2017 Bored of Studies Trial Examinations

Mathematics Extension 2

9th October 2017

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 1 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 6 – 21

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box on the Multiple Choice answer sheet provided.

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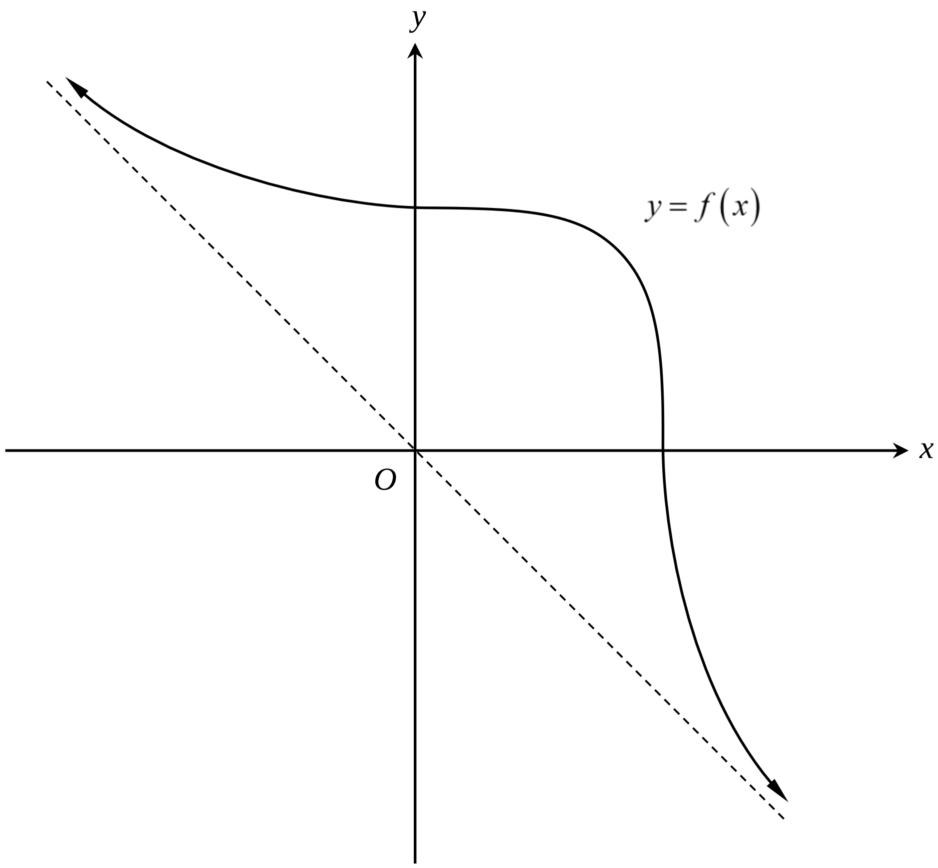
1 The diagram below shows a sketch of the function $y = f(x)$.



Which of the following is the most suitable expression for $\dfrac{dy}{dx}$?

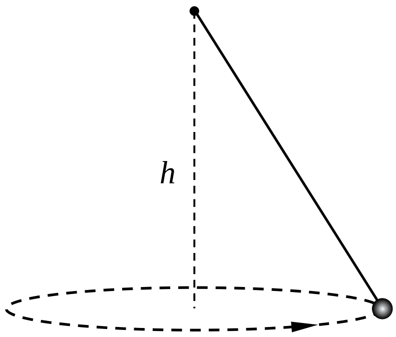
(A) $\dfrac{x}{y}$.

(B) $-\dfrac{x}{y}$.

(C) $\dfrac{x^2}{y^2}$.

(D) $-\dfrac{x^2}{y^2}$.

2 An object of mass m is attached to a fixed point by a taut inelastic string. It rotates in a circle with constant angular velocity ω, tracing out a cone with the string. Let h be the vertical distance between the particle and the fixed point, as shown in the diagram below.



When the angular velocity is doubled, the value of h becomes $k \times h$ for some constant k.

Which of the following is the value of k?

(A) $\frac{1}{4}$.

(B) $\frac{1}{2}$.

(C) 2.

(D) 4.

3 For what values of k does the locus of $|z+1|+|z-1|=k$ describe an ellipse?

(A) $k<2$.

(B) $k \leq 2$.

(C) $k>2$.

(D) $k \geq 2$.

4 What is the value of the obtuse angle between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where *a* and *b* are positive values and $a>b$?

(A) $\tan^{-1}\left(\frac{b}{a}\right)$.

(B) $\tan^{-1}\left(\frac{a}{b}\right)$.

(C) $2\tan^{-1}\left(\frac{b}{a}\right)$.

(D) $2\tan^{-1}\left(\frac{a}{b}\right)$.

5 Let $P(z)$ be a polynomial with real coefficients such that $P(\alpha)=\beta$, where α and β are *non-real* numbers.

Which of the following is always true?

(A) When $P(z)$ is divided by $(z+\alpha)$, the remainder is $-\beta$.

(B) When $P(z)$ is divided by $(z+\bar{\alpha})$, the remainder is $-\bar{\beta}$.

(C) When $P(z)$ is divided by $(z-\bar{\alpha})$, the remainder is $\bar{\beta}$.

(D) The factor $(z-\alpha)$ is the only one that will yield a remainder of β.

6 The integral $\int_0^1 \frac{3x^2+3x+2}{\left(x^2+1\right)\left(x+1\right)}\,dx$ can be expressed in the form

$$\ln 2+\int_0^1 \frac{f\left(x\right)}{\left(x^2+1\right)\left(x+1\right)}\,dx .$$

Which of the following is a possible expression for $f\left(x\right)$?

(A) x^2+x+2

(B) $3x^2-1$

(C) $3x^2+x$

(D) $4x^2-3x+4$

7 Let $w=\frac{az+bz^{-1}}{bz+az^{-1}}$, where $\left|z\right|=1$. What is the value of $\left|w\right|$?

(A) 1.

(B) $\frac{a}{b}$.

(C) $\frac{b}{a}$.

(D) It depends on $\arg\left(z\right)$.

8 Suppose $\left(\sqrt{3}-i\right)^n$ is a real number. Which of the following is not always an integer?

(A) n.

(B) $n/2$.

(C) $n/3$.

(D) $n/4$.

9 The graph of $y=\dfrac{x^3+2}{x}$ has a stationary point at $(1,3)$. Let $P(x)=x^3-kx+2$.

For what values of k will $P(x)$ have 3 distinct real roots?

(A) $k<0$.

(B) $0<k<1$.

(C) $1<k<3$.

(D) $k>3$.

10 Let z be a variable complex number with unit modulus, other than $\pm i$ and ± 1.

Which of the following expressions has the locus equation $x=0$?

(A) $\dfrac{z+1}{z-1}$.

(B) $\dfrac{z-i}{z+1}$.

(C) $\dfrac{z+i}{z-1}$.

(D) $\dfrac{z-i}{z-1}$.

Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find **2**

$$\int \frac{dx}{\sin x\sqrt{\sin 2x}}.$$

(b) Consider the polynomial $P(z) = z^{n+1} - 1$. Let ω be any root of $P(z)$.

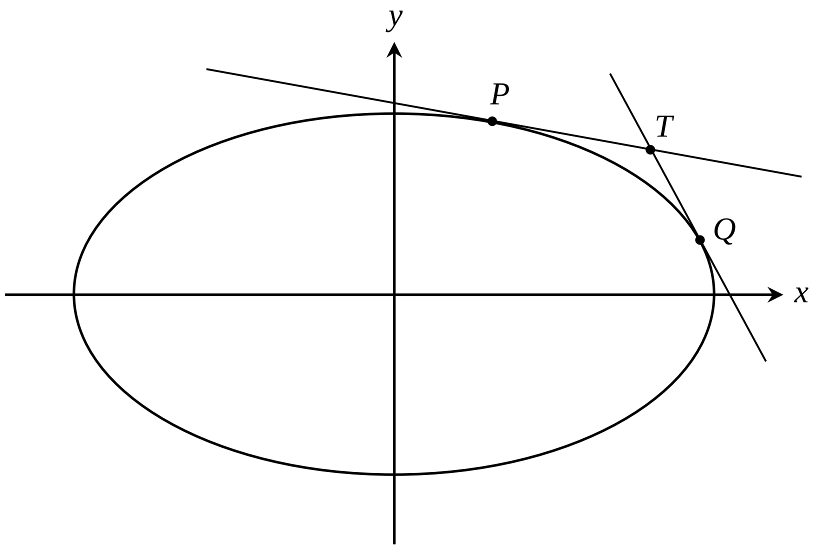
(i) Show that any integer power of ω is a root of the polynomial. **1**

(ii) By considering $P\left(1-\dfrac{1}{z}\right)$, or otherwise, show that **2**

$$\frac{1}{1-\omega} + \frac{1}{1-\omega^2} + \frac{1}{1-\omega^3} + \cdots + \frac{1}{1-\omega^n} = \frac{n}{2}.$$

Question 11 continues on page 7

(c) The diagram below shows two points $P(a\cos\theta, b\sin\theta)$ and $Q(a\sin\theta, b\cos\theta)$ on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. From P and Q, tangents are drawn to intersect at T.



The equation of the tangent drawn from P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \quad \textbf{(Do NOT prove this)}$$

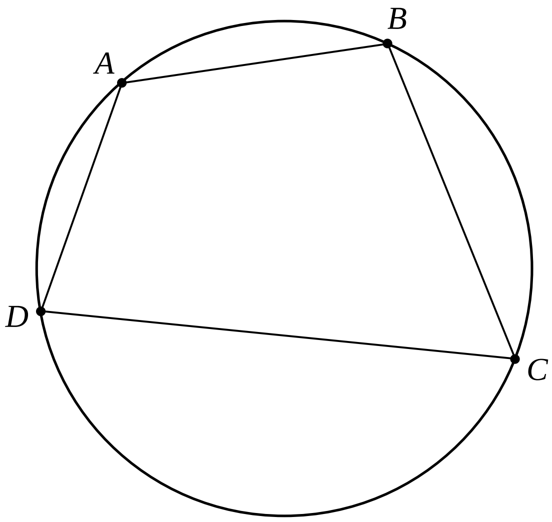
(i) Show that T has coordinates **2**

$$\left(\frac{a}{\sin\theta+\cos\theta}, \frac{b}{\sin\theta+\cos\theta}\right).$$

(ii) Hence, find the locus of T, stating restrictions. **2**

Question 11 continues on page 8

(d) Let A, B, C and D be concyclic points on the Argand diagram, representing complex numbers a, b, c and d respectively as shown in the diagram below.



(i) Show that **2**

$$|(a-b)(c-d)|+|(a-d)(b-c)| \geq |(a-c)(b-d)|.$$

(ii) Show that $\dfrac{(a-b)(c-d)}{(a-d)(b-c)}$ is real. **2**

(iii) Let A, B, C and D be points on the Argand diagram representing complex numbers a, b, c and d respectively. **2**

Show that $AB \times CD + BC \times AD = AC \times BD$.

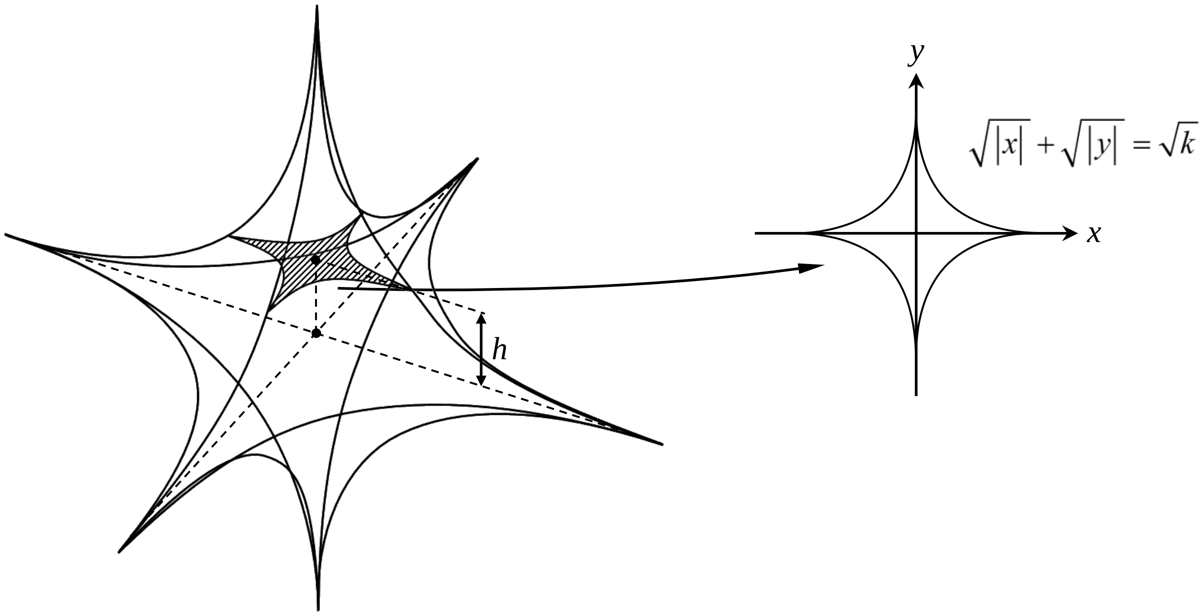
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A solid has twelve curved edges, each following the shape of the curve $\sqrt{x}+\sqrt{y}=1$.

Cross sections taken perpendicular to the vertical axis of symmetry satisfy the relation

$$\sqrt{|x|}+\sqrt{|y|}=\sqrt{k}\ .$$



(i) Show that $\sqrt{k}=1-\sqrt{h}$. **1**

(ii) Hence, find the volume of the solid. **2**

(b) (i) Sketch the graph of $y=\dfrac{x}{x^2-1}$. **2**

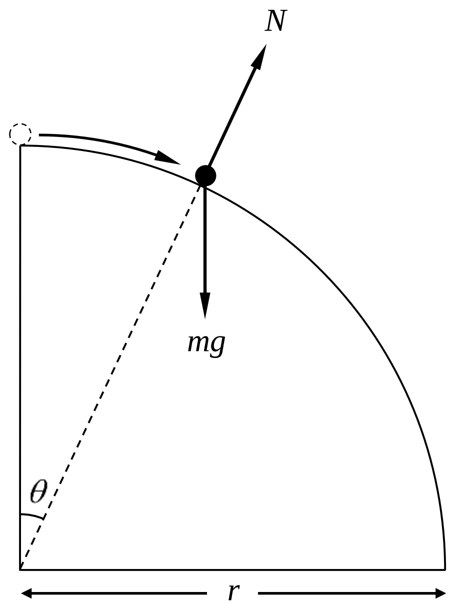
(ii) Hence, or otherwise, sketch the graph of $y=\dfrac{e^x}{e^{2x}-1}$. **2**

Question 12 continues on page 10

Question 12 (continued)

(c) A marble of mass m is initially at rest on top of a smooth circular path with radius r. It is given a push with a negligible amount of force to initiate the movement along the circular path.

Let N be the normal force of the circular path acting on the marble, and g be the magnitude of acceleration due to gravity. Let the angular displacement of the particle be θ, as shown in the diagram below.



(i) By resolving the forces in the tangential and normal direction of the marble, or otherwise, show that **2**

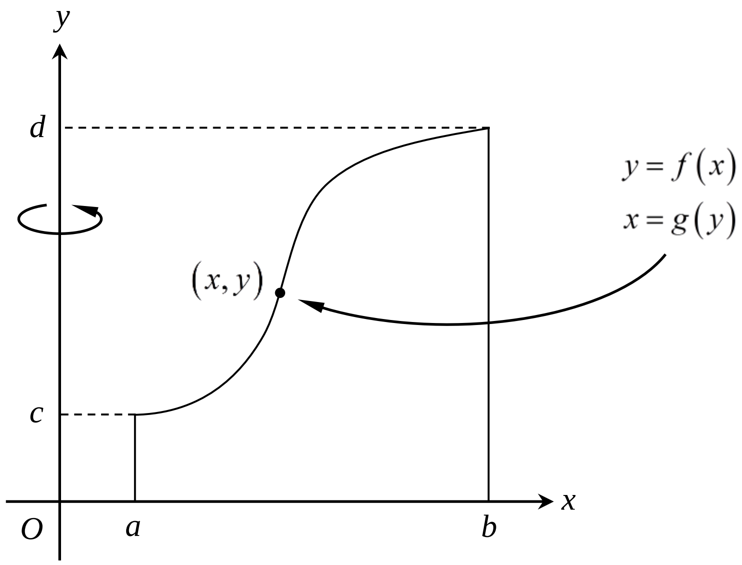
$$N = mg\left(3\cos\theta - 2\right).$$

(ii) When $\theta = \alpha$, the marble leaves the circular surface. **1**

Find the exact value of α.

Question 12 continues on page 11

(d) The diagram below shows the graph of $f(x)$, which is increasing in the domain $a \le x \le b$. Let the inverse function be $x = g(y)$.



The region bound by $y = f(x)$ and the *x* axis in the domain $a \le x \le b$ is rotated about the *y* axis, forming a solid of revolution.

(i) Use the slicing method to show that the volume of the solid is **2**

$$V = \pi\left(b^2 d - a^2 c\right) - \pi \int_c^d g^2(y)\, dy.$$

(ii) Use the substitution $y = f(x)$ and integration by parts to show that the expression in part (i) is equivalent to **2**

$$2\pi \int_a^b x\, f(x)\, dx.$$

(iii) Explain the significance of the result in part (ii). **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Harin enters a casino with \$$k$, where k is an integer, and goes to the roulette wheel. If the ball lands on red, which has probability $0 < p < 1/2$, then he wins \$1, otherwise he loses \$1. Harin has no self-control, so he continues until he leaves either disappointed or satisfied. If he loses all \$$k$, he leaves the casino disappointed, but if he obtains a total of \$$K$, including the money he came in with where $K \geq k$, then he leaves satisfied.

Let P_k be the probability that he eventually leaves disappointed after starting with \$$k$.

(i) Explain why $P_k = pP_{k+1} + (1-p)P_{k-1}$. **1**

(ii) Let $\alpha = \dfrac{1-p}{p}$ where $\alpha \neq 1$. **1**

Show that $P_{k+1} - P_k = \alpha\left(P_k - P_{k-1}\right)$.

(iii) Hence show that $P_{k+1} - P_k = \alpha^k\left(P_1 - 1\right)$. **1**

(iv) Deduce that $P_k = 1 + \left(P_1 - 1\right)\left(\dfrac{\alpha^k - 1}{\alpha - 1}\right)$. **1**

(v) By considering a similar expression for P_K and then eliminating P_1, show that **2**

$$P_k = \frac{\alpha^K - \alpha^k}{\alpha^K - 1}.$$

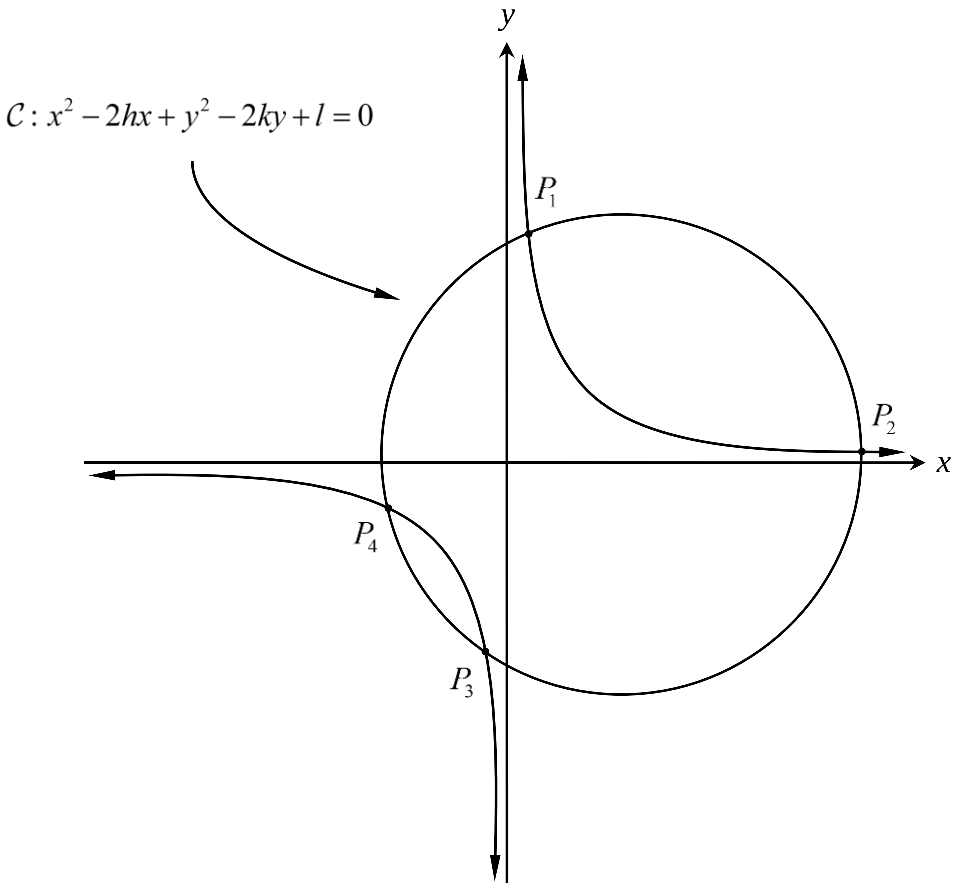
(vi) Find the probability of leaving satisfied as $K \to \infty$. **2**

Explain the significance of your result.

Question 13 continues on page 13

Question 13 (continued)

(b) The diagram below shows a circle $\mathcal{C}: x^2 - 2hx + y^2 - 2ky + l = 0$ intersecting the hyperbola $xy = c^2$ at four points $P_i\left(cp_i, \dfrac{c}{p_i}\right)$, where $i = 1, 2, 3, 4$. **3**



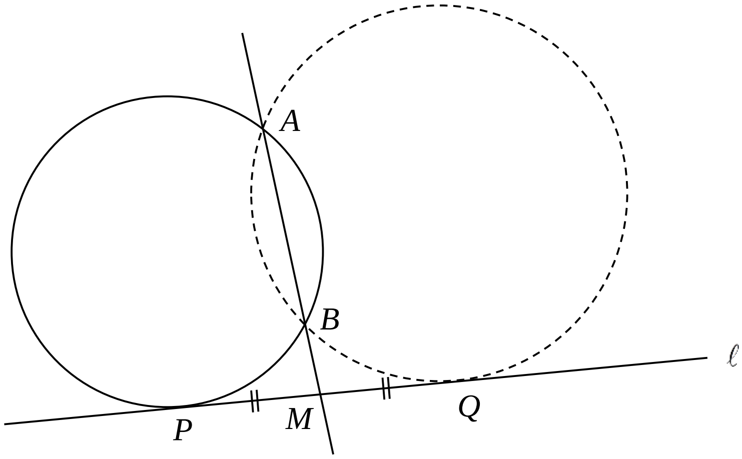
The *mean point* of *n* points (x_i, y_i), where $i = 1, 2, 3, \ldots, n$, is defined to be

$$M\left(\frac{1}{n}\sum_{i=1}^{n} x_i,\ \frac{1}{n}\sum_{i=1}^{n} y_i\right).$$

Show that the origin, the centre of $\mathcal{C}$ and the mean point of P_1, P_2, P_3 and P_4 are collinear.

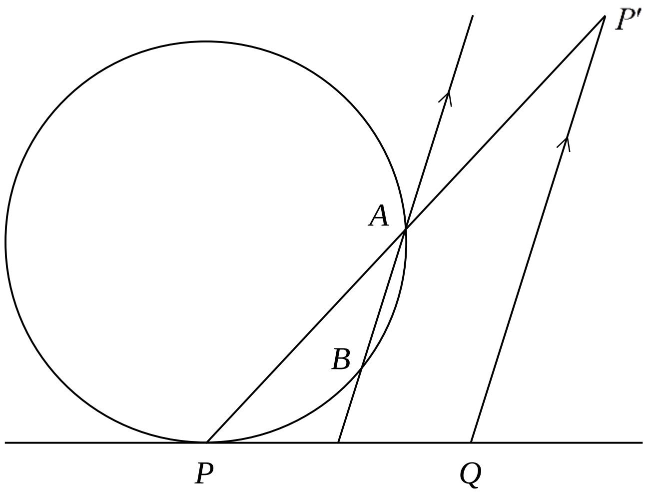
Question 13 continues on page 14

(c) Let M be intersection of a tangent ℓ and secant MBA, as shown in the diagram below. Let Q be the point on ℓ such that M is the midpoint of PQ. **2**



Show that the circle passing through A, B and Q is tangential to ℓ.

(d) Let PQ be a tangent to a circle at P. Let A be any point on the circle and P' be on PA produced so that $AP = AP'$, as shown in the diagram below. Let B be the point on the circle such that $AB \parallel P'Q$. **2**



Use part (c) to show that $\angle BQP = \angle AQP'$.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Two players A and B compete against each other in a coin-tossing game. If the coin lands on heads, then A scores a point and if the coin lands on tails, then B scores a point. The first person to have a lead of two points wins the game. If nobody has a lead of two points after the $2n^{\text{th}}$ toss is made, then the game is tied. **2**

How many possible ways can the games be played?

(b) Let a_k and b_k be real for $k = 1, 2, \ldots, n$. **3**

Use mathematical induction to prove that

$$\left(a_1b_1 + a_2b_2 + \cdots + a_nb_n\right)^2 \le \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

(c) Consider the functions $f\left(x\right) = x - \dfrac{a^2}{x}$ and $g\left(x\right) = 2a\ln\left(\dfrac{x}{a}\right)$.

(i) Show that $f'\left(x\right) \ge g'\left(x\right)$ when $x \ge a$. **1**

(ii) Hence, explain why $f\left(x\right) \ge g\left(x\right)$ when $x \ge a$. **1**

(iii) Sketch the graph of $y = f\left(x\right) - g\left(x\right)$. **1**

Question 14 continues on page 16

(d) An object of unit mass is projected vertically from the origin with initial speed u. It experiences resistance force of magnitude kv, where k is a positive constant and v is the speed of the object. Let $x(t)$ be the height of the object from the ground at time t seconds, and let g be acceleration due to gravity. After attaining a maximum height, the object falls towards the ground and experiences a resistance force of the same magnitude as the upwards journey.

(i) Show that **3**

$$x(t) = \frac{1}{k^2}(g+ku)\left(1-e^{-kt}\right) - \frac{gt}{k}.$$

(ii) Find the time T taken for the object to reach its maximum height. **1**

(iii) Show that when $t = 2T$ **2**

$$x(2T) = \frac{1}{k^2}\left[g + ku - \frac{g^2}{g+ku} - 2g\ln\left(\frac{g+ku}{g}\right)\right].$$

(iv) Hence, using the result in part (c), or otherwise, explain why the object takes longer on the downwards journey than the upwards journey. **1**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $\sin x \le x \le \cos x$ for $0 \le x \le \dfrac{\pi}{6}$. **3**

(b) Suppose that

$$I_n = \int_0^{\frac{\pi}{6}} \frac{\sin^{2n} x}{\cos x}\, dx,$$

for some integer $n \ge 0$.

(i) Show that **3**

$$I_n = \frac{\ln 3}{2} - \sum_{k=1}^{n} \frac{1}{(2k-1)\,2^{2k-1}}.$$

(ii) Using the result in part (a), or otherwise, show that **2**

$$0 \le I_n \le \frac{1}{2n}\left(\frac{\pi}{6}\right)^{2n}.$$

(iii) Hence, find the limiting value of **1**

$$1 + \frac{1}{3\times 2^2} + \frac{1}{5\times 2^4} + \frac{1}{7\times 2^6} + \cdots$$

Question 15 continues on page 18

(c) Suppose the polynomial $P(z) = z^3 + \beta z^2 + k$ has two non-real roots α and $\bar{\alpha}$, where $\alpha = x + iy$, and a real root β.

(i) Show that $\beta = -x$. **1**

(ii) Show that $|\alpha|^2 = 2\beta^2$. **2**

(iii) Show that α and $\bar{\alpha}$ lie on the lines $y = \pm x$ in the complex plane. **1**

(iv) The roots of $P(z)$ form a triangle in the complex plane. **2**

Show that the area of the triangle is $|\alpha|^2$.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

A *prime number* is a positive integer divisible only by 1 and itself. Let p be a prime number that is larger than another fixed positive integer n.

Let $Q(x)$ be a polynomial of degree r, with equation

$$Q(x)=\frac{x^{p-1}(1-x)^p(2-x)^p(3-x)^p\ldots(n-x)^p}{(p-1)!}.$$

Define

$$T(x)=Q(x)+Q^{(1)}(x)+Q^{(2)}(x)+\ldots+Q^{(r)}(x),$$

where $Q^{(i)}(x)=\frac{d^i}{dx^i}\left[Q(x)\right]$, that is the i-th derivative of $Q(x)$ for $i=1, 2, \ldots, r$.

(a) Since $Q(x)$ is a polynomial of degree r with lowest-degree term x^{p-1}, it may be expressed as **1**

$$Q(x)=a_r x^r+a_{r-1}x^{r-1}+\ldots+a_{p-1}x^{p-1},$$

for some integers $a_{p-1}, a_p, \ldots, a_r$.

Explain why for any integer $i\geq p$ that

$$Q^{(i)}(x)=pR_i(x),$$

where $R_i(x)$ is some polynomial with integer coefficients.

(b) Show that $Q^{(p-1)}(0)=(n!)^p$. **2**

(c) Hence, explain why $\frac{Q^{(p-1)}(0)}{p}$ is not an integer. **1**

Question 16 continues on page 20

Question 16 (continued)

(d) Explain why for any integer $1 \le m \le n$, **1**

$$T(m) = Q^{(p)}(m) + Q^{(p+1)}(m) + Q^{(p+2)}(m) + \ldots + Q^{(r)}(m),$$

and

$$T(0) = Q^{(p-1)}(0) + Q^{(p)}(0) + Q^{(p+1)}(0) + \ldots + Q^{(r)}(0).$$

(e) Let $c_0,\ c_1, \ldots,\ c_n$ be any set of integers such that $0 < |c_0| < p$. **2**

Explain why

$$\frac{c_0 T(0) + c_1 T(1) + c_2 T(2) + \ldots + c_n T(n)}{p},$$

is not an integer.

(f) Consider the function $f(x) = e^{-x} T(x)$. **1**

Show that $f'(x) = -e^{-x} Q(x)$.

(g) For any real values $a < b$, there exists some $0 < \alpha < 1$ such that **1**

$$f'(a + \alpha(b-a)) = \frac{f(b) - f(a)}{b-a} \quad \textbf{(Do NOT prove this)}$$

Use this result to show that for any, $1 \le k \le n$, there exists $0 < \alpha < 1$ such that

$$T(k) - e^k T(0) = -k e^{k(1-\alpha)} Q(k\alpha).$$

Question 16 continues on page 21

Question 16 (continued)

(h) Define $\beta_k = -ke^{k(1-\alpha)}Q(k\alpha)$. Show that $|\beta_k| \le \dfrac{e^n n^p (n!)^p}{(p-1)!}$. **2**

(i) Let $P(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n$ be a polynomial with integer coefficients such that $0 < |c_0| < p$. **2**

Suppose that $P(e) = 0$, show that

$$c_0 T(0) + c_1 T(1) + c_2 T(2) + ... + c_n T(n) = c_1\beta_1 + c_2\beta_2 + c_3\beta_3 + ... + c_n\beta_n.$$

(j) It can be shown that if *y* is fixed, then $\dfrac{y^p}{p!} \to 0$ as *p* gets large. **1**

Explain why for each *k*, there exists some *p* such that $|c_k\beta_k| < \dfrac{1}{n}$.

(k) Hence, prove that *e* cannot be a root of any polynomial with integer coefficients. **1**

End of Exam

2017 Bored of Studies Trial Examinations

Mathematics Extension 2

Solutions

Multiple Choice

1. D
2. A
3. C
4. D
5. C
6. A
7. A
8. D
9. D
10. A

Brief Explanations

Question 1 The graph is non-increasing, so the derivative must be non-negative throughout the domain. Hence, the answer is (D).

Question 2 Let θ be the angle between the string and the vertical. From the force components along the string and gravity, we have:

$$T\cos\theta = mg$$
$$T\sin\theta = mr\omega^2$$
$$\tan\theta = \frac{r}{h}$$

Dividing the equations and equating the cotangents yields $h = \dfrac{g}{\omega^2}$.

If ω is doubled, the new value of h is $\dfrac{1}{4}$ of the original value, hence the answer is (A).

Question 3 From the triangle inequality $|z+1|+|z-1| = |z+1|+|1-z| \geq 2$. This means that k must be at least 2 for the locus to exist. However, for $k=2$, the locus is a line segment with endpoints $(-1,0)$ and $(1,0)$, which is not an ellipse. For $k>2$, the locus has the form $PS+PS'=2a$ where S and S' are the points $(1,0)$ and $(-1,0)$ respectively, which is a sufficient condition to describe an ellipse. Hence, the correct answer is (C).

Question 4 The equations of the asymptotes are $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$. The acute angle between the two lines is twice the angle between $y=\frac{b}{a}x$ and the positive x-axis. Hence, the obtuse angle must be $\pi-2\tan^{-1}\left(\frac{b}{a}\right)$. But recall that $\tan^{-1}\left(\frac{b}{a}\right)=\frac{\pi}{2}-\tan^{-1}\left(\frac{a}{b}\right)$, so the answer is $2\tan^{-1}\left(\frac{a}{b}\right)$, which is (D).

Question 5 The correct answer is (C). An exhaustive solution follows below.
For the counterexamples, consider $P(z)=z^2$ so $P(\alpha)=\alpha^2=\beta$.

(A) is not always true because $P(-\alpha)=\alpha^2=\beta\neq-\beta$

(B) is not always true because $P(-\bar{\alpha})=\bar{\alpha}^2=\bar{\beta}\neq-\bar{\beta}$

(D) is not always true as per counterexample for (A) and the Remainder Theorem.

For (C), $P(\bar{z})=\overline{P(z)}$ if $P(z)$ has real coefficients. Using this result in conjunction with the Remainder Theorem, the claim given is true.

Question 6 Note that the equivalent goal is to find an appropriate $f(x)$ such that

$$\int_0^1 \frac{3x^2+3x+2-f(x)}{(x+1)(x^2+1)}\,dx=\ln 2$$

For (B), the integrand becomes $\frac{3}{x^2+1}$ which evaluates to $\frac{3\pi}{4}$

For (C), the integrand becomes $\frac{2}{x^2+1}$ which evaluates to $\frac{\pi}{2}$

For (D), the integrand becomes $\frac{-x^2+6x-2}{(x+1)(x^2+1)}$, we need to reduce it to the form

$$\frac{-x^2+6x-2}{(x+1)(x^2+1)}=\frac{a}{x+1}+\frac{bx}{x^2+1}+\frac{c}{x^2+1}$$

The integral of $\frac{a}{x+1}+\frac{bx}{x^2+1}$ evaluates to logarithmic terms but integral of $\frac{c}{x^2+1}$ must evaluate to $\frac{\pi c}{4}$ (a quick substitution allows us to verify c is not zero). A result with $\frac{\pi c}{4}$ in it cannot simplify with other rational terms to produce $\ln 2$.

For (A), the integrand becomes $\frac{2x}{x^2+1}$ which evaluates to $\ln 2$ and is thus the correct answer.

Alternatively, by splitting the numerator as $2\left(x^2+x\right)+x^2+x+2$ and cancelling out $(x+1)$ from the first term, the integral is equivalent to: $\int_0^1 \frac{2x}{x^2+1}\,dx+\int_0^1 \frac{x^2+x+2}{(x+1)\left(x^2+1\right)}\,dx$

The first integral evaluates to $\ln 2$, hence the correct answer is (A).

Question 7 Note that a and b are real, thus unaffected by conjugation. Noting that $|z|=1$ and hence $\bar{z}=z^{-1}$ then

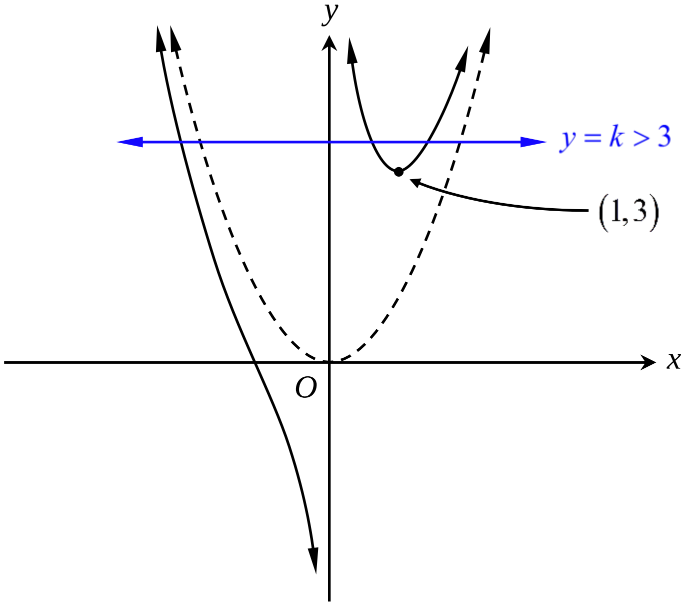
$$\begin{aligned} w &= \frac{az+bz^{-1}}{bz+az^{-1}} \\ &= \frac{az+b\bar{z}}{a\bar{z}+bz} \end{aligned}$$

Notice that the numerator and denominator are conjugates of each other. Since a complex number and the conjugate have equal modulus, then $|w|=1$ so the correct answer is (A).

Question 8 $\left(\sqrt{3}-i\right)^n=2^n \operatorname{cis}\left(\frac{n\pi}{6}\right)$.

For the expression to be an integer, n must be a multiple of 6, which is not necessarily divisible by 4, hence answer is (D).

Question 9 Using standard techniques of sketching rational functions, or otherwise, the graph of $y = \dfrac{x^3+2}{x}$ can be sketched as shown:



For the polynomial $P(x) = x^3 - kx + 2$ to have three distinct roots, the graphs of $y = \dfrac{x^3+2}{x}$ and $y = k$ must have three points of intersection. From the diagram above, this occurs when $k > 3$, hence the answer is (D).

Question 10 Consider the argument of (A), and note that z lies on the unit circle.

$$\begin{aligned}\arg\left(\frac{z+1}{z-1}\right) &= \arg(z+1) - \arg(z-1) \\ &= \text{Angle between vectors connecting } \pm 1 \text{ to } z \\ &= \pm\frac{\pi}{2} \quad \ldots(\text{angle in a semi-circle})\end{aligned}$$

This means that $\dfrac{z+1}{z-1}$ is a purely imaginary number and therefore has locus $x = 0$.

The other options will not yield a semi-circle, so the angle can not be $\pm\dfrac{\pi}{2}$, so they can not work.

Written Response

Question 11 (a)

$$\begin{aligned}\int \frac{dx}{\sin x\sqrt{\sin 2x}} &= \int \frac{dx}{\sin x\sqrt{2\sin x\cos x}} \\ &= \int \frac{dx}{\sin^2 x\sqrt{2\cot x}} \\ &= \frac{1}{\sqrt{2}}\int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot x}}\,dx \\ &= -\sqrt{2\cot x}+C\end{aligned}$$

Question 11 (b) (i)

From the definition of ω, we have $\omega^{n+1}=1$. Consider ω^k for some integer *k*.

$$\begin{aligned}P\left(\omega^k\right) &= \left(\omega^k\right)^{n+1}-1 \\ &= \left(\omega^{n+1}\right)^k-1 \\ &= 1^k-1 \\ &= 0\end{aligned}$$

Hence, any integer power of ω is a root of $P\left(z\right)$.

Question 11 (b) (ii)

Let $x=\dfrac{1}{1-z}$ (noting that $z\neq 1$), which implies $z=1-\dfrac{1}{x}$.

Forming a new polynomial equation $P\left(1-\dfrac{1}{x}\right)=0$

$$\begin{aligned}\left(1-\frac{1}{x}\right)^{n+1}-1&=0 \\ \frac{\left(x-1\right)^{n+1}-x^{n+1}}{x^{n+1}}&=0 \\ \left(x-1\right)^{n+1}-x^{n+1}&=0 \\ -\binom{n+1}{1}x^n+\binom{n+1}{2}x^{n-1}-\cdots+\left(-1\right)^{n+1}&=0\end{aligned}$$

Since the original polynomial has roots $\omega^1, \omega^2, \omega^3, \ldots, \omega^n$ then the new polynomial has roots $\frac{1}{1-\omega}, \frac{1}{1-\omega^2}, \frac{1}{1-\omega^3}, \ldots, \frac{1}{1-\omega^n}$.

Taking the sum of the roots of this new polynomial

$$\begin{aligned}\frac{1}{1-\omega}+\frac{1}{1-\omega^2}+\frac{1}{1-\omega^3}+\ldots+\frac{1}{1-\omega^n} &= \frac{-\binom{n+1}{2}}{-\binom{n+1}{1}} \\ &= \frac{\frac{n(n+1)}{2}}{n+1} \\ &= \frac{n}{2}\end{aligned}$$

Alternative method: Pairwise Summation

Observe that:

$$\begin{aligned}\frac{1}{1-\omega^k}+\frac{1}{1-\omega^{n-k+1}} &= \frac{1}{1-\omega^k}+\frac{1}{1-\omega^{-k}} \\ &= \frac{1}{1-\omega^k}+\frac{\omega^k}{\omega^k-1} \\ &= \frac{1-\omega^k}{1-\omega^k} \\ &= 1\end{aligned}$$

Now sum this over $k=1,2,3,\ldots,n$ to get

$$\left(\frac{1}{1-\omega}+\frac{1}{1-\omega^n}\right)+\left(\frac{1}{1-\omega^2}+\frac{1}{1-\omega^{n-1}}\right)+\cdots+\left(\frac{1}{1-\omega^{n-1}}+\frac{1}{1-\omega^2}\right)+\left(\frac{1}{1-\omega^n}+\frac{1}{1-\omega}\right)$$

$$\begin{aligned}&= \underbrace{1+1+\ldots+1+1}_{n \text{ terms}} \\ &= n\end{aligned}$$

Hence, we have

$$\frac{1}{1-\omega}+\frac{1}{1-\omega^2}+\ldots+\frac{1}{1-\omega^{n-1}}+\frac{1}{1-\omega^n}=\frac{n}{2}.$$

Question 11 (c) (i)

It suffices to show that T satisfies the equations of the tangents at P and Q so it must be a point of intersection. For the tangent at P:

$$\begin{aligned}
LHS &= \frac{x\sin\theta}{a}+\frac{y\cos\theta}{b}\\
&= \frac{\sin\theta}{a}\left(\frac{a}{\sin\theta+\cos\theta}\right)+\frac{\cos\theta}{b}\left(\frac{b}{\sin\theta+\cos\theta}\right)\\
&= \frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\\
&= 1\\
&= RHS
\end{aligned}$$

The tangent from Q will have a near-identical equation to the tangent from P, with θ replaced by $\frac{\pi}{2}-\theta$. We need to show that T also satisfies the equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

$$\begin{aligned}
LHS &= \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}\\
&= \frac{\cos\theta}{a}\left(\frac{a}{\sin\theta+\cos\theta}\right)+\frac{\sin\theta}{b}\left(\frac{b}{\sin\theta+\cos\theta}\right)\\
&= \frac{\cos\theta+\sin\theta}{\sin\theta+\cos\theta}\\
&= 1\\
&= RHS
\end{aligned}$$

Hence, the intersection of the tangents T has the coordinates $\left(\frac{a}{\sin\theta+\cos\theta},\frac{b}{\sin\theta+\cos\theta}\right)$

Question 11 (c) (ii)

From (i), we have $x=\frac{a}{\sin\theta+\cos\theta}$ and $y=\frac{b}{\sin\theta+\cos\theta}$, so clearly $y=\frac{b}{a}x$.

But note that $\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$, which means that $-\sqrt{2}\le\sin\theta+\cos\theta\le\sqrt{2}$ and therefore $\frac{1}{|\sin\theta+\cos\theta|}\ge\frac{1}{\sqrt{2}}$. From this, we deduce that $|x|\ge\frac{a}{\sqrt{2}}$ and $|y|\ge\frac{b}{\sqrt{2}}$.

Question 11 (d) (i)

$$\begin{aligned}\left|(a-b)(c-d)\right|+\left|(a-d)(b-c)\right| &= \left|ac-ad-bc+bd\right|+\left|ab-ac-bd+cd\right| \\ &\geq \left|ac-ad-bc+bd+ab-ac-bd+cd\right| \qquad \ldots(\text{Triangle Inequality}) \\ &= \left|ab-bc-ad+cd\right| \\ &= \left|(a-c)(b-d)\right|\end{aligned}$$

Question 11 (d) (ii)

By rotation of vectors, $\dfrac{b-a}{\left|\overrightarrow{AB}\right|}=\dfrac{d-a}{\left|\overrightarrow{AD}\right|}\operatorname{cis}(\angle BAD)$ and $\dfrac{d-c}{\left|\overrightarrow{CD}\right|}=\dfrac{b-c}{\left|\overrightarrow{CB}\right|}\operatorname{cis}(\angle DCB)$.

Multiplying the two equations together

$$\frac{(b-a)(d-c)}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}=\frac{(d-a)(b-c)}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{CB}\right|}\operatorname{cis}(\angle BAD)\times\operatorname{cis}(\angle DCB)$$

$$\frac{(b-a)(d-c)}{(d-a)(b-c)}=\frac{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{CB}\right|}\operatorname{cis}(\angle BAD+\angle DCB)$$

$$-\frac{(a-b)(c-d)}{(a-d)(b-c)}=\frac{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{CB}\right|}\operatorname{cis}(\pi) \quad \ldots(\text{opposite angles in a cyclic quadrilateral are supplementary})$$

$$\frac{(a-b)(c-d)}{(a-d)(b-c)}=\frac{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{CB}\right|}$$

Hence $\dfrac{(a-b)(c-d)}{(a-d)(b-c)}$ is a real positive number.

Question 11 (d) (iii)

The triangle inequality states that $|z|+|w|\ge|z+w|$. When $\dfrac{z}{w}$ is positive and real, equality is attained because $\arg z-\arg w=\pm\pi$ which geometrically represents a line segment (i.e. the 'triangle' formed by the complex numbers *z*, *w* and the origin degenerates into a line).

From (ii), $\dfrac{(a-b)(c-d)}{(a-d)(b-c)}$ was shown to be positive and real.

Hence, the condition for equality in (i) is satisfied. Therefore,

$$|(a-b)(c-d)|+|(a-d)(b-c)|=|(a-c)(b-d)|$$

$$\begin{aligned}\left|\overrightarrow{AB}\times\overrightarrow{CD}\right|+\left|\overrightarrow{AD}\times\overrightarrow{BC}\right|&=\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|\\ AB\times CD+AD\times BC&=AC\times BD\end{aligned}$$

Question 12 (a) (i)

The point (h,k), on the vertical central plane, lies on the curved edge of the solid. It therefore satisfies the relation of the curve. Hence:

$$\begin{aligned}\sqrt{k}+\sqrt{h}&=1\\ \sqrt{k}&=1-\sqrt{h}\end{aligned}$$

Question 12 (a) (ii)

The area of the slice in terms of *k* can be found by integrating the *y*-expression of the slice.

By symmetry, we only need to find the area of the first quadrant of the slice and quadruple the resultant value.

$$\begin{aligned}A(k)&=4\int_0^k y\,dx\\ &=4\int_0^k\left(\sqrt{k}-\sqrt{x}\right)^2dx\\ &=4\int_0^k\left(k-2\sqrt{kx}+x\right)dx\\ &=4\left(k^2-\frac{4}{3}k^2+\frac{k^2}{2}\right)\\ &=\frac{2}{3}k^2\end{aligned}$$

The integral is evaluated with respect to *h*, so the relation from part (i) is required.

By symmetry, we only need to find the volume of the upper half of the solid and double the resultant value.

$$\begin{aligned}V&=2\int_0^1 A(k)\,dh\\ &=\frac{4}{3}\int_0^1 k^2\,dh\\ &=\frac{4}{3}\int_0^1\left(1-\sqrt{h}\right)^4dh\end{aligned}$$

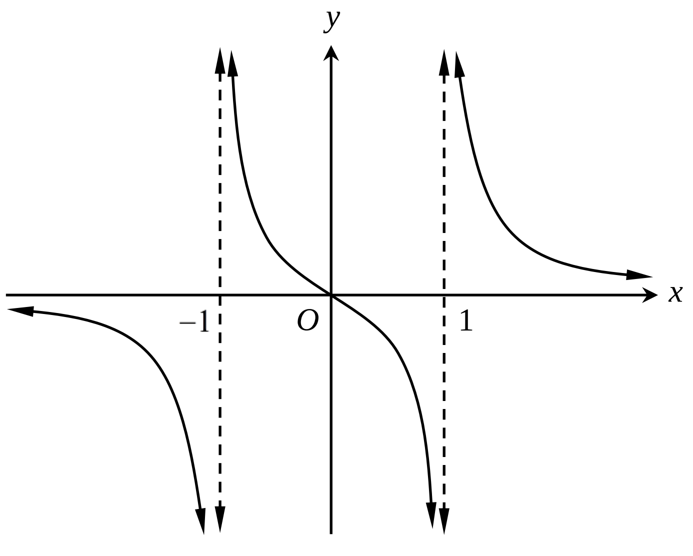
Let $u = 1 - \sqrt{h}$ to simplify the integral through substitution. Note the bounds remain the same but swap places.

$$h = (u-1)^2 \Rightarrow dh = 2(u-1)\,du$$

$$\begin{aligned}
V &= \frac{4}{3}\int_0^1 \left(1-\sqrt{h}\right)^4 dh \\
&= \frac{8}{3}\int_1^0 u^4 (u-1)\,du \\
&= \frac{8}{3}\int_0^1 \left(u^4 - u^5\right) du \\
&= \frac{8}{3}\left[\frac{u^5}{5} - \frac{u^6}{6}\right]_0^1 \\
&= \frac{8}{3}\left(\frac{1}{5} - \frac{1}{6}\right) \\
&= \frac{4}{45}
\end{aligned}$$

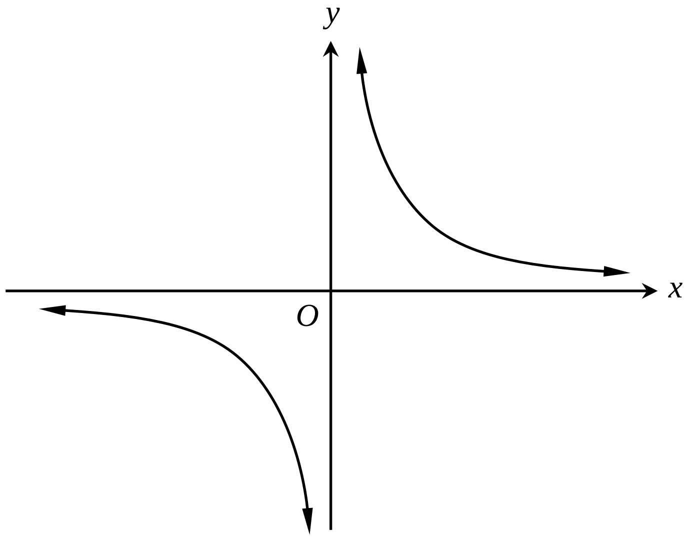
Question 12 (b) (i)

The asymptotes are $x = -1$ and $x = 1$. When $x \to 1^-$ then $y \to -\infty$ and when $x \to 1^+$ then $y \to \infty$. When $x \to \infty$ then $y \to 0^+$. Using these features and noting that the function is odd, we can obtain the following graph:



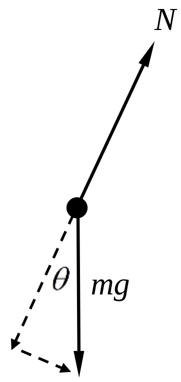
Question 12 (b) (ii)

The y-axis is the asymptote. When $x \to \infty$ then $y \to 0^+$. When $x \to -\infty$ then $y \to 0^-$. When $x \to 0^-$ then $y \to -\infty$ and when $x \to 0^+$ then $y \to \infty$.



Question 12 (c) (i)

Resolving the forces in the normal and tangential direction



$$mg\cos\theta - N = mr\omega^2$$
$$mg\sin\theta = mr\dot{\omega}$$

where $\dot{\omega}$ is the angular acceleration, but $\dot{\omega} = \dfrac{d}{d\theta}\left(\dfrac{\omega^2}{2}\right)$ so

$$mg\sin\theta = mr\frac{d}{d\theta}\left(\frac{\omega^2}{2}\right)$$
$$-mg\cos\theta + C = \frac{mr\omega^2}{2}$$
$$2mg(1-\cos\theta) = mr\omega^2$$
$$2mg(1-\cos\theta) = mg\cos\theta - N$$
$$N = mg(3\cos\theta - 2)$$

To find C, we use the fact that $\theta = 0$ when $\omega = 0$, so $C = mg$.

Question 12 (c) (ii)

When $\theta = \alpha$ then $N = 0$ as marble leaves the circular surface, hence $mg\left(3\cos\alpha - 2\right) = 0$.

Since α is acute then it must be that $\alpha = \cos^{-1}\left(2/3\right)$.

Question 12 (d) (i)

For the curved section of the solid, let the thickness of an arbitrary slice parallel to the x-axis be δy, where y is the function value taken at x. The volume of the slice is $\pi\left(b^2 - x^2\right)\delta y$. The above only considers the curved section of the solid. For the vertical line at $x = a$, the volume of the cylindrical annulus is $\pi\left(b^2 - a^2\right)c$. The total volume is:

$$
\begin{aligned}
V &= \pi\left(b^2c - a^2c\right) + \pi\int_c^d \left(b^2 - x^2\right)dy \\
&= \pi\left(b^2c - a^2c\right) + \pi\left(b^2d - b^2c\right) - \pi\int_c^d x^2\,dy \\
&= \pi\left(b^2d - a^2c\right) - \pi\int_c^d g^2\left(y\right)dy
\end{aligned}
$$

Question 12 (d) (ii)

From the question, the function is increasing and is one-to-one so $g\left(c\right) = a$ and $g\left(d\right) = b$.

Using the substitution $y = f\left(x\right)$ and $dy = f'\left(x\right)dx$ we have

$$
\begin{aligned}
V &= \pi\left(b^2d - a^2c\right) - \pi\int_c^d g^2\left(y\right)dy \\
&= \pi\left(b^2d - a^2c\right) - \pi\int_a^b x^2 f'\left(x\right)dx \\
&= \pi\left(b^2d - a^2c\right) - \pi\left[x^2 f\left(x\right)\right]_a^b + \pi\int_a^b 2x\,f\left(x\right)dx \quad \ldots\left(\text{IBP}\right) \\
&= \pi\left(b^2d - a^2c\right) - \pi\left(b^2d - a^2c\right) + 2\pi\int_a^b x\,f\left(x\right)dx \\
&= 2\pi\int_a^b x\,f\left(x\right)dx
\end{aligned}
$$

Question 12 (d) (iii)

The volume of the solid of revolution of any one-to-one function can be found using either the method of slices or method of cylindrical shells, and are mathematically equivalent.

Question 13 (a) (i)

When Harin plays, there are two possible scenarios.

Case #1:

Probability p of winning the first turn of the wheel. At this point he has won \$1 and it is as if he entered the casino with $\$(k+1)$. Hence, we have pP_{k+1}.

Case #2:

Probability $1-p$ of losing the first turn of the wheel. At this point he has lost \$1 and it is as if he entered the casino with $\$(k-1)$. Hence, we have $(1-p)P_{k-1}$.

Therefore, we have $P_k = pP_{k+1} + (1-p)P_{k-1}$.

Question 13 (a) (ii)

Using (i), we have

$$\begin{aligned}
P_k &= pP_{k+1} + (1-p)P_{k-1} \\
pP_{k+1} - P_k &= (p-1)P_{k-1} \\
pP_{k+1} - pP_k - (1-p)P_k &= (p-1)P_{k-1} \\
p(P_{k+1} - P_k) &= (1-p)P_k + (p-1)P_{k-1} \\
&= (1-p)(P_k - P_{k-1}) \\
P_{k+1} - P_k &= \left(\frac{1-p}{p}\right)(P_k - P_{k-1}) \\
&= \alpha(P_k - P_{k-1})
\end{aligned}$$

Question 13 (a) (iii)

$$\begin{aligned}
P_{k+1} - P_k &= \alpha(P_k - P_{k-1}) \\
&= \alpha^2(P_{k-1} - P_{k-2}) \\
&= \alpha^3(P_{k-2} - P_{k-3}) \\
&\vdots \\
&= \alpha^k(P_1 - P_0)
\end{aligned}$$

But if Harin enters the casino with \$0, then he is guaranteed to leave disappointed. Hence, $P_0 = 1$ and hence $P_{k+1} - P_k = \alpha^k(P_1 - 1)$.

Question 13 (a) (iv)

From part (iii), we have the following sets of equations for differing values of *k*.

$$\begin{aligned}
P_k - P_{k-1} &= \alpha^{k-1}\left(P_1 - 1\right) \\
P_{k-1} - P_{k-2} &= \alpha^{k-2}\left(P_1 - 1\right) \\
P_{k-2} - P_{k-3} &= \alpha^{k-3}\left(P_1 - 1\right) \\
&\vdots \\
P_2 - P_1 &= \alpha^{1}\left(P_1 - 1\right) \\
P_1 - P_0 &= \alpha^{0}\left(P_1 - 1\right)
\end{aligned}$$

Sum them all up and recognise that we have the sum of a geometric series.

$$\begin{aligned}
P_k - P_0 &= \left(P_1 - 1\right)\left(\alpha^{k-1} + \alpha^{k-2} + \alpha^{k-3} + \ldots + \alpha + 1\right) \\
P_k &= P_0 + \left(P_1 - 1\right)\left(\frac{\alpha^k - 1}{\alpha - 1}\right) \\
&= 1 + \left(P_1 - 1\right)\left(\frac{\alpha^k - 1}{\alpha - 1}\right)
\end{aligned}$$

Question 13 (a) (v)

When $k = K$, we have $P_K = 1 + \left(P_1 - 1\right)\left(\frac{\alpha^K - 1}{\alpha - 1}\right)$

But $P_K = 0$ because if Harin enters the casino with \$*K*, then he will immediately leave satisfied since he has already reached his satisfaction threshold (without even doing anything!). So, his chance of leaving disappointed is zero. Hence $P_1 - 1 = -\left(\frac{\alpha - 1}{\alpha^K - 1}\right)$.

Substituting this into the general expression for P_k:

$$\begin{aligned}
P_k &= 1 + \left(P_1 - 1\right)\left(\frac{\alpha^k - 1}{\alpha - 1}\right) \\
&= 1 - \left(\frac{\alpha - 1}{\alpha^K - 1}\right)\left(\frac{\alpha^k - 1}{\alpha - 1}\right) \\
&= 1 - \frac{\alpha^k - 1}{\alpha^K - 1} \\
&= \frac{\alpha^K - \alpha^k}{\alpha^K - 1}
\end{aligned}$$

Question 13 (a) (vi)

Note that $\alpha > 1$, so $\alpha^K \to \infty$ as $K \to \infty$. So for a fixed k we have $P_k \to 1$. So the probability of leaving satisfied approaches zero.

This means that as the threshold for leaving satisfied gets larger, the chance of leaving satisfied starting with a fixed finite \$$k$ goes to zero. Know your limits. Stay within it.

Question 13 (b)

Substitute $y = \dfrac{c^2}{x}$ into the equation of the circle $x^2 - 2hx + y^2 - 2ky + l = 0$

$$\begin{aligned} x^2 - 2hx + y^2 - 2ky + l &= 0 \\ x^2 - 2hx + \frac{c^4}{x^2} - \frac{2kc^2}{x} + l &= 0 \\ x^4 - 2hx^3 + lx^2 - 2kc^2x + c^4 &= 0 \end{aligned}$$

The roots of the quartic are the x coordinates of the intersections between the circle and the hyperbola which are $x_1,\ x_2,\ x_3,\ x_4$.

The x-value of the mean point is $\dfrac{x_1 + x_2 + x_3 + x_4}{4}$, and by the sum of roots, is equal to $\dfrac{h}{2}$.

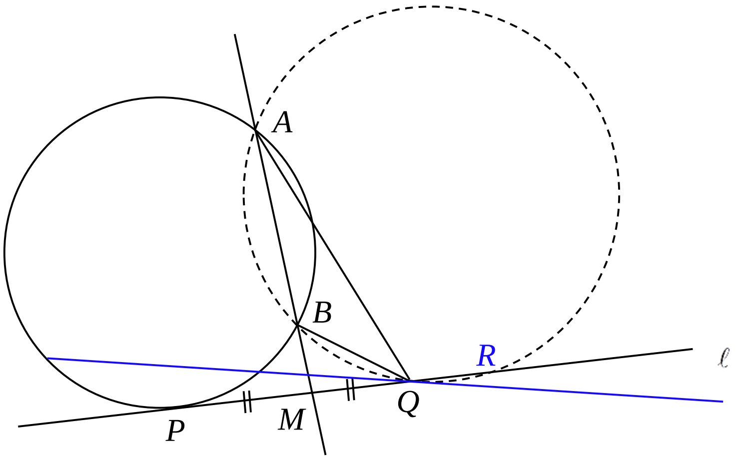
By symmetry, we can deduce that the y coordinate will be similar but with k instead of h.

Hence, the mean point has coordinates $\left(\dfrac{h}{2}, \dfrac{k}{2}\right)$.

The equation of the circle can be rewritten as $(x-h)^2 + (y-k)^2 = h^2 + k^2 - l$ so the coordinates of its centre is (h, k).

Note that the mean point is in fact the midpoint of the centre of the circle and the origin along the line $y = \dfrac{k}{h}x$. Hence, the mean point, centre of the circle and the origin are collinear.

Question 13 (c)



The goal is to show that the circle passing through A, B and Q intersects the line ℓ only once at Q, which will confirm that Q is a tangent to that circle.

Suppose that the line ℓ passes that circle at two points Q and say point R.

$$MP^2 = MB \times MA \text{ (secant-tangent ratio)}$$

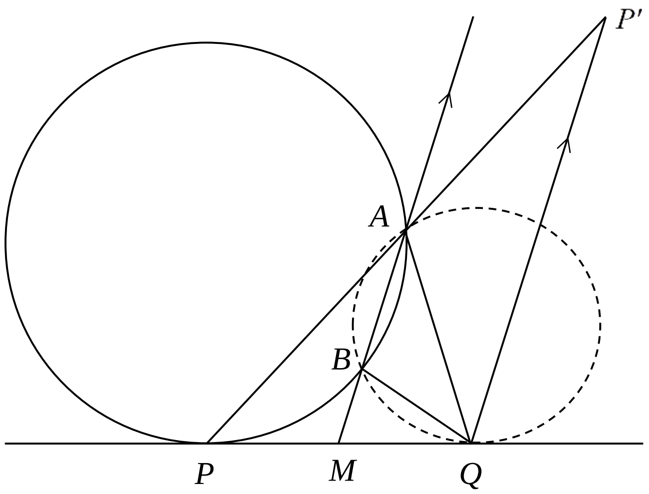
Since $MP = MQ$ then $MQ^2 = MB \times MA$. This means that $\dfrac{MQ}{MA} = \dfrac{MB}{MQ}$.

Hence ΔMQB and ΔMAQ are similar, as sides are in proportion with included angle $\angle AMQ$ being common. As a result, $\angle MQB = \angle MAQ$.

If a tangent is constructed to the circle at Q (see blue line above), the acute angle between this tangent and the chord AQ must be equal to angle in the alternate segment (i.e. $\angle MAQ$) and therefore equal to $\angle MQB$.

However, this is not possible unless the tangent and the line ℓ coincide. Therefore, it must be that the circle passing through A, B and Q is tangential to ℓ.

Question 13 (d)



Let M be the intersection point of AB and PQ. Since $AP = AP'$ and AB is parallel to PQ then

$MP = MQ$ (ratio of intercepts of parallel lines)

From part (c), a circle can be constructed passing through points A, B and Q such that PQ is a tangent to that circle.

$\angle BQP = \angle BAQ$ (angle between tangent and chord equals angle in alternate segment)

$\angle AQP' = \angle BAQ$ (alternate angles as AB and PQ are parallel)

$\therefore \angle BQP = \angle AQP'$

Question 14 (a)

Consider a possible sequence, where *H* wins: *HTTHTHHH*.
The main thing to observe here is that we can sort them into pairs as such:
(HT)(TH)(TH)[HH].

Note that no matter how we swap the individual terms in each pair, the sequence will always last exactly 8 tosses. So, the number of possible sequences consisting of 8 tosses is 2^4. This is because we have 2^3 for the first three pairs of *HT* and then 2 options for the 'terminating pair', since it could be *HH* or *TT*.

So, for a sequence lasting $2n$ games, we have 2^n sequences. But sequences last anywhere from 2 tosses to 2*n* tosses, so we have a sum of geometric series $2+2^2+...+2^n=2^{n+1}-2$.

This is assuming that somebody wins. However, we must also consider the case where nobody wins after $2n$ tosses. This can happen in 2^n ways. Simply create *n* pairs of *(HT)* and swap around the terms in each pair.

Hence, the total number of possible games is $2^{n+1}+2^n-2$.

Question 14 (b)

When $n = 1$,

$$\begin{aligned} LHS &= \left(a_1 b_1\right)^2 \\ &= \left(a_1{}^2\right)\left(b_1{}^2\right) \\ &\leq RHS \end{aligned}$$

Hence the statement is true for $n = 1$.

Assume the statement is true for $n = k$

$$\left(a_1 b_1 + a_2 b_2 + \cdots + a_k b_k\right)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_k^2\right)\left(b_1^2 + b_2^2 + \cdots + b_k^2\right)$$

Required to prove

$$\left(a_1 b_1 + a_2 b_2 + \cdots + a_{k+1} b_{k+1}\right)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_{k+1}^2\right)\left(b_1^2 + b_2^2 + \cdots + b_{k+1}^2\right)$$

Using the assumption and the well-known inequality $x^2 + y^2 \geq 2xy$.

$$\begin{aligned} RHS &= \left(a_1^2 + a_2^2 + \cdots + a_{k+1}^2\right)\left(b_1^2 + b_2^2 + \cdots + b_{k+1}^2\right) \\ &= \left(a_1^2 + a_2^2 + \cdots + a_k^2\right)\left(b_1^2 + b_2^2 + \cdots + b_k^2\right) + a_{k+1}^2\left(b_1^2 + b_2^2 + \cdots + b_k^2\right) + b_{k+1}^2\left(a_1^2 + a_2^2 + \cdots + a_k^2\right) + a_{k+1}^2 b_{k+1}^2 \\ &\geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_k b_k\right)^2 + a_{k+1}^2\left(b_1^2 + b_2^2 + \cdots + b_k^2\right) + b_{k+1}^2\left(a_1^2 + a_2^2 + \cdots + a_k^2\right) + a_{k+1}^2 b_{k+1}^2 \\ &= \left(a_1 b_1 + a_2 b_2 + \cdots + a_k b_k\right)^2 + \left(a_{k+1}^2 b_1^2 + b_{k+1}^2 a_1^2\right) + \left(a_{k+1}^2 b_2^2 + b_{k+1}^2 a_2^2\right) + \ldots + \left(a_{k+1}^2 b_k^2 + b_{k+1}^2 a_k^2\right) + a_{k+1}^2 b_{k+1}^2 \\ &\geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_k b_k\right)^2 + 2a_{k+1} b_{k+1}\left(a_1 b_1 + a_2 b_2 + \ldots + a_k b_k\right) + a_{k+1}^2 b_{k+1}^2 \\ &= \left(a_1 b_1 + a_2 b_2 + \cdots + a_{k+1} b_{k+1}\right)^2 \\ &= LHS \end{aligned}$$

Hence, the statement is true by the principle of mathematical induction.

Question 14 (c) (i)

$$f(x) = x - \frac{a^2}{x} \Rightarrow f'(x) = 1 + \frac{a^2}{x^2}$$
$$g(x) = 2a \ln\left(\frac{x}{a}\right) \Rightarrow g'(x) = \frac{2a}{x}$$

Consider the difference of the two derivatives

$$\begin{aligned} f'(x) - g'(x) &= 1 + \frac{a^2}{x^2} - \frac{2a}{x} \\ &= \frac{x^2 - 2ax + a^2}{x^2} \\ &= \left(\frac{x-a}{x}\right)^2 \\ &\geq 0 \end{aligned}$$

Hence $f'(x) \geq g'(x)$ in the domain $x \geq a$.

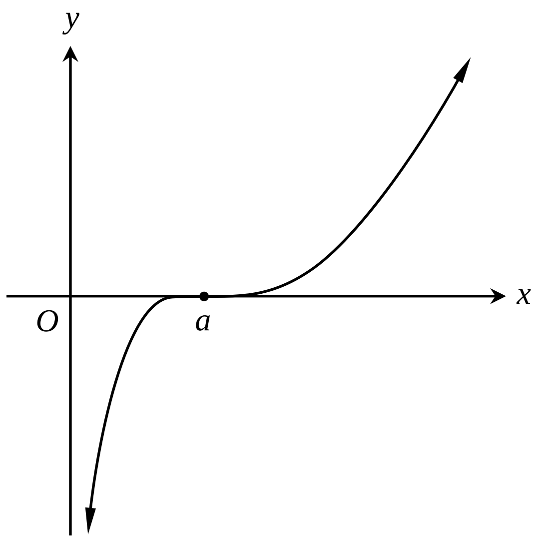
Question 14 (c) (ii)

From (i), we have $f'(t) \geq g'(t)$. Integrate both sides over the domain $a \leq t \leq x$.

$$f(x) - f(a) \geq g(x) - g(a)$$

But $f(a) = g(a) = 0$, so $f(x) \geq g(x)$ for $x \geq a$.

Question 14 (c) (iii)



Question 14 (d) (i)

The acceleration equation is $\ddot{x} = -g - kv$ so

$$\begin{aligned} \frac{dv}{dt} &= -(g+kv) \\ t &= -\int \frac{dv}{g+kv} \\ &= -\frac{1}{k}\ln(g+kv) + C_1 \end{aligned}$$

When $t = 0$, $v = u$ hence $C_1 = \frac{1}{k}\ln(g+ku)$

$$\begin{aligned} t &= \frac{1}{k}\ln\left(\frac{g+ku}{g+kv}\right) \\ g+kv &= (g+ku)e^{-kt} \\ \frac{dx}{dt} &= \frac{1}{k}\left[(g+ku)e^{-kt} - g\right] \\ x &= \frac{1}{k}\left[-\frac{1}{k}(g+ku)e^{-kt} - gt + C_2\right] \end{aligned}$$

When $t = 0$, $x = 0$ hence $C_2 = \frac{1}{k}(g+ku)$

$$\begin{aligned} x &= \frac{1}{k}\left[-\frac{1}{k}(g+ku)e^{-kt} - gt + \frac{1}{k}(g+ku)\right] \\ &= \frac{1}{k^2}(g+ku)\left(1-e^{-kt}\right) - \frac{gt}{k} \end{aligned}$$

Question 14 (d) (ii)

Since $t = \frac{1}{k}\ln\left(\frac{g+ku}{g+kv}\right)$ and maximum height occurs when $v = 0$ so $T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$.

Question 14 (d) (iii)

When $t = 2T$ and $T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$ then

$$\begin{aligned}
x &= \frac{1}{k^2}(g+ku)(1-e^{-2kT}) - \frac{2gT}{k} \\
&= \frac{1}{k^2}(g+ku)\left(1-\left(\frac{g}{g+ku}\right)^2\right) - \frac{2g}{k^2}\ln\left(\frac{g+ku}{g}\right) \\
&= \frac{1}{k^2}\left[g+ku-\frac{g^2}{g+ku}-2g\ln\left(\frac{g+ku}{g}\right)\right]
\end{aligned}$$

Question 14 (d) (iv)

From (c) (ii), it was shown that $x-\frac{a^2}{x}-2a\ln\left(\frac{x}{a}\right)\geq 0$ for $x \geq a$. If $a = g$ and $x = g+ku$ in this result, then since $x > a$,

$$g+ku-\frac{g^2}{g+ku}-2g\ln\left(\frac{g+ku}{g}\right) > 0$$

Hence, $x(2T) > 0$. This means that at time $t = 2T$, the object is still above the ground. Since the upwards journey takes a total of *T* seconds, then it must take more than *T* seconds to complete the downwards journey. Thus, the object takes longer on the downwards journey than the upwards journey.

Remark:

If *x* is defined as the height of the object at time *t* relative to the ground, then for both the upwards and downwards journey the origin and co-ordinate axes remain the same. Thus, the equations of motion relating acceleration, velocity and displacement are really the same. To see this, note that if *v* was defined as velocity rather than speed (which was the case in the question), then *v* is negative on the downwards journey which causes the acceleration equation to be the same as the upwards journey.

Question 15 (a)

In the domain $0 < t < \dfrac{\pi}{6}$, note that $\cos t \le 1$. Integrating this inequality from 0 to some x gives

$$\int_0^x \cos t \, dt \le \int_0^x 1 \, dt$$

Which leads to $\sin x \le x$ which is the left side of the required inequality.

To obtain the right side of the inequality, consider the function $f(x) = \cos x - x$. Note that

$$f'(x) = -\sin x - 1$$

Given that $\sin x \ge -1$, then the function is non-decreasing across its entire domain. Since

$$f(0) = 1$$

$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

Then $f(x) \ge 0$ in the domain $0 < x < \dfrac{\pi}{6}$. This proves the right side of inequality $\cos \ge x$.

Question 15 (b) (i)

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{6}} \frac{\sin^{2n} x}{\cos x} dx \\
&= \int_0^{\frac{\pi}{6}} \frac{\sin^{2n-2} x\left(1 - \cos^2 x\right)}{\cos x} dx \\
&= \int_0^{\frac{\pi}{6}} \frac{\sin^{2n-2} x}{\cos x} dx \quad - \int_0^{\frac{\pi}{6}} \sin^{2n-2} x \cos x \, dx \\
&= I_{n-1} - \left[\frac{\sin^{2n-1} x}{2n-1} \right]_0^{\frac{\pi}{6}} \\
&= I_{n-1} - \frac{1}{(2n-1)2^{2n-1}}
\end{aligned}$$

Applying the recurrence multiple times

$$\begin{aligned}
I_n &= I_{n-2} - \frac{1}{(2n-3)2^{2n-3}} - \frac{1}{(2n-1)2^{2n-1}} \\
&= I_{n-3} - \frac{1}{(2n-5)2^{2n-5}} - \frac{1}{(2n-3)2^{2n-3}} - \frac{1}{(2n-1)2^{2n-1}} \\
&\vdots \\
&= I_0 - \frac{1}{(1)2^1} - \frac{1}{(3)2^3} - \ldots\ldots - \frac{1}{(2n-3)2^{2n-3}} - \frac{1}{(2n-1)2^{2n-1}} \\
&= \int_0^{\frac{\pi}{6}} \frac{1}{\cos x}\,dx - \sum_{k=1}^{n} \frac{1}{(2k-1)2^{2k-1}} \\
&= \Big[\ln\left|\sec x + \tan x\right|\Big]_0^{\frac{\pi}{6}} - \sum_{k=1}^{n} \frac{1}{(2k-1)2^{2k-1}} \\
&= \frac{\ln 3}{2} - \sum_{k=1}^{n} \frac{1}{(2k-1)2^{2k-1}}
\end{aligned}$$

Alternative method: Identifying a geometric series

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{6}} \frac{\sin^{2n} x}{\cos x}\,dx \\
&= \int_0^{\frac{\pi}{6}} \frac{\sin^{2n} x + 1 - 1}{\cos x}\,dx \\
&= \int_0^{\frac{\pi}{6}} \frac{1}{\cos x}\,dx \quad + \int_0^{\frac{\pi}{6}} \frac{\cos x\left(\sin^{2n} x - 1\right)}{\cos^2 x}\,dx \\
&= \Big[\ln\left|\sec x + \tan x\right|\Big]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{\cos x\left(\sin^{2n} x - 1\right)}{\sin^2 x - 1}\,dx \\
&= \frac{\ln 3}{2} - \int_0^{\frac{\pi}{6}} \left(\cos x + \sin^2 x\cos x + \sin^4 x\cos x + \ldots + \sin^{2n-2} x\cos x\right)dx \\
&= \frac{\ln 3}{2} - \left[\sin x + \frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + \ldots + \frac{\sin^{2n-1} x}{2n-1}\right]_0^{\frac{\pi}{6}} \\
&= \frac{\ln 3}{2} - \left[\frac{1}{2} + \frac{1}{3\times 2^3} + \frac{1}{5\times 2^5} + \ldots + \frac{1}{(2n+1)2^{2n+1}}\right] \\
&= \frac{\ln 3}{2} - \sum_{k=1}^{n} \frac{1}{(2k-1)2^{2k-1}}
\end{aligned}$$

Question 15 (b) (ii)

Since $\dfrac{\sin^{2n} x}{\cos x} \geq 0$ for the domain $0 < x < \dfrac{\pi}{6}$, then $I_n \geq 0$.

From (i), the left side of the inequality leads to the result $\sin^{2n} x \leq x^{2n}$

The right side of the inequality leads to the result $\dfrac{1}{\cos x} \leq \dfrac{1}{x}$

Multiplying the two inequalities leads to

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \frac{\sin^{2n} x}{\cos x}\,dx &\leq \int_0^{\frac{\pi}{6}} x^{2n-1}\,dx \\ &= \left[\frac{x^{2n}}{2n}\right]_0^{\frac{\pi}{6}} \\ &= \frac{1}{2n}\left(\frac{\pi}{6}\right)^{2n}\end{aligned}$$

Hence, $0 \leq I_n \leq \dfrac{1}{2n}\left(\dfrac{\pi}{6}\right)^{2n}$.

Question 15 (b) (iii)

Since, $\dfrac{\pi}{6} < 1$ then $\left(\dfrac{\pi}{6}\right)^{2n} \to 0$ and $\dfrac{1}{2n} \to 0$ when $n \to \infty$, so the right side of the inequality in (ii), approaches zero.

This means that $I_n \to 0$ when $n \to \infty$ so $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{(2k-1)\,2^{2k-1}} = \frac{\ln 3}{2}$

or equivalently $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{(2k-1)\,2^{2k-2}} = \ln 3$.

Thus, $1 + \dfrac{1}{3\times 2^2} + \dfrac{1}{5\times 2^4} + \dfrac{1}{7\times 2^6} + \cdots = \ln 3$.

Question 15 (c) (i)

By sum of roots

$$\begin{aligned}\alpha+\bar{\alpha}+\beta&=-\beta\\ x+iy+x-iy&=-2\beta\\ \beta&=-x\end{aligned}$$

Question 15 (c) (ii)

Sum of roots in pairs

$$\begin{aligned}\alpha\beta+\bar{\alpha}\beta+\alpha\bar{\alpha}&=0\\ \beta(x+iy+x-iy)+|\alpha|^2&=0\\ |\alpha|^2&=-2x\beta\\ |\alpha|^2&=2\beta^2\end{aligned}$$

Question 15 (c) (iii)

From part (ii)

$$\begin{aligned}|\alpha|^2&=2\beta^2\\ x^2+y^2&=2x^2\\ y^2&=x^2\\ y&=\pm x\end{aligned}$$

This means that α and $\bar{\alpha}$ have the complex numbers $x\pm ix$ which lies on the lines $y=\pm x$.

Question 15 (c) (iv)

The vertical distance between α and $\bar{\alpha}$ on the Argand diagram is $2|y|$ or equivalently $2|x|$, which forms the base of the triangle. Since $\beta=-x$ then the height of the triangle is $2|x|$. Hence the area is given by

$$A=\frac{1}{2}\times 2|x|\times 2|x|=2x^2=2\beta^2=|\alpha|^2$$

Question 16 (a)

Note that the lowest-degree term has power $p-1$. If we differentiate it $i \geq p$ times, the last term disappears and the other terms get their powers brought down *i* times. Since $i \geq p$, each term will eventually have either *p* itself or a factor of *p* being brought down. Hence, each term of $Q^{(i)}(x)$ will have a factor of *p* in it, thus $Q^{(i)}(x) = pR_i(x)$, where $R_i(x)$ is some polynomial with integer coefficients.

Remark:

Below is a more detailed explanation for the reader, but not necessary to obtain the mark.

Consider a general x^n for some integer $n \geq i$. Differentiating this *i* times gives

$$\begin{aligned}\frac{d^i}{dx^i}\left(x^n\right) &= n(n-1)(n-2)...(n-i+1)x^{n-i} \\ &= \frac{n(n-1)(n-2)...(n-i+1)(n-i)!}{i!(n-i)!} \times i!x^{n-i} \\ &= \binom{n}{i} i!x^{n-i}\end{aligned}$$

Thus for any $p \leq i$ there exists a factor of *p* in this derivative expression (within the $i!$) and the remaining factors are integers. Note that for $n < i$, the i^{th} derivative is zero.

Since $Q(x) = a_r x^r + a_{r-1}x^{r-1} + ... + a_p x^p + a_{p-1}x^{p-1}$ then differentiating each term $i \geq p$ times will yield a factor of *p* (any terms containing powers of less than *i* will be differentiated to zero). Hence the i^{th} derivative can be expressed in the form

$$Q^{(i)}(x) = pR_i(x)$$

where $R_i(x)$ is some polynomial with integer coefficients.

Question 16 (b)

Since $Q(x) = a_r x^r + a_{r-1}x^{r-1} + ... + a_p x^p + a_{p-1}x^{p-1}$ then the $(p-1)$th derivative is a polynomial with the constant term $a_{p-1}(p-1)!$. When this polynomial is evaluated at zero, it is equal to that constant term. Hence $Q^{(p-1)}(0) = a_{p-1}(p-1)!$.

To find a_{p-1} we need to evaluate the coefficient of x^{p-1} from the original definition of $Q(x)$

$$Q(x)=\frac{x^{p-1}(1-x)^p(2-x)^p(3-x)^p\ldots(n-x)^p}{(p-1)!}$$

We can deduce that $a_{p-1}=\dfrac{1^p\times2^p\times\ldots\times n^p}{(p-1)!}$ so $Q^{(p-1)}(0)=(n!)^p$

Question 16 (c)

Since $p>n$ then there does not exist a factor of *p* within $n!$ and since *p* is a prime number $(n!)^p$ does not have a factor of *p* within it either. Hence, $\dfrac{Q^{(p-1)}(0)}{p}$ cannot be an integer.

Question 16 (d)

Substituting in $x=m$, we have

$$T(m)=Q(m)+Q^{(1)}(m)+\cdots+Q^{(p)}(m)+\cdots+Q^{(r)}(m).$$

where

$$Q(x)=\frac{x^{p-1}(1-x)^p(2-x)^p(3-x)^p\ldots(n-x)^p}{(p-1)!}$$

But since *m* is some integer between 1 and *n* inclusive, there exists a factor $(m-x)^p$ in the numerator of $Q(x)$ so $x=m$ is a root of multiplicity *p* of $Q(x)$.

By the multiple root theorem, this means that $Q(m)=Q^{(1)}(m)=Q^{(2)}(m)=\ldots=Q^{(p-1)}(m)=0$
Hence, what we end up with is

$$T(m)=Q^{(p)}(m)+Q^{(p+1)}(m)+Q^{(p+2)}(m)+\cdots+Q^{(r)}(m).$$

Similarly, $x=0$ is a root of multiplicity $(p-1)$ of $Q(x)$ so by the multiple root theorem

$Q(0)=Q^{(1)}(0)=Q^{(2)}(0)=\ldots=Q^{(p-2)}(0)=0$, which means

$$T(0)=Q^{(p-1)}(0)+Q^{(p)}(0)+Q^{(p+1)}(0)+\cdots+Q^{(r)}(0).$$

Question 16 (e)

From (a), $Q^{(i)}(x) = pR_i(x)$ for any $i \geq p$. From (d), this means that $T(m)$ is a product of p and some integer all $m = 1,2,3,..,n$ so $\dfrac{T(m)}{p}$ is an integer.

From (c), $\dfrac{Q^{(p-1)}(0)}{p}$ is not an integer which implies that $\dfrac{T(0)}{p}$ is not an integer.

This means that $\dfrac{c_0T(0)+c_1T(1)+c_2T(2)+...+c_nT(n)}{p}$ cannot be an integer for any set of integers c_0, $c_1, \ldots, c_n$.

Question 16 (f)

$$
\begin{aligned}
f(x) &= e^{-x}\,T(x) \\
f'(x) &= e^{-x}\,T'(x) - e^{-x}\,T(x) \\
&= e^{-x}\left[Q^{(1)}(x)+Q^{(2)}(x)+Q^{(3)}(x)+...+Q^{(r+1)}(x)\right] - e^{-x}\left[Q(x)+Q^{(1)}(x)+Q^{(2)}(x)+...+Q^{(r)}(x)\right] \\
&= -e^{-x}Q(x)+e^{-x}Q^{(r+1)}(x) \\
&= -e^{-x}Q(x)
\end{aligned}
$$

Since $Q(x)$ is a polynomial of degree r so $Q^{(r+1)}(x)=0$

Question 16 (g)

Let $a=0$ and $b=k$, then there exists some $0<\alpha<1$ such that

$$
\begin{aligned}
f'(\alpha k) &= \frac{f(k)-f(0)}{k} \\
-ke^{-\alpha k}Q(\alpha k) &= e^{-k}\,T(k)-T(0) \\
T(k)-e^{k}T(0) &= -ke^{k(1-\alpha)}Q(\alpha k)
\end{aligned}
$$

Question 16 (h)

Since $\beta_k = -ke^{k(1-\alpha)}Q(k\alpha)$ then $|\beta_k| = ke^{k(1-\alpha)}|Q(k\alpha)|$. Note the following

$$k(1-\alpha) \le k \le n \Rightarrow e^{k(1-\alpha)} \le e^n$$

$$\begin{aligned}
k|Q(k\alpha)| &= \frac{k\left|(k\alpha)^{p-1}(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-k\alpha)^p\right|}{(p-1)!} \\
&\le \frac{n^p \times 1^p \times 2^p \times \ldots \times n^p}{(p-1)!} \\
&= \frac{n^p(n!)^p}{(p-1)!}
\end{aligned}$$

Multiplying the inequalities altogether we get $|\beta_k| \le \dfrac{e^n n^p(n!)^p}{(p-1)!}$

Remark:

Since $x - k\alpha \le x$ given $k\alpha > 0$. Also, since $k \le n$ and $\alpha < 1$ we have $k(k\alpha)^{p-1} \le n^p$, hence

$$k(k\alpha)^{p-1}(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-k\alpha)^p \le n^p(n!)^p$$

If $k(k\alpha)^{p-1}(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-k\alpha)^p$ is non-negative then

$$k\left|(k\alpha)^{p-1}(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-k\alpha)^p\right| \le n^p(n!)^p$$

If $k(k\alpha)^{p-1}(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-k\alpha)^p$ is negative then

$$\begin{aligned}
&k\left|(k\alpha)^{p-1}(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-k\alpha)^p\right| \\
&= -k(k\alpha)^{p-1}(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-1-k\alpha)^p(n-k\alpha)^p \\
&= k(k\alpha)^{p-1}(k\alpha-n)(1-k\alpha)^p(2-k\alpha)^p(3-k\alpha)^p\ldots(n-1-k\alpha)^p(n-k\alpha)^{p-1} \\
&\le n^p \times 1 \times 1^p \times 2^p \times \ldots \times n^{p-1} \\
&\le n^p \times 1^p \times 2^p \times \ldots \times n^p \\
&= n^p(n!)^p
\end{aligned}$$

Question 16 (i)

If $P(e)=0$ then $c_0+c_1e+c_2e^2+...+c_ne^n=0$

Since $\beta_k=T(k)-e^kT(0)$ as defined in (g) and (h), then

$$\begin{aligned}c_1\beta_1+c_2\beta_2+c_3\beta_3+...+c_n\beta_n&=c_1T(1)+c_2T(2)+c_3T(3)+...+c_nT(n)-T(0)\left(c_1e^1+c_2e^2+c_3e^3+...+c_ne^n\right)\\&=c_1T(1)+c_2T(2)+c_3T(3)+...+c_nT(n)-T(0)(-c_0)\\&=c_0T(0)+c_1T(1)+c_2T(2)+c_3T(3)+...+c_nT(n)\end{aligned}$$

Question 16 (j)

Since $|c_k\beta_k|\le\frac{|c_k|e^nn^p(n!)^p}{(p-1)!}$ then as p increases to a large value $\frac{|c_k|e^nn^p(n!)^p}{(p-1)!}$ will decrease towards zero. As p gets larger and the upper bound decreases there is some point where the upper bound passes the value $\frac{1}{n}$. In other words, there exists some p such that $|c_k\beta_k|<\frac{1}{n}$.

Question 16 (k)

$$\begin{aligned}\left|c_0T(0)+c_1T(1)+c_2T(2)+...+c_nT(n)\right|&=\left|c_1\beta_1+c_2\beta_2+c_3\beta_3+...+c_n\beta_n\right|\\&\le|c_1\beta_1|+|c_2\beta_2|+|c_3\beta_3|+...+|c_n\beta_n|\\&\le\frac{1}{n}\times n\\&=1\end{aligned}$$

This suggests that $c_0T(0)+c_1T(1)+c_2T(2)+...+c_nT(n)$ is not an integer or is zero.

If $c_0T(0)+c_1T(1)+c_2T(2)+...+c_nT(n)$ is not an integer, we have a contradiction because $T(0),T(1),T(2),...,T(n)$ and c_0, $c_1,\ldots,$ c_n are integers.

If $c_0T(0)+c_1T(1)+c_2T(2)+...+c_nT(n)$ is zero then $\frac{c_0T(0)+c_1T(1)+c_2T(2)+...+c_nT(n)}{p}$ is an integer, which is a contradiction to the result in (e).

Hence, the initial assumption of $P(e)=0$ cannot be true, so e cannot be a root of any polynomial with integer coefficients.