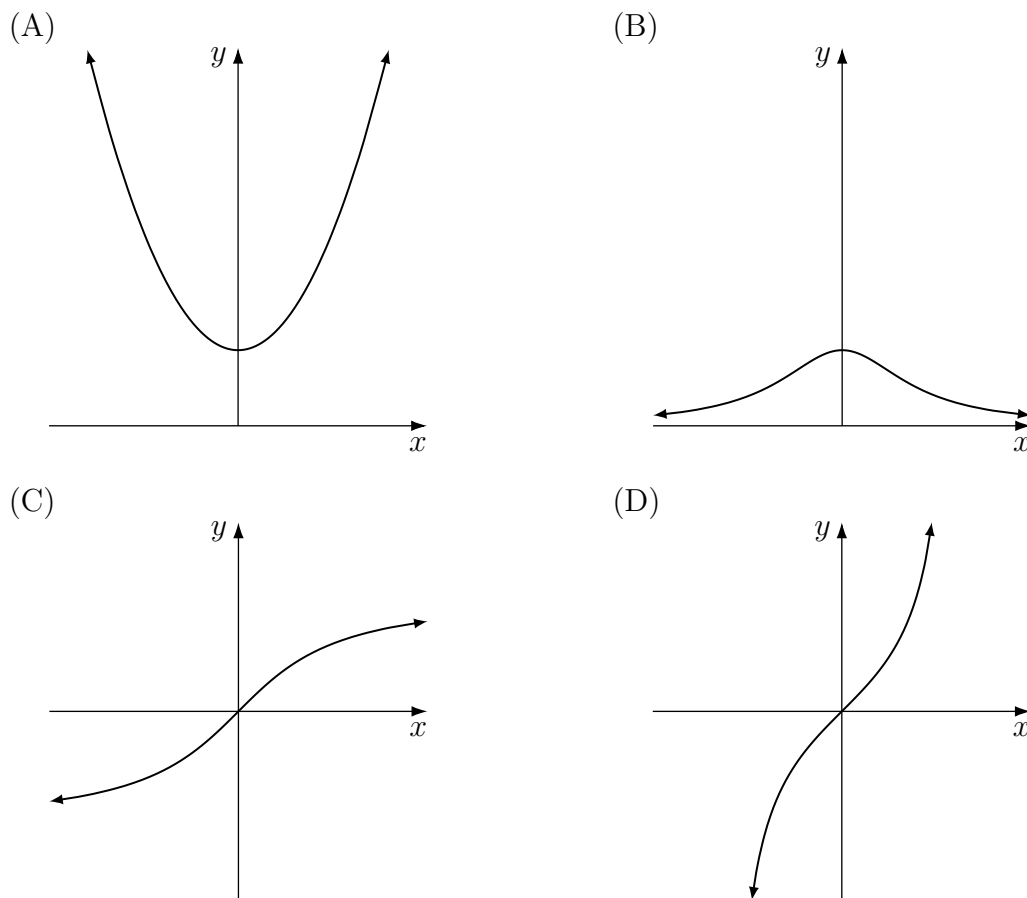
- 1 Which of the following differential equations best represents the following direction field?



- (A) $\frac{dy}{dx} = \cos x$ (B) $\frac{dy}{dx} = 1 - x^2$ (C) $\frac{dy}{dx} = e^{-x^2}$ (D) $\frac{dy}{dx} = \ln(x^2 + e)$
- 2 What is the angle between the vectors $\begin{pmatrix} 9 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$?
- (A) $\sin^{-1}(0.08)$ (B) $\cos^{-1}(0.08)$ (C) $\sin^{-1}(0.8)$ (D) $\cos^{-1}(0.8)$
- 3 A monic cubic polynomial $P(x)$ has a maximum turning point located at the origin. Which quadrant does the other turning point lie in?
- (A) 1st quadrant (B) 2nd quadrant (C) 3rd quadrant (D) 4th quadrant

- 4 Which of the following graphs represents a solution to the differential equation below?

$$\frac{dy}{dx} = 1 + y^2$$



- 5 Let $P(x) = x^2 + ax + b$ for some non-zero real values a and b . Suppose that

$$\int \frac{dx}{P(x)} = K \tan^{-1}(Ax + B) + C$$

for some real constants A, B, C and K .

Which of the following must always be true?

- (A) $a > 0$ (B) $a < 0$ (C) $b > 0$ (D) $b < 0$

- 6 An object is subject to two forces. One force acts in the direction of $\underline{j}$ with a magnitude of 1 newton. The other force acts in the direction of $\sqrt{3}\underline{i} - \underline{j}$ with a magnitude of 4 newtons.

What is the magnitude of the total force on the object, in newtons?

- (A) $\sqrt{3}$ (B) $\sqrt{13}$ (C) 3 (D) 5

- 7 A curve in the x - y plane is represented by the graph of $y = \frac{P(x)}{Q(x)}$ where $P(x)$ is a cubic polynomial and $Q(x)$ is a quadratic polynomial. $P(x)$ and $Q(x)$ have no common factors. What is the minimum number of asymptotes that this curve can have?

- (A) 0 (B) 1 (C) 2 (D) 3

- 8 What is the solution set to $|x - 3| \geq \frac{1}{x - 1}$?

- (A) $\{x < -2, x = \sqrt{2} - 2, x > 1\}$ (B) $\{x > 1, x = -2, x \leq -2 - \sqrt{2}\}$
(C) $\{x \leq -1, x = 2 - \sqrt{2}, x \geq 2\}$ (D) $\{x < 1, x = 2, x \geq 2 + \sqrt{2}\}$

- 9 A particle moves along a number plane at time t according to the displacement vector

$$\underline{r} = (t \sin t)\underline{i} - (t \cos t)\underline{j}.$$

The particle is initially at the origin. Let θ be the acute angle at which the path of the particle first crosses the x -axis after leaving the origin. What is the value of θ ?

- (A) $\frac{\pi}{2}$ (B) π (C) $\tan^{-1} \frac{\pi}{2}$ (D) $\tan^{-1} \pi$

- 10 A student tosses a fair coin 100 times. Which of the following is the best estimate for the probability of getting either:

- 60 or more tosses showing heads; or
- 60 or more tosses showing tails.

- (A) 0.05 (B) 0.32 (C) 0.68 (D) 0.95

Section II

60 marks

Attempt Questions 11—14

Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Evaluate $\int \cos^2 4x \cos^2 x \, dx$. **2**

- (b) A particle is launched from the ground with speed V at an angle of projection of θ . **3**
Let g be the acceleration due to gravity. The equations of motion at time t are

$$x = Vt \cos \theta, \quad y = Vt \sin \theta - \frac{gt^2}{2} \quad (\text{Do NOT prove this})$$

Let D be the distance between the particle and its initial launch point.

Show that if D increases throughout the particle's entire trajectory then $\sin^2 \theta < \frac{8}{9}$.

- (c) Using the substitution $u = x\sqrt{x}$, show that **3**

$$\int_0^\infty \sqrt{\frac{x}{e^{x^3}}} \, dx = \frac{\sqrt{2\pi}}{3}.$$

Question 11 continues on page 6

Question 11 (continued)

- (d) A model which approximates the spread of a virus in a population over time can be described by the differential equation

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I$$

where

- I is the number of infected people which varies over time t
- S is the number of non-infected people which varies over time t
- N is the total constant population where $N = S + I$
- β and γ are positive constants and $\beta \neq \gamma$.

- (i) Show that

1

$$\frac{dI}{dt} = \frac{1}{\beta - \gamma} \left(\frac{1}{I} + \frac{\beta}{N(\beta - \gamma) - \beta I} \right).$$

- (ii) Hence, use integration to show that the general solution is given by

3

$$I = \frac{N(\beta - \gamma)}{\beta + Ae^{-(\beta - \gamma)t}}.$$

where A is some constant.

- (iii) On two separate sets of axes, sketch the graphs of the number of infected people I over time for the two separate cases when $\beta > \gamma$ and $\beta < \gamma$. Indicate any intercepts and asymptotes in terms of A, N, β and γ .

2

- (iv) Hence, explain the physical significance of the ratio $\frac{\beta}{\gamma}$ with regards to how it affects the number of infected people in the population over time.

1

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

- (a) Two resistors A and B in circuit are connected in parallel such that their effective resistance R_E is given by **2**

$$\frac{1}{R_E} = \frac{1}{R_A} + \frac{1}{R_B}$$

where R_A and R_B are the resistances of A and B respectively.

Suppose that the resistance of A is increasing at a constant rate of 1 ohm per minute and the resistance of B is decreasing at a constant rate of 1 ohm per minute.

Show that the rate of change of the effective resistance (in ohms per minute) is given by

$$\frac{dR_E}{dt} = R_E \left(\frac{1}{R_A} - \frac{1}{R_B} \right).$$

- (b) Suppose that X is a continuous random variable with a cumulative distribution function $F(x)$. Let $Y = F(X)$ be another random variable in terms of X . **2**

Show that the probability density function of Y represents a uniform distribution.

- (c) Recall that a polyhedron is a solid composed of polygonal faces and straight edges. A cube is an example of a polyhedron. **2**

Let N be the largest number of edges of any given face of a given polyhedron.

By using the pigeonhole principle, or otherwise, prove that there exists at least two faces of the given polyhedron with the same number of edges.

- (d) Show that **3**

$$\frac{\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \dots + \sin 44^\circ}{\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \dots + \cos 44^\circ} = \sqrt{2} - 1.$$

Question 12 continues on page 8

Question 12 (continued)

- (e) Consider the distinct points A and B on the x - y plane with position vectors $\underline{a}$ and $\underline{b}$ respectively. Let P be a point that lies strictly within the interval AB such that

$$\frac{AP}{BP} = \frac{1 - \mu}{\mu},$$

for some $0 < \mu < 1$.

- (i) Show that the position vector of the point P is represented by **1**

$$\underline{p} = \mu \underline{a} + (1 - \mu) \underline{b}.$$

- (ii) Let two other distinct points C and D lie in the plane with position vectors $\underline{c}$ and $\underline{d}$ respectively. Suppose that no three points of A, B, C and D are collinear and the intervals AB and CD have a point of intersection. **2**

Using the result in (i), show that there exists some non-zero real numbers $\lambda_1, \lambda_2, \lambda_3$ and λ_4 such that

$$\lambda_1 \underline{a} + \lambda_2 \underline{b} + \lambda_3 \underline{c} + \lambda_4 \underline{d} = \underline{0},$$

where $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ and $\underline{0} = 0\underline{i} + 0\underline{j}$.

- (f) Suppose that the trigonometric equation $a \sin 4\theta + b \cos 4\theta = c$ has distinct solutions $\theta_1, \theta_2, \theta_3$ and θ_4 for non-zero constants a, b and c . Define **3**

$$S = \tan \theta_1 + \tan \theta_2 + \tan \theta_3 + \tan \theta_4$$

$$T = \tan \theta_2 \tan \theta_3 \tan \theta_4 + \tan \theta_1 \tan \theta_3 \tan \theta_4 + \tan \theta_1 \tan \theta_2 \tan \theta_4 + \tan \theta_1 \tan \theta_2 \tan \theta_3.$$

Show that

$$S + T = 0.$$

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Let $f(x) = \tan^{-1} x$ and $g(x) = \frac{x}{x^2 + 1}$.

(i) Show that $f'(x) > g'(x)$ for the domain $x > 0$. **2**

(ii) Hence, show that $\frac{\sqrt{\tan^{-1} x}}{x} > \frac{1}{\sqrt{x(x^2 + 1)}}$ for the domain $x > 0$. **2**

(iii) Using calculus to investigate the behaviour of each of the curves, sketch **4**

$$y = \frac{1}{\sqrt{x(x^2 + 1)}} \quad \text{and} \quad y = \frac{\sqrt{\tan^{-1} x}}{x}$$

on the same set of axes.

(iv) The region bounded by the curves $y = \frac{1}{\sqrt{x(x^2 + 1)}}$ and $y = \frac{\sqrt{\tan^{-1} x}}{x}$ over the domain $[1, \sqrt{3}]$ is rotated about the x -axis to form a solid of revolution. **3**

Find the volume of this solid.

(b) Let a, b, p and q be real constants and let x_n be a sequence defined by the following relation for integers $n \geq 2$.

$$x_n = ax_{n-1} + bx_{n-2},$$

where $x_1 = p$ and $x_0 = q$.

(i) Prove by mathematical induction for integers $n \geq 0$ **3**

$$\alpha^n(p - \beta q) = x_{n+1} - \beta x_n$$

where α and β are the real roots of the equation $x^2 = ax + b$ and $\alpha \neq \beta$.

(ii) Hence, show that **1**

$$x_n = \frac{(\alpha^n - \beta^n)p + (\alpha^{n-1} - \beta^{n-1})bq}{\alpha - \beta}.$$

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

- (a) Suppose that X is a non-negative discrete random variable defined on some interval $[a, b]$ with expected value $E(X)$ and variance $\text{Var}(X)$.

- (i) Show that for any $r > 0$

3

$$\text{Var}(X) \geq rP(|X - E(X)| \geq \sqrt{r}).$$

- (ii) In a large country, a survey was conducted on a random sample of n people. It was found that $100\hat{p}$ percent of them own a bike. Let p be the true proportion of the country's population that own a bike. For any $\varepsilon > 0$, it can be shown that

1

$$P(|\hat{p} - p| < \varepsilon) \geq L.$$

Use part (i) to find an appropriate value of L in terms of n, p and ε .

- (iii) Deduce that

2

$$\lim_{n \rightarrow \infty} P(|\hat{p} - p| < \varepsilon) = 1$$

and explain the significance of this result for the survey.

- (b) Consider a polyhedron which has n faces and is assigned a distinct colour to each face. Define a “colouring” as a specific arrangement of colours on the faces of the solid. Any rotations of a solid with this specific arrangement are considered the same “colouring”.

- (i) Find the total number of colourings for a cube.

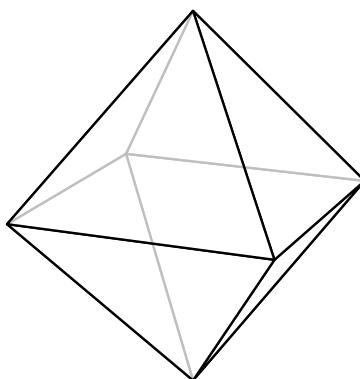
2

- (ii) Find the total number of colourings for a regular octahedron.

2

For reference, a diagram of a regular octahedron is provided below.

All 8 faces are equilateral triangles, and all 12 edges are equal in length.



Question 14 continues on page 11

Question 14 (continued)

- (c) Leonardo is in St. Peter's Basilica and encounters a staircase consisting of n stairs. He is able to "step" up one stair at a time, or "lunge" up two stairs at a time. Let $\psi(n)$ be the number of ways Leonardo can ascend an n -stair staircase.

(i) Explain why $\psi(n) = \psi(n-1) + \psi(n-2)$ for $n \geq 3$. 1

(ii) The *Fibonacci sequence* F_k is defined by the following relation 1

$$F_k = F_{k-1} + F_{k-2},$$

where $F_0 = 0$ and $F_1 = 1$.

Show that $\psi(n) = F_{n+1}$.

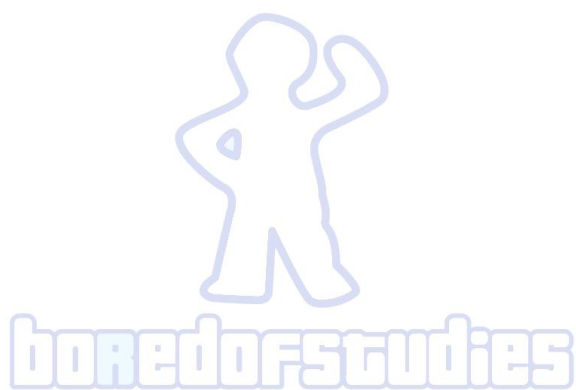
(iii) Leonardo wishes to lunge exactly k times, where k is an integer satisfying $0 \leq k \leq \frac{n}{2}$ for even n , and $0 \leq k \leq \frac{n-1}{2}$ for odd n . 1

Find the number of ways he can do this for a fixed value of k .

(iv) Deduce that 2

$$F_{n+1} = \begin{cases} \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n}{2}}{\frac{n}{2}} & \text{if } n \text{ is even.} \\ \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

End of paper



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2020

BORED OF STUDIES TRIAL EXAMINATION

Mathematics Extension 1

Solutions

Section I

Answers

- | | | | |
|---|---|----|---|
| 1 | C | 6 | B |
| 2 | C | 7 | B |
| 3 | D | 8 | D |
| 4 | D | 9 | C |
| 5 | C | 10 | A |

Brief explanations

- 1 When $x \rightarrow \pm\infty$, the slope is very close to 0. The only option with this property is (C).
- 2 Using $\theta = \cos^{-1} \left(\frac{u \cdot v}{|u||v|} \right) \Rightarrow \theta = \cos^{-1} \left(\frac{3}{5} \right)$.
This is equivalent to $\sin^{-1} \left(\frac{4}{5} \right)$, so the answer is (C).
- 3 Since the polynomial is monic and has a maximum turning point at the origin then it has the form $P(x) = x^2(x - \alpha)$ for some constant α . Since the leading coefficient of $P(x)$ is positive, the maximum turning point of the curve must occur at a smaller x -value than the minimum turning point. Also, since the minimum turning point cannot be higher in y -value than the maximum turning point, then it must be located in the 4th quadrant. Hence, the answer is (D).
- 4 Since $y^2 \geq 0$ then $\frac{dy}{dx} \geq 1$. When $y = 0$, $\frac{dy}{dx} = 1$.
This means that the curve has the smallest gradient when $y = 0$ and the gradient is larger elsewhere. The only option with this property is (D).
- 5 The only quadratic polynomials with reciprocals that integrate to inverse tangent functions are those that have no real solution. Hence, the discriminant of $P(x)$ needs to be negative, so $a^2 - 4b < 0$. This implies that $b > \frac{a^2}{4}$. Since $a^2 > 0$ for non-zero values of a , then it is necessary that $b > 0$. Hence, the answer is (C).

- 6 The unit vector corresponding to the direction $\sqrt{3}\underline{i} - \underline{j}$ is $\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$.

The net force vector is $\underline{j} + 4\left(\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}\right)$, which is equivalent to $2\sqrt{3}\underline{i} - \underline{j}$.

The magnitude of the net force is $\sqrt{(2\sqrt{3})^2 + 1^2} = \sqrt{13}$, so the answer is (B).

- 7 By polynomial long division, $\frac{P(x)}{Q(x)}$ can be rewritten as $A(x) + \frac{R(x)}{Q(x)}$, where $A(x)$ and $R(x)$ are polynomials of degree 1 or 0. Regardless of the degrees of $A(x)$ and $R(x)$, the polynomial $A(x)$ is an asymptote of the graph.

Hence, the minimum number of asymptotes is 1, so the answer is (B).

- 8 If $x < 1$ then $|x - 3| > 0 > \frac{1}{x - 1}$, so $x < 1$ is part of the set. Hence, the answer is (D).

- 9 Note that for $0 < t < \frac{\pi}{2}$, the y -value of the displacement is negative.

After leaving the origin, the next time the particle crosses the x -axis is $t = \frac{\pi}{2}$.

The acute angle at which the particle crosses the x -axis is given by $\tan \theta = \left| \frac{\dot{y}}{\dot{x}} \right|$.

When $t = \frac{\pi}{2}$ then $\tan \theta = \left| \frac{\frac{\pi}{2} \sin \frac{\pi}{2} - \cos \frac{\pi}{2}}{\sin \frac{\pi}{2} + \frac{\pi}{2} \cos \frac{\pi}{2}} \right|$. Hence, $\tan \theta = \frac{\pi}{2}$ or $\theta = \tan^{-1} \frac{\pi}{2}$,

so the answer is (C).

- 10 Let X be the number of heads. It is binomial distributed with

$$E(X) = 50 \quad \text{and} \quad \text{Var}(X) = 25$$

The probability of getting at least 60 heads or at least 60 tails is given by

$$\begin{aligned} P(X \geq 60) + P(X \leq 40) &= P\left(\frac{X - 50}{\sqrt{25}} \geq \frac{60 - 50}{\sqrt{25}}\right) + P\left(\frac{X - 50}{\sqrt{25}} \leq \frac{40 - 50}{\sqrt{25}}\right) \\ &= P(Z \geq 2) + P(Z \leq -2) \end{aligned}$$

Since the number of trials is large, X can be approximated by a normal distribution, where $Z = \frac{X - np}{\sqrt{np(1-p)}}$ follows a standard normal distribution.

From the reference sheet, approximately 95% of z -scores lie between -2 and 2 . This implies that $P(-2 < Z < 2) \approx 0.95$.

Since $P(Z \geq 2) + P(Z \leq -2) = 1 - P(-2 < Z < 2)$ then the probability required is approximately 0.05. Hence, the answer is (A).

Question 11

- (a) Using the cosine double angle formula and converting products of trig functions to sums

$$\begin{aligned}\int \cos^2 4x \cos^2 x \, dx &= \int \left(\frac{1 + \cos 8x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right) dx \\&= \frac{1}{4} \int (1 + \cos 2x \cos 8x + \cos 2x + \cos 8x) \, dx \\&= \frac{1}{4} \int \left(1 + \frac{\cos 10x + \cos 6x}{2} + \cos 2x + \cos 8x \right) dx \\&= \frac{1}{4} \left(x + \frac{\sin 10x}{20} + \frac{\sin 6x}{12} + \frac{\sin 2x}{2} + \frac{\sin 8x}{8} \right) + c\end{aligned}$$

- (b) The distance from the launch point $(0, 0)$ to a given position (x, y) is found by

$$\begin{aligned}D^2 &= x^2 + y^2 \\&= V^2 t^2 \cos^2 \theta + \left(Vt \sin \theta - \frac{gt^2}{2} \right)^2 \\&= V^2 t^2 \cos^2 \theta + V^2 t^2 \sin^2 \theta - gVt^3 \sin \theta + \frac{g^2 t^4}{4} \\&= V^2 t^2 - gVt^3 \sin \theta + \frac{g^2 t^4}{4}\end{aligned}$$

Using the chain rule

$$\begin{aligned}\frac{dD^2}{dt} &= 2V^2 t - 3gVt^2 \sin \theta + g^2 t^3 \\ \frac{dD^2}{dD} \times \frac{dD}{dt} &= t(g^2 t^2 - 3gVt \sin \theta + 2V^2) \\ \frac{dD}{dt} &= \frac{t(g^2 t^2 - 3gVt \sin \theta + 2V^2)}{2D}\end{aligned}$$

If D is increasing then $\frac{dD}{dt} > 0$. Since $t > 0$ and $D > 0$ then $g^2 t^2 - 3gVt \sin \theta + 2V^2 > 0$ is needed to ensure this condition holds. This is a quadratic polynomial in t , so the leading coefficient is needed to be positive and the discriminant is needed to be negative.

Note that $g^2 > 0$ and the discriminant is given by

$$\begin{aligned}\Delta &= 9g^2 V^2 \sin^2 \theta - 8g^2 V^2 \\&= g^2 V^2 (9 \sin^2 \theta - 8)\end{aligned}$$

For $\Delta < 0$ to hold, we must have $\sin^2 \theta < \frac{8}{9}$.

- (c) Let $u = x\sqrt{x}$, then $du = \frac{3}{2}\sqrt{x} dx \Rightarrow \sqrt{x} dx = \frac{2}{3}du$.
When $x = 0, u = 0$ and when $x \rightarrow \infty, u \rightarrow \infty$.

$$\begin{aligned}\int_0^\infty \sqrt{\frac{x}{e^{x^3}}} dx &= \int_0^\infty \frac{1}{\sqrt{e^{u^2}}} \times \frac{2}{3} du \\ &= \frac{2}{3} \int_0^\infty e^{-\frac{u^2}{2}} du \\ &= \frac{1}{3} \int_{-\infty}^\infty e^{-\frac{u^2}{2}} du \\ &= \frac{\sqrt{2\pi}}{3}\end{aligned}$$

Since $e^{-\frac{u^2}{2}}$ is an even function then $\int_{-\infty}^\infty e^{-\frac{u^2}{2}} du = 2 \int_0^\infty e^{-\frac{u^2}{2}} du$.

Also, recall that the standard normal probability density function is $f(u) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}$.
Hence, it follows that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{u^2}{2}} du = 1.$$

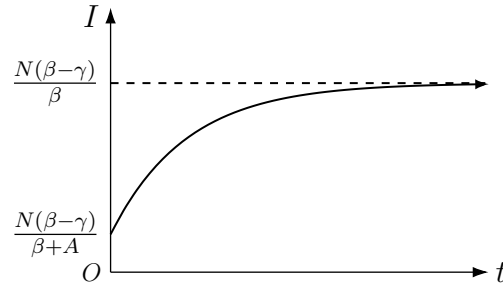
- (d) (i) Substitute $S = N - I$

$$\begin{aligned}\frac{dI}{dt} &= \frac{\beta SI}{N} - \gamma I \\ &= \frac{\beta I(N - I) - \gamma NI}{N} \\ \frac{dt}{dI} &= \frac{N}{I((\beta - \gamma)N - \beta I)} \\ &= \frac{1}{\beta - \gamma} \times \frac{(\beta - \gamma)N - \beta I + \beta I}{I((\beta - \gamma)N - \beta I)} \\ &= \frac{1}{\beta - \gamma} \left(\frac{1}{I} + \frac{\beta}{N(\beta - \gamma) - \beta I} \right)\end{aligned}$$

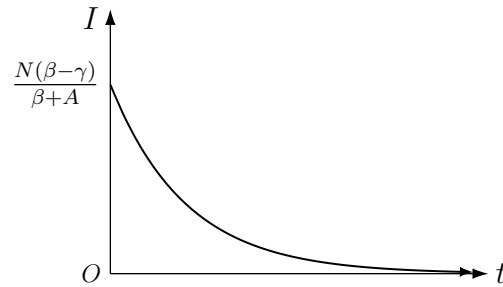
- (ii) By integration of part (i)

$$\begin{aligned}t &= \frac{1}{\beta - \gamma} \int \left(\frac{1}{I} + \frac{\beta}{N(\beta - \gamma) - \beta I} \right) dI \\ &= \frac{1}{\beta - \gamma} [\ln I - \ln (N(\beta - \gamma) - \beta I)] + c \\ (\beta - \gamma)(t - c) &= \ln \left(\frac{I}{N(\beta - \gamma) - \beta I} \right) \\ e^{-(\beta - \gamma)(t - c)} &= \frac{N(\beta - \gamma) - \beta I}{I} \\ \beta + e^{(\beta - \gamma)c} e^{-(\beta - \gamma)t} &= \frac{N(\beta - \gamma)}{I} \\ I &= \frac{N(\beta - \gamma)}{\beta + A e^{-(\beta - \gamma)t}} \quad \text{where } A = e^{(\beta - \gamma)c} \text{ which is a constant}\end{aligned}$$

- (iii) Let $k = \beta - \gamma$. When $k > 0$ then as $t \rightarrow \infty$, $e^{-kt} \rightarrow 0$, hence $I \rightarrow \frac{N(\beta-\gamma)}{\beta}$. This means that I increases from its initial value $\frac{N(\beta-\gamma)}{\beta+A}$ to its limiting value $\frac{N(\beta-\gamma)}{\beta}$.



When $k < 0$ then as $t \rightarrow \infty$, $e^{-kt} \rightarrow \infty$, hence $I \rightarrow 0$. This means that I decreases from its initial value $\frac{N(\beta-\gamma)}{\beta+A}$ towards its limiting value of zero.



- (iv) When $\frac{\beta}{\gamma} > 1$ or $\beta > \gamma$, the number of infections increases rapidly before stabilising towards a population infection number close to $\frac{N(\beta-\gamma)}{\beta}$.

However, when $\frac{\beta}{\gamma} < 1$ or $\beta < \gamma$, the number of infections decreases rapidly before stabilising close to zero infections.

Remark: The ratio $\frac{\beta}{\gamma}$ represents the reproductive number (or R_0) of the virus for this population. As shown above, whether R_0 is less than or greater than 1 leads to drastically different long term infection outcomes for the population.

Question 12

- (a) Differentiating both sides of the relation with respect to t and using the chain rule

$$\begin{aligned}\frac{1}{R_E} &= \frac{1}{R_A} + \frac{1}{R_B} \\ \frac{d}{dt} \left(\frac{1}{R_E} \right) &= \frac{d}{dt} \left(\frac{1}{R_A} \right) + \frac{d}{dt} \left(\frac{1}{R_B} \right) \\ \frac{d}{dR_E} \left(\frac{1}{R_E} \right) \frac{dR_E}{dt} &= \frac{d}{dR_A} \left(\frac{1}{R_A} \right) \frac{dR_A}{dt} + \frac{d}{dR_B} \left(\frac{1}{R_B} \right) \frac{dR_B}{dt} \\ -\frac{1}{R_E^2} \frac{dR_E}{dt} &= -\frac{1}{R_A^2} \frac{dR_A}{dt} - \frac{1}{R_B^2} \frac{dR_B}{dt}\end{aligned}$$

But $\frac{dR_A}{dt} = 1$ and $\frac{dR_B}{dt} = -1$ so

$$\begin{aligned}-\frac{1}{R_E^2} \frac{dR_E}{dt} &= -\frac{1}{R_A^2} + \frac{1}{R_B^2} \\ \frac{dR_E}{dt} &= R_E^2 \left(\frac{1}{R_A^2} - \frac{1}{R_B^2} \right) \\ &= R_E^2 \left(\frac{1}{R_A} + \frac{1}{R_B} \right) \left(\frac{1}{R_A} - \frac{1}{R_B} \right) \quad \text{but } \frac{1}{R_E} = \frac{1}{R_A} + \frac{1}{R_B} \\ &= R_E \left(\frac{1}{R_A} - \frac{1}{R_B} \right)\end{aligned}$$

- (b) Let $G(y)$ and $g(y)$ be the cumulative distribution and probability density functions of Y respectively. Since $Y = F(X)$ then $y \in [0, 1]$ as $F(x) \in [0, 1]$. Also, note that cumulative distribution functions are non-decreasing and so have a unique inverse.

$$\begin{aligned}G(y) &= P(Y \leq y) \\ &= P(F(X) \leq y) \\ &= P(X \leq F^{-1}(y)) \quad \text{but } F(x) = P(X \leq x) \\ &= F(F^{-1}(y)) \\ &= y \\ g(y) &= G'(y) \\ &= 1\end{aligned}$$

Hence, the probability density function of Y represents a uniform distribution.

- (c) Consider a reference face with N edges. Each edge is shared with another face. Hence, there must be at least N other faces in the polyhedron. These N other faces have a number of edges which can range from 3 to N . By the pigeonhole principle, there exists at least two faces with the same number of edges.

Remark: A stronger conclusion that can be drawn from the same information is that there exists at least three faces with the same number of edges OR there exists at least two faces with k_1 edges, and at least two faces with k_2 edges, for distinct integers k_1, k_2 .

- (d) **Method 1:** Consider the sum of the numerator and denominator.
Use the following auxiliary angle transformation

$$\begin{aligned}\sin \theta + \cos \theta &= \sqrt{2} \left(\cos \theta \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sin \theta \right) \\ &= \sqrt{2} \cos(45^\circ - \theta)\end{aligned}$$

$$\begin{aligned}\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 44^\circ + \cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 44^\circ \\ &= (\sin 1^\circ + \cos 1^\circ) + (\sin 2^\circ + \cos 2^\circ) + (\sin 3^\circ + \cos 3^\circ) + \cdots + (\sin 44^\circ + \cos 44^\circ) \\ &= \sqrt{2} \cos 44^\circ + \sqrt{2} \cos 43^\circ + \sqrt{2} \cos 42^\circ + \cdots + \sqrt{2} \cos 1^\circ\end{aligned}$$

Hence

$$\begin{aligned}\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 44^\circ &= (\sqrt{2} - 1)(\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 44^\circ) \\ \frac{\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 44^\circ}{\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 44^\circ} &= \sqrt{2} - 1\end{aligned}$$

Method 2: Converting the trigonometric sums to trigonometric products using

$$\begin{aligned}\sin \theta + \sin(45^\circ - \theta) &= 2 \sin 22.5^\circ \cos(22.5^\circ - \theta) \\ \cos \theta + \cos(45^\circ - \theta) &= 2 \cos 22.5^\circ \cos(22.5^\circ - \theta)\end{aligned}$$

From the numerator

$$\begin{aligned}\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 44^\circ \\ &= (\sin 1^\circ + \sin 44^\circ) + (\sin 2^\circ + \sin 43^\circ) + (\sin 3^\circ + \sin 42^\circ) + \cdots + (\sin 22^\circ + \sin 23^\circ) \\ &= 2 \sin 22.5^\circ [\cos(22.5^\circ - 1^\circ) + \cos(22.5^\circ - 2^\circ) + \cos(22.5^\circ - 3^\circ) + \cdots + \cos(22.5^\circ - 22^\circ)]\end{aligned}$$

Similarly for the denominator

$$\begin{aligned}\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 44^\circ \\ &= 2 \cos 22.5^\circ [\cos(22.5^\circ - 1^\circ) + \cos(22.5^\circ - 2^\circ) + \cos(22.5^\circ - 3^\circ) + \cdots + \cos(22.5^\circ - 22^\circ)]\end{aligned}$$

This implies that

$$\frac{\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 44^\circ}{\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 44^\circ} = \tan 22.5^\circ$$

Using the tangent double angle

$$\begin{aligned}\tan 45^\circ &= \frac{2 \tan 22.5^\circ}{1 - \tan^2 22.5^\circ} \\ \tan^2 22.5^\circ + 2 \tan 22.5^\circ - 1 &= 0 \\ (\tan 22.5^\circ + 1)^2 &= 2 \\ \tan 22.5^\circ + 1 &= \sqrt{2} \quad \text{noting that } \tan 22.5^\circ + 1 > 0 \\ \tan 22.5^\circ &= \sqrt{2} - 1\end{aligned}$$

Hence

$$\frac{\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \cdots + \sin 44^\circ}{\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \cdots + \cos 44^\circ} = \sqrt{2} - 1$$

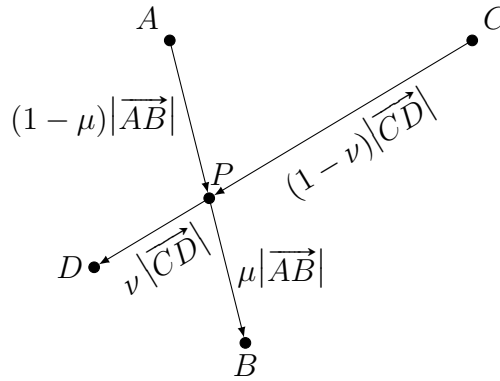
- (e) (i) Since P is a point on the interval AB then $\overrightarrow{AP}$ and $\overrightarrow{AB}$ point in the same direction but have different lengths.

$$\begin{aligned}\overrightarrow{AP} &= (1 - \mu)\overrightarrow{AB} \\ \underline{p} - \underline{a} &= (1 - \mu)(\underline{b} - \underline{a}) \\ \underline{p} &= (1 - \mu)\underline{b} - (1 - \mu)\underline{a} + \underline{a} \\ \underline{p} &= \mu\underline{a} + (1 - \mu)\underline{b}\end{aligned}$$

- (ii) Since $0 < \mu < 1$ then the ratio $\frac{1-\mu}{\mu}$ can be any positive number. In fact, μ represents the percentage of the interval AB that the sub-interval PB takes up.

Since P can be any point on the interval AB , let it also be the point where AB and CD intersect. Since P divides both AB and CD there exists some parameters μ and say ν such that

$$\frac{AP}{PB} = \frac{1 - \mu}{\mu} \quad \text{and} \quad \frac{CP}{PD} = \frac{1 - \nu}{\nu}$$



From part (i) this means that

$$\begin{aligned}\underline{p} &= \mu\underline{a} + (1 - \mu)\underline{b} \\ \underline{p} &= \nu\underline{c} + (1 - \nu)\underline{d} \\ \mu\underline{a} + (1 - \mu)\underline{b} - \nu\underline{c} - (1 - \nu)\underline{d} &= \underline{0}\end{aligned}$$

Let $\lambda_1 = \mu$, $\lambda_2 = 1 - \mu$, $\lambda_3 = -\nu$ and $\lambda_4 = -(1 - \nu)$. Notice that

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= \mu + 1 - \mu - \nu - (1 - \nu) \\ &= 0\end{aligned}$$

(f) Let $t = \tan \theta$

$$\begin{aligned}
a \sin 4\theta + b \cos 4\theta &= c \\
2a \sin 2\theta \cos 2\theta + b(2 \cos^2 2\theta - 1) &= c \\
2a \left(\frac{2t}{1+t^2} \right) \left(\frac{1-t^2}{1+t^2} \right) + b \left(2 \left(\frac{1-t^2}{1+t^2} \right)^2 - 1 \right) &= c \\
4at(1-t^2) + 2b(1-t^2)^2 - b(1+t^2)^2 &= c(1+t^2)^2 \\
4at - 4at^3 + 2bt^4 - 4bt^2 + 2b - bt^4 - 2bt^2 - b &= ct^4 + 2ct^2 + c \\
(b-c)t^4 - 4at^3 - 2(3b+c)t^2 + 4at + b - c &= 0
\end{aligned}$$

This is a quartic equation in $t = \tan \theta$ with the roots $\tan \theta_1, \tan \theta_2, \tan \theta_3$ and $\tan \theta_4$. Using the relationships between roots and coefficients

$$\begin{aligned}
S &= \tan \theta_1 + \tan \theta_2 + \tan \theta_3 + \tan \theta_4 \\
&= \frac{4a}{b-c} \\
T &= \tan \theta_2 \tan \theta_3 \tan \theta_4 + \tan \theta_1 \tan \theta_3 \tan \theta_4 + \tan \theta_1 \tan \theta_2 \tan \theta_4 + \tan \theta_1 \tan \theta_2 \tan \theta_3 \\
&= -\frac{4a}{b-c}
\end{aligned}$$

Hence $S + T = 0$.

Question 13

(a) (i)

$$\begin{aligned}
 f(x) &= \tan^{-1} x \\
 f'(x) &= \frac{1}{x^2 + 1} \\
 g(x) &= \frac{x}{x^2 + 1} \\
 g'(x) &= \frac{(1)(x^2 + 1) - x(2x)}{(x^2 + 1)^2} \\
 &= \frac{1 - x^2}{(x^2 + 1)^2}
 \end{aligned}$$

Consider

$$\begin{aligned}
 f'(x) - g'(x) &= \frac{1}{x^2 + 1} - \frac{1 - x^2}{(x^2 + 1)^2} \\
 &= \frac{x^2 + 1 - 1 + x^2}{(x^2 + 1)^2} \\
 &= \frac{2x^2}{(x^2 + 1)^2} \\
 &> 0 \quad \text{when } x > 0
 \end{aligned}$$

Hence $f'(x) > g'(x)$ for the domain $x > 0$.

(ii) First note that $f(0) = 0$ and $g(0) = 0$. Since $f(x)$ and $g(x)$ share a common point at the origin and $f'(x) > g'(x)$ for $x > 0$ then it must be that $f(x) > g(x)$, since $f(x)$ has a larger gradient than $g(x)$. Hence

$$\begin{aligned}
 \tan^{-1} x &> \frac{x}{x^2 + 1} \quad \text{note that } f(x) > 0 \text{ and } g(x) > 0 \text{ for } x > 0 \\
 \sqrt{\tan^{-1} x} &> \sqrt{\frac{x}{x^2 + 1}} \\
 \frac{\sqrt{\tan^{-1} x}}{x} &> \frac{1}{\sqrt{x(x^2 + 1)}}
 \end{aligned}$$

Noting that taking the square root of both sides of the inequality preserves the sign as it is an increasing function.

Remark: Another way to think of it is as follows.

Let a and b be positive numbers such that $a > b$, so

$$\begin{aligned}
 a - b &> 0 \\
 (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) &> 0 \\
 \sqrt{a} - \sqrt{b} &> 0 \quad \text{noting that } \sqrt{a} + \sqrt{b} > 0. \\
 \sqrt{a} &> \sqrt{b}
 \end{aligned}$$

(iii) Investigating $y = \frac{\sqrt{\tan^{-1} x}}{x}$ noting the domain of $x > 0$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{x}{2(1+x^2)\sqrt{\tan^{-1} x}} - \sqrt{\tan^{-1} x}}{x^2} \\ &= \frac{x - 2(1+x^2)\tan^{-1} x}{2x^2(1+x^2)\sqrt{\tan^{-1} x}}\end{aligned}$$

From part (ii)

$$\begin{aligned}\frac{\sqrt{\tan^{-1} x}}{x} &> \frac{1}{\sqrt{x(x^2+1)}} \\ \frac{(1+x^2)\tan^{-1} x}{x} &> 1 \\ 2(1+x^2)\tan^{-1} x &> 2x \\ x - 2(1+x^2)\tan^{-1} x &< -x\end{aligned}$$

But $x > 0$ is the domain so $-x < 0$ which implies that $x - 2(1+x^2)\tan^{-1} x < 0$. Since $2x^2(1+x^2)\sqrt{\tan^{-1} x} > 0$ for $x > 0$ then the curve is decreasing.

As $x \rightarrow \infty$ then $\frac{\sqrt{\tan^{-1} x}}{x} \rightarrow 0$ since $\tan^{-1} x \rightarrow \frac{\pi}{2}$.

Note that as $x \rightarrow 0$ then $\frac{1}{\sqrt{x(x^2+1)}} \rightarrow \infty$.

Since the curve $y = \frac{\sqrt{\tan^{-1} x}}{x}$ is always above $y = \frac{1}{\sqrt{x(x^2+1)}}$ from part (ii),

then it must be that $\frac{\sqrt{\tan^{-1} x}}{x} \rightarrow \infty$ as $x \rightarrow 0$ as well.

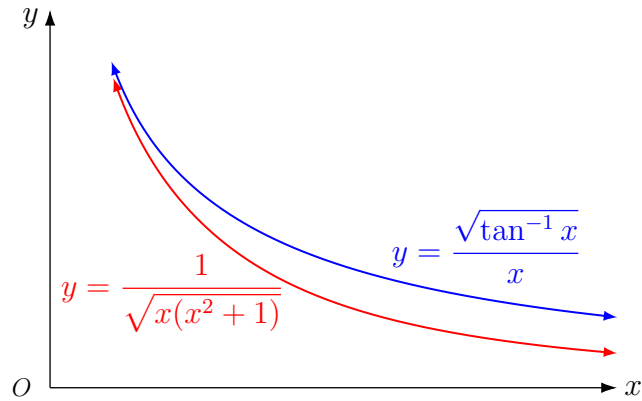
Investigating $y = \frac{1}{\sqrt{x(x^2+1)}}$ noting the domain of $x > 0$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{2}(x^3+x)^{-\frac{3}{2}}(3x^2+1) \\ &= -\frac{3x^2+1}{2(x^3+x)^{\frac{3}{2}}} \\ &< 0 \quad \text{since } 2(x^3+x)^{\frac{3}{2}} > 0 \text{ and } 3x^2+1 > 0 \text{ for } x > 0\end{aligned}$$

This implies that the curve is decreasing for $x > 0$.

Also as $x \rightarrow \infty$ then $\frac{1}{\sqrt{x(x^2+1)}} \rightarrow 0$ and as $x \rightarrow 0$ then $\frac{1}{\sqrt{x(x^2+1)}} \rightarrow \infty$.

Using this information to sketch the curves gives the following.



(iv) Let V be the volume of the solid

$$\begin{aligned} V &= \pi \int_1^{\sqrt{3}} \left(\frac{\sqrt{\tan^{-1} x}}{x} \right)^2 dx - \pi \int_1^{\sqrt{3}} \left(\frac{1}{\sqrt{x(x^2 + 1)}} \right)^2 dx \\ &= \pi \int_1^{\sqrt{3}} \left(\frac{\tan^{-1} x}{x^2} - \frac{1}{x(x^2 + 1)} \right) dx \end{aligned}$$

But $\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$ and $\frac{d}{dx} (\tan^{-1} x) = \frac{1}{x^2 + 1}$ so

$$\begin{aligned} V &= \pi \int_1^{\sqrt{3}} \left(-\tan^{-1} x \frac{d}{dx} \left(\frac{1}{x} \right) - \frac{1}{x} \frac{d}{dx} (\tan^{-1} x) \right) dx \\ &= \pi \int_1^{\sqrt{3}} \frac{d}{dx} \left(-\frac{1}{x} \times \tan^{-1} x \right) dx \\ &= \pi \left[-\frac{\tan^{-1} x}{x} \right]_1^{\sqrt{3}} \\ &= \pi \left(\frac{\pi}{4} - \frac{\frac{\pi}{3}}{\sqrt{3}} \right) \\ &= \pi^2 \left(\frac{1}{4} - \frac{1}{3\sqrt{3}} \right) \text{ cubic units} \end{aligned}$$

(b) (i) For $n = 0$

$$\begin{aligned}
 LHS &= \alpha^0(p - \beta q) \\
 &= p - \beta q \\
 RHS &= x_1 - \beta x_0 \quad \text{but } x_0 = q, x_1 = p \\
 &= p - \beta q
 \end{aligned}$$

$LHS = RHS$ so the statement is true for $n = 0$.

Assume the statement is true $n = k$ so

$$\alpha^k(p - \beta q) = x_{k+1} - \beta x_k$$

Required to prove the statement is true for $n = k + 1$

$$\alpha^{k+1}(p - \beta q) = x_{k+2} - \beta x_{k+1}$$

Since α and β are the roots of the quadratic equation $x^2 = ax + b$ then

$$\alpha + \beta = a \quad \text{and} \quad \alpha\beta = -b$$

Using this result and the assumption for the induction proof

$$\begin{aligned}
 LHS &= \alpha^{k+1}(p - \beta q) \\
 &= \alpha(x_{k+1} - \beta x_k) \quad \text{by assumption} \\
 &= \alpha x_{k+1} - \alpha\beta x_k \\
 &= (a - \beta)x_{k+1} + bx_k \\
 &= ax_{k+1} + bx_k - \beta x_{k+1} \quad \text{but } x_n = ax_{n-1} + bx_{n-2} \\
 &= x_{k+2} - \beta x_{k+1} \\
 &= RHS
 \end{aligned}$$

Hence, by induction the statement is true for all integers $n \geq 0$.

(ii) For part (i), the roots α and β are interchangeable because the relation makes no distinction on which particular root is used (it just uses one of the roots). Hence

$$\begin{aligned}
 \alpha^n(p - \beta q) &= x_{n+1} - \beta x_n \\
 \beta^n(p - \alpha q) &= x_{n+1} - \alpha x_n \\
 \alpha^n(p - \beta q) - \beta^n(p - \alpha q) &= (\alpha - \beta)x_n \\
 (\alpha^n - \beta^n)p + (\alpha\beta^n - \alpha^n\beta)q &= (\alpha - \beta)x_n \quad \text{but } \alpha\beta = -b \\
 x_n &= \frac{(\alpha^n - \beta^n)p + (\alpha^{n-1} - \beta^{n-1})bq}{\alpha - \beta}
 \end{aligned}$$

Question 14

- (a) (i) Since $\text{Var}(X) = E[X - E(X)]^2$ then the expression to prove can be rewritten as

$$E[X - E(X)]^2 \geq rP([X - E(X)]^2 \geq r)$$

Let $Y = [X - E(X)]^2$ and it suffices to prove first that $E(Y) \geq rP(Y \geq r)$.

Let the outcomes of Y be $\{y_1, y_2, \dots, y_n\}$ where $y_1 \leq y_2 \leq y_3 \leq \dots \leq y_n$ (without loss of generality) and their respective probabilities of occurring are $\{p_1, p_2, \dots, p_n\}$.

First consider the case where $y_1 \leq r \leq y_n$.

Suppose that $y_1 \leq \dots \leq y_{r-1} \leq r \leq y_r \leq \dots \leq y_n$ for some values of y_r and y_{r-1} . This means that

$$\begin{aligned} E(Y) &= p_1y_1 + \dots + p_{r-1}y_{r-1} + p_r y_r + \dots + p_n y_n \\ &\geq 0 \times y_1 + \dots + 0 \times y_{r-1} + p_r \times r + \dots + p_n \times r \\ &= r(p_r + p_{r+1} + \dots + p_n) \\ &= rP(Y \geq r) \end{aligned}$$

Since the probabilities $\{p_1, p_2, \dots, p_{r-1}\}$ are all non-negative and the values of $\{y_r, y_{r+1}, \dots, y_n\}$ are all greater than or equal to r . Also, the sum $p_r + p_{r+1} + \dots + p_n$ is the probability that Y is at least r .

Consider the case where $r > y_n$ then $P(Y \geq r) = 0$. Since Y is non-negative then $E(Y) \geq 0$. Hence, $E(Y) \geq rP(Y \geq r)$ for $r > y_n$.

Consider the case where $r < y_1$ then $P(Y \geq r) = 1$. This also suggests that $E(Y) \geq r$ since the average value of Y cannot be larger than a number below its minimum value. Hence, $E(Y) \geq rP(Y \geq r)$ for $r < y_1$.

By letting $Y = [X - E(X)]^2$ and noting that $r > 0$ then

$$\text{Var}(X) \geq rP(|X - E(X)| \geq \sqrt{r})$$

- (ii) Let $X = \hat{p}$ where $\hat{p}$ is the proportion of people in the sample who own a bike. It is known that

$$E(\hat{p}) = p \quad \text{and} \quad \text{Var}(\hat{p}) = \frac{p(1-p)}{n}$$

Using part (i), let $\epsilon = \sqrt{r}$ and note that $P(|\hat{p} - p| \geq \epsilon) = 1 - P(|\hat{p} - p| < \epsilon)$

$$\frac{p(1-p)}{n} \geq \epsilon^2 P(|\hat{p} - p| \geq \epsilon)$$

$$P(|\hat{p} - p| < \epsilon) \geq 1 - \frac{p(1-p)}{n\epsilon^2}$$

Hence, an appropriate value for L is $\left(1 - \frac{p(1-p)}{n\epsilon^2}\right)$.

- (iii) When $n \rightarrow \infty$ then $\frac{p(1-p)}{n\epsilon^2} \rightarrow 0$, hence $L \rightarrow 1$. However, $P(|\hat{p} - p| \geq \epsilon) \leq 1$ as it is a probability.

Since the lower bound is increasing towards 1 then it must be that

$$\lim_{n \rightarrow \infty} P(|\hat{p} - p| < \epsilon) = 1$$

Note that ϵ can be any positive number. This means that as the sample size n grows larger, the probability that the absolute difference between the sample proportion $\hat{p}$ and the population proportion p being less than *every* positive value (no matter how large or small) is almost certain.

This implies that the bigger the sample size, the more likely that the sample estimate will accurately represent the population proportion.

Remark: This limit result is a mathematical expression of the *law of large numbers*, which basically says that the bigger the sample size, the more likely the sample will be similar to the population.

- (b) (i) There are $6!$ ways to colour the 6 faces of the cube. This is an overcount because identical configurations from rotating the cube are being counted more than once.

Consider a reference face which is in a fixed position on the top of the cube. This cube is the same cube after rotating about each of the 4 square faces facing the sides. Altogether, there are 6×4 identical configurations of the cube.

Hence, it can be deduced that there are $\frac{6!}{6 \times 4} = 30$ colourings.

Alternative Method: Select one of the 6 faces to colour first, but the face can be rotated to any of the 5 other faces. Further note that the second colour can be in any of the 5 other faces, but can be rotated to 4 locations only. Finally, the other 4 faces can be coloured but their location is now fixed, so there are a total of $\frac{6}{6} \times \frac{5}{4} \times 4! = 30$ colourings.

- (ii) Applying a similar analogy to the cube, there are $8!$ ways to colour the 8 faces of the octahedron. This is an overcount because identical configurations from rotating the octahedron are being counted more than once.

Consider a reference vertex which is in a fixed position on the top of the octahedron. The octahedron is the same octahedron after rotating about each of the 4 equilateral triangle faces facing the sides. There are 6 vertices, so there are 6×4 identical configurations of the octahedron.

Hence, it can be deduced that there are $\frac{8!}{6 \times 4} = 1680$ colourings.

Alternative Method 1: Select one of the 8 faces to colour first, but the face can be rotated to any of the 7 other faces. Further note that the second colour can be in any of the 7 other faces, but can be rotated to 3 locations only. Finally, the other 6 faces can be coloured and their location is now fixed, so there are a total of $\frac{8}{8} \times \frac{7}{3} \times 6! = 1680$ colourings.

Alternative Method 2: Suppose instead of colouring each face, colour the centre of each face. These centres happen to form a cube and the problem is equivalent to colouring the vertices of a cube.

Select one of the 8 vertices to colour first, but the vertex can be rotated to any of the 7 other vertices. Note that the second colour can be in any of the 7 other vertices, but can be rotated to 3 locations only. Finally, the other 6 vertices can be coloured and their location is now fixed, so there are a total of $\frac{8}{8} \times \frac{7}{3} \times 6! = 1680$ colourings.

- (c) (i) For each path that gets to the $(n-1)^{\text{th}}$ step, the only way to finish is with a “step”. For each path that gets to the $(n-2)^{\text{th}}$ step, the only way to finish is with a “lunge”, since taking two steps will result in a path that passes through the $(n-1)^{\text{th}}$ step, which has already been counted.

Hence, $\psi(n) = \psi(n-1) + \psi(n-2)$ for all $n \geq 3$.

- (ii) For $n = 1$, there is only 1 way to climb the staircase (one “step”), so $\psi(1) = 1$. For $n = 2$, there are 2 ways (two “steps” or one “lunge”), so $\psi(2) = 2$.

From the definition of F_k , it follows that $F_0 = 0, F_1 = 1, F_2 = 1$ and $F_3 = 2$. Hence, $\psi(1) = F_2$ and $\psi(2) = F_3$.

As $\psi(n)$ and F_k share the same general relation (the sum of the last two values) but n and k are related by $k = n + 1$, then it can be concluded that $\psi(n) = F_{n+1}$.

- (iii) If Leonardo wishes to lunge exactly k times then he will cover $2k$ out of the n stairs. The remaining $n - 2k$ are covered by a “step”. The number of ways of arranging this sequence of $(n-2k)$ “steps” and k “lunges” is $\frac{(n-2k+k)!}{(n-2k)!k!}$, or equivalently, $\binom{n-k}{k}$.

- (iv) Consider the case where n is even. The minimum number of lunges is 0 and the maximum number of lunges is $\frac{n}{2}$. Hence, using part (iii) the total number of ways Leonardo can ascend an n -stair staircase is given by

$$\begin{aligned}\psi(n) &= \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-\frac{n}{2}}{\frac{n}{2}} \\ &= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n}{2}}{\frac{n}{2}}\end{aligned}$$

When n is odd, the minimum number of lunges is also 0 and the maximum number of lunges is $\frac{n-1}{2}$. Hence, using part (iii) the total number of ways Leonardo can ascend an n -stair staircase is given by

$$\begin{aligned}\psi(n) &= \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-\frac{n-1}{2}}{\frac{n-1}{2}} \\ &= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}}\end{aligned}$$

Since $\psi(n) = F_{n+1}$ then

$$F_{n+1} = \begin{cases} \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n}{2}}{\frac{n}{2}} & \text{if } n \text{ is even.} \\ \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$