2014 Bored of Studies Trial Examinations

Mathematics Extension 2

Written by Carrotsticks & Trebla

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 1 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 6 – 18

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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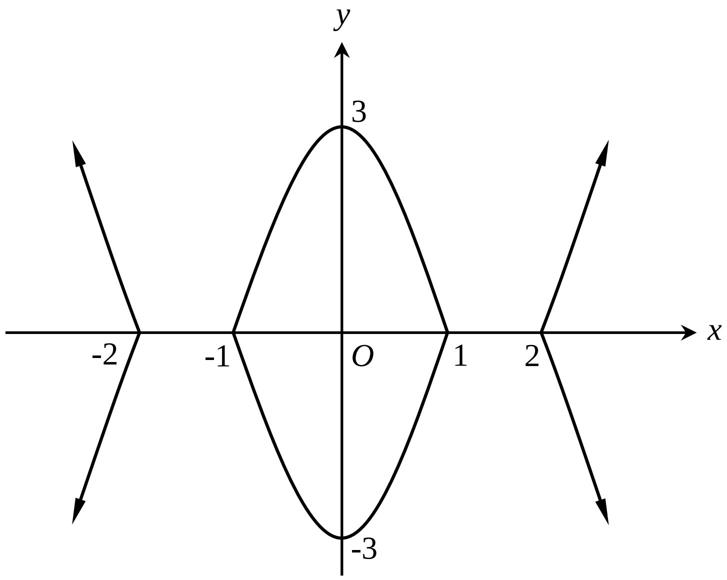
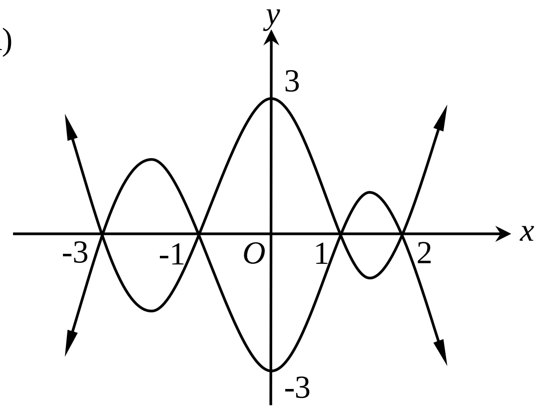
1 The following diagram shows the sketch of $|y| = f\left(|x|\right)$.

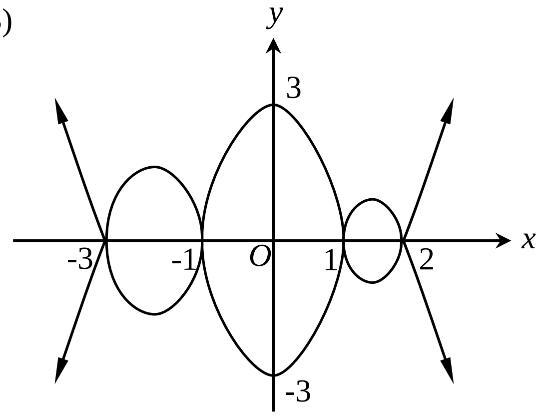
Which of the following curves is a possible sketch of $y^2 = \left[f\left(x\right)\right]^2$?



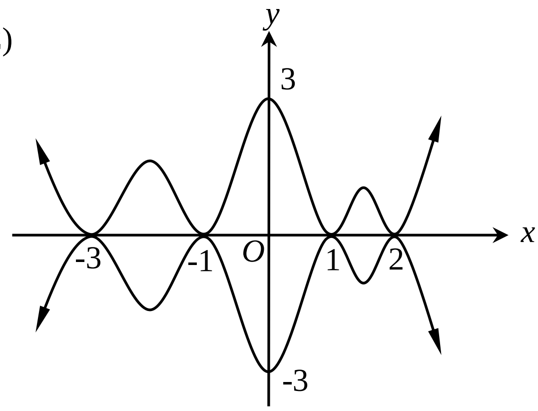
(A)



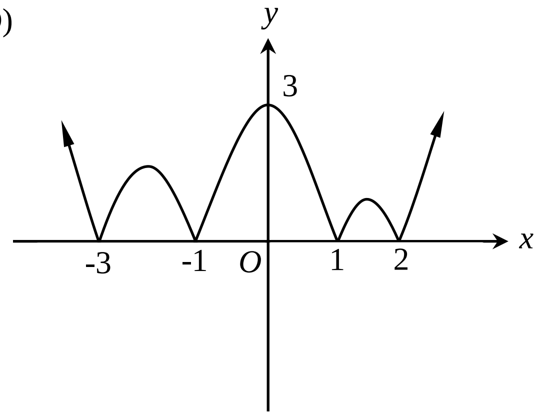
(B)



(C)



(D)



2 Consider an ellipse with the equation $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$ and a hyperbola with the equation $\frac{x^2}{a_2^2}-\frac{y^2}{b_2^2}=1$ for some positive constants a_1, a_2, b_1 and b_2. The ellipse and hyperbola share the same foci. Which of the following statements is true?

(A) The x-intercepts of the ellipse are greater than those of the hyperbola

(B) The x-intercepts of the ellipse are less than those of the hyperbola

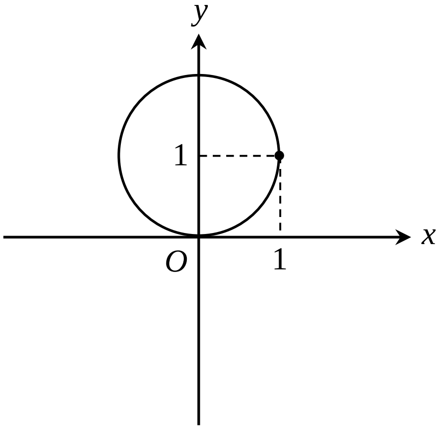
(C) The ellipse and hyperbola share the same x-intercepts

(D) The ellipse and hyperbola both do not have any x-intercepts

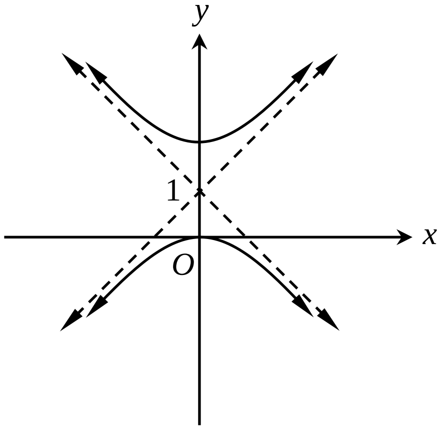
3 The implicit derivative of a curve is $\frac{dy}{dx}=\frac{x}{y-1}$.

Which of the following sketches is a possible graph of the curve?

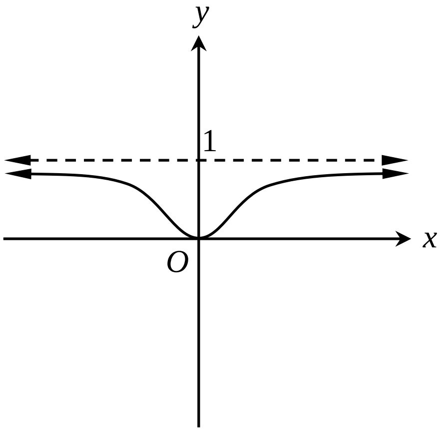
(A)



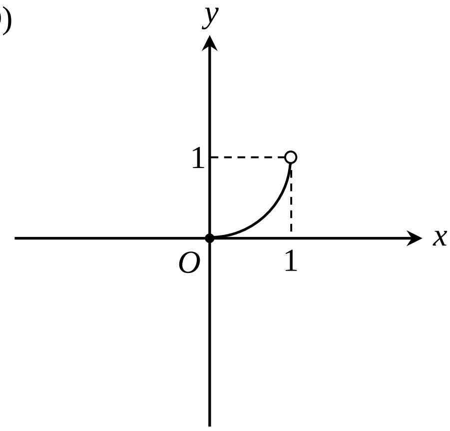
(B)



(C)



(D)



4 Suppose that $P(x)$ is a polynomial of degree n over the set of complex numbers where $n \neq 0$. What does the Fundamental Theorem of Algebra state about this polynomial?

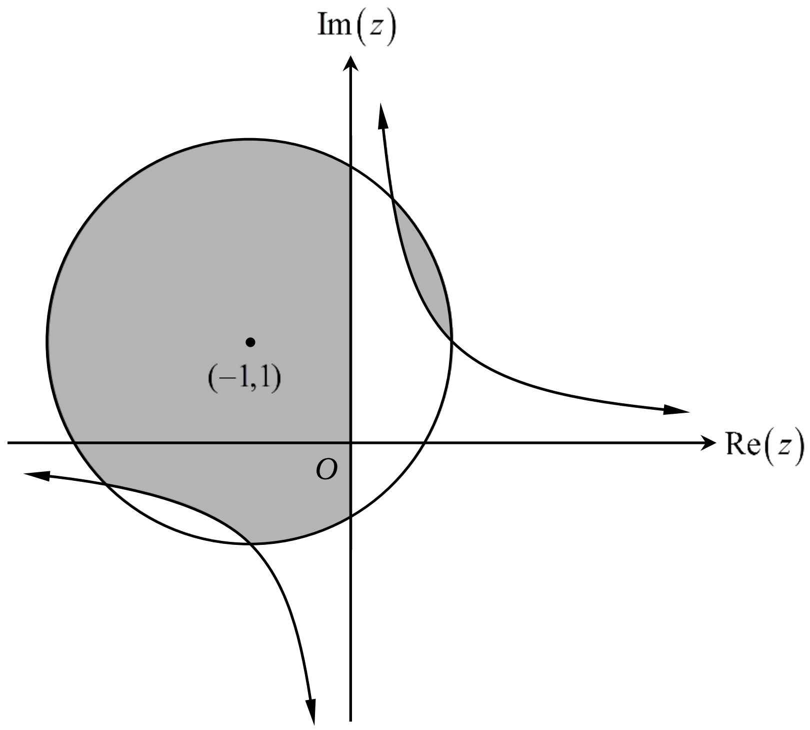
(A) $P(x)$ has at least one complex root

(B) $P(x)$ has at least one real root

(C) $P(x)$ has exactly n complex roots

(D) $P(x)$ has exactly n real roots

5 Which of the following inequalities best define the region shown below?



(A) $|z+1-i| \leq 2$ and $i\,\mathrm{Re}(z)(\bar{z}-z) \leq 2$

(B) $|z+1-i| \leq 2$ and $i\,\mathrm{Re}(z)(\bar{z}-z) \geq 2$

(C) $|z+1-i| \leq 2$ and $i(\bar{z}-z) \leq \dfrac{2}{\mathrm{Re}(z)}$

(D) $|z+1-i| \leq 2$ and $i(\bar{z}-z) \geq \dfrac{2}{\mathrm{Re}(z)}$

6 Which of the following is the correct simplification of $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^4$?

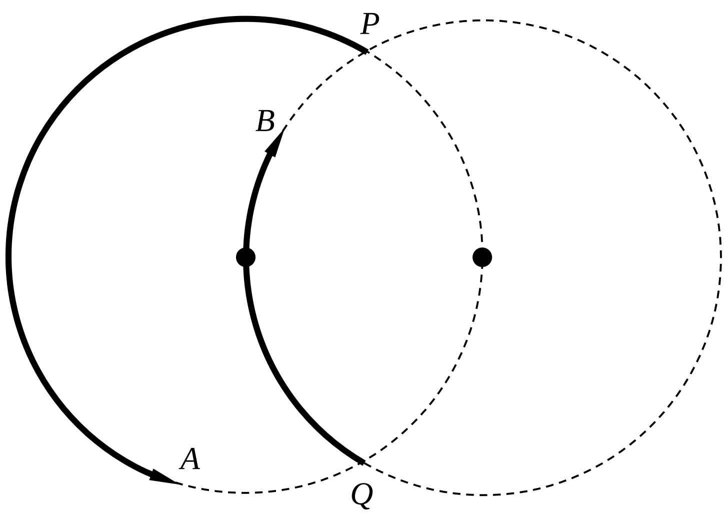
(A) $\sin 4\theta + i\cos 4\theta$

(B) $\sin 4\theta - i\cos 4\theta$

(C) $\cos 4\theta + i\sin 4\theta$

(D) $\cos 4\theta - i\sin 4\theta$

7 Two particles A and B travel along intersecting circles with equal radii, where the centre of one circle lies on the other circle as shown in the diagram below. Particle A is released from point P and moves anti-clockwise towards point Q along the major arc of one circle with constant angular velocity ω_A. Simultaneously, particle B is released from point Q and moves clockwise towards P along the minor arc of the other circle with constant angular velocity ω_B.



If the particles collide at point P, which of the following best describes the relationship between their angular velocities?

(A) $\omega_A = 2\omega_B$

(B) $\omega_B = 2\omega_A$

(C) $\omega_A = 3\omega_B$

(D) $\omega_B = 3\omega_A$

8 Let $P(x)$ be a non-zero polynomial with degree $n \geq 2$.

Which of the following statements is always true?

(A) $P(x)$ has real quadratic factors if n is even.

(B) $P(x)$ has at least one real root if n is odd.

(C) $P(x)$ has only real roots if it is a real polynomial.

(D) $P(x)$ has at least one real quadratic factor if it is a real polynomial.

9 Which of the following inequalities holds for all real x ?

(A) $\sin x \leq x$

(B) $\tan^{-1} x < e^x$

(C) $2\left|\sin\left(\frac{x}{2}\right)\right| < \sin x$

(D) $|x| > \tan^{-1} x$

10 Let w be an n^{th} root of 1, where $w \neq 1$.

Which of the following is the exact value of

$$1+2\left(w+\overline{w}\right)+3\left(w^2+\overline{w}^2\right)+4\left(w^3+\overline{w}^3\right)+\ldots+n\left(w^{n-1}+\overline{w}^{n-1}\right)?$$

(A) n

(B) $-n$

(C) $1-n$

(D) $-1-n$

Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = a + ib$ and $w = \frac{1}{a} + \frac{1}{b}i$, where $a > b > 0$.

(i) Calculate zw. **1**

(ii) Deduce that $\tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{2}$. **1**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(1+\cos x)(\sin x + \cos x)}}$. **3**

(c) Let $F(x)$ be a curve such that $F'(x) = \frac{x^3}{x^3+1}$.

(i) Sketch the curve $y = F'(x)$. **2**

(ii) Hence, sketch a possible graph of $y = F(x)$. **2**

(d) Consider the polynomial $P(x) = x^4 + ax^3 + bx + c$, where *a*, *b* and *c* are non-zero constants. **3**

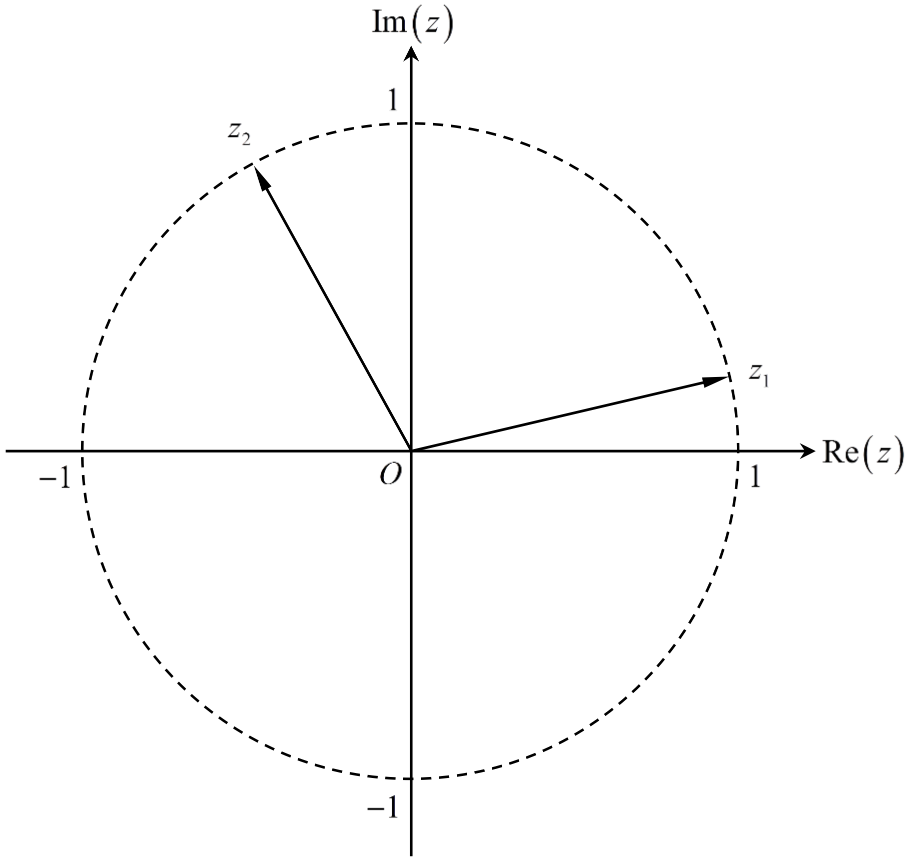
Prove that if $P(x)$ has a triple root, then $ab = 4c$.

(e) Find the equation of the locus of *z* if $\arg z = 2\arg(z+1)$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Let z_1 and z_2 be two non-zero complex numbers on the unit circle such that $z_1 z_2 \neq -1$.



By using vectors, or otherwise, prove that **3**

$$\frac{z_1+z_2}{1+z_1 z_2}$$

is a real number.

(b) The real polynomial $P(x)=x^3+px+q$ has roots α, β and γ. **3**

Find the polynomial with roots $\left(\frac{1}{\alpha}+\frac{1}{\beta}\right), \left(\frac{1}{\beta}+\frac{1}{\gamma}\right)$ and $\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)$

Question 12 continues on page 8

(c) A section of flat road has a circular curve of radius r. A car moves along this curve and the tyres provide a frictional force F. The value of F is a maximum when the car travels with velocity V_0. **4**

Suppose the same road surface is now banked at some angle θ from the horizontal, where $0 < \theta < \frac{\pi}{2}$.

When the same car travels along the bank with velocity V_2, it experiences no lateral forces. The value of the frictional force from the tyres is a maximum when the car travels with velocity V_1.

Show that $\cos\theta = \dfrac{V_0^2}{V_1^2 - V_2^2}$.

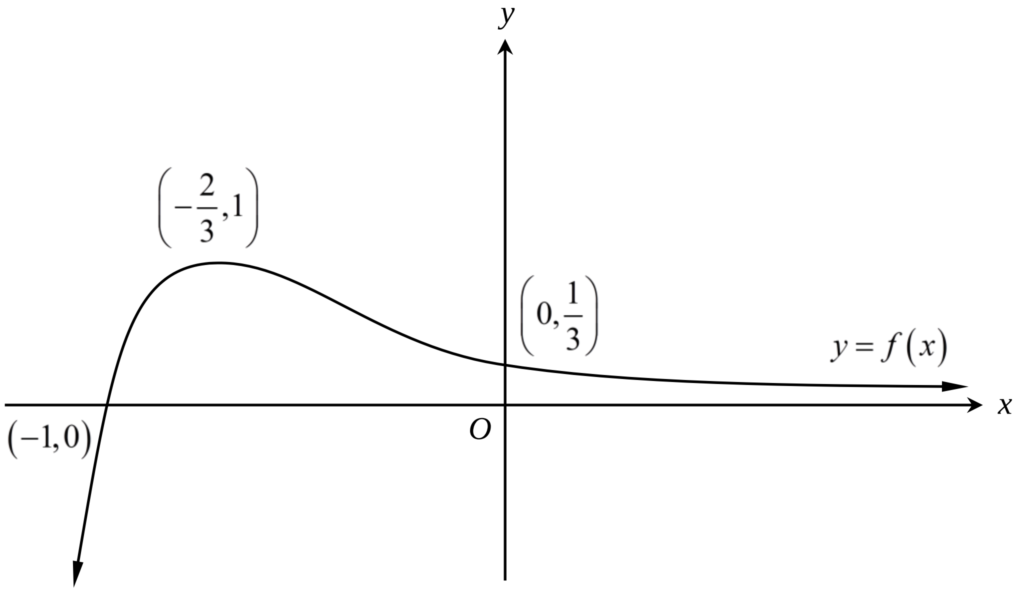
(d) Let z_1, z_2 and z_3 be three complex numbers with modulus 1. **2**

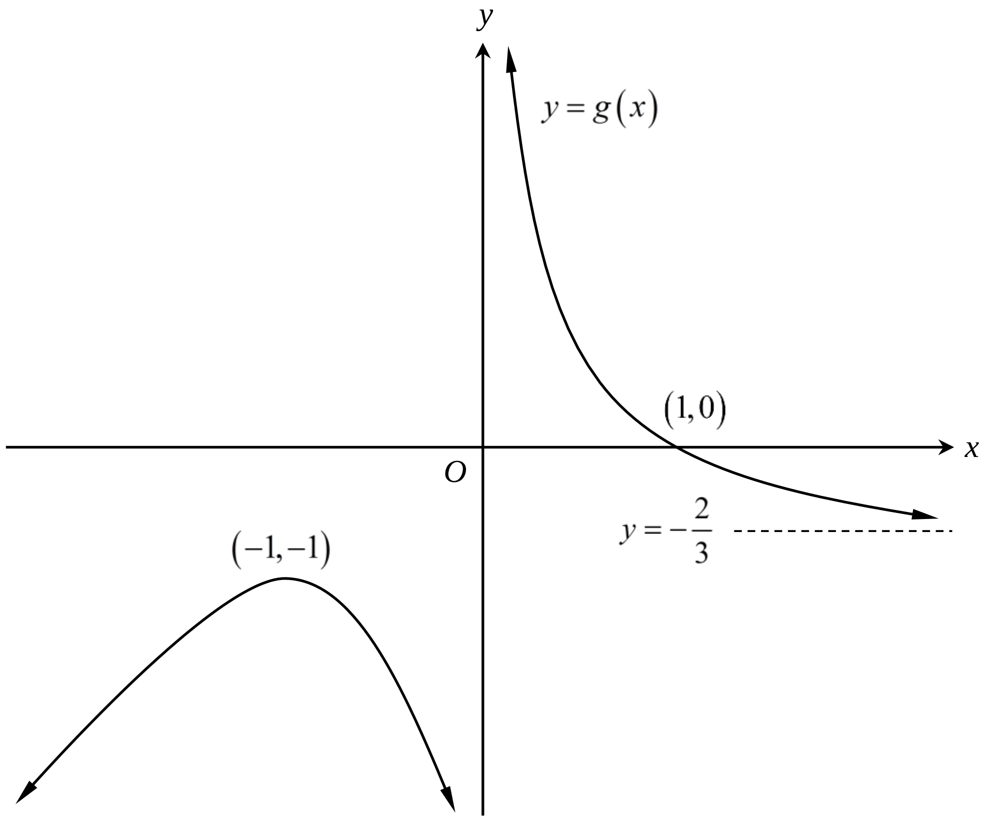
Prove that $\left|z_1 z_2 + z_1 z_3 + z_2 z_3\right| = \left|z_1 + z_2 + z_3\right|$.

Question 12 continues on page 9

(e) The diagrams below show the graphs of $y = f(x)$ and $y = g(x)$. **3**

Sketch a one-third page graph of $y = f(g(x))$.





End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Let $P(z)=\sum_{k=0}^{n} a_k z^k$ be a monic non-constant polynomial with real coefficients, where $0<|P(i)|<1$.

(i) Suppose $P(z)$ has p real roots. Let r_j be any real root. **1**

Show that $|r_j - i| \geq 1$, for any $j=1,2,3,\ \ldots\ ,p$.

(ii) Suppose $P(z)$ has q non-real roots c_k, where $k=1,2,\ \ldots\ ,q$. **2**

Prove that $|c_1 - i|\times|c_2 - i|\times\ldots\times|c_q - i|<1$

(iii) Deduce that at least one non-real root α satisfies $|\alpha + i|<1$. **2**

Question 13 continues on page 11

(b) A particle A is dropped from a weather balloon. The equation of motion of particle A is

$$\ddot{x} = g - kv_A$$

where g is the acceleration due to gravity, k is a positive constant and v_A is the velocity of particle A.

T seconds later, an identical particle B is projected downwards from the same weather balloon with initial velocity $u\,\text{ms}^{-1}$. The equation of motion of particle B is

$$\ddot{x} = g - kv_B$$

where v_B is the velocity of particle B.

(i) Show that **4**

$$T = -\frac{1}{k}\ln\left(\frac{g-ku}{g} \times \frac{g-kv_A}{g-kv_B}\right).$$

(ii) Show that particle B's displacement x_B is given by **3**

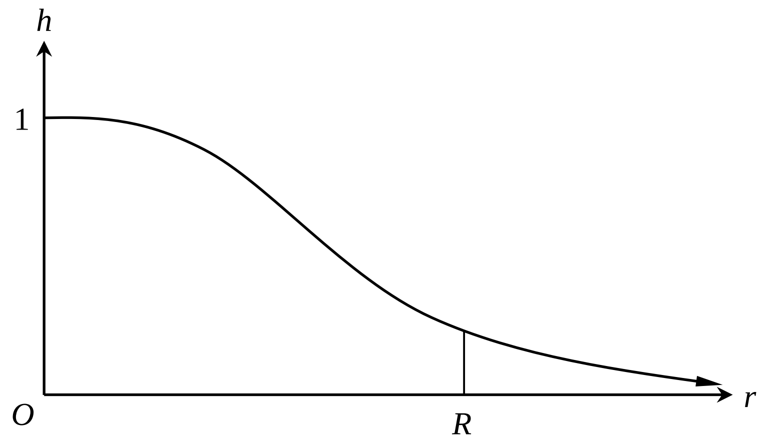
$$x_B = \frac{1}{k}\left[u - v_B + \frac{g}{k}\ln\left(\frac{g-ku}{g-kv_B}\right)\right].$$

(iii) Deduce that if particle B catches up with particle A, then particle B must have been released no more than $\dfrac{u}{g}$ seconds after particle A. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

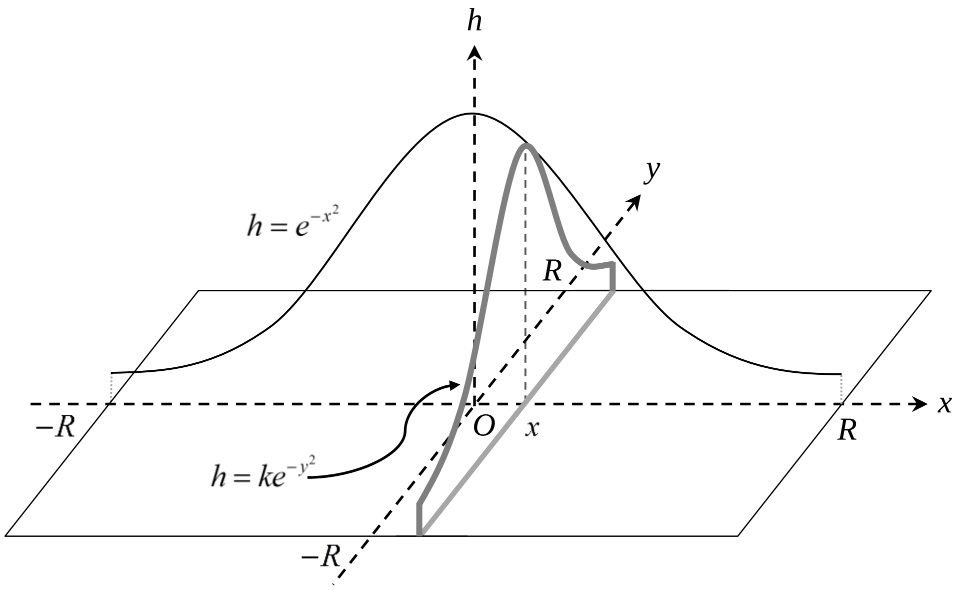
(a) The following diagram shows the curve $h = e^{-r^2}$, where $r \geq 0$. **2**



Use cylindrical shells to find the volume V_A of the solid formed when the region bounded by the curve, the coordinate axes and the line $r = R$ is rotated about the vertical axis.

Question 14 continues on page 13

(b) The following diagram shows a square base on the xy – plane bounded by $-R \le x \le R$ and $-R \le y \le R$.



For a given value of x on the x axis, a thin slice is taken perpendicular to the square base and parallel to the y axis. The shape of the slice is determined by the equation $h = ke^{-y^2}$, for some positive constant k, and is made such that its maximum point lies on the curve $h = e^{-x^2}$ as shown in the diagram above. Let $I = \int_{-R}^{R} e^{-t^2}\,dt$.

(i) Show that the area of this slice is given by $e^{-x^2} I$. **2**

(ii) Slices are taken continuously along the x axis and then added together, forming a solid. Show that this solid has volume $V_B = I^2$. **2**

(iii) Consider the solid formed in part (a) with volume V_A. Suppose that the rotated region is now bounded by the coordinate axis and the line $r = 2R$. Write down the volume V_C of the solid formed. **1**

(iv) Explain why $V_A < V_B < V_C$. **2**

(v) Deduce that $\lim_{R\to\infty} \int_{-R}^{R} e^{-t^2}\,dt = \sqrt{\pi}$. **1**

Question 14 continues on page 14

(c)

(i) Show that for $x \neq 1$ **2**

$$1+2x+3x^2+.....+nx^{n-1}=\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}.$$

(ii) A game is played where the probability of winning is p. **1**

Let $P(k)$ represent the probability of winning for the first time on the k^{th} game.

Write down an expression for $P(k)$ in terms of p and k.

(iii) Hence, prove that **2**

$$\lim_{n\to\infty}\sum_{k=1}^{n}kP(k)=\frac{1}{p}$$

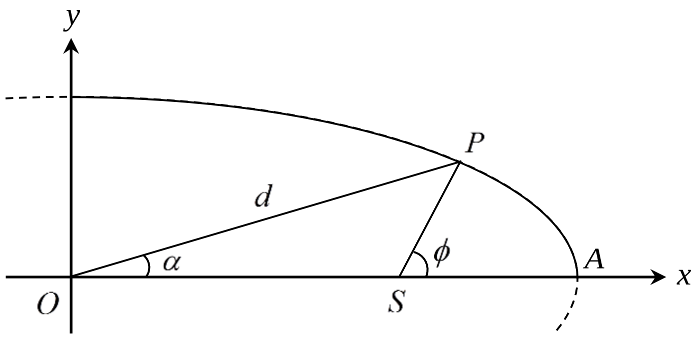
End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int \frac{x^2}{(x\sin x+\cos x)(x\cos x-\sin x)}\,dx$. **4**

(b) The diagram below shows a point P on the first quadrant of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a>b$. Let A be the positive x-intercept of the ellipse.

Let d be the distance from P to the origin O, and let $\angle POS=\alpha$ and $\angle PSA=\phi$ where S is the focus of the ellipse.



(i) Show that $PS=\dfrac{a\left(1-e^2\right)}{1+e\cos\phi}$ **2**

(ii) Show that $d\sin\alpha=\dfrac{a\left(1-e^2\right)\sin\phi}{1+e\cos\phi}$. **1**

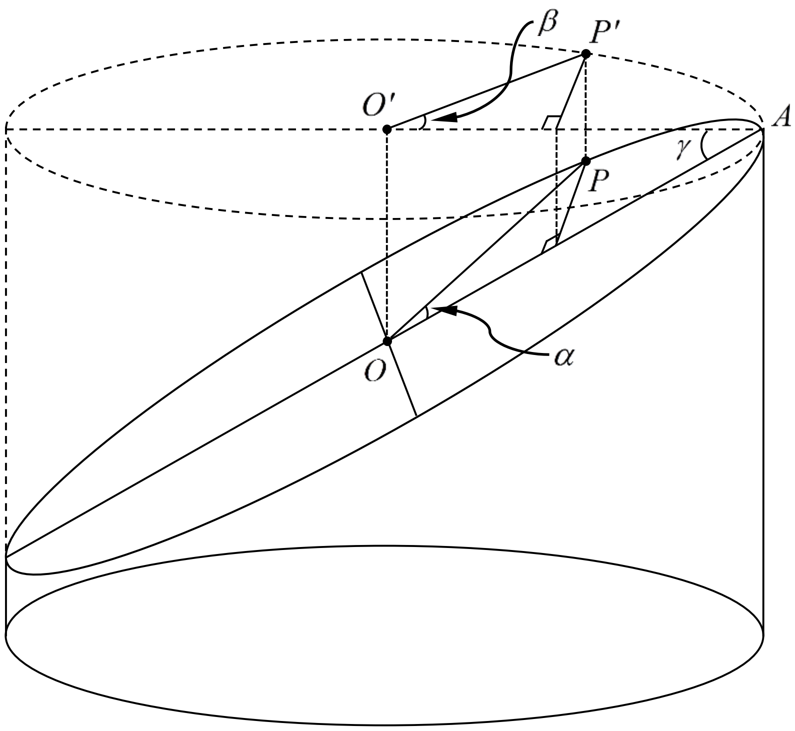
(iii) Show that $d\cos\alpha=\dfrac{a\left(e+\cos\phi\right)}{1+e\cos\phi}$. **1**

(iv) Hence, prove that **3**

$$d=\frac{a}{\sqrt{\dfrac{\sin^2\alpha}{1-e^2}+\cos^2\alpha}}.$$

Question 15 continues on page 16

(c) The diagram below shows a cylinder that has been sliced at some angle γ from the horizontal, forming a solid with its top surface being an ellipse.



Consider a point P' on the edge of the top circular surface of the cylinder, which is directly above point P on the ellipse. The diameter of the top circular surface on the cylinder is directly above the major axis of the ellipse.

Let β be the angle subtended at the circle's centre by the minor arc AP'.

(i) Using part (b), show that **2**

$$\cos\beta = \frac{\cos\alpha}{\sqrt{\dfrac{\sin^2\alpha}{1-e^2}+\cos^2\alpha}}.$$

where e is the eccentricity of the ellipse at the top surface of the solid

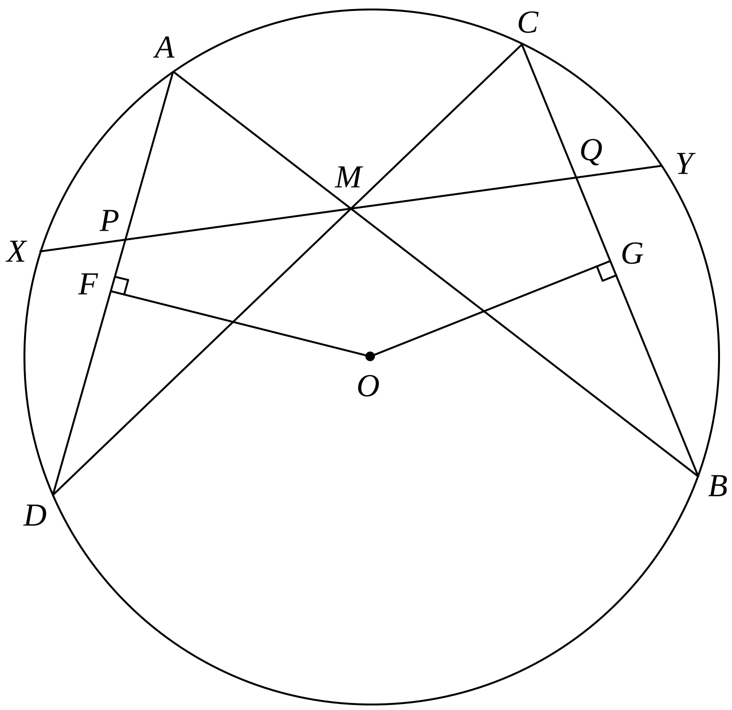
(ii) Hence, or otherwise, prove that $\tan\alpha = \tan\beta\cos\gamma$. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A chord XY with midpoint M is constructed in a circle with centre O. Two arbitrary chords AB and CD are drawn to pass through M. Let P and Q be the intersection points of AD and CB onto XY respectively.

Let F and G be the foot of the perpendiculars from O onto AD and CB respectively.



(i) Prove that $\Delta FMA \parallel\!\mid \Delta GMC$. **3**

(ii) Hence, or otherwise, prove that M is the midpoint of PQ. **3**

Question 16 continues on page 18

(b) Let $I_n = (-1)^{n+1} \int_0^1 x^n e^x \, dx$, where n is a positive integer.

(i) Use mathematical induction to prove that **3**

$$I_n = n!\left[1 - e\sum_{r=0}^{n} \frac{(-1)^r}{r!}\right]$$

(ii) Prove that $0 < \int_0^1 x^n e^x \, dx < \dfrac{e}{n+1}$. **1**

(c) Consider a group of n people sitting in chairs labelled 1, 2, 3, … , n and a pile of n cards also labelled 1, 2, 3, … , n. Each person picks a card randomly without returning it to the pile. A person has a matching card if the number on the card is the same as the number on their chair.

Let the number of ways where nobody receives a matching card be D_n, where $n \geq 2$.

(i) Explain why $D_n = (n-1)(D_{n-1} + D_{n-2})$. **1**

(ii) Show that the expression **2**

$$D_n = n! \times \sum_{r=0}^{n} \frac{(-1)^r}{r!}$$

satisfies the equation in (i).

(iii) Use part (b) to prove that when n is large, the probability of nobody picking their own card is approximately $\dfrac{1}{e}$. **2**

End of Exam

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2014 Bored of Studies Trial Examinations

Mathematics Extension 2

Solutions

Section I

1. A	3. B	5. D	7. C	9. B
2. A	4. A	6. D	8. D	10. D

Working/Justification

Question 1

Note that $y^2 = [f(x)]^2$ is equivalent to $y = \pm f(x)$ so the graph is simply $y = f(x)$ and its reflection about the x-axis. One can deduce some parts of $y = f(x)$ by observing the first quadrant of the given graph of $|y| = f(|x|)$. One key point to note is that there are no critical points at the x-intercepts unless the original function had critical points at the x-intercept in the first place. This is because from $y^2 = [f(x)]^2$ this implies that $y\dfrac{dy}{dx} = f'(x)f(x)$. If $f(x) = 0$ then $y = 0$ for the graph $y^2 = [f(x)]^2$ which says nothing about $\dfrac{dy}{dx}$.

Question 2

If the foci are equal then $a_1e_1 = a_2e_2$ where e_1 and e_2 are the eccentricities of the ellipse and hyperbola respectively. This implies that $\dfrac{a_1}{a_2} = \dfrac{e_2}{e_1} > 1$ since $e_1 < 1$ and $e_2 > 1$ which leads to $a_1 > a_2$.

Question 3

(A) is false because if we take a point on the curve where $0 < x < 1$ and $y < 1$ then it has a positive slope. However, such a point makes $\dfrac{dy}{dx} < 0$ from the expression.

(C) is false because $y < 1$ for the entire curve so when x is positive, $\dfrac{dy}{dx} < 0$ from the expression but the graph shows a positive slope.

(D) is false using a similar argument for (A).

(B) is true because when $y > 1$, the sign of $\dfrac{dy}{dx}$ is the same sign as the x-value on the curve and when $y < 1$ the sign of $\dfrac{dy}{dx}$ is the opposite sign as x-value on the curve, which is consistent with the expression.

Question 4

The Fundamental Theorem of Algebra states that every non-constant polynomial over the complex numbers has at least one root. Note that $P(x)$ having n complex roots is a consequence of the Fundamental Theorem of Algebra but is not the theorem itself.

Question 5

Substituting $z = x + iy$ the choices in relation to the graph provided we are looking for the inequalities $(x+1)^2 + (y-1)^2 \leq 4$ and $y \geq \dfrac{1}{x}$. Note that the point $(-1, 1)$ satisfies $y \geq \dfrac{1}{x}$ but not $xy \geq 1$ (which would have been option (B))

Question 6

Let $z = \sin\theta + i\cos\theta$. Note that $|z| = 1$ and

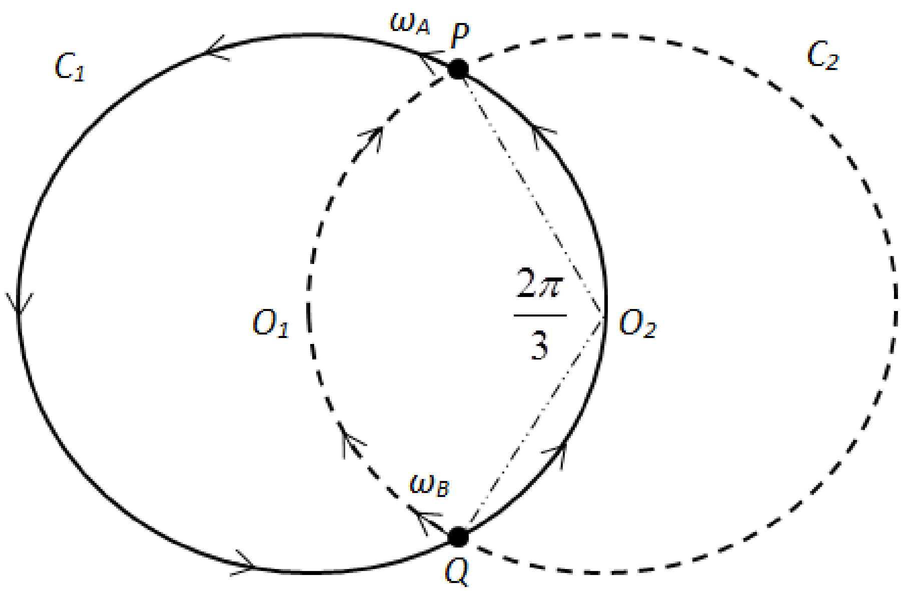
$$\begin{aligned}
\frac{1+z}{1+\bar{z}} &= \frac{z\bar{z}+z}{1+\bar{z}} \quad \text{since } z\bar{z} = |z|^2 = 1 \\
&= \frac{z(1+\bar{z})}{1+\bar{z}} \\
&= z \quad \text{noting that } \bar{z} \neq -1
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^4 &= \left(\frac{1+z}{1+\bar{z}}\right)^4 \\
&= z^4 \\
&= (\sin\theta + i\cos\theta)^4 \\
&= \left(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right)^4 \\
&= \cos(2\pi - 4\theta) + i\sin(2\pi - 4\theta) \\
&= \cos 4\theta - i\sin 4\theta
\end{aligned}$$

Question 7

Consider the motion in terms of angular displacement. Note that we can easily show that ΔPO_1O_2 and ΔQO_1O_2 are equilateral so $\angle PO_2Q$ has size $\frac{2\pi}{3}$.



For particle A to return to point P it must travel an angular displacement of 2π. For particle B to hit point P after being released from point Q, it must travel an angular displacement of $\dfrac{2\pi}{3}$.

Since the angular velocities are constant then $\omega_A = \dfrac{2\pi}{t}$ and $\omega_B = \dfrac{2\pi}{3t}$ where t is the time travelled to reach point P.

Equating the times then we get $\omega_A = 3\omega_B$.

Question 8

The key caveat here is that the polynomial $P(x)$ is not necessarily a real polynomial. Counter examples can be found for (A) (e.g. $x^2(x^2+2ix-1)$), (B) (e.g. x^3-i^3) and (C) (e.g. x^2+1). For (D), note that a real polynomial would have any non-real roots occur in conjugate pairs, thus a quadratic factor derived from the linear factors of any non-real roots will always be real.

Question 9

(A) is clearly not true because $\sin x \geq -1$ so this option would suggest that $x \geq -1$ which is not true for all real x

(C) is not true because when say $x = \pi$ then $2\left|\sin\left(\frac{\pi}{2}\right)\right| < \sin\pi$ which suggests $2 < 0$ which is false

(D) is not true because equality occurs at $x = 0$

(B) is true from observing the graph of the two curves $y = e^x$ and $y = \tan^{-1} x$.

We note that $e^x > 0$ for all real x and since $\tan^{-1} x < 0$ for $x < 0$ then clearly $e^x > \tan^{-1} x$ when $x < 0$. When $x = 0$, $e^x = 1$ and $\tan^{-1} x = 0$ so the inequality still holds.

When $x > 0$ both e^x and $\tan^{-1} x$ are positive, so we use the following argument to deduce the inequality. The gradients of $y = e^x$ and $y = \tan^{-1} x$ are both 1 at $x = 0$ but for $x > 0$, $y = e^x$ is concave up and $y = \tan^{-1} x$ is concave down (can easily verify this by the second derivative) thus the slope of $y = e^x$ gets steeper as x increases but for $y = \tan^{-1} x$ it gets flatter as x increases. Since $y = e^x$ is already above $y = \tan^{-1} x$ at $x = 0$ and $y = e^x$ increases in value much more quickly than $y = \tan^{-1} x$ then the graph of $y = e^x$ will always be above $y = \tan^{-1} x$ for $x > 0$.

Question 10

The series can be rewritten as

$$1 + 2w + 3w^2 + ... + nw^{n-1} + 2\bar{w} + 3\bar{w}^2 + ... + n\bar{w}^{n-1}$$

First consider $1 + 2w + 3w^2 + ... + nw^{n-1}$ and let

$$S = 1 + 2w + 3w^2 + ... + nw^{n-1}$$

$$wS = w + 2w^2 + 3w^3 + ... + nw^n$$

$$S - wS = 1 + w + w^2 + ... + w^{n-1} - nw^n$$

$$(1 - w)S = \frac{1 - w^n}{1 - w} - nw^n \qquad \text{but } w^n = 1$$

$$S = -\frac{n}{1 - w}$$

Similarly $2\bar{w} + 3\bar{w}^2 + ... + n\bar{w}^{n-1} = -\dfrac{n}{1 - \bar{w}} - 1$

Hence

$$
\begin{aligned}
1 + 2(w + \bar{w}) + 3(w^2 + \bar{w}^2) + ... + n(w^{n-1} + \bar{w}^{n-1}) &= n\left(\frac{1}{w-1} + \frac{1}{\bar{w}-1}\right) - 1 \\
&= \frac{n(w+\bar{w}-2)}{(w-1)(\bar{w}-1)} - 1 \\
&= \frac{n(w+\bar{w}-2)}{w\bar{w} - w - \bar{w} + 1} - 1 \qquad \text{but } w\bar{w} = |w|^2 = 1 \\
&= \frac{n(w+\bar{w}-2)}{-(w+\bar{w}-2)} - 1 \\
&= -1 - n
\end{aligned}
$$

Section II

Question 11

(a) (i)

$$zw = (a+ib)\left(\frac{1}{a}+\frac{1}{b}i\right)$$

$$= \left(\frac{a}{b}+\frac{b}{a}\right)i$$

(ii) We note that since $a > b > 0$ then $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ and $\arg(w) = \tan^{-1}\left(\frac{\frac{1}{b}}{\frac{1}{a}}\right) = \tan^{-1}\left(\frac{a}{b}\right)$

Also, we note that zw is purely imaginary from part (i), so $\arg(zw) = \dfrac{\pi}{2}$ but we also have

$$\arg(zw) = \arg(z) + \arg(w)$$

$$= \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{a}{b}\right)$$

Hence $\tan^{-1}\left(\dfrac{a}{b}\right) + \tan^{-1}\left(\dfrac{b}{a}\right) = \dfrac{\pi}{2}$

(b) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ and $dx = \dfrac{2}{1+t^2}\,dt$. When $x = \frac{\pi}{2}$ then $t = 1$ and when $x = 0$ then $t = 0$.

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{(1+\cos x)(\sin x+\cos x)}} = \int_0^1 \frac{1}{\sqrt{\left(1+\frac{1-t^2}{1+t^2}\right)\left(\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\right)}} \times \frac{2}{1+t^2}\,dt$$

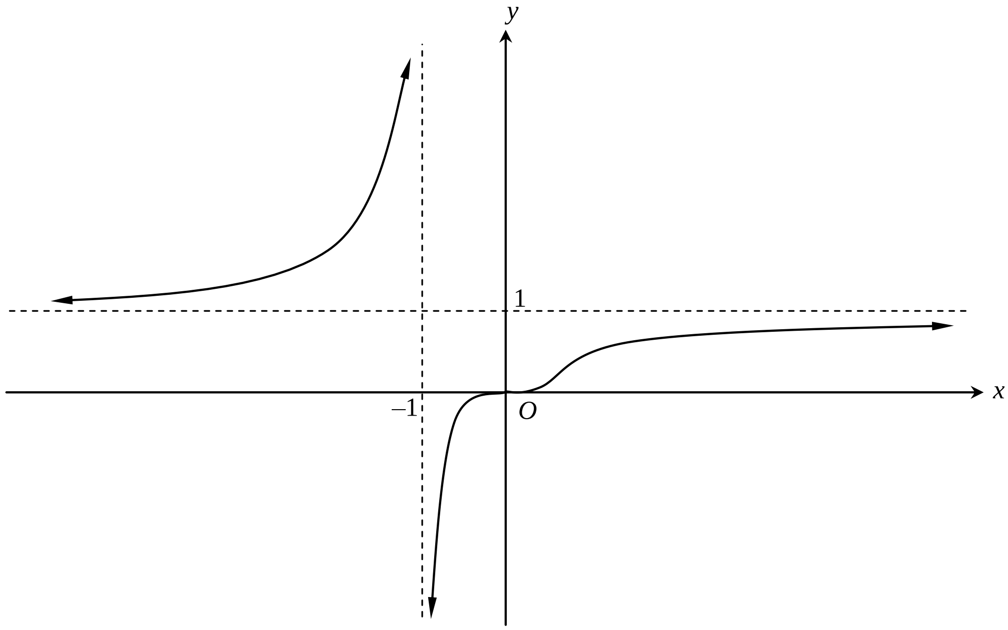
$$= \int_0^1 \frac{2}{\sqrt{\left(1+t^2+1-t^2\right)\left(1+2t-t^2\right)}}\,dt$$

$$= \int_0^1 \frac{\sqrt{2}}{\sqrt{2-(t-1)^2}}\,dt$$

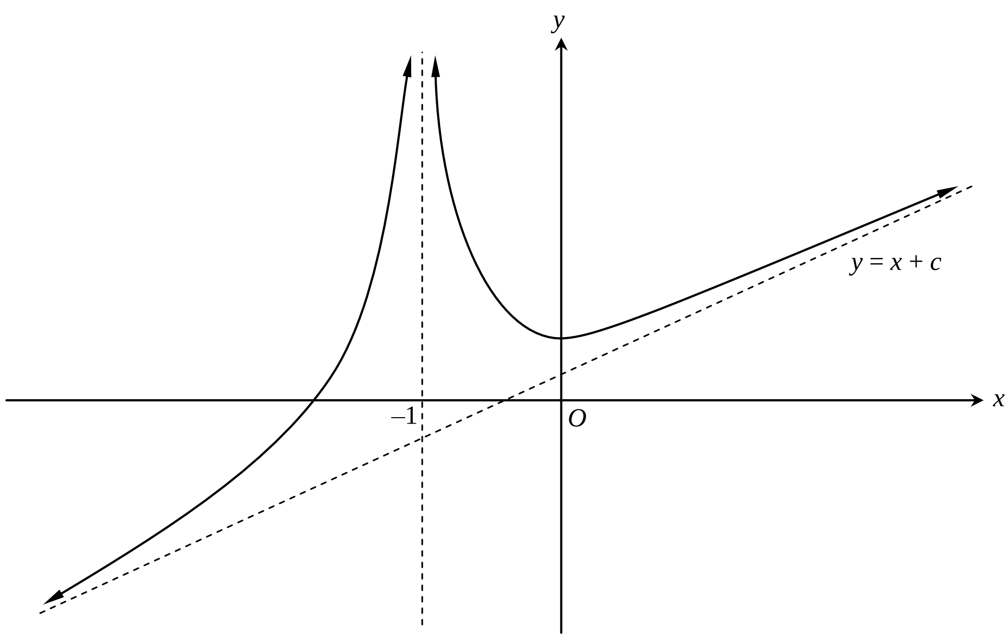
$$= \sqrt{2}\left[\sin^{-1}\left(\frac{t-1}{\sqrt{2}}\right)\right]_0^1$$

$$= \frac{\sqrt{2}}{4}\pi$$

(c) (i) Note that there is a triple root at the origin of $y = F'(x)$, a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = 1$ (as $F'(x) \to 1$ when $x \to \infty$ or $x \to -\infty$



(ii) From (i), when $x \to \infty$ the gradient of $F(x)$ approaches 1 but is less steep than a gradient of 1. When $x \to -1^-$ the gradient of $F(x)$ approaches infinity. When $x \to -1^+$ the gradient of $F(x)$ approaches negative infinity. Note that there is an oblique asymptote with gradient 1 and a stationary point at the y-axis.



(d) If $P(x)$ has a triple root say at $x=\alpha$ then $P(\alpha)=P'(\alpha)=P''(\alpha)=0$

$$P(x)=x^4+ax^3+bx+c$$

$$P'(x)=4x^3+3ax^2+b$$

$$P''(x)=12x^2+6ax$$

At $x=\alpha$

$$12\alpha^2+6a\alpha=0$$

$$6\alpha(2\alpha+a)=0$$

$$\alpha=0,-\frac{a}{2}$$

But $P(0)=c\neq 0$ so the triple root should be $-\frac{a}{2}$ so that $P'\left(-\frac{a}{2}\right)=0$ so

$$4\left(-\frac{a}{2}\right)^3+3a\left(-\frac{a}{2}\right)^2+b=0$$

$$-\frac{a^3}{2}+\frac{3a^3}{4}+b=0$$

$$a^3=-4b$$

Also $P\left(-\frac{a}{2}\right)=0$ so

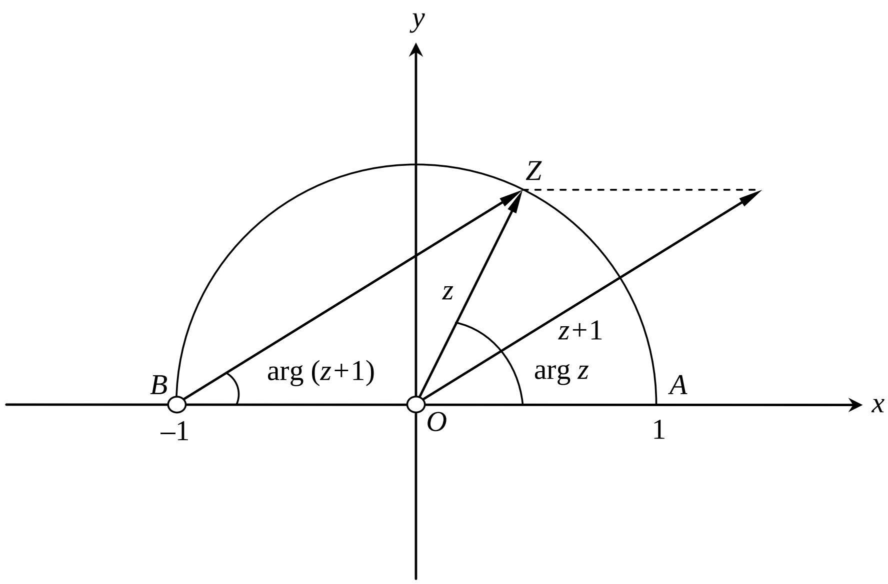
$$\left(-\frac{a}{2}\right)^4+a\left(-\frac{a}{2}\right)^3+b\left(-\frac{a}{2}\right)+c=0$$

$$-\frac{a^4}{16}-\frac{ab}{2}+c=0 \quad \text{but } a^3=-4b$$

$$-a(-4b)-8ab+16c=0$$

$$\therefore ab=4c$$

(e) Consider the vectors representing z and $z+1$. Let A and B be the points $(1,0)$ and $(-1,0)$ on the Argand diagram. Let Z be the point on the Argand diagram represented by the complex number z. The vector BZ is parallel to the vector represented by the complex at number $z+1$.



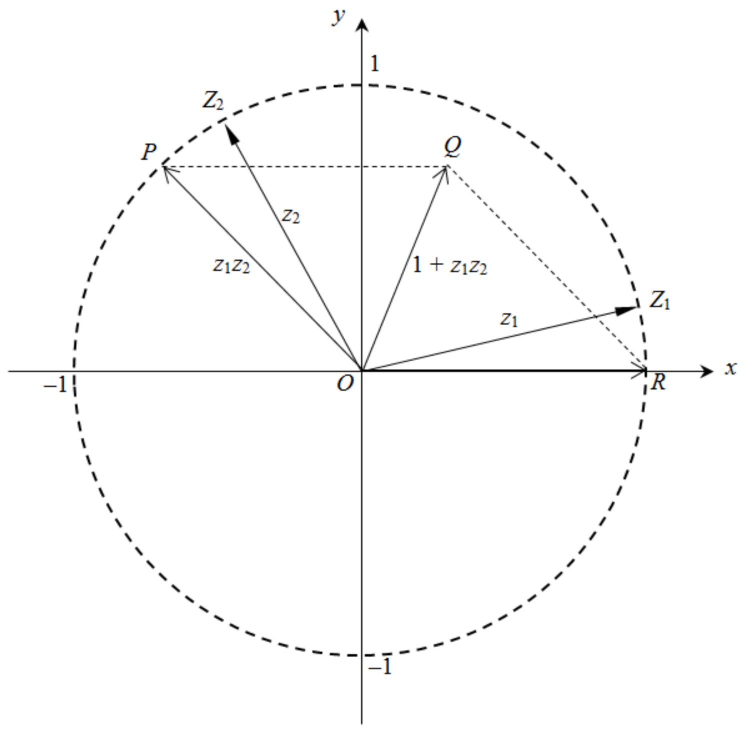
Since $\angle ZOA = 2\angle ZBA$ then B, Z and A are concyclic due to the angle at centre O being twice the angle at the circumference standing on the same arc AZ. Thus, one part of the locus of z is the circle $x^2+y^2=1$ but excluding the point $(-1,0)$ since the arg(0) is undefined.

Another component to the locus is the case where $\arg(z) = \arg(z+1) = 0$ which also satisfies the condition specified. This occurs when z is a purely positive real number.

Thus, the locus of z is $x^2+y^2=1$ in the domain $-1 < x \leq 1$ AND $y=0$ for $x>0$.

Question 12

(a) Let Z_1 and Z_2 be the points on the Argand diagram represented by the complex numbers z_1 and z_2. Let P and Q be the points represented by the complex numbers z_1z_2 and $1 + z_1z_2$ respectively. Let R be the positive x-intercept of the unit circle.



Since $|z_1| = |z_2| = 1$ then $|z_1z_2| = 1$ so z_1z_2 also lies on the unit circle. Also, OQ is the diagonal of the rhombus $OPQR$ from the parallelogram rule and noting that $|\vec{OP}| = |\vec{OR}| = 1$ as equal adjacent sides. Hence $\angle POQ = \angle ROQ$.

But $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) \Rightarrow \arg(z_1z_2) - \arg(z_2) = \arg(z_1)$

This means that $\angle POZ_2 = \angle ROZ_1$ but since $\angle POQ = \angle POZ_2 + \angle QOZ_2$ and $\angle QOR = \angle ROZ_1 + \angle QOZ_1$ then $\angle QOZ_1 = \angle QOZ_2$ hence OQ bisects $\angle Z_1OZ_2$.

However, we also know from the parallelogram rule that the vector represented by $z_1 + z_2$ is a diagonal which also bisects $\angle Z_1OZ_2$, thus $\arg(1 + z_1z_2) = \arg(z_1 + z_2)$ for this to occur.

Note that since $\arg(1 + z_1z_2) = \arg(z_1 + z_2)$ then $\arg(1 + z_1z_2) - \arg(z_1 + z_2) = 0$ or $\arg\left(\dfrac{1 + z_1z_2}{z_1 + z_2}\right) = 0$.

A similar argument can be applied in consideration of the case for different positions of z_1 and z_2 which gives the more general result that $\arg(1 + z_1z_2) - \arg(z_1 + z_2) = k\pi$ for some integer k. This means that $\dfrac{1 + z_1z_2}{z_1 + z_2}$ must be a purely real number.

An alternative approach to show this alegbraically is to note that

$$\frac{z_1+z_2}{1+z_1z_2} = \frac{z_1+z_2}{1+z_1z_2} \times \frac{1+\overline{z_1z_2}}{1+\overline{z_1z_2}}$$

$$= \frac{z_1+z_2+z_1\overline{z_1z_2}+z_2\overline{z_1z_2}}{1+z_1z_2+\overline{z_1z_2}+z_1z_2\overline{z_1z_2}}$$

$$= \frac{z_1+z_2+|z_1|^2\overline{z_2}+|z_2|^2\overline{z_1}}{1+z_1z_2+\overline{z_1z_2}+|z_1z_2|^2} \quad \text{but } |z_i|=1 \quad \text{and } |z_i||z_j|=1$$

$$= \frac{z_1+\overline{z_1}+z_2+\overline{z_2}}{2+z_1z_2+\overline{z_1z_2}}$$

This is purely real because any complex number z added to its conjugate $\bar{z}$ gives a real value.

(b) First observe that $\alpha+\beta+\gamma=0$ and $\alpha\beta\gamma=-q$

$$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$

$$= \frac{-\gamma}{\left(\frac{-q}{\gamma}\right)}$$

$$= \frac{\gamma^2}{q}$$

Hence we are really looking for a polynomial equation with the roots $\frac{\alpha^2}{q}, \frac{\beta^2}{q}$ and $\frac{\gamma^2}{q}$.

Let $y=\frac{x^2}{q} \Rightarrow x^2=qy$ so we arrive at the polynomial equation as follows

$$x^3+px+q=0$$

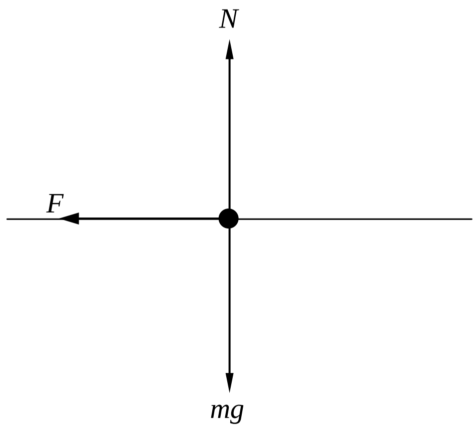
$$x(x^2+p)=-q$$

$$x^2(x^2+p)^2=q^2$$

$$qy(qy+p)^2=q^2$$

$$q^2y^3+2pqy^2+p^2y-q=0$$

(c) Consider a flat track and a banked track

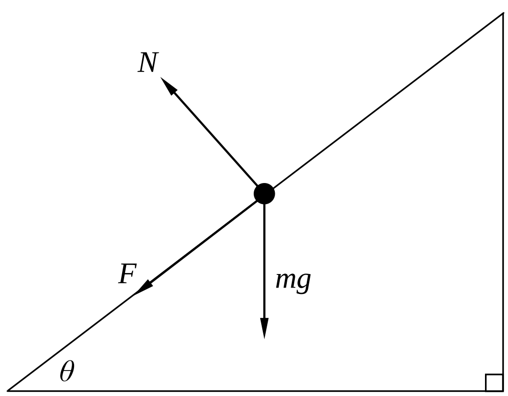


$$N = mg$$

$$F = \frac{mv^2}{r}$$

When $v = V_0$ the value of F is a maximum hence

$$F_{max} = \frac{mV_0^2}{r}$$



$$N \sin\theta + F\cos\theta = \frac{mv^2}{r} \quad (1)$$

$$N\cos\theta - F\sin\theta = mg \quad (2)$$

$(1)\cos\theta - (2)\sin\theta$

$$N\sin\theta\cos\theta + F\cos^2\theta - N\sin\theta\cos\theta + F\sin^2\theta = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$

$$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$

When $v = V_2$ then $F = 0$ hence

$$\frac{mV_2^2}{r}\cos\theta - mg\sin\theta = 0 \quad (3)$$

When $v = V_1$ then $F = F_{max}$ hence

$$\frac{mV_1^2}{r}\cos\theta - mg\sin\theta = F_{max} \quad (4)$$

$(4)-(3)$

$$F_{max} = \frac{mV_1^2}{r}\cos\theta - \frac{mV_2^2}{r}\cos\theta \quad \text{but } F_{max} = \frac{mV_0^2}{r}$$

$$\frac{mV_0^2}{r} = \frac{m\cos\theta(V_1^2 - V_2^2)}{r}$$

$$\cos\theta = \frac{V_0^2}{V_1^2 - V_2^2}$$

(d) First note that $|z_i| = \sqrt{z_i\bar{z_i}} = 1$ for $i = 1, 2, 3$.

$$\text{LHS} = |z_1z_2 + z_1z_3 + z_2z_3|$$

$$= \frac{|z_1z_2 + z_1z_3 + z_2z_3|}{|z_1||z_2||z_3|} \quad \text{since } |z_i| = 1$$

$$= \left|\frac{z_1z_2 + z_1z_3 + z_2z_3}{z_1z_2z_3}\right|$$

$$= \left|\frac{1}{z_3} + \frac{1}{z_2} + \frac{1}{z_1}\right|$$

$$= \left|\frac{\bar{z_3}}{\bar{z_3}z_3} + \frac{\bar{z_2}}{\bar{z_2}z_2} + \frac{\bar{z_1}}{\bar{z_1}z_1}\right|$$

$$= \left|\frac{\bar{z_3}}{|z_3|^2} + \frac{\bar{z_2}}{|z_2|^2} + \frac{\bar{z_1}}{|z_1|^2}\right| \quad \text{since } |z_i| = \sqrt{z_i\bar{z_i}}$$

$$= |\bar{z_1} + \bar{z_2} + \bar{z_3}|$$

$$= |\overline{z_1 + z_2 + z_3}| \quad \text{but when } z = x + iy \quad \text{then } |z| = |\bar{z}| = \sqrt{x^2 + y^2}$$

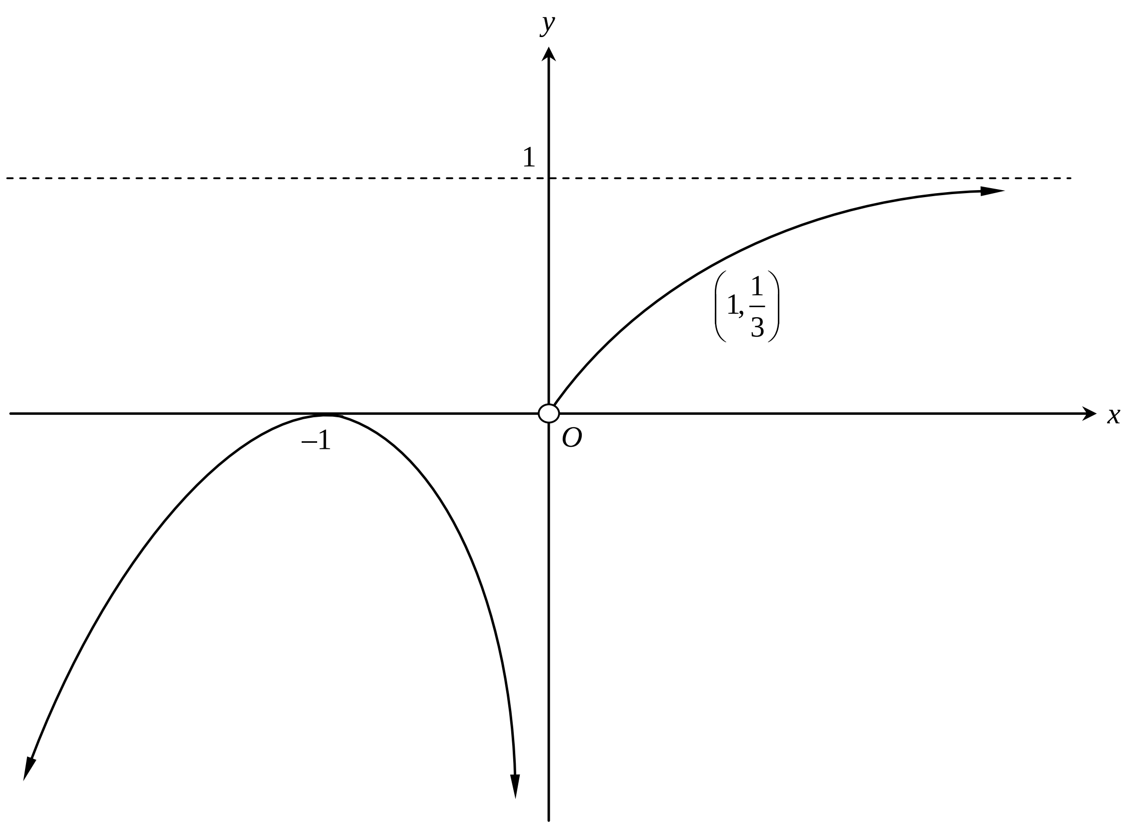
$$= |z_1 + z_2 + z_3|$$

$$= \text{RHS}$$

(e) Consider the following features of the graph when replacing x with $g(x)$ in $f(x)$

x	$g(x)$	$f(g(x))$
$\to -\infty$	$\to -\infty$	$\to -\infty$
-1	-1	0
$\to 0^-$	$\to -\infty$	$\to -\infty$
$\to 0^+$	$\to \infty$	$\to 0^+$
1	0	$\frac{1}{3}$
$\to \infty$	$\to -\frac{2}{3}^+$	$\to 1^-$

Translate this into the sketch below



Question 13

(a) (i) If r_k is a real value for $k = 1, 2, 3, ..., p$ then

$$LHS = |r_k - i|$$

$$= \sqrt{r_k^2 + 1} \quad \text{but } r_k^2 \geq 0$$

$$\geq \sqrt{1}$$

$$= 1$$

$$= RHS$$

(ii) Since $r_1, r_2, ..., r_p, c_1, c_2, ..., c_q$ are the roots of the monic polynomial $P(z)$ then

$$P(z) = (z - r_1)(z - r_2)...(z - r_p)(z - c_1)(z - c_2)...(z - c_q)$$

$$|P(i)| = |i - r_1||i - r_2|...|i - r_p||i - c_1||i - c_2|...|i - c_q|$$

$$= |r_1 - i||r_2 - i|...|r_p - i||c_1 - i||c_2 - i|...|c_q - i|$$

But $|r_1 - i||r_2 - i|...|r_p - i| \geq 1$ since $|r_k - i| \geq 1$ for $k = 1, 2, ..., p$ from part (i), so the only way we can have $|P(i)| < 1$ is to have $|c_1 - i||c_2 - i|...|c_q - i| < 1$

(iii) For the inequality to hold in part (ii), there must exist at least one non-real root c_k such that $|c_k - i| < 1$. However, the modulus of a complex number equals the modulus of its conjugate so

$$|\overline{c_k - i}| < 1$$

$$\Rightarrow |\bar{c_k} + i| < 1$$

But, $\alpha = \bar{c_k}$ is also a root of $P(x)$ since $P(x)$ has real coefficients. Hence $|\alpha + i| < 1$

(b) (i) For particle B, if we define the direction of motion to be the positive direction, the acceleration equation is

$$m\ddot{x} = mg - mkv_B$$

$$\frac{dv_B}{dt} = g - kv_B$$

$$t = \int \frac{dv_B}{g - kv_B}$$

$$= -\frac{1}{k}\ln(g - kv_B) + c$$

But at $t = T, v_B = u \Rightarrow c = T + \frac{1}{k}\ln(g - ku)$

$$t = -\frac{1}{k}\ln(g - kv_B) + \frac{1}{k}\ln(g - ku) + T$$

$$= T - \frac{1}{k}\ln\left(\frac{g - kv_B}{g - ku}\right)$$

If particle B is projected T seconds after particle A then the time-velocity equation for particle A is similar except replace v_B with v_A as well as set $u = 0, T = 0$.

$$t = -\frac{1}{k}\ln\left(\frac{g - kv_A}{g}\right)$$

Eliminating t and we get

$$T = \frac{1}{k}\ln\left(\frac{g - kv_B}{g - ku}\right) - \frac{1}{k}\ln\left(\frac{g - kv_A}{g}\right)$$

$$= -\frac{1}{k}\ln\left(\frac{g - ku}{g} \times \frac{g - kv_A}{g - kv_B}\right)$$

(ii) For particle B

$$v_B \frac{dv_B}{dx_B} = g - kv_B$$

$$x_B = \int \frac{v_B}{g - kv_B}\, dv_B$$

$$= -\frac{1}{k} \int \frac{g - kv_B - g}{g - kv_B}\, dv_B$$

$$= -\frac{1}{k} \int \left(1 - \frac{g}{g - kv_B}\right) dv_B$$

$$= -\frac{1}{k} \left[v_B + \frac{g}{k} \ln (g - kv_B)\right] + c$$

When $x_B = 0$, $v_B = u$ so $c = \frac{1}{k} \left[u + \frac{g}{k} \ln (g - ku)\right]$

$$x_B = -\frac{1}{k} \left[v_B + \frac{g}{k} \ln (g - kv_B)\right] + \frac{1}{k} \left[u + \frac{g}{k} \ln (g - ku)\right]$$

$$= \frac{1}{k} \left[u - v_B + \frac{g}{k} \ln \left(\frac{g - ku}{g - kv_B}\right)\right]$$

(iii) For particle B to catch up to particle A, there must be a point where the displacements are equal. We have the displacement function for particle B in part (ii), and similarly the displacement function for particle A is found by replacing x_B with x_A, v_B with v_A and setting $u = 0$.

$$x_A = \frac{1}{k} \left[-v_A + \frac{g}{k} \ln \left(\frac{g}{g - kv_A}\right)\right]$$

For particle B to catch up with particle A, there exists a point where $x_A = x_B$

$$\frac{1}{k} \left[u - v_B + \frac{g}{k} \ln \left(\frac{g - ku}{g - kv_B}\right)\right] = \frac{1}{k} \left[-v_A + \frac{g}{k} \ln \left(\frac{g}{g - kv_A}\right)\right]$$

$$u + v_A - v_B = \frac{g}{k} \ln \left(\frac{g}{g - kv_A} \times \frac{g - kv_B}{g - ku}\right)$$

$$gT = u + v_A - v_B \quad \text{from part (i)}$$

$$T = \frac{v_A - v_B + u}{g}$$

For particle B to catch up to particle A, we require $v_B > v_A$ which implies that

$$\frac{v_A - v_B + u}{g} < \frac{u}{g}$$

$$\Rightarrow T < \frac{u}{g}$$

Hence particle B must be released no more than $\dfrac{u}{g}$ seconds after particle A.

Question 14

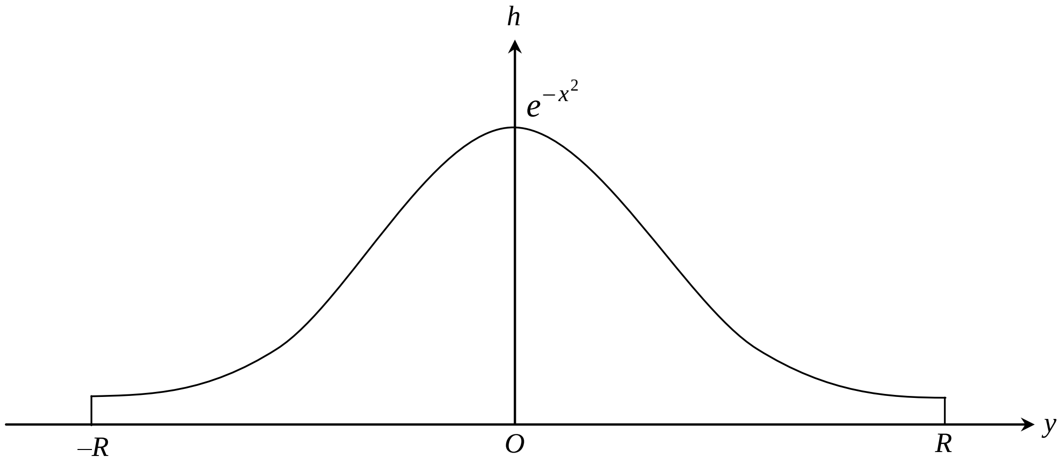
(a) From the cylindrical shells approach, we have $\delta V_A = 2\pi r h\, \delta r$ so

$$V_A = \pi \int_0^R 2re^{-r^2}\, dr$$

$$= -\pi \left[e^{-r^2} \right]_0^R$$

$$= \pi(1 - e^{-R^2})$$

(b) (i) Let the area of the slice be A, which is also the area under the curve $h = ke^{-y^2}$ from $y = -R$ and $y = R$.



$$A = k \int_{-R}^{R} e^{-y^2}\, dy$$

$$= k \int_{-R}^{R} e^{-t^2}\, dt \qquad \text{as } t \text{ can be treated as a dummy variable}$$

$$= kI$$

However, note that for this slice that when $y = 0$, $h = e^{-x^2}$ for a given fixed value of x, thus $k = e^{-x^2}$ and $A = e^{-x^2} I$

(ii) Now we allow x to vary across the slices so that

$$V_B = \lim_{\delta x \to 0} \sum_{-R \le x \le R} e^{-x^2} I \, \delta x$$

$$= \int_{-R}^{R} e^{-x^2} I \, dx$$

$$= I \int_{-R}^{R} e^{-x^2} \, dx \quad \text{since } I \text{ is a constant}$$

$$= I \int_{-R}^{R} e^{-t^2} \, dt \quad \text{as } x \text{ can be treated as a dummy variable}$$

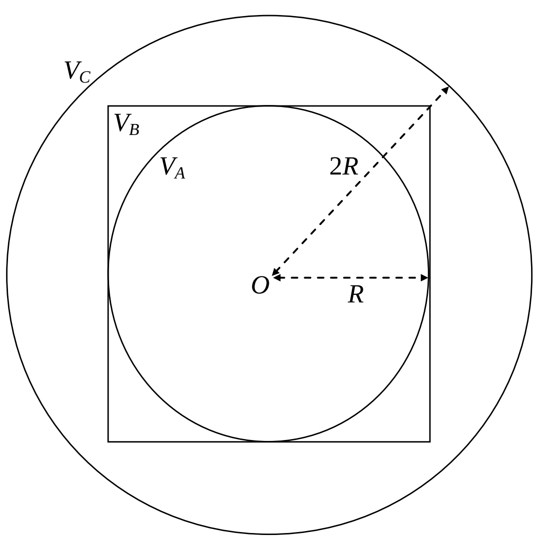
$$= I^2$$

(iii) Replace R with $2R$ we get $V_C = \pi(1 - e^{-4R^2})$

(iv) Note that the volumes V_A, V_B and V_C are very similar except for different bases which affect the volume.

$V_C > V_B$ because the base of the solid in V_C is circular with radius $2R$ whereas for the solid in V_B the base is a square that is fully enclosed by the larger circular base.

$V_A < V_B$ because the base of the solid in V_A is circular with radius R whereas for the solid in V_B the base is a square with each side R, which can enclose the circular base. The diagram below shows the bird's-eye view of the solid.



(v) Note that $I = \displaystyle\int_{-R}^{R} e^{-t^2}\,dt$

Substituting the results into the inequality

$$\pi(1 - e^{-R^2}) < I^2 < \pi(1 - e^{-4R^2})$$

When $R \to \infty$ then $\pi(1 - e^{-R^2}) \to \pi$ and $\pi(1 - e^{-4R^2}) \to \pi$ so $I^2 \to \pi$

This means that since $e^{-t^2} > 0$ we get

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-t^2}\,dt = \sqrt{\pi}$$

(c) (i) Consider the geometric series

$$1 + x + x^2 + ... + x^n = \frac{x^{n+1} - 1}{x - 1}$$

Differentiate both sides with respect to x

$$1 + 2x + 3x^2 + ... + nx^{n-1} = \frac{(n+1)x^n(x-1) - (x^{n+1} - 1)}{(x-1)^2}$$

$$= \frac{(n+1)x^{n+1} - (n+1)x^n - x^{n+1} + 1}{(x-1)^2}$$

$$= \frac{(n+1)x^{n+1} - (n+1)x^n - x^{n+1} + 1}{(x-1)^2}$$

$$= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

Alternatively, let

$$S = 1 + 2x + 3x^2 + ... + nx^{n-1}$$

$$xS = x + 2x^2 + 3x^3 + ... + nx^n$$

$$S - xS = 1 + x + x^2 + ... + x^{n-1} - nx^n$$

$$(1-x)S = \frac{1-x^n}{1-x} - nx^n$$

$$S = \frac{1 - x^n - nx^n + nx^{n+1}}{(1-x)^2}$$

$$= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

(ii) The first win occurs on the k-th game. For this to happen, the first $k-1$ games must be losses.

Hence $P(k) = (1-p)^{k-1}p$

(iii)

$$LHS = \lim_{n\to\infty} \sum_{k=1}^{n} kP(k)$$

$$= p \lim_{n\to\infty} \sum_{k=1}^{n} k(1-p)^{k-1} \quad \text{from (ii)}$$

$$= p \lim_{n\to\infty} \left(\frac{n(1-p)^{n+1} - (n+1)(1-p)^n + 1}{((1-p)-1)^2} \right) \quad \text{using (i) noting that } x = 1-p$$

$$= p \times \frac{1}{p^2}$$

$$= \frac{1}{p}$$

Since $0 < 1-p < 1$, then when $n \to \infty$ we have $(1-p)^n \to 0$ also noting that $n(1-p)^{n+1} \to 0$

Question 15

(a) Note that $x^2 = x^2(\cos^2 x + \sin^2 x) + x\sin x\cos x - x\sin x\cos x$ so

$$\int \frac{x^2}{(x\sin x + \cos x)(x\cos x - \sin x)}\,dx = \int \frac{x^2(\cos^2 x + \sin^2 x) + x\sin x\cos x - x\sin x\cos x}{(x\sin x + \cos x)(x\cos x - \sin x)}\,dx$$

$$= \int \frac{x\cos x(x\cos x - \sin x) + x\sin x(x\sin x + \cos x)}{(x\sin x + \cos x)(x\cos x - \sin x)}\,dx$$

$$= \int \left(\frac{x\cos x}{x\sin x + \cos x} + \frac{x\sin x}{x\cos x - \sin x}\right)\,dx$$

Let $u = x\sin x + \cos x$ and $v = x\cos x - \sin x$ so that $du = x\cos x\,dx$ and $dv = -x\sin x\,dx$ so

$$\int \frac{x^2}{(x\sin x + \cos x)(x\cos x - \sin x)}\,dx = \int \frac{1}{u}\,du - \int \frac{1}{v}\,dv$$

$$= \ln u - \ln v + c$$

$$= \ln(x\sin x + \cos x) - \ln(x\cos x - \sin x) + c$$

$$= \ln\left(\frac{x\sin x + \cos x}{x\cos x - \sin x}\right) + c$$

(b) (i) Let M be the foot of the perpendicular from P onto the positive directrix, where P has the coordinates $(a\cos\theta, b\sin\theta)$. First note that

$$\frac{PS}{PM} = e$$

$$PS = ePM$$

$$= e\left(\frac{a}{e} - a\cos\theta\right)$$

$$= a(1 - e\cos\theta)$$

Construct a vertical line from P to the positive x-axis at point R which forms a right-angled triangle PSR.

Deducing the lengths of the sides and applying trigonometry we can say that

$$\cos\phi = \frac{RS}{PS}$$

$$= \frac{a\cos\theta - ae}{PS}$$

$$\cos\theta = \frac{PS\cos\phi + ae}{a} \quad \text{substitute this into the expression for } PS$$

$$PS = a\left(1 - \frac{PS.e\cos\phi + ae^2}{a}\right)$$

$$PS(1 + e\cos\phi) = a - ae^2$$

$$PS = \frac{a(1-e^2)}{1+e\cos\phi}$$

(ii) In ΔPOR

$$\sin\alpha = \frac{PR}{PO} \quad \text{but in } \Delta PSR, \frac{PR}{PS} = \sin\phi$$

$$= \frac{PS\sin\phi}{d}$$

$$d\sin\alpha = \frac{a(1-e^2)\sin\phi}{1+e\cos\phi} \quad \text{from (i)}$$

(iii) In ΔPOR

$$\cos\alpha = \frac{RO}{PO}$$

$$= \frac{OS+RS}{PO} \qquad \text{but in } \Delta PSR, \frac{RS}{RS} = \cos\phi$$

$$= \frac{PS\cos\phi + ae}{d}$$

$$d\cos\alpha = \frac{a(1-e^2)\cos\phi}{1+e\cos\phi} + ae \qquad \text{from (i)}$$

$$= \frac{a\cos\phi - ae^2\cos\phi + ae + ae^2\cos\phi}{1+e\cos\phi}$$

$$= \frac{a(e+\cos\phi)}{1+e\cos\phi}$$

(iv) From (ii)

$$d\sin\alpha = \frac{a(1-e^2)\sin\phi}{1+e\cos\phi}$$

$$d^2\sin^2\alpha = \frac{a^2(1-e^2)^2\sin^2\phi}{(1+e\cos\phi)^2}$$

$$\frac{d^2\sin^2\alpha}{1-e^2} = \frac{a^2(1-e^2)\sin^2\phi}{(1+e\cos\phi)^2}$$

From (iii)

$$d\cos\alpha = \frac{a(e+\cos\phi)}{1+e\cos\phi}$$

$$d^2\cos^2\alpha = \frac{a^2(e+\cos\phi)^2}{(1+e\cos\phi)^2}$$

Hence

$$d^2\left(\frac{\sin^2\alpha}{1-e^2}+\cos^2\alpha\right) = \frac{a^2(1-e^2)\sin^2\phi + a^2(e+\cos\phi)^2}{(1+e\cos\phi)^2}$$

$$= \frac{a^2(\sin^2\phi - e^2\sin^2\phi + e^2 + 2e\cos\phi + \cos^2\phi)}{(1+e\cos\phi)^2}$$

$$= \frac{a^2(1+e^2(1-\sin^2\phi)+2e\cos\phi)}{(1+e\cos\phi)^2}$$

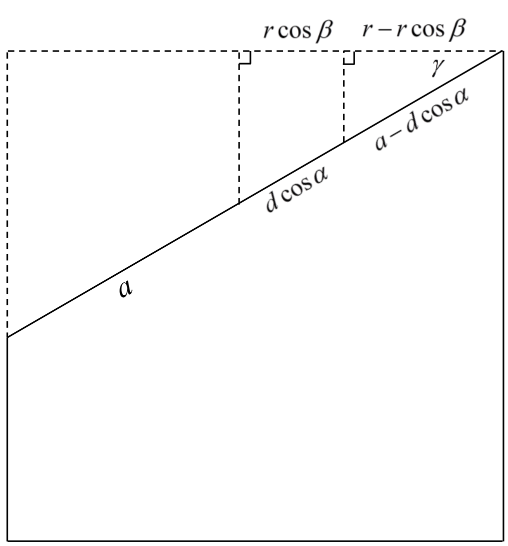
$$= \frac{a^2(e^2\cos^2\phi + 2e\cos\phi + 1)}{(1+e\cos\phi)^2}$$

$$= \frac{a^2(1+e\cos\phi)^2}{(1+e\cos\phi)^2}$$

$$d^2 = \frac{a^2}{\dfrac{\sin^2\alpha}{1-e^2}+\cos^2\alpha}$$

$$d = \frac{a}{\sqrt{\dfrac{\sin^2\alpha}{1-e^2}+\cos^2\alpha}}$$

(c) (i) Let r be the radius of the cylinder. From the ratio of corresponding sides of similar triangles



$$\frac{r-r\cos\beta}{r} = \frac{a-d\cos\alpha}{a}$$

$$d\cos\alpha = a\cos\beta$$

$$\cos\beta = \frac{d}{a}\cos\alpha \quad \text{sub into (b)(iv)}$$

$$\cos\beta = \frac{\cos\alpha}{\sqrt{\dfrac{\sin^2\alpha}{1-e^2}+\cos^2\alpha}}$$

(ii) Note that

$$\cos\gamma = \frac{r}{a} \quad \text{but } r = b \text{ since the minor axis is directly below the circular surface}$$

$$= \frac{b}{a} \quad \text{but } b^2 = a^2(1-e^2)$$

$$\cos^2\gamma = 1 - e^2$$

Sub into result in part (i) and noting that $0 \le \gamma \le \frac{\pi}{2}$

$$\cos\beta = \frac{\cos\alpha}{\sqrt{\dfrac{\sin^2\alpha}{\cos^2\gamma} + \cos^2\alpha}}$$

$$= \frac{\cos\alpha\cos\gamma}{\sqrt{\cos^2\alpha(\tan^2\alpha + \cos^2\gamma)}}$$

$$\sec^2\beta = \frac{\tan^2\alpha + \cos^2\gamma}{\cos^2\gamma}$$

$$\tan^2\beta = \frac{\tan^2\alpha}{\cos^2\gamma}$$

$$\tan\alpha = \tan\beta\cos\gamma$$

Question 16

(a) (i) First note that

$$\angle DAM = \angle MCB \quad \text{(angles in the same segment)*}$$

$$\angle AMD = \angle CMB \quad \text{(vertically opposite angles)}$$

$$\therefore \Delta AMD ||| \Delta CMB \quad \text{(equiangular)}$$

$$\Rightarrow \frac{AM}{CM} = \frac{AD}{CB} \quad \text{(corresponding sides of similar triangles in proportion)}$$

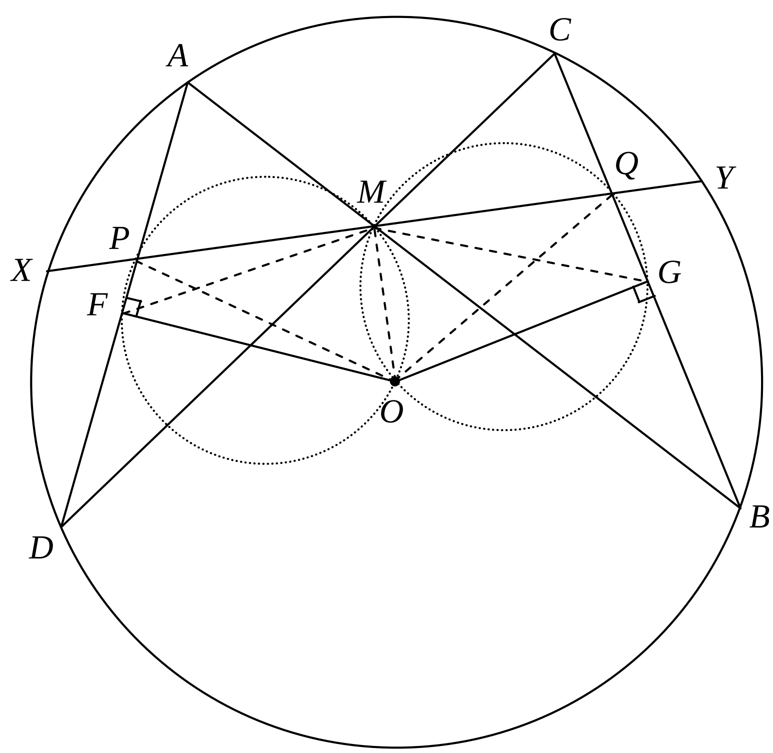
But $OF \perp AD$ which means that $AD = 2AF$ as the perpendicular from the centre of a circle to a chord bisects the chord. Similarly, $CG = 2CB$ so substituting these into the above equation we get

$$\frac{AM}{CM} = \frac{\frac{1}{2}AF}{\frac{1}{2}CB}$$

$$= \frac{AF}{CB}$$

From * and this we can deduce that $\Delta FMA ||| \Delta GMC$ (two corresponding sides in same proportion and corresponding included angles are equal)

(ii)



Since M is the midpoint of chord XY then $OM \perp XY$. However, this means that $\angle PFO + \angle PMO = 180^\circ$ hence $OFPM$ is a cyclic quadrilaterial as opposite angles are supplementary. Using a similar argument, $OGQM$ is also a cyclic quadrilateral.

From this, we can say that

$\angle MFP = \angle POM$ (angles in the same segment for the circle enclosing $OFPM$)

Similarly, $\angle MGQ = \angle QOM$

However, since $\Delta FMA ||| \Delta GMC$ then $\angle MFP = \angle MGQ$ due to corresponding angles being equal.

This implies that $\angle POM = \angle QOM$

Also, $\angle PMO = \angle QMO = 90^\circ$

OM is common

$\therefore \Delta PMO \equiv \Delta QMO$ (AAS)

Thus, $PM = QM$ due to corresponding sides of congruent triangles. In other words, M is the midpoint of PQ.

(b) (i) When $n = 0$

$LHS = I_0$

$$= (-1)^1 \int_0^1 e^x \, dx$$

$$= -\int_0^1 e^x \, dx$$

$$= -[e^x]_0^1$$

$$= 1 - e$$

$$RHS = 0! \left[1 - e\left(\frac{1}{0!}\right)\right]$$

$$= 1 - e$$

$LHS = RHS$ hence the statement is true for $n = 1$.

Assume that statement is true for $n = k$

$$I_k = k!\left[1 - e\sum_{r=0}^{k}\frac{(-1)^r}{r!}\right]$$

Required to prove that the statement is true for $n = k+1$ which is that

$$I_{k+1} = (k+1)!\left[1 - e\sum_{r=0}^{k+1}\frac{(-1)^r}{r!}\right]$$

$$I_{k+1} = (-1)^{k+2}\int_0^t x^{k+1}e^x\,dx$$

$$= (-1)^{k+2}\left\{[x^{k+1}e^x]_0^1 - (k+1)\int_0^1 x^k e^x\,dx\right\} \qquad \text{using integration by parts}$$

$$= (-1)^{k+2}e + (k+1)(-1)^{k+1}\int_0^1 x^k e^x\,dx$$

$$= (-1)^{k+2}e + (k+1)I_k$$

$$= (-1)^{k+2}e + (k+1)k!\left[1 - e\sum_{r=0}^{k}\frac{(-1)^r}{r!}\right] \qquad \text{by assumption}$$

$$= -(k+1)!(-1)^{k+1}e\frac{1}{(k+1)!} + (k+1)!\left[1 - e\sum_{r=0}^{k}\frac{(-1)^r}{r!}\right]$$

$$= (k+1)!\left[1 - e\sum_{r=0}^{k}\frac{(-1)^r}{r!} - e\frac{(-1)^{k+1}}{(k+1)!}\right]$$

$$= (k+1)!\left[1 - e\sum_{r=0}^{k+1}\frac{(-1)^r}{r!}\right]$$

$$= RHS$$

If the statement is true for $n = k$ then it is also true for $n = k+1$. Since the statement is true for $n = 0$ then by induction it is also true for all integers $n \geq 0$ and hence also for all positive integers n.

(ii) Note that for $0 \leq x \leq 1$ we have $x^n e^x \geq 0$ hence $\int_0^1 x^n e^x \, dx \geq 0$.

Also, for $0 \leq x \leq 1$ we have

$$e^x \leq e$$

$$x^n e^x \leq x^n e \quad \text{for } x \geq 0$$

$$\int_0^1 x^n e^x \, dx \leq \int_0^1 x^n e \, dx$$

$$= e \left[\frac{x^n}{n+1} \right]_0^1$$

$$= \frac{e}{n+1}$$

Hence

$$0 \leq \int_0^1 x^n e^x \, dx \leq \frac{e}{n+1}$$

(c) (i) Consider a case where person i draws card j. There are two cases to consider:

1. Person i draws card j and person j draws card i. For a *fixed* or given j then the problem becomes having to sort the remaining $n-2$ people such that nobody receives a matching card. This is because the two people have each other's cards and are effectively 'isolated' from being dependent on the outcomes of the rest of the group. There are $(n-1)$ ways of doing this because j can vary between values 1 to n except for i (note that varying i instead of j is equivalent). Thus, there are $(n-1)D_{n-2}$ ways for this case to occur.

2. Person i draws card j and person j does not draw card i. For a *fixed* or given j then the problem becomes having to sort $n-1$ people such that nobody receives a matching card. There are $(n-1)$ ways of doing this because j can vary between values 1 to n except for i. Thus, there are $(n-1)D_{n-1}$ ways for this case to occur.

Hence, the total number of ways in which nobody in a group of n people receive a matching card is given by

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

(ii) Substituting directly into (i) we get

$$
\begin{aligned}
RHS &= (n-1)(D_{n-1} + D_{n-2}) \\
&= (n-1)\left((n-1)!\sum_{r=0}^{n-1}\frac{(-1)^r}{r!} + (n-2)!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!}\right) \\
&= (n-1)\left((n-1)(n-2)!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!} + (n-1)!\frac{(-1)^{n-1}}{(n-1)!} + (n-2)!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!}\right) \\
&= (n-1)\left((n-2)!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!}(n-1+1) + (-1)^{n-1}\right) \\
&= n(n-1)(n-2)!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!} + (n-1)(-1)^{n-1} \\
&= n!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!} + (n-1)(-1)^{n-1} \\
&= n!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!} + n(-1)^{n-1} + (-1)^n \\
&= n!\sum_{r=0}^{n-2}\frac{(-1)^r}{r!} + n(n-1)!\frac{(-1)^{n-1}}{(n-1)!} + n!\frac{(-1)^n}{n!} \\
&= n!\sum_{r=0}^{n}\frac{(-1)^r}{r!} \\
&= D_n \\
&= LHS
\end{aligned}
$$

(iii) From (b)(ii), when $n \to \infty$ then $\displaystyle\int_0^1 x^n e^x \, dx \to 0$

Since $I_n = (-1)^{n+1} \displaystyle\int_0^1 x^n e^x \, dx$ then $I_n \to 0$

From part (i), $I_n = n! \left[1 - e \displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{r!} \right]$ which can only approach zero if $1 - e \displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{r!} \to 0$

i.e. $\displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{r!} \to \frac{1}{e}$

Back to the problem and the probability of nobody picking their own card is $\dfrac{D_n}{n!}$ since there are $n!$ ways of arranging n cards across n people.

From part (ii), this probability is also $\displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{r!}$ which approaches $\dfrac{1}{e}$ as $n \to \infty$.

2014 Bored Of Studies

Mathematics Extension 2 Trial Examination
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	A
3	B
4	A
5	D
6	D
7	C
8	D
9	B
10	D

Section II

Question 11 (a) (i)

Criteria	Marks
• Correct answer	1

Question 11 (a) (ii)

Criteria	Marks
• Correct solution	1

Question 11 (b)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Uses t formula substitutions, or equivalent merit	1

Question 11 (c) (i)

Criteria	Marks
• Correct sketch	2
• Indicates some important features of the graph	1

Question 11 (c) (ii)

Criteria	Marks
• Correct sketch	2
• Indicates some important features of the graph	1

Question 11 (d)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Identifies that the triple root is $x = -\dfrac{a}{2}$, or equivalent merit	1

Question 11 (e)

Criteria	Marks
• Correct solution	3
• Identifies part of the locus	2
• Identifies relevant circle geometry property, or equivalent merit	1

Question 12 (a)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Attempts to use an appropriate vector or algebraic argument	1

Question 12 (b)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Identifies that the polynomial has roots $\dfrac{\alpha^2}{q}, \dfrac{\beta^2}{q}, \dfrac{\gamma^2}{q}$	1

Question 12 (c)

Criteria	Marks
• Correct solution	4
• Makes substantial progress	3
• Identifies that $F = \dfrac{mv^2}{r}\cos\theta - mg\sin\theta$, or equivalent merit	2
• Resolves forces for the flat OR banked scenario	1

Question 12 (d)

Criteria	Marks
• Correct answer	2
• Makes appropriate use of $\lvert z_k \rvert = 1$, where $k = 1, 2, 3$, OR relevant conjugate properties.	1

Question 12 (e)

Criteria	Marks
• Correct sketch	3
• Indicates most important features of the graph	2
• Indicates one important feature of the graph	1

Question 13 (a) (i)

Criteria	Marks
• Correct solution	1

Question 13 (a) (ii)

Criteria	Marks
• Correct solution	2
• Identifies that the polynomial can be expressed as the product of n linear factors.	1

Question 13 (a) (iii)

Criteria	Marks
• Correct solution	2
• Identifies that at least one non-real root lies satisfies $\lvert z-i\rvert<1$.	1

Question 13 (b) (i)

Criteria	Marks
• Correct solution	4
• Makes substantial progress	3
• Obtains correct velocity-time equations for both particles	2
• Obtains correct velocity-time equations for either particle A OR B	1

Question 13 (b) (ii)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Obtains correct integral expression of x_B	1

Question 13 (b) (iii)

Criteria	Marks
• Correct solution	3
• Shows that $T = \dfrac{v_A - v_B + u}{g}$, or equivalent merit	2
• Obtains correct expression for the displacement of particle A	1

Question 14 (a)

Criteria	Marks
• Correct answer	2
• Obtains a correct integral	1

Question 14 (b) (i)

Criteria	Marks
• Correct solution.	2
• Identifies that $k = e^{-x^2}$ OR evaluates the area in terms of k.	1

Question 14 (b) (ii)

Criteria	Marks
• Correct solution.	2
• Obtains the integral $\int_{-R}^{R} e^{-x^2} I \, dx$	1

Question 14 (b) (iii)

Criteria	Marks
• Correct answer	1

Question 14 (b) (iv)

Criteria	Marks
• Correct explanation	2
• Explains part of the inequality	1

Question 14 (b) (v)

Criteria	Marks
• Correct deduction	1

Question 14 (c) (i)

Criteria	Marks
• Correct solution	2
• Attempts to make use of a geometric series, or equivalent merit	1

Question 14 (c) (ii)

Criteria	Marks
• Correct answer AND explanation	1

Question 14 (c) (iii)

Criteria	Marks
• Correct solution	2
• Correctly links the sum to part (i)	1

Question 15 (a) (i)

Criteria	Marks
• Correct solution	4
• Makes substantial progress	3
• Correctly splits the fraction, or equivalent merit	2
• Attempts to split the fraction, or equivalent merit	1

Question 15 (b) (i)

Criteria	Marks
• Correct solution	2
• Attempts to use the focus directrix property, or equivalent merit	1

Question 15 (b) (ii)

Criteria	Marks
• Correct solution	1

Question 15 (b) (iii)

Criteria	Marks
• Correct solution	1

Question 15 (b) (iv)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Attempts to use simultaneous equations with parts (ii) and (iii)	1

Question 15 (c) (i)

Criteria	Marks
• Correct solution	2
• Identifies similar triangle ratios	1

Question 15 (c) (ii)

Criteria	Marks
• Correct solution	2
• Identifies that $\cos\gamma = \dfrac{b}{a}$	1

Question 16 (a) (i)

Criteria	Marks
• Correct proof	3
• Makes substantial progress	2
• Identifies that $\Delta AMD \,\|\|\, \Delta CMB$ OR $AF = \frac{1}{2}AD$	1

Question 16 (a) (ii)

Criteria	Marks
• Correct proof	3
• Makes substantial progress	2
• Shows that either $OFPM$ & $OGQM$ are cyclic, OR $\angle POM = \angle QOM$	1

Question 16 (b) (i)

Criteria	Marks
• Correct proof	3
• Correctly substitutes the assumption in the inductive step	2
• Checks that equality holds for $n = 1$	1

Question 16 (b) (ii)

Criteria	Marks
• Correct proof	1

Question 16 (c) (i)

Criteria	Marks
• Correct explanation	1

Question 16 (c) (ii)

Criteria	Marks
• Correct solution	2
• Substitutes D_n into the expression from (i) AND substantially simplifies expression	1

Question 16 (c) (iii)

Criteria	Marks
• Correct proof	2
• Uses part (b) to prove that $I_n \to 0$ as $n \to \infty$, OR identifies that the probability is $\frac{D_n}{n!}$.	1