2013 Bored of Studies Trial Examinations

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 1 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 5 – 19

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

---

1 Let $P(x)$ be a cubic polynomial. Consider the following conditions.

(I) $\alpha^2+\beta^2+\gamma^2>0$.
(II) $\alpha^2+\beta^2+\gamma^2<0$.
(III) Has a root of multiplicity two.

Which of the following statements are always true?

(A) If (I) is true, then all three roots are real.

(B) If (II) is true, then there are at least two non-real roots.

(C) If (III) is true, then all three roots are real.

(D) None of the above.

2 Which of the following statements is always correct?

(A) If $z=a+ib$ is in the first quadrant, then $\arg(z)=\tan^{-1}\left(-\frac{b}{a}\right)$.

(B) If $z=a+ib$ is in the second quadrant, then $\arg(z)=\tan^{-1}\left(\frac{b}{a}\right)$.

(C) If $z=a+ib$ is in the third quadrant, then $\arg(z)=\tan^{-1}\left(\frac{b}{a}\right)$.

(D) If $z=a+ib$ is in the fourth quadrant, then $\arg(z)=\tan^{-1}\left(\frac{b}{a}\right)$.

3 A directrix of an ellipse has the equation $x = \frac{25}{4}$ and one of its foci has the coordinates $(-4, 0)$. What is the equation of this ellipse?

(A) $\frac{x^2}{5} + \frac{y^2}{3} = 1.$

(B) $\frac{x^2}{3} + \frac{y^2}{5} = 1.$

(C) $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

(D) $\frac{x^2}{9} + \frac{y^2}{25} = 1.$

4 A convex polygon has 44 diagonals. How many sides does the polygon have?

(A) 8.

(B) 9.

(C) 10.

(D) 11.

5 Which of the following integrals is NOT equivalent to the others?

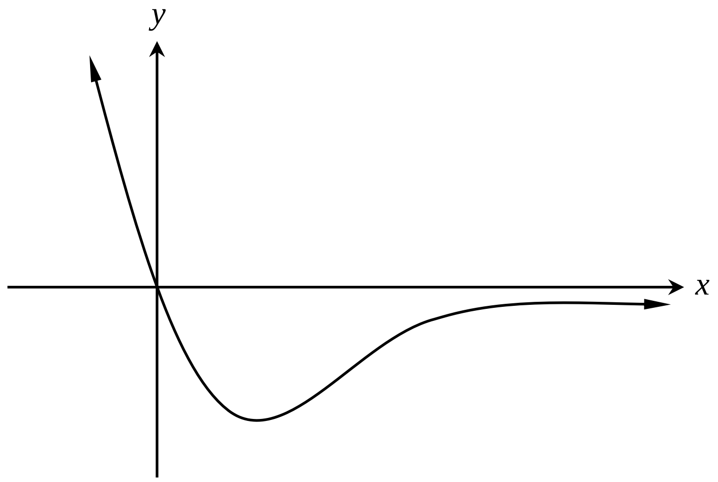
(A) $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx.$

(B) $\int_0^1 \frac{x}{2(x+1)} dx.$

(C) $\int_1^e \frac{e^{2x}}{1+e^{-2x}} dx.$

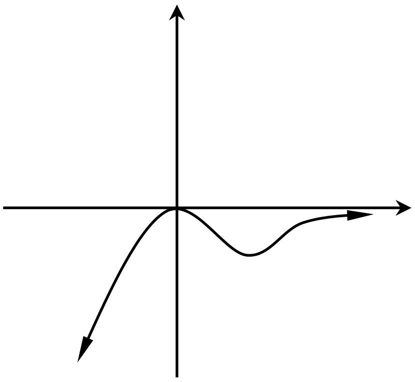
(D) $\int_0^1 \frac{x^3}{x^2+1} dx.$

6 The diagram shows a sketch of the curve $y = x\,f(x)$.

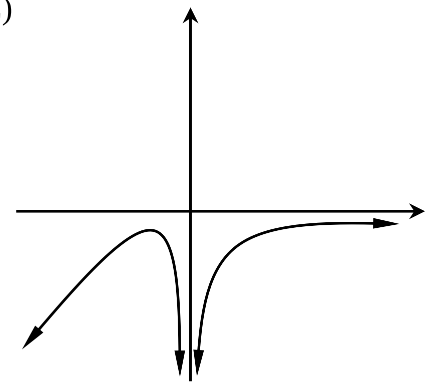


Which of the following best represents $y = f(x)$?

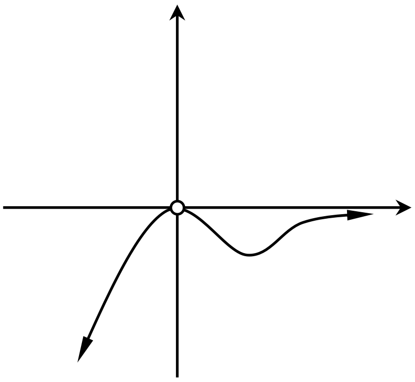
(A)



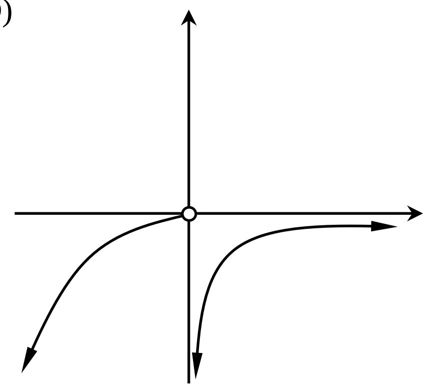
(C)



(B)



(D)



7 Which of the following polynomials is divisible by x^2-2x+1?

(A) $P(x)=x^n-n(x-1)-1$.

(B) $P(x)=x^n+n(x-1)-1$.

(C) $P(x)=x^n-n(x-1)+1$.

(D) $P(x)=x^n+n(x-1)+1$.

8 A particle of mass m is undergoing circular motion in a circle of radius r and with angular velocity ω. Let the particle's tangential velocity be V and let its tangential acceleration be a. Which of the following expressions is correct?

(A) $a=mr\omega^2$.

(B) $a=r\omega^2$.

(C) $V=mr\omega$.

(D) $V=r\omega$.

9 The area bounded by the curve $y=\sin x$, the x axis and $x=\dfrac{\pi}{2}$ is rotated about the y axis. Which of the following is an expression for the volume of the solid formed?

(A) $\pi\int_0^{\frac{\pi}{2}} x^2\cos x\,dx$.

(B) $2\pi\int_0^{\frac{\pi}{2}} x(1-\sin x)\,dx$.

(C) $\pi\int_0^{\frac{\pi}{2}} x(\pi-x)\sin x\,dx$.

(D) $2\pi\int_0^{\frac{\pi}{2}} x\cos x\,dx$.

10 Suppose $f(x)$ is a continuous smooth function over $a \leq x \leq b$ and $g(x)$ is a continuous function over $c \leq x \leq d$. Which of the following integrals is always greater than or equal to the other choices?

(A) $\displaystyle \int_a^b f(x)\,dx + \int_c^d g(x)\,dx.$

(B) $\displaystyle \int_a^b |f(x)|\,dx + \int_c^d |g(x)|\,dx.$

(C) $\displaystyle \left| \int_a^b f(x)\,dx \right| + \left| \int_c^d g(x)\,dx \right|.$

(D) $\displaystyle \left| \int_a^b f(x)\,dx + \int_c^d g(x)\,dx \right|.$

Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{a+b\sin x}{(b+a\sin x)^2}\,dx$. **2**

(b) Use implicit differentiation to show that $x^2 - y^2 = a^2$ and $xy = c^2$, where $a,\ c \neq 0$, always intersect at right angles. **2**

(c) Let the origin, z_1, z_2 and z_3 be four points in the complex plane that form a cyclic quadrilateral with centre c.

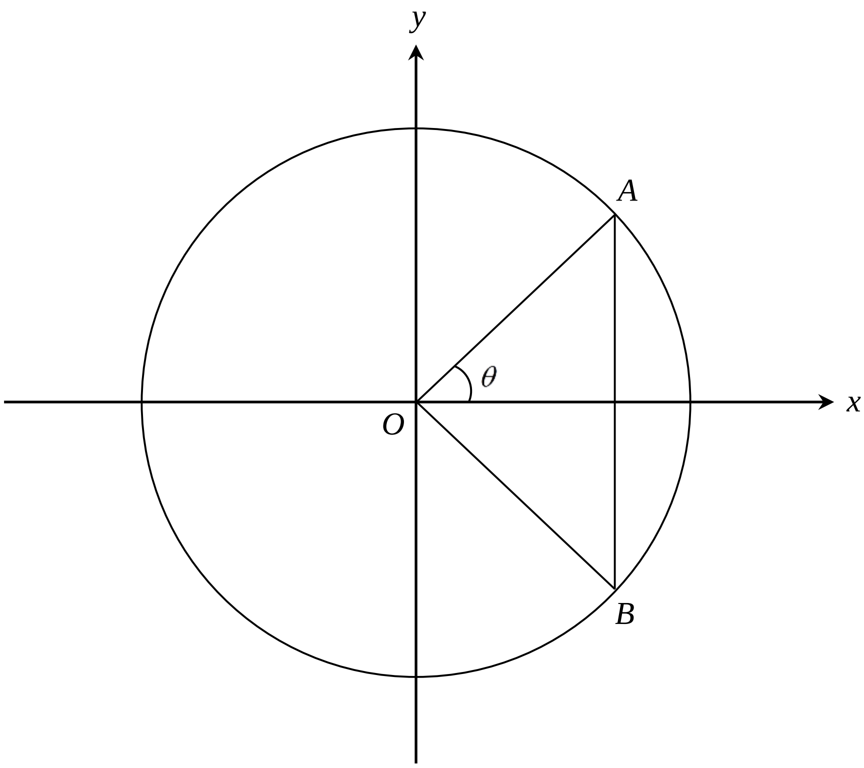
(i) Describe the set of complex numbers w satisfying $|w-k| = |w|$, where k is a fixed complex number. **1**

(ii) Prove that $\frac{1}{z_1}$, $\frac{1}{z_2}$ and $\frac{1}{z_3}$ are collinear. **2**

(d) Sketch the curve $y = \frac{2x^3-1}{2x^2-1}$. **3**

Question 11 continues on page 7

(e) In the circle $x^2 + y^2 = r^2$, an angle of $-\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$ subtends sector *OAB*, which is symmetrical about the *x* axis. The areas described by ΔOAB and the minor segment *AB* are rotated about the *y* axis, forming solids with volumes V_1 and V_2 respectively.



(i) Find V_1 in terms of θ. **2**

(ii) Hence, find the value(s) of θ such that $V_1 = V_2$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Define the sequence $a_{n+1} = 1 + a_1 a_2 \dots a_n$ for all integers $n \ge 1$, where $a_1 = 1$.

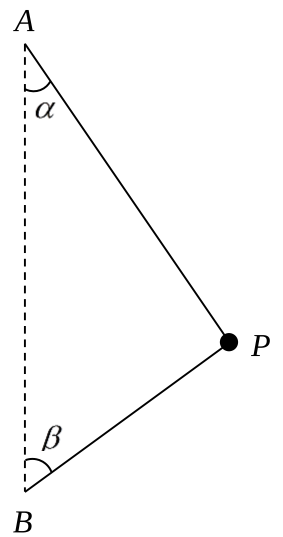
(i) Use Mathematical Induction to prove that **3**

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} = 2 - \frac{1}{a_1 a_2 a_3 \dots a_n}.$$

(ii) Prove that $a_{n+1} > a_n$, for all $n \ge 1$. **1**

(iii) Hence, find the limiting value of $\sum_{k=1}^{n} \frac{1}{a_k}$ as $n \to \infty$. **1**

(b) A particle P of mass m spins with angular velocity ω in a circle of radius r, and is suspended by two light inextensible strings making angles from the vertical of α and β, where $0 < \alpha < \beta < \frac{\pi}{2}$. Let A be the point from which the top string is suspended from, and let B be the point where the bottom string is attached. **4**



The strings AP and BP experience tensions of T_1 and T_2 respectively.

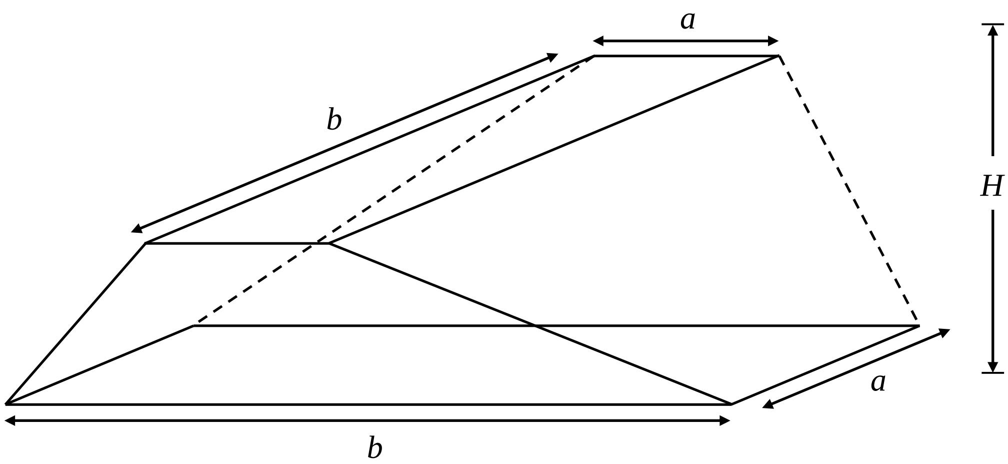
Prove that if $T_1 > T_2$, then $\omega^2 < \frac{g}{r}\left(\frac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta}\right)$.

Question 12 continues on page 9

(c) Two sheets of paper with dimensions a and b, $a < b$, are stacked on a flat surface. One sheet of paper is raised up by a height H, and then rotated 90° about its centre.

The vertices of the bottom sheet and top sheet are connected, forming a solid.

Cross-sections taken parallel to the sheets are rectangles. Let the height of a typical cross section from the floor be h.



(i) Let δV be the volume of a cross section with thickness δh.

Show that **4**

$$\delta V = \frac{1}{H^2}\left[abH^2 + (b-a)^2\left(hH - h^2\right)\right]\delta h$$

(ii) Hence, show that the volume of the solid is **2**

$$V = \frac{H}{6}\left(a^2 + 4ab + b^2\right)$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A pack of n cards consists of q different colours. Each colour has p different cards labeled from 1 to p, such that $n = pq$.

From the pack, k cards are drawn out, where $2 < k \le p < n$.

(i) Find the probability of exactly 2 of those cards being the same number. **2**

(ii) Show that if $p = q = k$, then the number of ways of having exactly two cards of the same number is $\frac{p^p}{2}(p-1)^2$. **1**

(b) A particle of mass m falls from rest and experiences a gravitational force of mg and a resistive force of mkv^2. Let its terminal velocity be denoted by u.

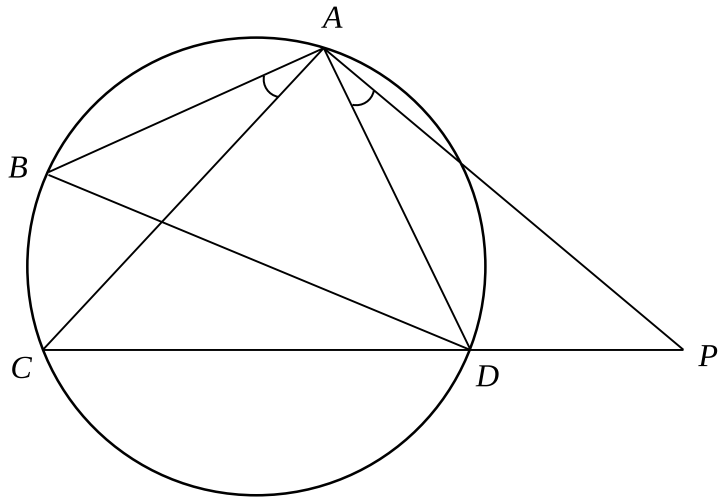
(i) Show that $u^2 = \frac{g}{k}$. **1**

(ii) Prove that the displacement of the particle at time t is **4**

$$x = \frac{1}{k}\ln\left(\frac{e^{\frac{gt}{u}} + e^{-\frac{gt}{u}}}{2}\right).$$

Question 13 continues on page 11

(c) Let $ABCD$ be a cyclic quadrilateral. A line is drawn through A to meet CD extended at P such that $\angle BAC = \angle DAP$.



(i) Prove that $\Delta BAC \parallel\!| \Delta DAP$ and hence show that $DP = \dfrac{AD \times BC}{AB}$. **2**

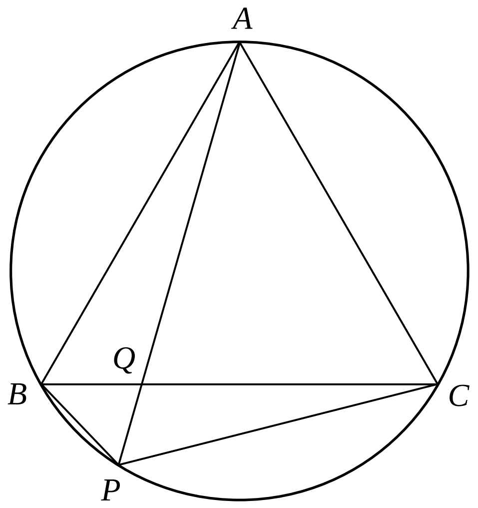
(ii) Prove that $\Delta ABD \parallel\!| \Delta ACP$ and hence show that $CP = \dfrac{AC \times BD}{AB}$. **2**

(iii) Prove that **1**

$$AC \times BD = AB \times CD + AD \times BC$$

Question 13 continues on page 12

Consider an equilateral triangle ABC inscribed within a circle. A line drawn from A passes through BC at Q and the circumference of the circle at P.



(iv) Use part (iii) to prove that $PA = PB + PC$. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let $P(z) = z^{n+1} - kz^n + kz - 1$, where $-1 \le k \le 1$, and n is a positive integer.

(i) Prove that if α is a root of $P(z)$, $\frac{1}{\alpha}$ is also a root. **1**

(ii) Suppose there exists $n+1$ distinct roots of $P(z)$ all with modulus not equal to one. **1**

Explain why there exists at least one root α, such that $|\alpha| > 1$.

(iii) Show that $|\alpha|^{2n} |\alpha - k|^2 = |1 - k\alpha|^2$ **1**

(iv) Deduce that $\left(|\alpha|^2 - 1\right)\left(1 - k^2\right) < 0$. **2**

(v) Hence, explain why all roots of $P(z)$ have modulus 1. **1**

(b) Consider an n sided regular polygon centred at the origin with vertices $P_1P_2P_3...P_n$, where P_1 has coordinates $(1, 0)$.

Let $\theta = \frac{2\pi}{n}$ and define $d_k = P_1P_{k+1}$.

(i) Show that $d_k^2 = (1 - \cos k\theta)^2 + \sin^2 k\theta$ **1**

(ii) Hence, show that **1**

$$d_k^2 = 2 - \omega^k - \omega^{n-k},$$

where ω is an n^{th} root of unity.

(iii) Deduce that $d_1^2 + d_2^2 + d_3^2 + ... + d_{n-1}^2 = 2n$. **2**

Question 14 continues on page 14

(c) Let a, b, A and B be positive numbers.

(i) Prove that $\dfrac{ab}{AB} \le \dfrac{1}{2}\left(\dfrac{a^2}{A^2} + \dfrac{b^2}{B^2}\right)$. **1**

(ii) Let $A = \sqrt{\sum_{k=1}^{n} a_k^2}$ and $B = \sqrt{\sum_{k=1}^{n} b_k^2}$, where a_k and b_k are positive real numbers. **2**

Use (i) to prove that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

(iii) Let $S = x_1 + x_2 + x_3 + ... + x_n$, where $x_k > 0$ for all $1 \le k \le n$. **2**

Use (ii) to prove that

$$\frac{S}{S - x_1} + \frac{S}{S - x_2} + \frac{S}{S - x_3} + ... + \frac{S}{S - x_n} \ge \frac{n^2}{n-1}$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Define the series $S_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx)$, where $0 \le x \le \pi$.

(i) Prove that $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha \sin\beta$ **1**

(ii) Prove that **2**

$$S_n(x) = \begin{cases} \dfrac{\sin\left[(2n+1)x/2\right]}{2\sin(x/2)}, & \text{if } x \neq 0. \\ n + \dfrac{1}{2}, & \text{if } x = 0. \end{cases}$$

(b) Prove that $\int_0^\pi x\cos(kx)\,dx = \frac{1}{k^2}\left[(-1)^k - 1\right]$, where k is a positive integer. **2**

(c) Define the integral $I_n = \lim_{a \to 0} \int_a^\pi x\, f_n(x)\,dx$, where

$$f_n(x) = \frac{\sin\left[(2n+1)x/2\right]}{2\sin(x/2)}$$

(i) Use (a) and (b), or otherwise, to prove that **1**

$$I_n = \frac{\pi^2}{4} + \sum_{k=1}^{n} \left[\frac{(-1)^k - 1}{k^2}\right].$$

(ii) Deduce that **1**

$$\frac{1}{2} I_{2n-1} = \frac{\pi^2}{8} - \sum_{k=1}^{n} \frac{1}{(2k-1)^2}.$$

Question 15 continues on page 16

Let $g(x) = \dfrac{d}{dx}\left(\dfrac{x/2}{\sin(x/2)}\right)$.

(iii) Using Integration by Parts, or otherwise, prove that **3**

$$I_{2n-1} = \frac{2}{4n-1}\left(1 + \lim_{a \to 0}\int_a^{\pi} g(x)\cos\left[\frac{(4n-1)x}{2}\right]dx\right).$$

(iv) Prove that **2**

$$1 - \frac{\pi}{2} \le \lim_{a \to 0}\int_a^{\pi} g(x)\cos\left[\frac{(4n-1)x}{2}\right]dx \le \frac{\pi}{2} - 1,$$

(v) Deduce that as $n \to \infty$, $I_{2n-1} \to 0$. **1**

(vi) Hence, prove that **2**

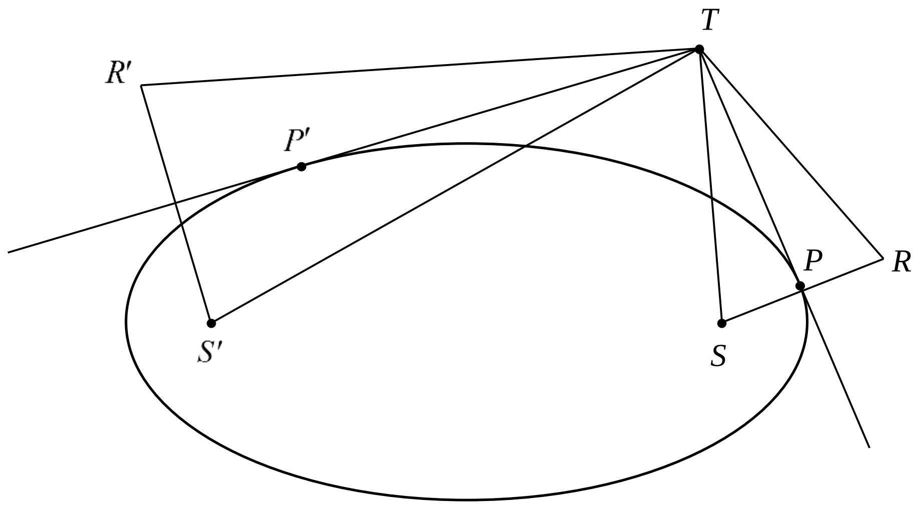
$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let T be an external point of an ellipse. From T, two tangents are drawn to the ellipse and let P and P' be the points of contact. Lines are drawn from T to both foci S and S'.

S and S' are reflected about the lines TP and TP' respectively, forming R and R' respectively.



(i) Prove that S, P' and R' are collinear. **2**

(ii) Hence, prove that $\Delta TR'S \equiv \Delta TS'R$. **2**

(iii) Deduce that $\angle STP = \angle S'TP'$. **1**

Question 16 continues on page 18

(b) Let $P(z)$ be a polynomial with real coefficients with exactly one real root. Its roots z_1, z_2 and z_3 form the vertices of a triangle Δ_P in the complex plane.

Define $T(z) = Az + B$, where A is a real non-zero constant, and B is a fixed complex number.

Let Δ_T be the triangle with vertices $T(z_k)$, where $k = 1, 2, 3$.

(i) Explain why $\Delta_T \,|||\, \Delta_P$. **1**

(ii) Let $P_T(z) = (z - T(z_1))(z - T(z_2))(z - T(z_3))$. **1**

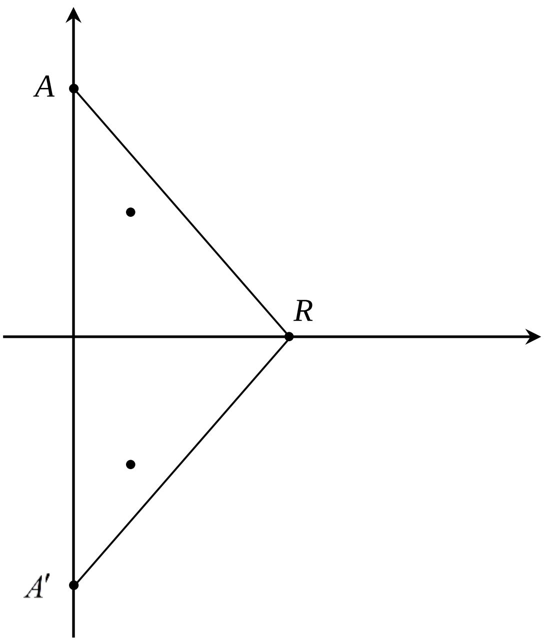
Prove that $P_T(T(z)) = A^3 P(z)$.

(iii) Deduce that if α is a root of $P'(z)$, then $T(\alpha)$ is a root of $P_T'(z)$. **1**

(iv) Describe the geometric significance of (iii). **1**

Question 16 continues on page 19

(c) Let $P(z)$ be a monic cubic polynomial with roots $i, -i, r$, where r is a positive non-zero real number. Let the triangle formed by the roots be T with vertices A, A', R.



(i) Show that $P'(z) = 3z^2 - 2rz + 1$. **1**

(ii) Suppose $P'(z)$ has no real roots. **3**

Prove that the roots of $P'(z)$ lie within T.

(d) Let $S(x)$ be a real cubic polynomial with two non-real roots such that the roots of $S'(x)$ are also non-real. Let the roots of $S(x)$ be z_1, z_2, z_3 and let the roots of $S'(x)$ be w_1 and w_2. **2**

Use (b) and (c) to show that w_1 and w_2 always lie within the triangle defined by z_1, z_2, z_3.

End of Exam

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2013 Bored of Studies Trial Examinations

Mathematics Extension 2

SOLUTIONS

Multiple Choice

1. D
2. D
3. C
4. D
5. C
6. A
7. A
8. D
9. C
10. B

Brief Explanations

Question 1 (I) is not necessarily true because the roots could be $i, -i, 4$.
If (II) is true, that means $\alpha^2+\beta^2+\gamma^2$ is a real number, since the real numbers can be ordered. However, just because $\alpha^2+\beta^2+\gamma^2$ is a real number, it doesn't necessarily mean that there are at least two non-real roots, since the roots could be $4i, 1, 2$. The root $4i$ doesn't have a conjugate because the polynomial may not be real, so the roots don't have to occur in conjugate pairs. For (III), again the roots may not be conjugate pairs. So a counter-example is a polynomial with roots i, i, 5. Hence, the answer is (D).

Question 2 Although (C) yields the correct 'answer' if the formula were blindly used, it would be incorrect because it only takes the solution in the principal range of $y=\tan^{-1}x$, which is $-\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2}$. However, since z is in the third quadrant, it clearly cannot exist within that domain. Hence, the answer is (D).

Question 3 Standard use of $x=\frac{a}{e}$ for directrix and $x=-ae$ for foci.

Question 4 To make a diagonal, we need 2 points out of n, so we have $\binom{n}{2}$ 'diagonals'. However, n of these 'diagonals' are actually sides of the polygon, so the total number of actual diagonals is $\binom{n}{2}-n$. Hence, we solve $\binom{n}{2}-n=44$, and the solution is (D).

Question 5 We can start from (D) and use various substitutions. To get (A), it looks like the substitution $x=\tan u$. To get (B), it looks like the substitution $x=u^2$. For (C), it looks like we make the substitution $x=e^u$, but the limits are incorrect. Hence, the answer is (C).

Question 6 To get the original curve, we multiply the given curve by $y = \frac{1}{x}$, noting that we actually do not have a discontinuity at the origin in the original curve, since we have in a sense 'introduced the discontinuity' in the process of multiplying the curve by $y = \frac{1}{x}$.

Question 7 Use the Multiple Root Theorem.

Question 8 Neither (A) nor (C) are correct since they have *m*'s, when the expressions should be independent of *m*. (B) is incorrect because it represents centripetal acceleration, not tangential acceleration. Hence, the answer is (D), which is tangential velocity.

Question 9 Using either the Washer or Cylindrical Shells method will not yield any of the options immediately. The correct answer (C) can be found by using the Washer Method and then the substitution $y = \sin x$ and $u = \frac{\pi}{2} - x$.

Question 10 We think about the situation geometrically. In (A), that is obviously not always $\geq$ the other choices because the functions could be negative. (C) cannot be the answer and to see this, consider the sine curve. (D) cannot be correct for similar reasons. However, (B) guarantees that the curve is above or on the *x* axis, thereby maximising the integral which will also be the area under the curve.

Written Response

Question 11 (a)

Divide top and bottom by $\cos^2 x$.

$$\int_0^{\frac{\pi}{2}} \frac{a + b\sin x}{(b + a\sin x)^2}\,dx = \int_0^{\frac{\pi}{2}} \frac{a\sec^2 x + b\sec x\tan x}{(b\sec x + a\tan x)^2}\,dx$$

$$= -\left.\frac{1}{b\sec x + a\tan x}\right|_0^{\frac{\pi}{2}}$$

$$= -\left.\frac{\cos x}{b + a\sin x}\right|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{b}$$

Question 11 (b)

Differentiating $x^2 - y^2 = a^2$ implicitly and substituting the point of intersection $P(x_0, y_0)$, we have

$$2x - 2yy' = 0$$

$$y'_1 = \frac{x_0}{y_0}$$

Similarly for $xy = c^2$, we have

$$y + xy' = 0$$

$$y'_2 = -\frac{y_0}{x_0}$$

And we see that $y'_1 \times y'_2 = -1$, hence the two curves intersect at right angles, regardless of the value of *a* and *c*.

Question 11 (c) (i)

The equation can be translated as "The set of all points *w* that are equidistant from *k* and the origin". The locus is then the perpendicular bisector of the interval from *k* to the origin. Hence, it is a straight line.

Question 11 (c) (ii)

Let $w = \frac{1}{z_i}$, where $i = 1, 2, 3$. Thus we have $z_i = \frac{1}{w}$.

Note that we have $|z_i - c| = |c|$. This is because *c* is the centre of the circle, so $|c|$ is the radius of the circle. However, all of the points z_i, where $i = 1, 2, 3$, must be equidistant from *c*, since they are points on a circle.

We now make the substitution $z_i = \frac{1}{w}$.

$$\left|\frac{1}{w} - c\right| = |c|$$
$$\left|\frac{1 - cw}{w}\right| = |c|$$
$$\frac{|1 - cw|}{|c|} = |w|$$
$$\left|\frac{1}{c} - w\right| = |w|$$
$$\left|w - \frac{1}{c}\right| = |w|$$

However, setting $\frac{1}{c} = k$, we have the expression from (i). Hence, the locus of *w* is a straight line, as we have found earlier.

However, we also have $w = \frac{1}{z_i}$, hence $\frac{1}{z_1}$, $\frac{1}{z_2}$ and $\frac{1}{z_3}$ are collinear. □

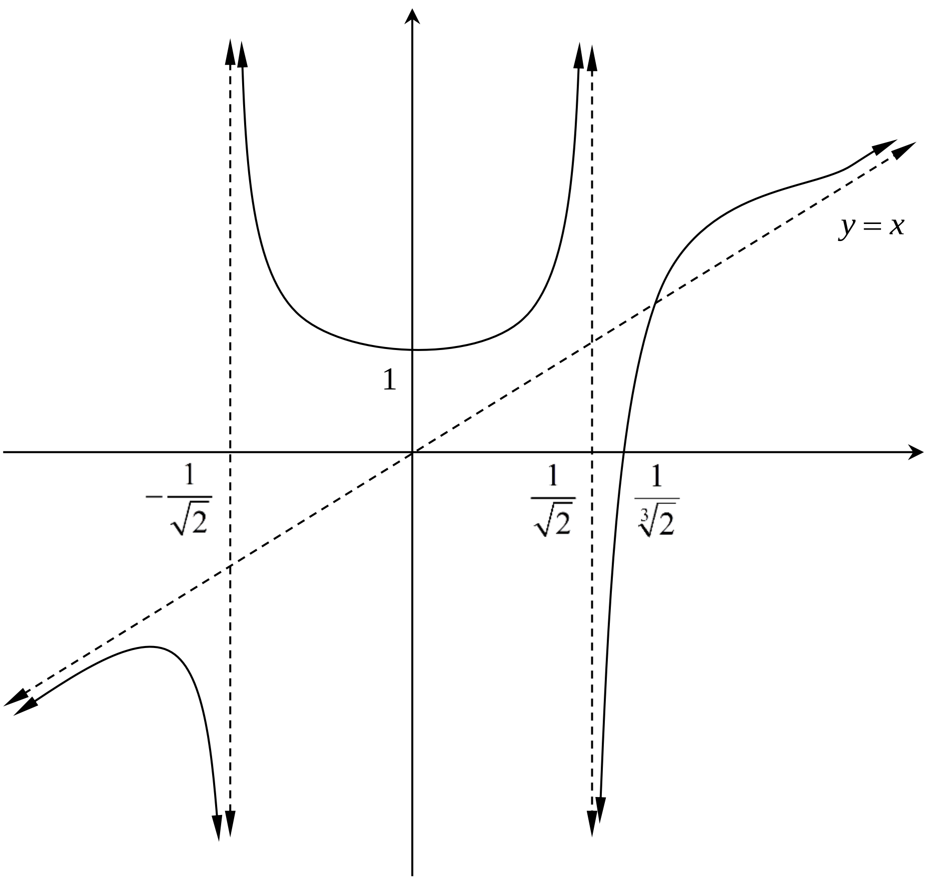
Question 11 (d)

Using polynomial long division, we have $\dfrac{2x^3-1}{2x^2-1}=x+\dfrac{x-1}{2x^2-1}$, so we can see that as $x\to\infty$, $y\to x^+$ and that as $x\to-\infty$, $y\to x^-$.

We know that the curve has vertical asymptotes where the denominator is zero, namely $x=\pm\dfrac{1}{\sqrt{2}}$.

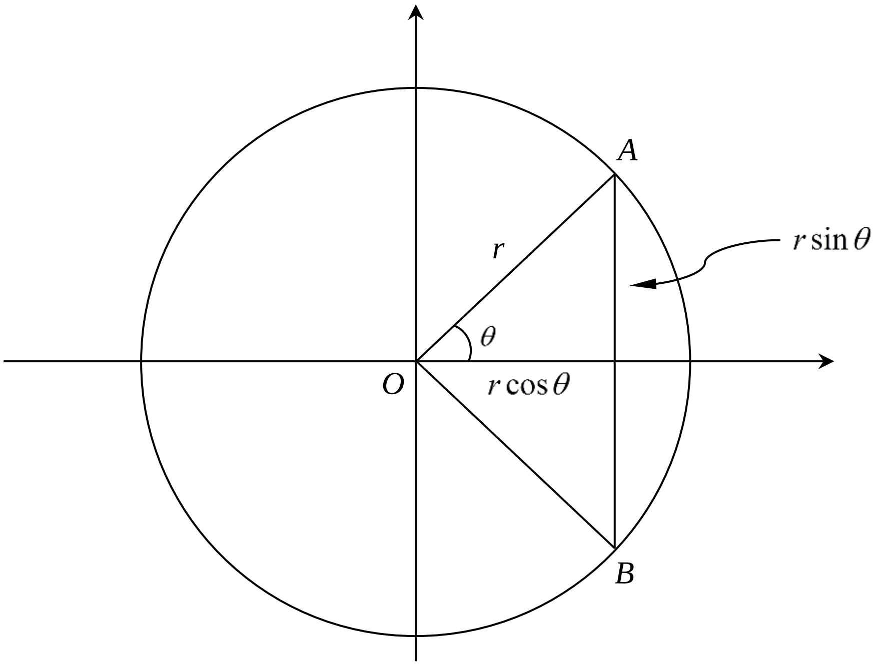
We also can easily find *x* and *y* intercepts, being $x=\dfrac{1}{\sqrt[3]{2}}$ and $y=1$ respectively.

Putting all the pieces together, we get the sketch of the curve.



Question 11 (e) (i)

The volume V_1 can be found by subtracting two identical cones from a larger cylinder.



We first find the volume of the cone.

$V_{\text{cone}} = \dfrac{\pi}{3}R^2h$, where $R = r\cos\theta$ and $h = r\sin\theta$.

$V_{\text{cone}} = \dfrac{\pi}{3}r^3\cos^2\theta\sin\theta$.

Now, to find the volume of the cylinder.

$V_{\text{cylinder}} = \pi R^2 H$, where $R = r\cos\theta$ and $H = 2r\sin\theta$.

$V_{\text{cylinder}} = 2\pi r^3\cos^2\theta\sin\theta$

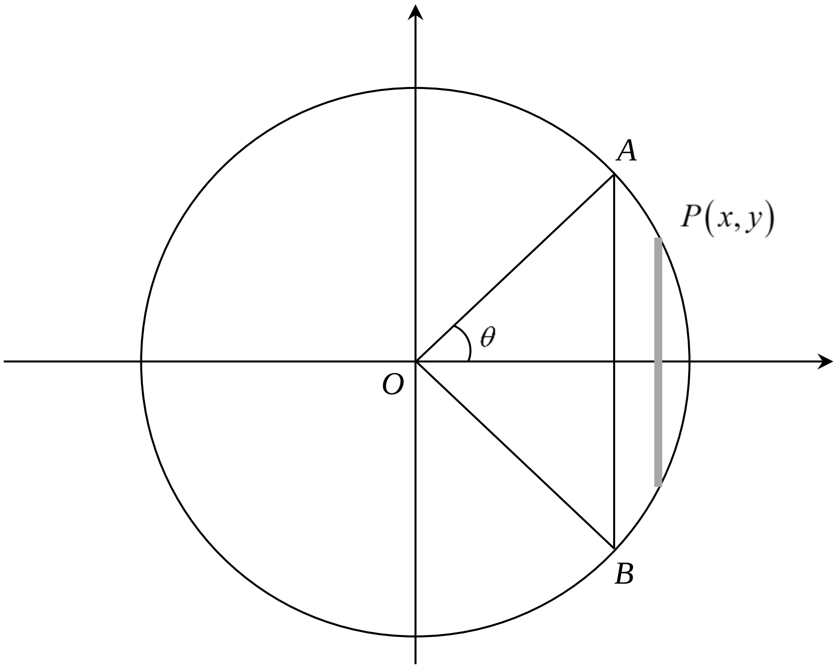
We now have enough to find V_1.

$$\begin{aligned}V_1 &= V_{\text{cylinder}} - 2\times V_{\text{cone}}\\ &= 2\pi r^3\cos^2\theta\sin\theta - \frac{2\pi}{3}r^3\cos^2\theta\sin\theta\\ &= \frac{4\pi}{3}r^3\cos^2\theta\sin\theta\end{aligned}$$

Question 11 (e) (ii)

We can either use cylindrical shells or the washer method. Here, we will use the method of cylindrical shells, though the washer method will of course yield the same result.

Construct a thin vertical strip of thickness δx from an arbitrary point $P(x, y)$ on the minor segment *AB*. Rotate this thin strip about the *y* axis to roughly form a cylinder. We now unfold the cylinder to acquire a rectangular prism with length $2\pi x$, height $2y$ and width δx.



$$\delta V = 4\pi xy\,\delta x$$
$$= 4\pi x\sqrt{r^2 - x^2}\,\delta x$$

$$V_2 = \lim_{\delta x \to 0} \sum_{r\cos\theta \le x \le r} 4\pi x\sqrt{r^2 - x^2}\,\delta x$$
$$= 4\pi \int_{r\cos\theta}^{r} x\sqrt{r^2 - x^2}\,dx$$
$$= -\frac{4\pi}{3}\left(r^2 - x^2\right)\sqrt{r^2 - x^2}\,\Big|_{r\cos\theta}^{r}$$
$$= \frac{4\pi}{3}\left(r^2 - r^2\cos^2\theta\right)\sqrt{r^2 - r^2\cos^2\theta}$$
$$= \frac{4\pi}{3} r^3\left(1 - \cos^2\theta\right)\sqrt{1 - \cos^2\theta}$$
$$= \frac{4\pi}{3} r^3 \sin^3\theta$$

Since $V_1 = V_2$, we have

$$\frac{4\pi}{3} r^3 \cos^2\theta \sin\theta = \frac{4\pi}{3} r^3 \sin^3\theta$$
$$\cos^2\theta = \sin^2\theta$$
$$\tan^2\theta = 1$$
$$\tan\theta = \pm 1$$
$$\theta = \pm\frac{\pi}{4}$$

Question 12 (a) (i)

Base Case: $n = 1$

$$LHS = \frac{1}{a_1} = 1 \qquad RHS = 2 - \frac{1}{a_1} = 2 - 1 = 1$$

Inductive Hypothesis: $n = k$

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_k} = 2 - \frac{1}{a_1 a_2 a_3 \ldots a_k}$$

Inductive Step: $k \Rightarrow k+1$

Required to prove: $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_k} + \frac{1}{a_{k+1}} = 2 - \frac{1}{a_1 a_2 a_3 \ldots a_k a_{k+1}}$.

$$\begin{aligned}
LHS &= \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_k} + \frac{1}{a_{k+1}} \\
&= 2 - \frac{1}{a_1 a_2 a_3 \ldots a_k} + \frac{1}{a_{k+1}} \\
&= 2 - \frac{a_{k+1} - a_1 a_2 a_3 \ldots a_k}{a_1 a_2 a_3 \ldots a_k a_{k+1}} \\
&= 2 - \frac{1}{a_1 a_2 a_3 \ldots a_k a_{k+1}}, \quad \text{since } a_{n+1} = 1 + a_1 a_2 \ldots a_n \\
&= RHS \qquad \square
\end{aligned}$$

Question 12 (a) (ii)

$$\begin{aligned}
a_{n+1} - a_n &= 1 + a_1 a_2 \ldots a_n - 1 - a_1 a_2 \ldots a_{n-1} \\
&= a_1 a_2 \ldots a_n - a_1 a_2 \ldots a_{n-1} \\
&= a_1 a_2 \ldots a_{n-1}\left(a_n - 1\right) \\
&= a_1 a_2 \ldots a_{n-1}\left(1 + a_1 a_2 \ldots a_{n-1} - 1\right) \\
&= a_1 a_2 \ldots a_{n-1}\left(a_1 a_2 \ldots a_{n-1}\right) \\
&= \left(a_1 a_2 \ldots a_{n-1}\right)^2 \\
&> 0
\end{aligned}$$

$\therefore a_{n+1} > a_n \qquad \square$

Question 12 (a) (iii)

Since $a_{n+1} > a_n$, we know that the sequence a_k is increasing for all $k \geq 1$.

Hence, since $a_1 = 1$, we can deduce that as $n \to \infty$, we have $a_1 a_2 a_3 \ldots a_n \to \infty$.

Therefore as $n \to \infty$, $\sum_{k=1}^{n} \frac{1}{a_k} \to 2$.

Question 12 (b)

Resolving forces vertically and horizontally, we have

Horizontally: $T_1 \sin\alpha + T_2 \sin\beta = mr\omega^2$
Vertically: $T_1 \cos\alpha - T_2 \cos\beta = mg$

Applying the inequality $T_1 > T_2$ to both expressions, we have

$$T_1 \sin\alpha + T_2 \sin\beta = mr\omega^2 < T_1 \sin\alpha + T_1 \sin\beta$$

$$T_1 \cos\alpha - T_2 \cos\beta = mg > T_1 \cos\alpha - T_1 \cos\beta$$

$$T_1 \cos\alpha - T_2 \cos\beta = \frac{1}{mg} < \frac{1}{T_1 \cos\alpha - T_1 \cos\beta}$$

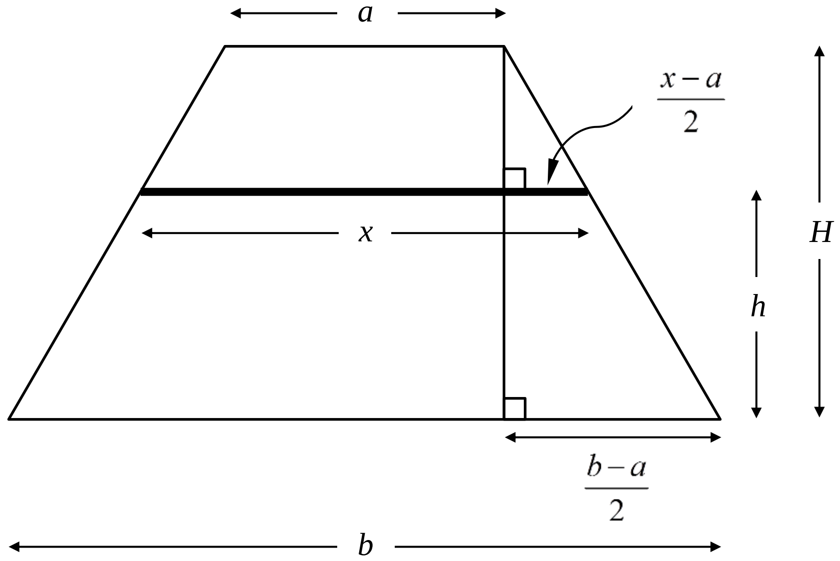
Since $0 < \alpha < \beta < \frac{\pi}{2} \Rightarrow \cos\alpha - \cos\beta > 0$, we know that all the quantities are positive.

Multiply both inequalities.

$$\begin{aligned} \frac{mr\omega^2}{mg} &< \frac{T_1 \sin\alpha + T_1 \sin\beta}{T_1 \cos\alpha - T_1 \cos\beta} \\ &= \frac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta} \\ \omega^2 &< \frac{g}{r}\left(\frac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta}\right) \qquad \square \end{aligned}$$

Question 12 (c) (i)

We view the slice from two perspectives, and construct lines such that we have similar triangles. The following is the front view of the solid.

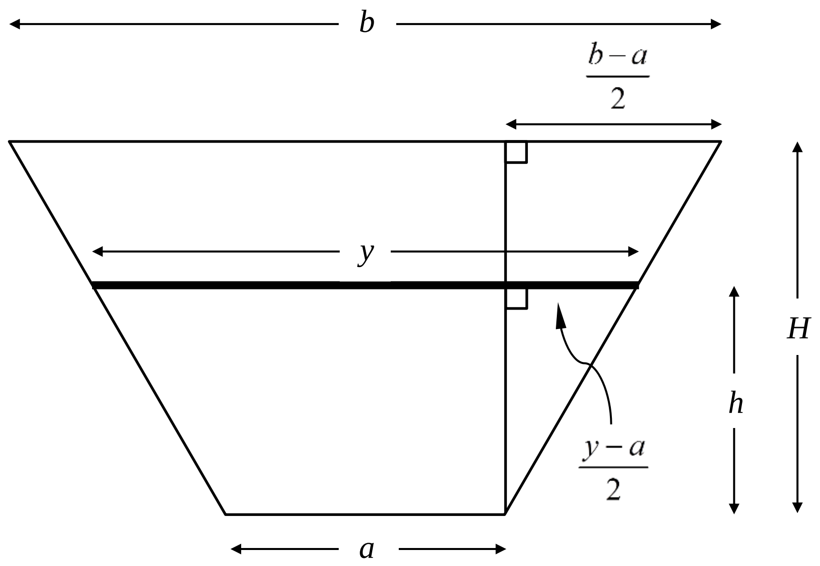


Using similar triangles, we have the following relation.

$$\frac{H}{H-h}=\frac{\frac{b-a}{2}}{\frac{x-a}{2}}=\frac{b-a}{x-a}$$

Re-arranging to make x the subject, we have $x=a+\frac{(b-a)(H-h)}{H}$.

Now, consider the side-view of the solid.



Using similar triangles, we have the following relation.

$$\frac{H}{h}=\frac{\frac{b-a}{2}}{\frac{y-a}{2}}=\frac{b-a}{y-a}\qquad\Rightarrow\qquad y=a+\frac{(b-a)h}{H}$$

But we also have $\delta V = xy\,\delta h$, so we put the pieces together.

$$\begin{aligned}
\delta V &= \left(a+\frac{(b-a)(H-h)}{H}\right)\left(a+\frac{(b-a)h}{H}\right)\delta h\\
&=\frac{1}{H^2}\Big[\big(aH-(b-a)h\big)+(b-a)H\Big]\Big[aH+(b-a)h\Big]\delta h\\
&=\frac{1}{H^2}\Big[a^2H^2-h^2(b-a)^2+(b-a)^2hH+a(b-a)H^2\Big]\delta h\\
&=\frac{1}{H^2}\Big[aH^2(a+b-a)+(b-a)^2\left(hH-h^2\right)\Big]\delta h\\
&=\frac{1}{H^2}\Big[abH^2+(b-a)^2\left(hH-h^2\right)\Big]\delta h \qquad \square
\end{aligned}$$

Question 12 (c) (ii)

To find the volume, we integrate with respect to *h*.

$$\begin{aligned}
V &= \lim_{\delta h\to 0}\sum_{0\le h\le H}\frac{1}{H^2}\Big[abH^2+(b-a)^2\left(hH-h^2\right)\Big]\delta h\\
&=\frac{1}{H^2}\int_0^H abH^2+(b-a)^2\left(hH-h^2\right)dh\\
&=\frac{1}{H^2}\int_0^H abH^2+H(b-a)^2h-(b-a)^2h^2dh\\
&=\frac{1}{H^2}\left[abhH^2+\frac{H(b-a)^2}{2}h^2-\frac{(b-a)^2}{3}h^3\right]_0^H\\
&=\frac{1}{H^2}\left[abH^3+\frac{H^3(b-a)^2}{2}-\frac{H^3(b-a)^2}{3}\right]\\
&=abH+\frac{H(b-a)^2}{6}\\
&=\frac{H}{6}\left(a^2+4ab+b^2\right) \qquad \square
\end{aligned}$$

Question 13 (a) (i)

We first pick a number to have exactly two of, so $\binom{p}{1}$.

We must now pick the colour of the two cards. There are q possible colours, so $\binom{q}{2}$.

Now, there remain $k-2$ cards to pick out from a remaining total of $p-1$ since none of them can be the same number as those which we have already chosen, so $\binom{p-1}{k-2}$.

We have now picked k cards, two of which have their colours chosen already. Each of the remaining $k-2$ cards needs a colour allocated to it. Hence, we must multiply by q^{k-2}, since each of the k cards have q possible colours.

The total number of ways of choosing k cards from n is $\binom{n}{k}$.

Thus, the probability is $\dfrac{\binom{p}{1}\binom{q}{2}\binom{p-1}{k-2}q^{k-2}}{\binom{n}{k}}$.

Question 13 (a) (ii)

Setting $p=q=k$, we have

$$\begin{aligned}\binom{p}{1}\binom{p}{2}\binom{p-1}{p-2}p^{p-2} &= \frac{p^2(p-1)}{2}\times\binom{p-1}{1}p^{p-2}\\ &= \frac{(p-1)^2}{2}p^p\end{aligned}$$

Question 13 (b) (i)

From Newton's 2nd Law, we have $ma = mg - mkv^2$, so $a = g - kv^2$.

Terminal velocity occurs as $a \to 0$, so $ku^2 = g \Rightarrow u^2 = \dfrac{g}{k}$ □

Question 13 (b) (ii)

Since the required expression has time, we use $a = \dfrac{dv}{dt}$.

$$\frac{dv}{dt} = g - kv^2$$

$$dt = \frac{dv}{g - kv^2}$$

We now integrate both sides with respect to their variables.

$$\begin{aligned}
\int_0^T dt \quad &= \quad \int_0^V \frac{dv}{g - kv^2} \\
&= \frac{1}{k} \int_0^V \frac{dv}{\frac{g}{k} - v^2} \\
&= \frac{1}{k} \int_0^V \frac{dv}{u^2 - v^2} \\
&= \frac{1}{2ku} \int_0^V \frac{1}{u+v} + \frac{1}{u-v}\, dv \\
&= \frac{1}{2ku} \Big[\ln(u+v) - \ln(u-v) \Big]_0^V \\
&= \frac{1}{2ku} \ln\left(\frac{u+v}{u-v} \right) \Bigg|_0^V \\
T &= \frac{1}{2ku} \ln\left(\frac{u+V}{u-V} \right)
\end{aligned}$$

We now re-arrange to make *v* the subject.

$$\begin{aligned}
e^{2ukT} &= \frac{u+V}{u-V} \\
(u-V)e^{2ukT} &= u+V \\
V\left(1+e^{2ukT}\right) &= u\left(e^{2ukT}-1\right) \\
V &= u\left(\frac{e^{2ukT}-1}{e^{2ukT}+1} \right) \\
\frac{dx}{dT} &= u\left(\frac{e^{2ukT}-1}{e^{2ukT}+1} \right) \\
dx &= u\left(\frac{e^{2ukT}-1}{e^{2ukT}+1} \right) dT
\end{aligned}$$

$$\int_0^X dx = \int_0^{T^*} u\left(\frac{e^{2ukT}-1}{e^{2ukT}+1}\right)dT$$

$$= u\int_0^{T^*} \frac{e^{ukT}-e^{-ukT}}{e^{ukt}+e^{-ukt}}dT$$

$$= \frac{1}{k}\ln\left(e^{ukT}+e^{-ukT}\right)\Big|_0^{T^*}$$

$$X = \frac{1}{k}\ln\left(\frac{e^{ukT^*}+e^{-ukT^*}}{2}\right)$$

But $u^2 = \frac{g}{k} \Rightarrow ku = \frac{g}{u}$, so we thus have

$$X = \frac{1}{k}\ln\left(\frac{e^{\frac{gT^*}{u}}+e^{-\frac{gT^*}{u}}}{2}\right) \qquad \square$$

Question 13 (c) (i)

$\angle BAC = \angle DAP$ ...(given)

$\angle CBA = \angle ADP$ ...(exterior $\angle$ of cyclic quad.)

$\therefore \Delta BAC \,|||\, \Delta DAP$ ...(equiangular) $\square$

By similar triangle ratios, we have $\frac{DP}{BC} = \frac{AD}{AB}$ and hence the result.

Question 13 (c) (ii)

$\angle ABD = \angle ACD$ ...(angles from same chord)

$\angle BAD = x + \angle CAD$ ...(where $x = \angle BAC = \angle DAP$)

$= \angle CAP$

$\therefore \Delta ABD \,|||\, \Delta ACP$ ...(equiangular) $\square$

By similar triangle ratios, we have $\frac{CP}{BD} = \frac{AC}{AB}$ and hence the result.

Question 13 (c) (iii)

$$\begin{aligned} AC \times BD &= AB \times CP && \ldots\text{(from (ii))} \\ &= AB(CD + DP) \\ &= AB \times CD + AB \times DP \\ &= AB \times CD + AD \times BC \quad \square && \ldots\text{(from (i))} \end{aligned}$$

Question 13 (c) (iv)

If we use the expression from (iii) for the quadrilateral *ABPC*, we have

$$PA \times BC = PB \times CA + PC \times BA$$

But the triangle is equilateral, so $AB = BC = CA$.

Substituting that in, we have the required result.

$$\begin{aligned} PA \times BC &= PB \times BC + PC \times BC \\ PA &= PB + PC \qquad \square \end{aligned}$$

Question 14 (a) (i)

$$\begin{aligned} P\left(\frac{1}{\alpha}\right) &= \left(\frac{1}{\alpha}\right)^{n+1} - k\left(\frac{1}{\alpha}\right)^{n} + k\left(\frac{1}{\alpha}\right) - 1 \\ &= \frac{1}{\alpha^{n+1}}\left[1 - k\alpha + k\alpha^{n} - \alpha^{n+1}\right] \\ &= \frac{1}{\alpha^{n+1}}(0), \text{ since } P(\alpha) = 0 \\ &= 0 \end{aligned}$$

Question 14 (a) (ii)

Suppose all roots exist inside the unit circle. Then all roots have modulus <1. However, we know from (i) that if α is a root of $P(z)$, $\frac{1}{\alpha}$ is also a root of $P(z)$. But if $|\alpha| < 1$, then $\left|\frac{1}{\alpha}\right| > 1$, which contradicts all roots having modulus <1. Hence, there exists at least one root α such that $|\alpha| > 1$.

Question 14 (a) (iii)

Substitute $z=\alpha$ and re-arrange our original polynomial.

$$\begin{aligned}\alpha^{n+1}-k\alpha^{n}+k\alpha-1&=0\\ |\alpha|^{n}|\alpha-k|&=|1-k\alpha|\\ |\alpha|^{2n}|\alpha-k|^{2}&=|1-k\alpha|^{2}\end{aligned}$$

Question 14 (a) (iv)

$$|\alpha|^{2n}|\alpha-k|^{2}=|1-k\alpha|^{2}$$

We now use the fact that $|\alpha|>1$

$$\begin{aligned}|\alpha|^{2n}|\alpha-k|^{2}&=|1-k\alpha|^{2}\\ |\alpha-k|^{2}&<|1-k\alpha|^{2}\end{aligned}$$

Let $\alpha=|\alpha|\operatorname{cis}\theta$, where $|\alpha|>1$.

$$\begin{aligned}\big||\alpha|\operatorname{cis}\theta-k\big|^{2}&<\big|1-k|\alpha|\operatorname{cis}\theta\big|^{2}\\ \big||\alpha|\cos\theta-k+i|\alpha|\sin\theta\big|^{2}&<\big|1-k|\alpha|\cos\theta-k|\alpha|i\sin\theta\big|^{2}\\ \left[\left(|\alpha|\cos\theta-k\right)^{2}+|\alpha|^{2}\sin^{2}\theta\right]&<\left(1-k|\alpha|\cos\theta\right)^{2}+k^{2}|\alpha|^{2}\sin^{2}\theta\\ \left[|\alpha|^{2}\left(\cos^{2}\theta+\sin^{2}\theta\right)-2k|\alpha|\cos\theta+k^{2}\right]&<1-2k|\alpha|\cos\theta+k^{2}|\alpha|^{2}\left(\cos^{2}\theta+\sin^{2}\theta\right)\\ |\alpha|^{2}-2k|\alpha|\cos\theta+k^{2}&<1-2k|\alpha|\cos\theta+k^{2}|\alpha|^{2}\\ |\alpha|^{2}+k^{2}&<1+k^{2}|\alpha|^{2}\\ |\alpha|^{2}-1+k^{2}-k^{2}|\alpha|^{2}&<0\\ \left(|\alpha|^{2}-1\right)\left(1-k^{2}\right)&<0 \qquad \square\end{aligned}$$

Question 14 (a) (v)

We know that $|\alpha| > 1 \Rightarrow |\alpha|^2 - 1 > 0$. Similarly, $-1 \le k \le 1 \Rightarrow 1 - k^2 \ge 0$, so we must have $\left(|\alpha|^2 - 1\right)\left(1 - k^2\right) \ge 0$.

We deduced (iv) from assuming that all roots have modulus $\ne 1$. However, this contradicts the fact that $\left(|\alpha|^2 - 1\right)\left(1 - k^2\right) \ge 0$. Hence, all the roots must have modulus 1.

Question 14 (b) (i)

Let $P_{k+1} = \text{cis}\, k\theta$, so $P_1 = 1$.

$$\begin{aligned}
d_k^2 &= |P_{k+1} - P_1|^2 \\
&= |\cos k\theta + i \sin k\theta - 1|^2 \\
&= (1 - \cos k\theta)^2 + \sin^2 k\theta
\end{aligned}$$

Question 14 (b) (ii)

$$\begin{aligned}
d_k^2 &= (1 - \cos k\theta)^2 + \sin^2 k\theta \\
&= 1 - 2\cos k\theta + \cos^2 k\theta + \sin^2 k\theta \\
&= 2 - 2\cos k\theta \\
&= 2 - \left(\omega^k + \overline{\omega}^k\right) \\
&= 2 - \left(\omega^k + \omega^{n-k}\right) \qquad \ldots\left(\text{since } \omega^n = 1\right) \\
&= 2 - \omega^k - \omega^{n-k} \qquad \square
\end{aligned}$$

Question 14 (b) (iii)

$$\sum_{k=1}^{n-1} d_k^2 = 2(n-1) - \left(\omega + \omega^2 + \ldots + \omega^{n-1}\right) - \left(\omega^{n-1} + \omega^{n-1} + \ldots + \omega\right)$$

But recall that $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$, so $\omega + \omega^2 + \ldots + \omega^{n-1} = -1$.

Hence, we have $\sum_{k=1}^{n-1} d_k^2 = 2(n-1) - (-1) - (-1) = 2n$.

Question 14 (c) (i)

Consider the following

$$(x-y)^2 \geq 0$$

$$x^2 - 2xy + y^2 \geq 0$$

$$xy \leq \frac{1}{2}\left(x^2 + y^2\right)$$

Setting $x = \dfrac{a}{A}$ and $y = \dfrac{b}{B}$ yields the result immediately. $\square$

Question 14 (c) (ii)

From (i), we know that $\displaystyle\sum_{k=1}^{n} \frac{a_k b_k}{AB} \leq \frac{1}{2}\sum_{k=1}^{n}\left(\frac{a_k^2}{A^2} + \frac{b_k^2}{B^2}\right) = \frac{1}{2}\left(\frac{\sum_{k=1}^{n} a_k^2}{A^2} + \frac{\sum_{k=1}^{n} b_k^2}{B^2}\right)$

But note that $A = \sqrt{\sum_{k=1}^{n} a_k^2}$, so $A^2 = \sum_{k=1}^{n} a_k^2$. Hence, we have

$$\sum_{k=1}^{n} \frac{a_k b_k}{AB} \leq \frac{1}{2}\left(\frac{\sum_{k=1}^{n} a_k^2}{A^2} + \frac{\sum_{k=1}^{n} b_k^2}{B^2}\right) = 1.$$

So our expression simplifies to

$$\sum_{k=1}^{n} a_k b_k \leq AB$$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq A^2 B^2$$

$$= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \quad \square$$

Question 14 (c) (iii)

The difficult part now is making the correct choice of a_k and b_k.

We notice that one of the terms has to be some form of $\frac{S}{S-x_k}$ in order to acquire the result.

However, we must take the square root because we notice that if we were to set $a_k = \frac{S}{S-x_k}$,

then we will be stuck with $\left(\frac{S}{S-x_k}\right)^2$, which is undesirable. So we set $a_k = \sqrt{\frac{S}{S-x_k}}$. We must now choose b_k. Notice that the required result's RHS is independent of *S* and x_k.
Hence, we must choose something that will result in a constant, so the appropriate choice would be $b_k = \sqrt{\frac{S-x_k}{S}}$.

Setting $a_k = \sqrt{\frac{S}{S-x_k}}$ and $b_k = \sqrt{\frac{S-x_k}{S}}$

$$\left(\sum_{k=1}^{n}\sqrt{\frac{S}{S-x_k}}\times\sqrt{\frac{S-x_k}{S}}\right)^2 \le \left(\sum_{k=1}^{n}\left(\sqrt{\frac{S}{S-x_k}}\right)^2\right)\left(\sum_{k=1}^{n}\left(\sqrt{\frac{S-x_k}{S}}\right)^2\right)$$

$$\begin{aligned}
\left(\sum_{k=1}^{n}1\right)^2 &\le \left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right)\left(\sum_{k=1}^{n}\left(1-\frac{x_k}{S}\right)\right)\\
n^2 &\le \left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right)\left(\sum_{k=1}^{n}1-\sum_{k=1}^{n}\frac{x_k}{S}\right)\\
&\le \left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right)(n-1)
\end{aligned}$$

$$\left(\sum_{k=1}^{n}\frac{S}{S-x_k}\right) \ge \frac{n^2}{n-1}$$

Question 15 (a) (i)

$$\begin{aligned}
\sin(\alpha+\beta)-\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\sin\beta\cos\alpha-\sin\alpha\cos\beta+\sin\beta\cos\alpha\\
&= 2\cos\alpha\sin\beta \qquad \square
\end{aligned}$$

Question 15 (a) (ii)

First, we re-arrange (i) and establish an identity.

$$\cos\alpha = \frac{1}{2\sin\beta}\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right)$$

To match $S_n(x)$, we let $\alpha = kx$. We must now make the appropriate choice of β, so we look at the required result. We see that in the denominator, we have $2\sin(x/2)$, so it implies that we must set $\beta = x/2$.

Hence, we have

$$\begin{aligned}\cos kx &= \frac{1}{2\sin\left(\frac{x}{2}\right)}\left(\sin\left(kx+\frac{x}{2}\right)-\sin\left(kx-\frac{x}{2}\right)\right)\\ &= \frac{1}{2\sin\left(\frac{x}{2}\right)}\left[\sin\left((2k+1)\frac{x}{2}\right)-\sin\left((2k-1)\frac{x}{2}\right)\right]\end{aligned}$$

Suppose $x \neq 0$. We sum the above expression and form a Telescoping Sum.

$$\begin{aligned}S_n(x) &= \frac{1}{2}+\sum_{k=1}^{n}\cos(kx)\\ &= \frac{1}{2}+\frac{1}{2\sin(x/2)}\sum_{k=1}^{n}\left[\sin((2k+1)x/2)-\sin((2k-1)x)\right]\\ &= \frac{1}{2}+\frac{\sin\left[(2n+1)x/2\right]-\sin(x/2)}{2\sin(x/2)} \quad \ldots\left(\text{since all terms except the first and last cancel}\right)\\ &= \frac{1}{2}+\frac{\sin\left[(2n+1)x/2\right]}{2\sin(x/2)}-\frac{1}{2}\\ &= \frac{\sin\left[(2n+1)x/2\right]}{2\sin(x/2)} \qquad \square\end{aligned}$$

Suppose $x = 0$. We use the original sum.

$$\begin{aligned}S_n(0) &= \frac{1}{2}+\sum_{k=1}^{n}\cos(0)\\ &= \frac{1}{2}+n \qquad \square\end{aligned}$$

Question 15 (b)

We use Integration by Parts, using $u = x$ and $dv = \cos kx$.

$$
\begin{aligned}
\int_0^{\pi} x\cos(kx)\,dx &= \frac{1}{k}x\sin kx\Big|_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin kx\,dx \\
&= -\frac{1}{k}\int_0^{\pi}\sin kx\,dx \\
&= \frac{1}{k^2}\cos kx\Big|_0^{\pi} \\
&= \frac{1}{k^2}\left[\left(\cos k\pi - 1\right)\right] \\
&= \frac{1}{k^2}\left[(-1)^k - 1\right]
\end{aligned}
$$

Note that we have acquired the last expression using the fact that the cosine of integer multiples of π can be 1 or -1, depending on *k*, so $\cos k\pi = (-1)^k$.

Question 15 (c) (i)

Notice that $f_n(x)$ is just $S_n(x)$, where $x \neq 0$.

$$
\begin{aligned}
I_n &= \lim_{a\to 0}\int_a^{\pi} x\,S_n(x)\,dx \\
&= \int_0^{\pi} x\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(kx)\right)dx \qquad \ldots\left(\text{since } S_n(x) \text{ is defined at } x = 0\right) \\
&= \int_0^{\pi} \frac{x}{2} + x\sum_{k=1}^{n}\cos(kx)\,dx \\
&= \frac{1}{4}x^2\Big|_0^{\pi} + \int_0^{\pi} x\sum_{k=1}^{n}\cos(kx)\,dx \\
&= \frac{\pi^2}{4} + \sum_{k=1}^{n}\int_0^{\pi} x\cos kx\,dx \\
&= \frac{\pi^2}{4} + \sum_{k=1}^{n}\frac{1}{k^2}\left[(-1)^k - 1\right] \qquad \square
\end{aligned}
$$

Question 15 (c) (ii)

$$\frac{1}{2}I_{2n-1}=\frac{\pi^2}{8}+\frac{1}{2}\sum_{k=1}^{2n-1}\left[\frac{(-1)^k-1}{k^2}\right]$$

$$=\frac{\pi^2}{8}+\frac{1}{2}\sum_{k=1}^{2n-1}\left[\frac{1}{k^2}\left((-1)^k-1\right)\right]$$

$$=\frac{\pi^2}{8}+\frac{1}{2}\left[\frac{1}{1^2}\left((-1)^1-1\right)+\frac{1}{2^2}\left((-1)^2-1\right)+\frac{1}{3^2}\left((-1)^3-1\right)+\ldots+\frac{1}{(2n-1)^2}\left((-1)^{2n-1}-1\right)\right]$$

$$=\frac{\pi^2}{8}+\frac{1}{2}\left[\frac{1}{1^2}(-2)+\frac{1}{3^2}(-2)+\ldots+\frac{1}{(2n-1)^2}(-2)\right]$$

$$=\frac{\pi^2}{8}+\frac{1}{2}\left(-2\times\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)$$

$$=\frac{\pi^2}{8}-\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\qquad\square$$

Question 15 (c) (iii)

$$I_{2n-1}=\lim_{a\to 0}\int_a^\pi x\,f_{2n-1}(x)\,dx$$

$$=\lim_{a\to 0}\int_a^\pi \frac{x}{2}\times\frac{\sin\left[(4n-1)x/2\right]}{\sin(x/2)}\,dx$$

$$=\lim_{a\to 0}\int_a^\pi \frac{x/2}{\sin(x/2)}\times\sin\left[(4n-1)x/2\right]dx$$

We use Integration by Parts with $u=\dfrac{x/2}{\sin(x/2)}$ and $dv=\sin\left[(4n-1)x/2\right]$. We make these choices because the question defined $g(x)=\dfrac{d}{dx}\left(\dfrac{x/2}{\sin(x/2)}\right)$, which implies that it has to be the term that is differentiated.

$$I_{2n-1}=-\frac{2}{4n-1}\lim_{a\to 0}\left[\cos\left[\frac{(4n-1)x}{2}\right]\frac{x/2}{\sin(x/2)}\right]_a^\pi+\frac{2}{4n-1}\lim_{a\to 0}\int_a^\pi g(x)\cos\left[\frac{(4n-1)x}{2}\right]dx$$

$$=-\frac{2}{4n-1}\left(-\lim_{a\to 0}\cos\left[\frac{(4n-1)a}{2}\right]\frac{a/2}{\sin(a/2)}\right)+\frac{2}{4n-1}\lim_{a\to 0}\int_a^\pi g(x)\cos\left[\frac{(4n-1)x}{2}\right]dx$$

Note that a large term cancelled in the substitutions because $\cos\left[\frac{(4n-1)\pi}{2}\right]=0$, since we are taking the cosine of an odd multiple of $\frac{\pi}{2}$. We also notice the opportunity to use the familiar limit $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ as well as $\lim_{a\to 0}\cos\left[\frac{(4n-1)a}{2}\right]=1$. Hence, we have the required result.

$$I_{2n-1}=\frac{2}{4n-1}\left(1+\lim_{a\to 0}\int_a^\pi g(x)\cos\left[\frac{(4n-1)x}{2}\right]dx\right)\quad\square$$

Question 15 (c) (iv)

Since we know that $g(x)\geq 0$ in the domain, we have

$$-1\leq\cos\left[\frac{(4n-1)x}{2}\right]\leq 1$$
$$-g(x)\leq g(x)\cos\left[\frac{(4n-1)x}{2}\right]\leq g(x)$$

Integrate both sides with respect to *x* from 0 to π. Note that the integral of $g(x)$ is $\frac{x/2}{\sin(x/2)}$ because $g(x)=\frac{d}{dx}\left(\frac{x/2}{\sin(x/2)}\right)$.

$$\lim_{a\to 0}-\frac{x/2}{\sin(x/2)}\Bigg|_a^\pi\leq g(x)\cos\left[\frac{(4n-1)x}{2}\right]\leq\lim_{a\to 0}\frac{x/2}{\sin(x/2)}\Bigg|_a^\pi$$

$$-\frac{\pi/2}{\sin(\pi/2)}+\lim_{a\to 0}\frac{a/2}{\sin(a/2)}\leq g(x)\cos\left[\frac{(4n-1)x}{2}\right]\leq\frac{\pi/2}{\sin(\pi/2)}-\lim_{a\to 0}\frac{a/2}{\sin(a/2)}$$

$$1-\frac{\pi}{2}\leq\lim_{a\to 0}\int_a^\pi g(x)\cos\left[\frac{(4n-1)x}{2}\right]dx\leq\frac{\pi}{2}-1\quad\square$$

Question 15 (c) (v)

From (iv), we have

$$1-\frac{\pi}{2}\le\frac{4n-1}{2}I_{2n-1}-1\le\frac{\pi}{2}-1$$

$$\frac{2}{4n-1}\left(2-\frac{\pi}{2}\right)\le I_{2n-1}\le\frac{2}{4n-1}\left(\frac{\pi}{2}\right)$$

Hence, as $n\to\infty$, I_{2n-1} is 'squeezed' between two terms, both of which approach 0. Hence, as $n\to\infty$, $I_{2n-1}\to 0$.

Question 15 (c) (vi)

We know from (ii) that $\frac{1}{2}I_{2n-1}=\frac{\pi^2}{8}-\sum_{k=1}^{n}\frac{1}{(2k-1)^2}$. But from (v), $I_{2n-1}\to 0$ as $n\to\infty$.

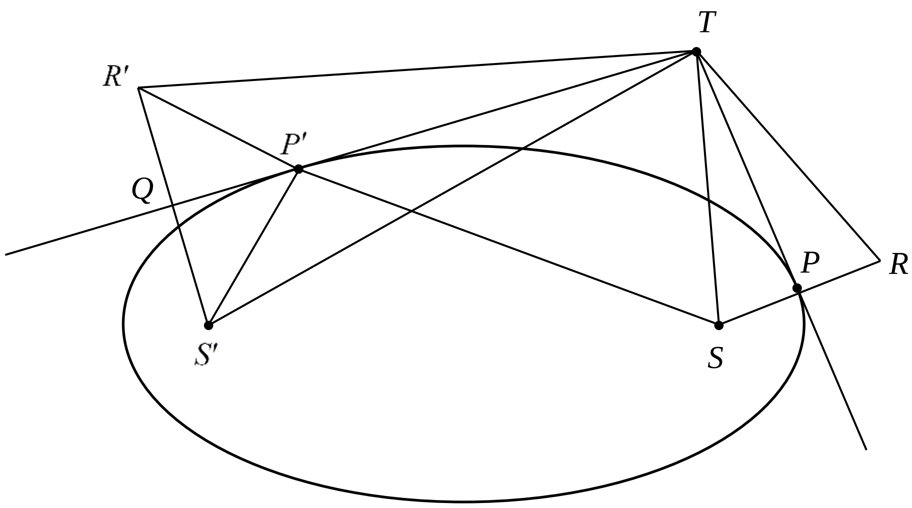
Hence, $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$. Consider the sum $S=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}$. For simplicity, assume *n* is even. This does not change anything since we are taking the limit as $n\to\infty$ anyway.

$$\begin{aligned}
S&=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2}\\
&=\lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}\right)\\
&=\lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots+\frac{1}{(n-1)^2}+\frac{1}{2^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}\right)\\
&=\lim_{n\to\infty}\left(\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots+\frac{1}{(n-1)^2}\right)+\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{(n/2)^2}\right)\right)\\
S&=\frac{\pi^2}{8}+\frac{S}{4}\\
\frac{3}{4}S&=\frac{\pi^2}{8}\\
S&=\frac{\pi^2}{6}\qquad\square
\end{aligned}$$

Question 16 (a) (i)

From the Reflection Property, we know that $\angle SP'T = \angle QP'S'$. But since $\angle QP'R'$ is a reflection of $\angle S'P'Q$ about the line QT, $\angle QP'R' = \angle QP'S'$, so therefore $\angle QP'R' = \angle TP'S$

Since $QP'T$ is a straight line, we have vertically opposite angles being equal, so therefore $SP'R'$ must be a straight line, so S, P' and R' are collinear. □



Question 16 (a) (ii)

We know that $TR' = TS'$ and $TR = TS$, since they are reflections of each other about TP' and TP respectively.

Recall the often forgotten identity $PS + PS' = 2a = k$ for some constant k as we do not know anything about the dimensions of this ellipse. Using that, we have $P'S + P'S' = k$. But $P'S' = P'R'$ so $P'S + P'R' = k$. Similarly, we have $PS + PS' = k = PR + PS'$. However, since we have proven that S, P' and R' are collinear from (i), we can conclude that $SR' = S'R$, which proves that the third side is equal. Hence, $\Delta TR'S \equiv \Delta TS'R$ since all three corresponding sides are equal.

Question 16 (a) (iii)

$\angle R'TS = \angle R'TS' + \angle S'TS$

$\angle RTS' = \angle RTS + \angle S'TS$

But from (ii), we know that $\angle R'TS = \angle RTS'$, since they are corresponding angles in congruent triangles. Hence, we can conclude that

$$\begin{aligned}\angle R'TS' + \angle S'TS &= \angle RTS + \angle S'TS \\ \angle R'TS' &= \angle RTS\end{aligned}$$

But note that by reflection about TP' and TP, we have $\angle R'TS' = 2\times\angle S'TP'$ and that $\angle RTS = 2\times\angle STP$. Thus, we have

$$\begin{aligned}2\times\angle STP &= 2\times\angle S'TP' \\ \angle STP &= \angle S'TP' \qquad \square\end{aligned}$$

Question 16 (b) (i)

Applying Az to all points scales the triangle and applying B to all points translates the triangle. Hence, the original triangle and the triangle with T applied to all points are similar.

Alternatively:

We will prove that any two sides are in the ratio A. By doing so, we prove that all sides are in the same ratio A and are therefore similar.

Let any two sides be z_i and z_j. So the length of that side is $|z_i - z_j|$. Now, apply T to each point.

$$\begin{aligned}\left|T(z_i) - T(z_j)\right| &= |Az_i + B - Az_i - B| \\ &= |Az_i - Az_i| \\ &= |A||z_i - z_j|\end{aligned}$$

Hence, we have proven similarity.

Question 16 (b) (ii)

$$\begin{aligned}
P_T(z) &= \left(z - T(z_1)\right)\left(z - T(z_2)\right)\left(z - T(z_3)\right) \\
P_T\left(T(z)\right) &= \left(T(z) - T(z_1)\right)\left(T(z) - T(z_2)\right)\left(T(z) - T(z_3)\right) \\
&= \left(Az + B - Az_1 - B\right)\left(Az + B - Az_2 - B\right)\left(Az + B - Az_3 - B\right) \\
&= \left(A(z - z_1)\right)\left(A(z - z_2)\right)\left(A(z - z_3)\right) \\
&= A^3 (z - z_1)(z - z_2)(z - z_3) \\
&= A^3 P(z) \qquad \square
\end{aligned}$$

Question 16 (b) (iii)

Differentiate both sides of the expression in (ii) with respect to *z*.

$$\begin{aligned}
P_T{}'\left(T(z)\right) \times A &= A^3 P'(z) \qquad \text{...(using the Chain Rule, where } T'(z) = A\text{)} \\
P_T{}'\left(T(z)\right) &= A^2 P'(z)
\end{aligned}$$

If α is a root of $P'(z)$, then $P'(\alpha) = 0$.

$$\begin{aligned}
P_T'\left(T(\alpha)\right) &= A^2 P'(\alpha) \\
&= 0
\end{aligned}$$

Question 16 (b) (iv)

Say we have some real cubic polynomial P with exactly one real root and thus two non-real roots that are also conjugates. These roots form Δ_P. If we apply *T* to all the roots, we get another similar triangle (from (i)) Δ_T, which are the roots of $P_T(z)$. This new polynomial has some derivative P_T'.

Now, P' has two roots (either of which denoted by α), since it is a quadratic. If we apply *T* to both roots, we get $T(\alpha)$.

What (iii) proved is that $T(\alpha)$ is also the same as the root of P_T'!

Geometrically, this means that when the roots of the cubic polynomial *P* are transformed by *T*, the roots of its derivative are also transformed by *T*.

Question 16 (c) (i)

Since the roots of $P(z)$ are $i, -i, r$, where P is monic, we can re-construct the cubic.

$$\begin{aligned} P(z) &= (z-i)(z+i)(z-r) \\ &= (z^2+1)(z-r) \\ &= z^3 - rz^2 + z - r \\ P'(z) &= 3z^2 - 2rz + 1 \qquad \square \end{aligned}$$

Question 16 (c) (ii)

Solving the quadratic explicitly, we have

$$\begin{aligned} z &= \frac{2r \pm \sqrt{4r^2 - 12}}{6} \\ &= \frac{r \pm \sqrt{r^2 - 3}}{3} \\ &= \frac{r \pm i\sqrt{3 - r^2}}{3} \end{aligned}$$

Since the real component is $x = \frac{r}{3}$, we know that the roots are between the *y* axis and *R*.

Now, we must prove that the positive imaginary component is below the line *AR* when $x = \frac{r}{3}$ and by proving the positive case, we also prove the negative case due to symmetry.

By inspecting the gradient and *y* intercept of *AR*, we can deduce its equation $y = -\frac{1}{r}x + 1$.

Setting $x=\frac{r}{3}$, we have

$$\begin{aligned} y &= -\frac{1}{r}\times\frac{r}{3}+1 \\ &= \frac{2}{3} \\ &= \frac{\sqrt{4}}{3} \\ &> \frac{\sqrt{3}}{3} \\ &> \frac{\sqrt{3-r^2}}{3} \end{aligned}$$

Thus, the roots of the derivative lie in the interior of the triangle. □

Question 16 (d)

In (c) (i), we proved a fairly specific case, for when the roots are $i, -i, r$, forming some isosceles triangle.

However, note that any real cubic polynomial with 3 distinct roots will have its roots forming an isosceles triangle with its axis of symmetry on the real axis, due to the Conjugate Root Theorem. But any isosceles triangle can have a similar image of it constructed from the points $i, -i, r$, when r is chosen appropriately

Since we have proven that the roots of P' lie within the interior of the triangle formed by the roots of P (for the case with $i, -i, r$), we have thus proven it for any isosceles triangle with two purely imaginary roots.

But from (b), we know that if the roots of P are transformed by T, so are the roots of P'. Hence, they stay in the same position relative to each other.

Thus, we have proven the case for ANY isosceles triangle, not just ones with two purely imaginary roots.