2020 **BORED OF STUDIES TRIAL EXAMINATION**

9th October

Mathematics Extension 2

General instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5–14)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the vector equations of the lines below.

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}.$$

Which of the following statements is correct?

(A) The lines are parallel with no point of intersection.

(B) The lines are not parallel with no point of intersection.

(C) The lines are parallel with a point of intersection.

(D) The lines are not parallel with a point of intersection.

2 Consider the region described by the following inequalities over the complex plane.

$$z + \bar{z} \leq 0 \quad \text{and} \quad -\frac{\pi}{6} \leq \arg(z + 1 - i) \leq \frac{\pi}{6}.$$

What is the area of the region?

(A) $\dfrac{2}{\sqrt{3}}$

(B) $\dfrac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $\dfrac{\sqrt{3}}{2}$

3 Let $f(x)$ and $g(x)$ be real-valued functions, over the domain $x \geq 0$.

Suppose that $f(x) \geq 0$, $\lim\limits_{x \to 0^+} f(x) = 0$ and $\lim\limits_{x \to 0^+} g(x) = 0$.

Further suppose that $\lim\limits_{x \to 0^+} (f(x))^{g(x)}$ exists, and is equal to L.

Which statement best describes the value of L?

(A) $L = 0$

(B) L can be any non-negative real number.

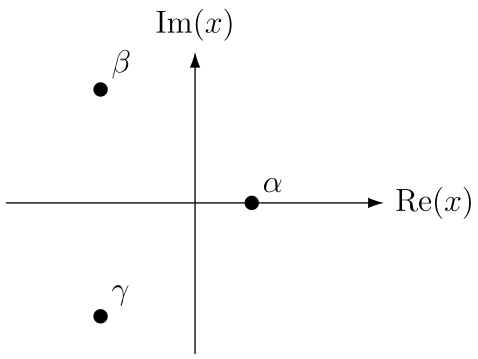
(C) $L = 1$

(D) L can be any real number.

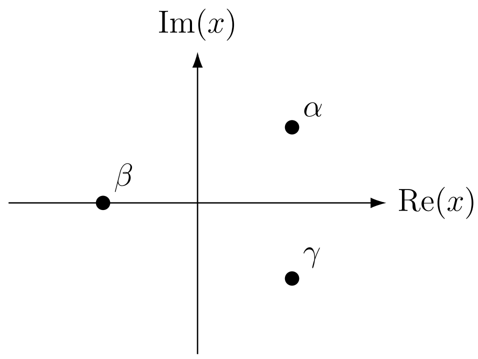
4 Consider the polynomial $P(x) = x^3 + ax^2 - bx + c$ where a, b and c are positive constants. The graph of $y = P(x)$ has the stationary points (x_1, y_1) and (x_2, y_2) in the first and second quadrant respectively.

Which diagram could represent the location of the roots of $P(x)$ in the complex plane?

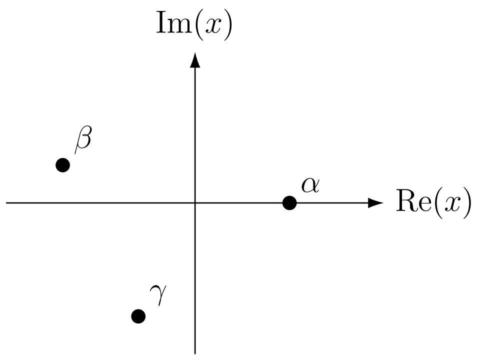
(A)



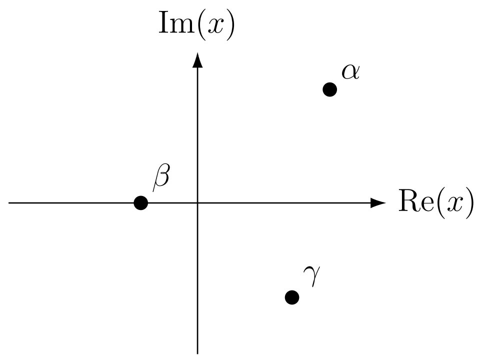
(B)



(C)



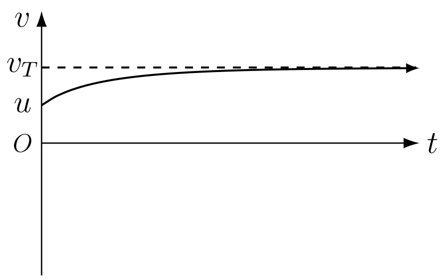
(D)



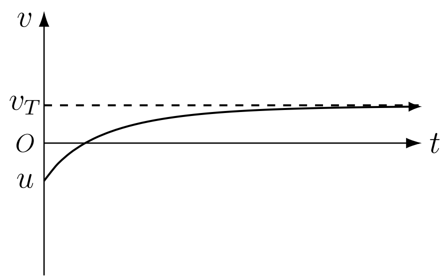
5 An object is projected directly upwards with an initial velocity of u.
It experiences a fixed gravitational force and air resistance during its trajectory.
The object eventually reaches a terminal velocity of v_T.

Let upwards be the positive direction.
Which of the following graphs best describes the object's velocity over time?

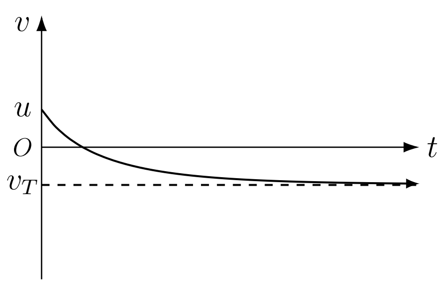
(A)



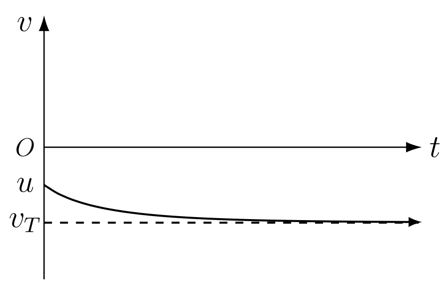
(B)



(C)



(D)



6 The vector equation of a line in three dimensional space for some scalar λ is given by

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{\hat{n}}$$

where $\underset{\sim}{a}$ is a position vector and $\underset{\sim}{\hat{n}}$ is a *unit* vector in the direction of the line.

Let $\underset{\sim}{p}$ be some point which does not lie on the line. Which of the following represents the shortest distance from $\underset{\sim}{p}$ to the line?

(A) $\left|((\underset{\sim}{a} - \underset{\sim}{p}) \cdot \underset{\sim}{\hat{n}})\underset{\sim}{\hat{n}}\right|$

(B) $\left|((\underset{\sim}{a} - \underset{\sim}{p}) \cdot \underset{\sim}{\hat{n}})\underset{\sim}{\hat{n}} - \underset{\sim}{\hat{n}}\right|$

(C) $\left|((\underset{\sim}{a} - \underset{\sim}{p}) \cdot \underset{\sim}{\hat{n}})\underset{\sim}{\hat{n}} - (\underset{\sim}{a} - \underset{\sim}{p})\right|$

(D) $\left|((\underset{\sim}{a} - \underset{\sim}{p}) \cdot \underset{\sim}{\hat{n}})\underset{\sim}{\hat{n}} - \underset{\sim}{p}\right|$

7 A particle is moving in simple harmonic motion according to the acceleration equation $\ddot{x} = -kx$ where k is a known positive constant and x is the particle's displacement at time t. Given that the particle is initially at the origin, and the information described above, which of the following *cannot* be determined?

(A) Period

(B) Amplitude

(C) Centre of motion

(D) Initial acceleration

8 Let $f(x)$ be a function over some domain D. Which of the following statements could imply that $\lim_{x \to 0} f(x) = 0$?

(A) $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in D, |x| < \delta \Rightarrow |f(x)| < \epsilon$

(B) $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in D, |x| < \delta \Rightarrow |f(x)| > \epsilon$

(C) $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in D, |x| > \delta \Rightarrow |f(x)| < \epsilon$

(D) $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in D, |x| > \delta \Rightarrow |f(x)| > \epsilon$

9 Let A, B and C be non-collinear points represented by complex numbers a, b and c respectively on the complex plane. Which condition implies $\triangle ABC$ is isosceles?

(A) $|b|^2 + a\overline{c} + \overline{a}c = |c|^2 + a\overline{b} + \overline{a}b$

(B) $|b|^2 + a\overline{b} + \overline{a}b + a\overline{c} + \overline{a}c = |c|^2$

(C) $|b|^2 = |c|^2 + a\overline{c} + \overline{a}c + a\overline{b} + \overline{a}b$

(D) $|b|^2 + a\overline{b} + \overline{a}b = |c|^2 + a\overline{c} + \overline{a}c$

10 Which of the following is equivalent to $\displaystyle\int_0^1 \frac{1 - x^2}{1 + x^2}\,dx$?

(A) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan x}{\tan 2x}\,dx$

(B) $\displaystyle\int_0^1 x \tan^{-1} x\,dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x \sin x}{\cos^3 x}\,dx$

(D) $\displaystyle\int_0^1 \left(\frac{1}{2}\ln(1 + x^2) + \tan^{-1} x\right)dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Use the substitution $x = \dfrac{1}{u}$ to evaluate **4**

$$\int_0^\infty \frac{1}{x^8 + x^4 + 1} dx.$$

You may assume, without proof, that

$$a^3 + 1 = (a+1)(a^2 - a + 1).$$

(b) Consider the quadratic polynomial $P(x) = ax^2 + bx + c$ where a, b and c are real. **2**
$P(x)$ has non-real solutions z_1 and z_2, which lie on the complex plane.

Find the equation of the circle with the smallest possible radius passing through both z_1 and z_2 in terms of a, b and c.

(c) Let $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ be arbitrary fixed position vectors in three dimensional space. **2**
Suppose that two distinct lines ℓ_1 and ℓ_2 are represented by the vector equations

$$\ell_1 : \underset{\sim}{r} = \underset{\sim}{a}_1 + \lambda_1 \underset{\sim}{b}_1$$

$$\ell_2 : \underset{\sim}{r} = \underset{\sim}{a}_2 + \lambda_2 \underset{\sim}{b}_2$$

where λ_1 and λ_2 are scalar constants.

Let $\underset{\sim}{n}$ be a vector that is simultaneously perpendicular to both $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$.
If the lines ℓ_1 and ℓ_2 have a point of intersection, show that

$$(\underset{\sim}{a}_1 - \underset{\sim}{a}_2) \cdot \underset{\sim}{n} = 0.$$

Question 11 continues on page 6

(d) An object is initially at the origin and moves horizontally in the positive direction of the x-axis with an initial speed of u. It is subjected to a resistance force such that its acceleration equation for some integer $n > 2$ is

$$\ddot{x} = -kv^n$$

where v is the object's speed at time t and k is a positive constant.

(i) Show that the displacement equation is **2**

$$x = \frac{u^{n-2} - v^{n-2}}{k(n-2)(uv)^{n-2}}.$$

(ii) By considering the average speed of the object from the origin to a particular positive displacement x, show that **3**

$$v < \frac{a_{n-1}}{a_{n-2}} < u$$

where $a_n = \dfrac{u^n - v^n}{n}$.

(e) Let $a_1, a_2, \ldots, a_{n-1}$ and a_n be distinct non-zero real numbers. Define the polynomial

$$P(x) = (x^2 + a_1^2)(x^2 + a_2^2)\ldots(x^2 + a_n^2).$$

(i) Show that there exists some real constants $c_1, c_2, \ldots, c_{n-1}$ and c_n such that **1**

$$\frac{1}{P(x)} = \frac{c_1}{x^2 + a_1^2} + \frac{c_2}{x^2 + a_2^2} + \cdots + \frac{c_n}{x^2 + a_n^2}.$$

(ii) Hence, express $c_1, c_2, \ldots, c_{n-1}$ and c_n in terms of $a_1, a_2, \ldots, a_{n-1}$ and a_n. **1**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Let $z_1, z_2, z_3, ..., z_n$ be the complex roots of $z^n = 1$ where n is a positive integer. Suppose that $\arg(z_{i+1}) - \arg(z_i) = \dfrac{2\pi}{n}$ for $i = 1, 2, 3, ..., n-1$. Show that **2**

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_4)^2 + ... + (z_{n-1} - z_n)^2 + (z_n - z_1)^2 = 0.$$

(b) A particle moves on the x-y plane according to the acceleration vector

$$\underset{\sim}{a} = -n^2(x\underset{\sim}{i} + y\underset{\sim}{j})$$

where n is a positive constant. The particle is initially at the point $(A, B\cos\theta)$ with velocity $\underset{\sim}{v} = -Bn\sin\theta\underset{\sim}{j}$ where A, B and θ are positive constants.

(i) Find the displacement position vector $\underset{\sim}{r}$ in terms of time t. **2**

(ii) Hence, show that the Cartesian equation of the path of the particle is **2**

$$\frac{x^2}{A^2} - \frac{2xy\cos\theta}{AB} + \frac{y^2}{B^2} = \sin^2\theta.$$

(iii) Describe the path of the particle and the *direction* of the motion on the number plane for $\theta = \dfrac{\pi}{2}$. **2**

(c) Let α, β and γ be the roots of the polynomial $P(x) = x^3 + bx^2 + cx + d$ where b, c and d are real, non-zero constants. It is known that **3**

- $\alpha + \beta = 2$; and
- All the roots of $P(x)$ also satisfy the equation $|x^3| = 8$.

Find the values of b, c and d.

(d) Let $\underset{\sim}{u}$ and $\underset{\sim}{v}$ be two distinct vectors in three dimensions. Let a_1 and a_2 be scalars.

Define $\underset{\sim}{0} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$ and suppose that $\underset{\sim}{u} \neq \underset{\sim}{0}$ and $\underset{\sim}{v} \neq \underset{\sim}{0}$.

Consider the following statement:

If $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ then the only solution to $a_1\underset{\sim}{u} + a_2\underset{\sim}{v} = \underset{\sim}{0}$ is $a_1 = a_2 = 0$.

(i) Prove that the above statement is always true. **2**

(ii) Is the converse of the statement always true? Justify your answer with either a proof or a counterexample. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Let z_1, z_2 and z_3 be distinct complex numbers such that for non-zero real values a and b

$$z_3 = \frac{az_1 + ibz_2}{a + ib}.$$

Suppose that A, B and C represent z_1, z_2 and z_3 respectively on the complex plane.

(i) Show that $\triangle ACB$ is a right-angled triangle. **2**

(ii) Suppose that $\triangle ACB$ is also an isosceles right-angled triangle. **1**
State the restrictions on the values of a and b.

(b) Suppose a, b and c are positive real numbers such that $ab + bc + ca = 3$.

(i) Show that **1**

$$abc \leq 1.$$

You may assume, without proof, that

$$\frac{x + y + z}{3} \geq (xyz)^{\frac{1}{3}}.$$

(ii) Hence, show that **2**

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(a + c)} + \frac{1}{1 + c^2(a + b)} \leq \frac{1}{abc}.$$

Question 13 continues on page 9

Question 13 (continued)

(c) Let $H_0(x) = 1$ and define

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

where n is a positive integer and $\frac{d^n}{dx^n}\left(e^{-x^2}\right)$ refers to the nth derivative of e^{-x^2}.

(i) Prove by mathematical induction that **3**

$$H_n(x) = 2xH_{n-1}(x) - \frac{d}{dx}H_{n-1}(x).$$

(ii) Hence, explain why $H_n(x)$ is a polynomial of degree n. **2**

(d) Let $\underset{\sim}{x}$ be a variable position vector X on a sphere with the vector equation

$$|\underset{\sim}{x} - \underset{\sim}{c}| = r$$

where $\underset{\sim}{c}$ is the position vector of the centre C and r is the radius of the sphere.

Let $\underset{\sim}{x}_0$ be a position vector of a fixed point P, lying on the surface of the sphere.

(i) Explain why the vector equation of the plane at P, perpendicular to CP is **1**

$$(\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot (\underset{\sim}{x}_0 - \underset{\sim}{c}) = 0$$

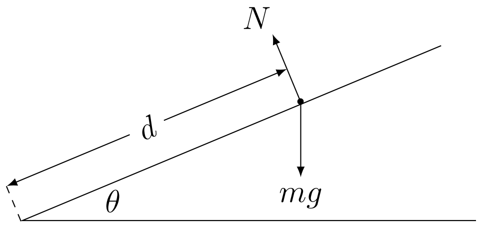
where $\underset{\sim}{x}$ is a variable point on the plane.

(ii) Hence, use the vector equations of the sphere and the plane to algebraically show that they touch only at the point P. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) An object of mass m is initially at rest on a frictionless surface at an angle of θ to the horizontal and a distance of d to the end of the slope. The two forces on the object are the normal force of magnitude N and the gravitational force of magnitude mg.



The direction of the net force on the object is parallel to the inclined surface.

(i) Explain why $N = mg\cos\theta$. **1**

(ii) Show that the speed of the object at the base of the inclined surface is **2**

$$v_d = \sqrt{2gd\sin\theta}.$$

(iii) Let t_d be the time taken for the object to reach the base of the inclined surface. **2**
Show that $d = \dfrac{v_d t_d}{2}$.

(b) Let $z = e^{i\theta}$ where $\theta \neq 2m\pi$ for any integer m.

(i) Show that for non-negative integers n **2**

$$\sum_{k=0}^{n} \mathrm{Re}(z^k) = \left(\frac{\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}}\right)\mathrm{Re}(z^{\frac{n}{2}}).$$

(ii) Hence, show that $\displaystyle\sum_{k=1}^{n} k\sin(k\theta) = \frac{(n+1)\sin(n\theta) - n\sin\left[(n+1)\theta\right]}{2(1-\cos\theta)}$. **3**

(c) Four distinct points A, B, C and D in three dimensional space are represented by the vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ and $\underset{\sim}{d}$ respectively. Suppose that no three points are collinear and $\angle ABC + \angle ADC = \pi$.

(i) If D lies on the same plane as A, B and C, making a quadrilateral $ABCD$, show that $|\underset{\sim}{b} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{c}| + |\underset{\sim}{d} - \underset{\sim}{a}||\underset{\sim}{c} - \underset{\sim}{b}| = |\underset{\sim}{c} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{b}|$. **4**

(ii) Suppose that D does not lie on the same plane as A, B and C. Explain why **1**

$$|\underset{\sim}{b} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{c}| + |\underset{\sim}{d} - \underset{\sim}{a}||\underset{\sim}{c} - \underset{\sim}{b}| > |\underset{\sim}{c} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{b}|.$$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ for non-negative integers n.

(i) Show that for $n \geq 2$ **2**

$$I_n = \frac{n-1}{n} I_{n-2}.$$

(ii) Hence show that **2**

$$I_{2n} = \frac{\pi}{2} \times \frac{(2n)!}{2^{2n}(n!)^2}.$$

(iii) Show that $\dfrac{2n+1}{2n+2} \leq \dfrac{I_{2n+1}}{I_{2n}} \leq 1$. **2**

(iv) It can be shown that **1**

$$I_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!} \quad \text{(Do NOT prove this).}$$

Use this result and the results in part (ii) and part (iii) to deduce that

$$\lim_{n \to \infty} \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} = \frac{\pi}{2}.$$

(v) Let $u_n = \dfrac{e^n n!}{n^{n+\frac{1}{2}}}$ be a sequence of real numbers for non-negative integers of n. **2**

It can be shown that as $n \to \infty$ then $u_n \to L$, where L is a positive constant.

Using the result in part (iv), show that $L = \sqrt{2\pi}$.

Question 15 continues on page 12

(b) Show that for $x > 0$ **2**

$$0 < \ln(1+x) - x + \frac{x^2}{2} < \frac{x^3}{3}.$$

(c) A random variable X which only takes positive values, based on a sample of size n, has a probability density function of

$$f(x) = \frac{x^n e^{-x}}{n!}$$

where n is a positive integer.

Let $Z = \dfrac{X-n}{\sqrt{n}}$ be a random variable with a probability density function $g(z)$.

(i) Show that **1**

$$g(z) = \sqrt{n}\, f(n + z\sqrt{n}).$$

(ii) Hence, using the results in parts (a)(v) and (b), show that for $z > 0$ **3**

$$\lim_{n \to \infty} g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) A student launches an object from the ground at an angle of θ to the horizontal with an initial speed of u. The object is subjected to air resistance such that the acceleration equations are

$$\ddot{x} = -\frac{g\dot{x}}{u}$$
$$\ddot{y} = -g\left(1 + \frac{\dot{y}}{u}\right)$$

where g is the acceleration due to gravity.

You may assume without proof that the displacement equations are

$$x = \frac{u^2}{g}\cos\theta\left(1 - e^{-\frac{gt}{u}}\right)$$
$$y = \frac{u^2}{g}(1 + \sin\theta)\left(1 - e^{-\frac{gt}{u}}\right) - ut.$$

(i) Let the range of the motion be R. Show that **2**

$$R = \frac{u^2}{g}\cos\theta\left(1 - e^{-f(\theta)}\right)$$

where $f(\theta) = \dfrac{gR}{u^2}(\sec\theta + \tan\theta)$.

(ii) The student now wishes to find the launch angle such that the range of the object's motion is maximised. Let the range be $R(\theta)$ as a function of θ. Show that the launch angle where $R'(\theta) = 0$ satisfies **4**

$$\sin\theta = \frac{1}{e-1}.$$

(iii) Hence, show that R is maximised by a launch angle of $\theta = \sin^{-1}\left(\dfrac{1}{e-1}\right)$. **2**

Question 16 continues on page 14

(b) Suppose that, for the purposes of this question, either $\sqrt[3]{2}$ or $\sqrt[3]{4}$ is rational.

(i) Prove the following statement. **2**

$$\sqrt[3]{2} \text{ is a rational number if and only if } \sqrt[3]{4} \text{ is a rational number.}$$

(ii) Let S_2 be the set of positive integers m such that $m\sqrt[3]{2}$ is an integer. **1**

Similarly, let S_4 be the set of positive integers n such that $n\sqrt[3]{4}$ is an integer.

Define $S = S_2 \cap S_4$. Show that there is at least one positive integer in S.

(iii) Let k be any positive integer where $k \in S$. Show that $[(\sqrt[3]{4} - 1)k] \in S$. **2**

(iv) Hence, prove that both $\sqrt[3]{2}$ and $\sqrt[3]{4}$ are, in fact, irrational. **2**

End of paper

2020 **BORED OF STUDIES TRIAL EXAMINATION**

Mathematics Extension 2

Solutions

Section I

Answers

1	(B)	**6**	(C)
2	(B)	**7**	(B)
3	(B)	**8**	(A)
4	(B)	**9**	(A)
5	(C)	**10**	(D)

Brief explanations

1 Equating the first and second components gives

$$1 + 2\lambda = 2 + 4\mu$$
$$2 + 4\lambda = 3 + 8\mu$$

Doubling the first equation and subtracting the second equation gives $0 = 1$ which is false, hence no solution exists and therefore no point of intersection exists.

Consider the direction vectors $\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$. The latter can be written as $2\begin{pmatrix} 2 \\ 4 \\ \frac{1}{2} \end{pmatrix}$ so the lines cannot be parallel. Hence, the answer is (B).

2 Let $z = x + iy$.

The first inequality gives $x \leq 0$. The second inequality gives $-\dfrac{1}{\sqrt{3}} \leq \dfrac{y-1}{x+1} \leq \dfrac{1}{\sqrt{3}}$.

This this forms a triangle with base length of 1 and a perpendicular height of $\dfrac{2}{\sqrt{3}}$, giving an area of $\dfrac{1}{\sqrt{3}}$. Hence, the answer is (B).

3 Consider the counterexample $f(x) = x$ and $g(x) = \dfrac{\ln a}{\ln x}$ for some positive constant a. $(f(x))^{g(x)}$ simplifies to just some positive constant a. Therefore, neither (A) nor (C) are candidate answers.

Any real power of a non-negative number is non-negative, so the answer cannot be (D). Hence, the answer is (B).

4 As the stationary points are in the first and second quadrants, there is only one real root.

Since $P(x)$ is monic then as $x \to -\infty$ then $P(x) \to -\infty$ and as $x \to \infty$ then $P(x) \to \infty$. However, at the stationary points $P(x) \geq 0$, which suggests that $P(x)$ has an x-intercept between the second and the third quadrant.

This means neither (A) nor (C) are correct. Since the remaining roots are non-real, they must occur in complex conjugates which is shown in (B).

5 The initial velocity is positive, as the object is projected upwards. This means neither (B) nor (D) are correct. Resistance and gravitational forces opposing the object's motion mean that the velocity decreases on the way up. It will eventually reach a maximum height at rest before falling back down. The conditions for terminal velocity do not occur until the object falls back down. Hence, the answer is (C).

6 Since $\underset{\sim}{a}$ is on the line and $\underset{\sim}{p}$ is outside the line, consider the projection of the vector $\underset{\sim}{a} - \underset{\sim}{p}$ in the direction on $\hat{\underset{\sim}{n}}$. This is the vector $\left[(\underset{\sim}{a} - \underset{\sim}{p}) \cdot \hat{\underset{\sim}{n}}\right] \hat{\underset{\sim}{n}}$. The shortest distance is the perpendicular distance from $\underset{\sim}{p}$ to the line, hence the vector representing that is $\left[(\underset{\sim}{a} - \underset{\sim}{p}) \cdot \hat{\underset{\sim}{n}}\right] \hat{\underset{\sim}{n}} - (\underset{\sim}{a} - \underset{\sim}{p})$. The perpendicular distance is the length of that vector, hence the answer is (C).

7 The general form of the acceleration equation is $\ddot{x} = -n^2(x - c)$ where c is the centre of motion and the period is $\frac{2\pi}{n}$. Comparing this to the given equation $\ddot{x} = -kx$, it can be deduced that the centre is the origin and the period is $\frac{2\pi}{\sqrt{k}}$ which is a known value.
Since the particle is initially at the origin, the initial acceleration can be deduced as zero.

Consider a general solution to the acceleration equation is $x = a\sin(\sqrt{k}\,t + \alpha)$. When $t = 0$, $x = 0$ so $\alpha = 0$, which means that $x = a\sin(\sqrt{k}\,t)$. This equation still satisfies the acceleration equation and initial conditions for any value of a. Hence, the amplitude cannot be specifically determined, so the answer is (B).

8 Translating the first part of the logic notation, which is identical for all choices

For every positive ϵ there exists a positive δ, such that for every x in the domain D...

Choices (B) and (D) are not correct because of the $|f(x)| > \epsilon$ component. This suggests that $|f(x)|$ is larger than every positive number in the domain. This does not imply it is approaching zero as x approaches zero, but rather potentially approaching $\pm infty$.

The remaining options are now based on deducing whether $|x| < \delta$ or $|x| > \delta$ is correct. Consider a simple example where $f(x) = x^2$ so this limit result is true. If (C) is correct then say for $\epsilon = 0.01$, we should be able to find a positive δ satisfying $|x| > \delta$ so that $x^2 < 0.01$. The latter inequality implies $-0.1 < x < 0.1$. However, it is not possible to find a positive δ which satisfies $|x| > \delta$ and gives $-0.1 < x < 0.1$.

Instead, it is possible to find a positive δ satisfying $|x| < \delta$ so that $x^2 < 0.01$ (namely $\delta = 0.1$). Hence, the answer is (A).

9 Use the fact that $z\overline{z} = |z|^2$ as well as $\overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2}$. After adding $a\overline{a}$ to both sides of the equality, each option can be rewritten as follows:

(A) $(a-b)(\overline{a-b}) = (a-c)(\overline{a-c})$ (B) $(a+b)(\overline{a+b}) = (a-c)(\overline{a-c})$

(C) $(a-b)(\overline{a-b}) = (a+c)(\overline{a+c})$ (D) $(a+b)(\overline{a+b}) = (a+c)(\overline{a+c})$

Using the fact that $\sqrt{z\overline{z}} = |z|$, these are equivalent to:

(A) $|a-b| = |a-c|$ (B) $|a+b| = |a-c|$

(C) $|a-b| = |a+c|$ (D) $|a+b| = |a+c|$

The only option that is related to the sides of the triangle formed by a, b and c is option (A) which implies that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal in length, or equivalently $\triangle ABC$ is isosceles.

10 The correct answer is (D). Note that the following choices are actually equivalent.

(A*) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2\tan x}{\tan 2x}\,dx$ (B*) $\displaystyle\int_0^1 2x\tan^{-1} x\,dx$

(C*) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{2x\sin x}{\cos^3 x}\,dx$ (D) $\displaystyle\int_0^1 \left(\frac{1}{2}\ln(1+x^2) + \tan^{-1} x\right) dx$

(A*) follows from substituting $x = \tan\theta$ and changing dummy variables, then manipulating with the tangent double angle identity.

(B*) follows from integrating by parts, using $u = 1 - x^2$ and $dv = \dfrac{1}{1+x^2}\,dx$.

(C*) follows from (B*) by substituting $x = \tan\theta$ and changing dummy variables.

(D) follows from using integration by parts using $u = 1 - x$ and $dv = \dfrac{1+x}{1+x^2}\,dx$.

$$\begin{aligned}\int_0^1 \frac{1-x^2}{1+x^2}\,dx &= \left[(1-x)\left(\tan^{-1} x + \frac{1}{2}\ln(1+x^2)\right)\right]_0^1 + \int_0^1 \left(\tan^{-1} x + \frac{1}{2}\ln(1+x^2)\right) dx \\ &= \int_0^1 \left(\tan^{-1} x + \frac{1}{2}\ln(1+x^2)\right) dx\end{aligned}$$

Question 11

(a) Let $x = \dfrac{1}{u} \Rightarrow dx = -\dfrac{1}{u^2}\,du$. As $x \to 0^+$, $u \to \infty$ and as $x \to \infty$, $u \to 0$.

Applying this substitution and letting

$$\begin{aligned}
I &= \int_0^\infty \frac{1}{x^8 + x^4 + 1}\,dx \\
&= -\int_\infty^0 \frac{1}{\frac{1}{u^8} + \frac{1}{u^4} + 1} \times \frac{1}{u^2}\,du \\
&= \int_0^\infty \frac{u^6}{u^8 + u^4 + 1}\,du \\
&= \int_0^\infty \frac{x^6}{x^8 + x^4 + 1}\,dx \quad \text{since } u \text{ is a dummy variable} \\
2I &= \int_0^\infty \frac{x^6}{x^8 + x^4 + 1}\,dx + \int_0^\infty \frac{1}{x^8 + x^4 + 1}\,dx \\
I &= \frac{1}{2}\int_0^\infty \frac{x^6 + 1}{x^8 + x^4 + 1}\,dx
\end{aligned}$$

The given identity can be used for $a = x^2$ to obtain $x^6 + 1 = (x^4 - x^2 + 1)(x^2 + 1)$. The denominator can also be manipulated as follows.

$$\begin{aligned}
x^8 + x^4 + 1 &= (x^8 + 2x^4 + 1) - x^4 \\
&= (x^4 + 1)^2 - (x^2)^2 \\
&= (x^4 - x^2 + 1)(x^4 + x^2 + 1)
\end{aligned}$$

This means that

$$\begin{aligned}
I &= \frac{1}{2}\int_0^\infty \frac{(x^2 + 1)(x^4 - x^2 + 1)}{(x^4 - x^2 + 1)(x^4 + x^2 + 1)}\,dx \\
&= \frac{1}{2}\int_0^\infty \frac{x^2 + 1}{x^4 + x^2 + 1}\,dx \\
&= \frac{1}{2}\int_0^\infty \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}}\,dx \\
&= \frac{1}{2}\int_0^\infty \frac{1 + \frac{1}{x^2}}{(x - \frac{1}{x})^2 + 3}\,dx
\end{aligned}$$

Consider the substitution $y = x - \dfrac{1}{x} \Rightarrow dy = \left(1 + \dfrac{1}{x^2}\right)\,dx$.

As $x \to 0^+, y \to -\infty$ and as $x \to \infty, y \to \infty$. Applying this substitution gives

$$\begin{aligned}
I &= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{y^2 + 3}\,dy \\
&= \frac{1}{2\sqrt{3}}\left[\tan^{-1}\frac{y}{\sqrt{3}}\right]_{-\infty}^\infty \\
&= \frac{\pi}{2\sqrt{3}}
\end{aligned}$$

Remark: Another approach to evaluating $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+x^2+1}\,dx$ could be to notice that

$$\begin{aligned}
x^4+x^2+1 &= (x^4+2x^2+1)-x^2\\
&= (x^2+1)^2-x^2\\
&= (x^2-x+1)(x^2+x+1)
\end{aligned}$$

Using some algebraic manipulation

$$\begin{aligned}
\int_0^\infty \frac{x^2+1}{x^4+x^2+1}\,dx &= \frac{1}{2}\int_0^\infty \frac{2x^2+2-x+x}{(x^2-x+1)(x^2+x+1)}\,dx\\
&= \frac{1}{2}\int_0^\infty \frac{x^2+x+1+x^2-x+1}{(x^2-x+1)(x^2+x+1)}\,dx\\
&= \frac{1}{2}\int_0^\infty \left(\frac{1}{x^2+x+1}+\frac{1}{x^2-x+1}\right)dx\\
&= \frac{1}{2}\int_0^\infty \left(\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right)dx\\
&= \frac{1}{2}\times\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^\infty\\
&= \frac{\pi}{\sqrt{3}}
\end{aligned}$$

$$\text{Hence}\qquad I = \frac{\pi}{2\sqrt{3}}$$

(b) Let A and B represent the complex numbers z_1 and z_2 respectively on the complex plane.

Since the coefficients are real then the non-real roots must occur in conjugate pairs on the complex plane. These roots are

$$x = \frac{-b\pm i\sqrt{4ac-b^2}}{2a}$$

The circle of the smallest radius that passes through A and B would be one where AB is the diameter of the circle. This means that the centre of the circle is the midpoint of AB and the radius is half the length of AB. Hence, the equation of the circle is

$$\left(x+\frac{b}{2a}\right)^2+y^2 = \frac{4ac-b^2}{4a^2}$$

(c) If the lines ℓ_1 and ℓ_2 have a point of intersection then these are the solutions to

$$\begin{aligned}
\underset{\sim}{a}_1+\lambda_1\underset{\sim}{b}_1 &= \underset{\sim}{a}_2+\lambda_2\underset{\sim}{b}_2\\
(\underset{\sim}{a}_1+\lambda_1\underset{\sim}{b}_1)\cdot\underset{\sim}{n} &= (\underset{\sim}{a}_2+\lambda_2\underset{\sim}{b}_2)\cdot\underset{\sim}{n}\\
\underset{\sim}{a}_1\cdot\underset{\sim}{n}+\lambda_1\underset{\sim}{b}_1\cdot\underset{\sim}{n} &= \underset{\sim}{a}_2\cdot\underset{\sim}{n}+\lambda_2\underset{\sim}{b}_2\cdot\underset{\sim}{n} \qquad \text{but } \underset{\sim}{n} \text{ is perpendicular to both } \underset{\sim}{b}_1 \text{ and } \underset{\sim}{b}_2\\
\underset{\sim}{a}_1\cdot\underset{\sim}{n} &= \underset{\sim}{a}_2\cdot\underset{\sim}{n}\\
(\underset{\sim}{a}_1-\underset{\sim}{a}_2)\cdot\underset{\sim}{n} &= 0
\end{aligned}$$

(d) (i) Starting from the acceleration equation

$$\begin{aligned}
\ddot{x} &= -kv^n \\
v\frac{dv}{dx} &= -kv^n \\
\int \frac{dv}{v^{n-1}} &= -k\int dx \\
-\frac{1}{(n-2)v^{n-2}} &= -kx + c \quad \text{when } x = 0, v = u \Rightarrow c = -\frac{1}{(n-2)u^{n-2}} \\
kx &= \frac{1}{(n-2)v^{n-2}} - \frac{1}{(n-2)u^{n-2}} \\
x &= \frac{u^{n-2} - v^{n-2}}{k(n-2)(uv)^{n-2}}
\end{aligned}$$

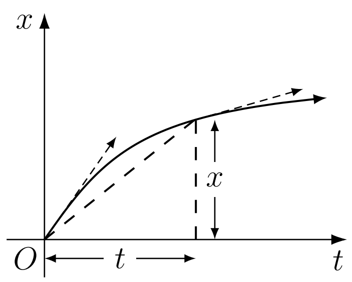
(ii) Applying the same approach similarly for the time derivative of acceleration

$$\begin{aligned}
\frac{dv}{dt} &= -kv^n \\
t &= \frac{u^{n-1} - v^{n-1}}{k(n-1)(uv)^{n-1}}
\end{aligned}$$

The average speed of the object from the origin to displacement x is

$$\begin{aligned}
v_{ave} &= \frac{x-0}{t-0} \\
&= \frac{u^{n-2} - v^{n-2}}{k(n-2)(uv)^{n-2}} \times \frac{k(n-1)(uv)^{n-1}}{u^{n-1} - v^{n-1}} \\
&= \frac{a_{n-2}(uv)}{a_{n-1}}
\end{aligned}$$

Note that the acceleration equation is always negative. This means that from the initial position, the object is slowing down while moving in the positive direction. This also means that a displacement-time graph of the object's motion will be concave down and increasing.



This means that the average speed (the gradient of the secant) at a given point is lower than the initial speed (slope of the tangent to the displacement curve at $t = 0$) but higher than the instantaneous speed (slope of the tangent to the displacement curve at time $t > 0$).

Hence

$$\begin{aligned} v < v_{ave} &< u \\ v < \frac{a_{n-2}(uv)}{a_{n-1}} &< u \\ \frac{1}{u} < \frac{a_{n-2}}{a_{n-1}} &< \frac{1}{v} \\ v < \frac{a_{n-1}}{a_{n-2}} &< u \end{aligned}$$

(e) (i) By the method for partial fractions, there exist pairs of real constants $(b_1, c_1), (b_2, c_2), \ldots, (b_{n-1}, c_{n-1})$ and (b_n, c_n), such that

$$\frac{1}{P(x)} = \frac{b_1x + c_1}{x^2 + a_1^2} + \frac{b_2x + c_2}{x^2 + a_2^2} + \cdots + \frac{b_nx + c_n}{x^2 + a_n^2}.$$

Notice that

$$\frac{1}{P(-x)} = \frac{-b_1x + c_1}{x^2 + a_1^2} + \frac{-b_2x + c_2}{x^2 + a_2^2} + \cdots + \frac{-b_nx + c_n}{x^2 + a_n^2}.$$

But $P(x)$ is an even function so $P(-x) = P(x)$, hence

$$\begin{aligned} \frac{1}{P(x)} + \frac{1}{P(-x)} &= \frac{2c_1}{x^2 + a_1^2} + \frac{2c_2}{x^2 + a_2^2} + \cdots + \frac{2c_n}{x^2 + a_n^2} \\ \frac{1}{P(x)} &= \frac{c_1}{x^2 + a_1^2} + \frac{c_2}{x^2 + a_2^2} + \cdots + \frac{c_n}{x^2 + a_n^2} \end{aligned}$$

(ii) Making c_1 the subject from part (i)

$$c_1 = \frac{1}{(x^2 + a_2^2)(x^2 + a_3^2)\ldots(x^2 + a_n^2)} - (x^2 + a_1^2)\left(\frac{c_2}{x^2 + a_2^2} + \frac{c_3}{x^2 + a_3^2} + \cdots + \frac{c_n}{x^2 + a_n^2}\right)$$

Let $x = ia_1$, which implies $x^2 + a_1^2 = 0$ hence

$$c_1 = \frac{1}{(a_2^2 - a_1^2)(a_3^2 - a_1^2)\ldots(a_n^2 - a_1^2)}$$

Applying this approach similarly for all the other constants gives

$$c_2 = \frac{1}{(a_1^2 - a_2^2)(a_3^2 - a_2^2)\ldots(a_n^2 - a_2^2)}$$

$$\vdots$$

$$c_n = \frac{1}{(a_1^2 - a_n^2)(a_2^2 - a_n^2)\ldots(a_{n-1}^2 - a_n^2)}$$

Question 12

(a) Let the principal root be $z_1 = e^{\frac{2\pi i}{n}}$, and note that $z_1^n = 1$ and $z_k = z_1^k$

$$
\begin{aligned}
&(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_4)^2 + ... + (z_{n-1} - z_n)^2 + (z_n - z_1)^2 \\
&= (z_1 - z_1^2)^2 + (z_1^2 - z_1^3)^2 + (z_1^3 - z_1^4)^2 + ... + (z_1^{n-1} - z_1^n)^2 + (z_1^n - z_1)^2 \\
&= z_1^2(1 - z_1)^2 + z_1^4(1 - z_1)^2 + z_1^6(1 - z_1)^2 + ... + z_1^{2(n-1)}(1 - z_1)^2 + (1 - z_1)^2 \\
&= (1 - z_1)^2(1 + z_1^2 + z_1^4 + ... + z_1^{2(n-1)}) \\
&= (1 - z_1)^2 \times \frac{z_1^{2n} - 1}{z_1^2 - 1} \\
&= 0 \quad \text{since } z_1^n = 1 \Rightarrow z_1^{2n} = 1
\end{aligned}
$$

(b) (i) Notice that each component of the acceleration shows simple harmonic motion about the origin ($a_x = -n^2 x$ and $a_y = -n^2 y$). Hence, the general solutions for the displacement equations can be used.

$$
\begin{aligned}
x &= \hat{a}\cos(nt + \alpha) \\
v_x &= -\hat{a}n\sin(nt + \alpha) \\
\text{when } t &= 0, x = A, v_x = 0 \\
A &= \hat{a}\cos\alpha \\
0 &= -\hat{a}n\sin\alpha
\end{aligned}
$$

Since $\hat{a} \neq 0$ and $n \neq 0$ then $\alpha = 0$, which means that $\hat{a} = A$.
Similarly, consider the general vertical displacement equation

$$
\begin{aligned}
y &= \hat{b}\cos(nt + \beta) \\
v_y &= -\hat{b}n\sin(nt + \beta) \\
\text{when } t &= 0, y = B\cos\theta, v_y = -Bn\sin\theta \\
B\cos\theta &= \hat{b}\cos\beta \\
B\sin\theta &= \hat{b}\sin\beta \\
B^2(\cos^2\theta + \sin^2\theta) &= \hat{b}^2(\cos^2\beta + \sin^2\beta) \\
\Rightarrow \hat{b} &= B \\
\tan\theta &= \tan\beta \\
\Rightarrow \beta &= \theta
\end{aligned}
$$

Hence the displacement vector is

$$\underset{\sim}{r} = A\cos(nt)\underset{\sim}{i} + B\cos(nt + \theta)\underset{\sim}{j}$$

Remark: Could have chosen $\hat{b} = -B$ instead, which would lead to a different value for β but lead to an equivalent equation.

(ii) From the displacement equations $x = A\cos(nt)$ and $y = A\cos(nt+\theta)$

$$\begin{aligned}
LHS &= \frac{x^2}{A^2} - \frac{2xy\cos\theta}{AB} + \frac{y^2}{B^2} \\
&= \frac{A^2\cos^2(nt)}{A^2} - \frac{2AB\cos(nt)\cos(nt+\theta)\cos\theta}{AB} + \frac{B^2\cos^2(nt+\theta)}{B^2} \\
&= \cos^2(nt) + \cos(nt+\theta)[\cos(nt+\theta) - 2\cos(nt)\cos\theta] \\
&= \cos^2(nt) + \cos(nt+\theta)[\cos(nt)\cos\theta - \sin(nt)\sin\theta - 2\cos(nt)\cos\theta] \\
&= \cos^2(nt) - \cos(nt+\theta)[\cos(nt)\cos\theta + \sin(nt)\sin\theta] \\
&= \cos^2(nt) - \cos(nt+\theta)\cos(nt-\theta) \\
&= \cos^2(nt) - [\cos(nt)\cos\theta - \sin(nt)\sin\theta][\cos(nt)\cos\theta + \sin(nt)\sin\theta] \\
&= \cos^2(nt) - \cos^2(nt)\cos^2\theta + \sin^2(nt)\sin^2\theta \\
&= \cos^2(nt)(1-\cos^2\theta) + \sin^2(nt)\sin^2\theta \\
&= \sin^2\theta(\cos^2(nt) + \sin^2(nt)) \\
&= \sin^2\theta \\
&= RHS
\end{aligned}$$

Hence, the Cartesian equation of the particle's path is

$$\frac{x^2}{A^2} - \frac{2xy\cos\theta}{AB} + \frac{y^2}{B^2} = \sin^2\theta.$$

(iii) When $\theta = \frac{\pi}{2}$, the particle's path follows the following Cartesian equation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

This curve has x-intercepts $(-A, 0)$ and $(A, 0)$, and y-intercepts $(0, B)$ and $(0, -B)$ (it is in fact an ellipse, which reduces to a circle if $A = B$).

From the conditions given in the question, the particle is initially at $(A, 0)$ with velocity $\underset{\sim}{v} = -Bn\underset{\sim}{j}$. This means that the particle starts at the positive x-intercept and is moving in the negative y direction. As it continues to move along the x-y plane, it will trace the path in a clockwise direction.

(c) Since the $P(x)$ has real coefficients, it can either have three real roots or two complex conjugate roots and one real root. However, there are only two real roots which satisfy $|x^3| = 8$ (namely $x = \pm 2$). This means that $P(x)$ must have two complex conjugate roots and one real root.

Given that $\alpha + \beta = 2$, the roots α and β must be the two complex conjugate roots, in order to sum to a real number. Let these be $1 \pm ia$ for some real a.

$$\begin{aligned}
|(1+ia)^3| &= 8 \\
|1+ia|^3 &= 8 \\
(\sqrt{1+a^2})^3 &= 8 \\
1 + a^2 &= 4 \quad \text{noting that } a \text{ is real} \\
a &= \pm\sqrt{3}
\end{aligned}$$

This means that the two complex conjugate roots of $P(x)$ are $1 \pm i\sqrt{3}$. Let the real root be γ. Taking the sum of the roots in pairs

$$\begin{aligned} \alpha\beta + \beta\gamma + \alpha\gamma &= c \\ (1+i\sqrt{3})(1-i\sqrt{3}) + \gamma(\alpha+\beta) &= c \\ 4 + 2\gamma &= c \end{aligned}$$

The only real roots to $|x^3| = 8$ are $x = \pm 2$. However, $\gamma \neq -2$ since c must be a non-zero coefficient. Hence, $\gamma = 2$ is the real root. This gives $c = 8$. To find the remaining coefficients b and d, use sum and product of roots.

$$\begin{aligned} \alpha + \beta + \gamma = -b &\Rightarrow b = -4 \\ \alpha\beta\gamma = -d &\Rightarrow d = -8 \end{aligned}$$

Hence $b = -4, c = 8$ and $d = -8$.

(d) (i) If $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ then consider

$$\begin{aligned} (a_1\underset{\sim}{u} + a_2\underset{\sim}{v}) \cdot \underset{\sim}{v} &= \underset{\sim}{0} \cdot \underset{\sim}{v} \\ a_1\underset{\sim}{u} \cdot \underset{\sim}{v} + a_2\underset{\sim}{v} \cdot v &= 0 \\ a_2|\underset{\sim}{v}|^2 &= 0 \end{aligned}$$

Since $\underset{\sim}{v} \neq \underset{\sim}{0}$ then $|\underset{\sim}{v}|^2 > 0$ so it must be that $a_2 = 0$. This, in turn, implies that

$$a_1\underset{\sim}{u} = \underset{\sim}{0}$$

Since $\underset{\sim}{u} \neq \underset{\sim}{0}$ then it must be that $a_1 = 0$. Hence, the statement is always true.

(ii) The converse statement is

If the only solution to $a_1\underset{\sim}{u} + a_2\underset{\sim}{v} = \underset{\sim}{0}$ is $a_1 = a_2 = 0$ then $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

Consider

$$\underset{\sim}{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

The vector equation $a_1\underset{\sim}{u} + a_2\underset{\sim}{v} = \underset{\sim}{0}$ becomes

$$\begin{pmatrix} a_1 + 2a_2 \\ a_1 + a_2 \\ a_1 + 2a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The **only** way this is possible is if $a_1 = a_2 = 0$ (i.e. there are no other possible values of a_1 and a_2 which could satisfy the equations). However, $\underset{\sim}{u} \cdot v = 5$ which is non-zero. This is a counterexample to the converse statement. Hence, the converse statement is not necessarily true.

Question 13

(a) (i) Rearranging the given relation

$$\begin{aligned} z_3 &= \frac{az_1 + ibz_2}{a + ib} \\ az_3 + ibz_3 &= az_1 + ibz_2 \\ -a(z_1 - z_3) &= ib(z_2 - z_3) \\ z_1 - z_3 &= -\frac{ib}{a}(z_2 - z_3) \end{aligned}$$

Since the vectors $\overrightarrow{CB}$ (represented by $z_2 - z_3$) and $\overrightarrow{CA}$ (represented by $z_1 - z_3$) are related by multiplication of i and a scalar, then CB and CA are right angles relative to each other. Hence, $\triangle ACB$ is a right-angled triangle.

(ii) For $\triangle ACB$ to be isosceles $CA = CB$ is required so

$$\begin{aligned} |z_1 - z_3| &= |z_2 - z_3| \\ \left| -\frac{ib}{a}(z_2 - z_3) \right| &= |z_2 - z_3| \\ |a| &= |b| \end{aligned}$$

Hence a and b must be equal in magnitude for $\triangle ACB$ to be isosceles.

(b) (i) Using the given inequality, let $x = ab, y = bc$ and $z = ca$

$$\begin{aligned} \frac{ab + bc + ca}{3} &\geq (abc)^{\frac{2}{3}} \quad \text{but } ab + bc + ca = 3 \\ (abc)^{\frac{2}{3}} &\leq 1 \\ abc &\leq 1 \end{aligned}$$

(ii) Since $abc \leq 1$ from part (i) and $ab + bc + ca = 3$

$$\begin{aligned} \frac{1}{1 + a^2(b + c)} &= \frac{1}{1 + a(ab + ac)} \\ &= \frac{1}{1 + a(3 - bc)} \\ &= \frac{1}{1 - abc + 3a} \\ &\leq \frac{1}{3a} \end{aligned}$$

Similarly applying this to the other fractions gives

$$\begin{aligned} \frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(a + c)} + \frac{1}{1 + c^2(a + b)} &\leq \frac{1}{3a} + \frac{1}{3b} + \frac{1}{3c} \\ &= \frac{ab + bc + ca}{3abc} \\ &= \frac{1}{abc} \end{aligned}$$

(c) (i) Using the definition of $H_1(x)$ for $n = 1$

$$\begin{aligned}
LHS &= H_1(x) \\
&= (-1)^1 e^{x^2} \frac{d}{dx}\left(e^{-x^2}\right) \\
&= -e^{x^2}\left(-2xe^{-x^2}\right) \\
&= 2x \\
RHS &= 2xH_0(x) - \frac{d}{dx}H_0(x) \quad \text{but } H_0(x) = 1 \\
&= 2x
\end{aligned}$$

$LHS = RHS$ so the statement is true for $n = 1$.

Assume the statement is true for $n = k$, that is

$$H_k(x) = 2xH_{k-1}(x) - \frac{d}{dx}H_{k-1}(x).$$

Required to prove that for $n = k + 1$

$$H_{k+1}(x) = 2xH_k(x) - \frac{d}{dx}H_k(x).$$

$$\begin{aligned}
LHS &= H_{k+1}(x) \\
&= (-1)^{k+1} e^{x^2} \frac{d^{k+1}}{dx^{k+1}}\left(e^{-x^2}\right) \\
&= (-1)^{k+1} e^{x^2} \frac{d}{dx}\left[\frac{d^k}{dx^k}\left(e^{-x^2}\right)\right] \\
&= (-1)^{k+1} e^{x^2} \frac{d}{dx}\left[\frac{H_k(x)}{(-1)^k e^{x^2}}\right] \\
&= -e^{x^2}\left[e^{-x^2}\frac{d}{dx}H_k(x) - 2xe^{-x^2}H_k(x)\right] \\
&= 2xH_k(x) - \frac{d}{dx}H_k(x) \\
&= RHS
\end{aligned}$$

Hence, the statement is true by induction for all positive integers n.

(ii) If $P(x)$ and $Q(x)$ are some polynomials, then $P(x) + Q'(x)$ is also a polynomial (since $Q'(x)$ is a polynomial). From part (i), $H_0(x)$ and $H_1(x)$ are polynomials, hence $H_n(x)$ must be a polynomial.

The degree of $2xH_{n-1}(x)$ is one more than the degree of $H_{n-1}(x)$ and the degree of $\frac{d}{dx}H_{n-1}(x)$ is at least one less than the degree of $H_{n-1}(x)$.

Therefore, the degree of $H_n(x) = 2xH_{n-1}(x) - \frac{d}{dx}H_{n-1}(x)$ must be one more than the degree of $H_{n-1}(x)$ since the power of the leading term of $2xH_{n-1}(x)$ will never coincide (or cancel out) with a term from $\frac{d}{dx}H_{n-1}(x)$. In other words, $\deg(H_n(x)) = \deg(H_{n-1}(x)) + 1$. Since $H_0(x)$ is of degree zero and $H_1(x)$ is degree one, then $H_n(x)$ must be degree n when applying this recurrence relationship.

(d) (i) If $\underset{\sim}{x}$ is some variable point on the plane and $\underset{\sim}{x}_0$ also lies on this plane, then the vector $\underset{\sim}{x} - \underset{\sim}{x}_0$ is parallel to the plane. This is perpendicular to the vector CP, which is represented by $\underset{\sim}{x}_0 - \underset{\sim}{c}$. Hence, their dot product must be zero.

$$(\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot (\underset{\sim}{x}_0 - \underset{\sim}{c}) = 0$$

(ii) From the equation of the sphere

$$\begin{aligned}
|\underset{\sim}{x} - \underset{\sim}{c}| &= r \\
(\underset{\sim}{x} - \underset{\sim}{c}) \cdot (\underset{\sim}{x} - \underset{\sim}{c}) &= r^2 \\
\underset{\sim}{x} \cdot \underset{\sim}{x} - 2\underset{\sim}{x} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c} &= r^2
\end{aligned}$$

From the equation of the plane

$$\begin{aligned}
(\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot (\underset{\sim}{x}_0 - \underset{\sim}{c}) &= 0 \\
\underset{\sim}{x} \cdot \underset{\sim}{x}_0 - \underset{\sim}{x} \cdot \underset{\sim}{c} - \underset{\sim}{x}_0 \cdot \underset{\sim}{x}_0 + \underset{\sim}{x}_0 \cdot \underset{\sim}{c} &= 0 \\
2\underset{\sim}{x} \cdot \underset{\sim}{x}_0 - 2\underset{\sim}{x} \cdot \underset{\sim}{c} - 2\underset{\sim}{x}_0 \cdot \underset{\sim}{x}_0 + 2\underset{\sim}{x}_0 \cdot \underset{\sim}{c} &= 0
\end{aligned}$$

Subtracting the two equations

$$\begin{aligned}
\underset{\sim}{x} \cdot \underset{\sim}{x} + \underset{\sim}{c} \cdot \underset{\sim}{c} - 2\underset{\sim}{x} \cdot \underset{\sim}{x}_0 + 2\underset{\sim}{x}_0 \cdot \underset{\sim}{x}_0 - 2\underset{\sim}{x}_0 \cdot \underset{\sim}{c} &= r^2 \\
\underset{\sim}{x} \cdot \underset{\sim}{x} - 2\underset{\sim}{x} \cdot \underset{\sim}{x}_0 + \underset{\sim}{x}_0\underset{\sim}{x}_0 + \underset{\sim}{x}_0\underset{\sim}{x}_0 - 2\underset{\sim}{x}_0 \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c} &= r^2 \\
(\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) + (\underset{\sim}{x}_0 - \underset{\sim}{c}) \cdot (\underset{\sim}{x}_0 - \underset{\sim}{c}) &= r^2 \\
|\underset{\sim}{x} - \underset{\sim}{x}_0|^2 + |\underset{\sim}{x}_0 - \underset{\sim}{c}|^2 &= r^2
\end{aligned}$$

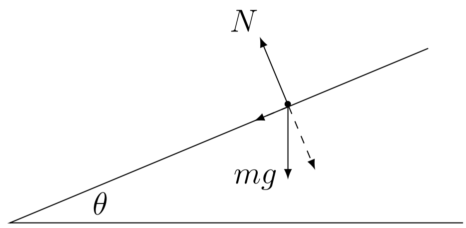
Since $\underset{\sim}{x}_0$ lies on the sphere then $|\underset{\sim}{x}_0 - \underset{\sim}{c}|^2 = r^2$, hence

$$|\underset{\sim}{x} - \underset{\sim}{x}_0|^2 = 0$$

The solution(s) to $\underset{\sim}{x}$ when solving the equations of the sphere and plane simultaneously should be the point(s) of intersection. From the above result there is only one solution for $\underset{\sim}{x}$, namely $\underset{\sim}{x}_0$. This means that the plane only touches the sphere at one point P (i.e. it is a tangent plane).

Question 14

(a) (i) Projecting the gravitational force vector in the direction of the normal force gives a vector of magnitude $mg\cos\theta$.



The direction of the net force on the object is parallel to the the inclined surface. This means that the net force perpendicular to the inclined surface must be zero. Hence, $N = mg\cos\theta$.

(ii) Let x, v and a be the displacement, velocity and acceleration of the object respectively relative to its initial point. Define the direction of motion as the positive direction.

$$\begin{aligned} ma &= mg\sin\theta \\ \frac{d}{dx}\left(\frac{v^2}{2}\right) &= g\sin\theta \\ \frac{v^2}{2} &= gx\sin\theta + c \quad \text{when } x = 0, v = 0 \Rightarrow c = 0 \\ v^2 &= 2gx\sin\theta \end{aligned}$$

When $x = d$, the speed of the object at the base of the inclined surface is

$$v_d = \sqrt{2gd\sin\theta}.$$

(iii) From the acceleration equation

$$\begin{aligned} \frac{dv}{dt} &= g\sin\theta \\ v &= gt\sin\theta + c \quad \text{when } t = 0, v = 0 \Rightarrow c = 0 \\ v &= gt\sin\theta \end{aligned}$$

When $t = t_d$ and $v = v_d$ so

$$\begin{aligned} v_d &= gt_d\sin\theta \\ v_d^2 &= gv_dt_d\sin\theta \quad \text{but } v_d = \sqrt{2gd\sin\theta} \\ 2gd\sin\theta &= gv_dt_d\sin\theta \\ d &= \frac{v_tt_d}{2} \end{aligned}$$

(b) (i) From the geometric series and the results $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

$$\begin{aligned}
\sum_{k=0}^{n} z^k &= \frac{z^{n+1}-1}{z-1} \times \frac{z^{-\frac{n+1}{2}}}{z^{-\frac{n+1}{2}}} \\
&= \frac{z^{\frac{n+1}{2}} - z^{-\frac{n+1}{2}}}{z^{-\frac{n}{2}}\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)} \\
&= \left(\frac{\cos\left(\frac{(n+1)\theta}{2}\right) + i\sin\left(\frac{(n+1)\theta}{2}\right) - \cos\left(-\frac{(n+1)\theta}{2}\right) - i\sin\left(-\frac{(n+1)\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right) - \cos\left(-\frac{\theta}{2}\right) - i\sin\left(-\frac{\theta}{2}\right)} \right) z^{\frac{n}{2}} \\
&= \left(\frac{2i\sin\left(\frac{(n+1)\theta}{2}\right)}{2i\sin\frac{\theta}{2}} \right) z^{\frac{n}{2}} \\
\sum_{k=0}^{n} \mathrm{Re}(z^k) &= \left(\frac{\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}} \right) \mathrm{Re}(z^{\frac{n}{2}})
\end{aligned}$$

Noting that $\mathrm{Re}\left(\sum_{k=0}^{n} z^k\right) = \sum_{k=0}^{n} \mathrm{Re}(z^k)$ when equating the real parts.

(ii) Evaluating the real parts in terms of real trigonometric functions and converting trigonometric products to sums

$$\begin{aligned}
\sum_{k=0}^{n} \mathrm{Re}(z^k) &= \left(\frac{\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}} \right) \mathrm{Re}(z^{\frac{n}{2}}) \\
\sum_{k=0}^{n} \cos(k\theta) &= \left(\frac{\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}} \right) \cos\frac{n\theta}{2} \\
&= \frac{\sin\left(\frac{(2n+1)\theta}{2}\right) + \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\
&= \frac{\sin\left(\frac{(2n+1)\theta}{2}\right)}{2\sin\frac{\theta}{2}} + \frac{1}{2}
\end{aligned}$$

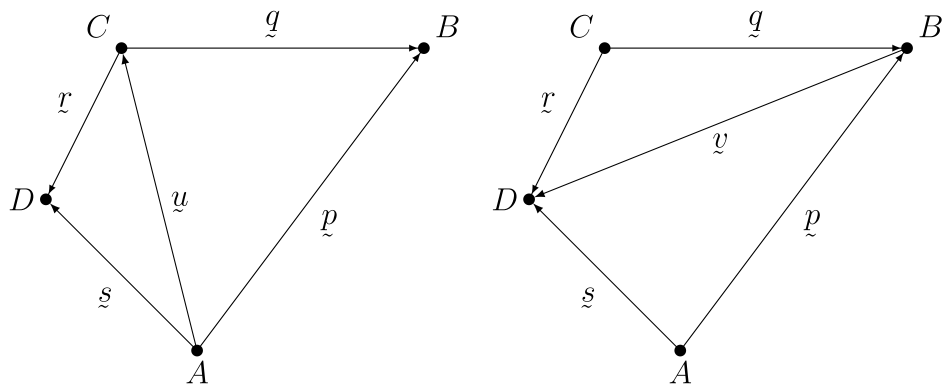
Differentiate both sides with respect to θ

$$\begin{aligned}
-\sum_{k=1}^{n} k\sin(k\theta) &= \frac{(2n+1)\cos\left(\frac{(2n+1)\theta}{2}\right)\sin\frac{\theta}{2} - \sin\left(\frac{(2n+1)\theta}{2}\right)\cos\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}} \\
\sum_{k=1}^{n} k\sin(k\theta) &= \frac{\left[\sin\left(\frac{(2n+1)\theta}{2}\right)\cos\frac{\theta}{2} - \cos\left(\frac{(2n+1)\theta}{2}\right)\sin\frac{\theta}{2}\right] - 2n\cos\left(\frac{(2n+1)\theta}{2}\right)\sin\frac{\theta}{2}}{2\left(2\sin^2\frac{\theta}{2}\right)} \\
&= \frac{\sin(n\theta) - n(\sin[(n+1)\theta] - \sin(n\theta))}{2(1-\cos\theta)} \\
&= \frac{(n+1)\sin(n\theta) - n\sin[(n+1)\theta]}{2(1-\cos\theta)}
\end{aligned}$$

(c) (i) Since $\angle ABC + \angle ADC = \pi$ then $\angle BCD + \angle BAD = \pi$, by angle sum of the quadrilateral $ABCD$.

For ease of notation, denote the following vectors as follows

$$\overrightarrow{AB} = \underset{\sim}{p} \quad \overrightarrow{CB} = \underset{\sim}{q} \quad \overrightarrow{CD} = \underset{\sim}{r} \quad \overrightarrow{AD} = \underset{\sim}{s} \quad \overrightarrow{AC} = \underset{\sim}{u} \quad \overrightarrow{BD} = \underset{\sim}{v}$$



From the cosine rule (or using the fact that $\underset{\sim}{u} = \underset{\sim}{q} - \underset{\sim}{p} \Rightarrow |\underset{\sim}{u}|^2 = (\underset{\sim}{q} - \underset{\sim}{p}) \cdot (\underset{\sim}{q} - \underset{\sim}{p})$)

$$\cos\angle ABC = \frac{|\underset{\sim}{p}|^2 + |\underset{\sim}{q}|^2 - |\underset{\sim}{u}|^2}{2|\underset{\sim}{p}||\underset{\sim}{q}|}$$

$$\text{Similarly} \qquad \cos\angle ADC = \frac{|\underset{\sim}{r}|^2 + |\underset{\sim}{s}|^2 - |\underset{\sim}{u}|^2}{2|\underset{\sim}{r}||\underset{\sim}{s}|}$$

Since $\angle ABC + \angle ADC = \pi$ and $\cos(\pi - x) = -\cos x$ then

$$\begin{aligned} \cos\angle ABC + \cos\angle ADC &= \cos\angle ABC + \cos(\pi - \angle ABC) \\ &= 0 \end{aligned}$$

Hence

$$\frac{|\underset{\sim}{p}|^2 + |\underset{\sim}{q}|^2 - |\underset{\sim}{u}|^2}{|\underset{\sim}{p}||\underset{\sim}{q}|} + \frac{|\underset{\sim}{r}|^2 + |\underset{\sim}{s}|^2 - |\underset{\sim}{u}|^2}{|\underset{\sim}{r}||\underset{\sim}{s}|} = 0$$

$$(|\underset{\sim}{p}|^2 + |\underset{\sim}{q}|^2)|\underset{\sim}{r}||\underset{\sim}{s}| + (|\underset{\sim}{r}|^2 + |\underset{\sim}{s}|^2)|\underset{\sim}{p}||\underset{\sim}{q}| = |\underset{\sim}{u}|^2(|\underset{\sim}{p}||\underset{\sim}{q}| + |\underset{\sim}{r}||\underset{\sim}{s}|)$$

Expanding and manipulating gives

$$\begin{aligned} |\underset{\sim}{u}|^2 &= \frac{|\underset{\sim}{p}||\underset{\sim}{s}|\left(|\underset{\sim}{p}||\underset{\sim}{r}| + |\underset{\sim}{q}||\underset{\sim}{s}|\right) + |\underset{\sim}{q}||\underset{\sim}{r}|\left(|\underset{\sim}{p}||\underset{\sim}{r}| + |\underset{\sim}{q}||\underset{\sim}{s}|\right)}{|\underset{\sim}{p}||\underset{\sim}{q}| + |\underset{\sim}{r}||\underset{\sim}{s}|} \\ &= \frac{\left(|\underset{\sim}{p}||\underset{\sim}{s}| + |\underset{\sim}{q}||\underset{\sim}{r}|\right)\left(|\underset{\sim}{p}||\underset{\sim}{r}| + |\underset{\sim}{q}||\underset{\sim}{s}|\right)}{|\underset{\sim}{p}||\underset{\sim}{q}| + |\underset{\sim}{r}||\underset{\sim}{s}|} \end{aligned}$$

Similarly applying the above derivation to the other diagonal gives

$$|\underset{\sim}{v}|^2 = \frac{\left(|\underset{\sim}{q}||\underset{\sim}{p}| + |\underset{\sim}{r}||\underset{\sim}{s}|\right)\left(|\underset{\sim}{q}||\underset{\sim}{s}| + |\underset{\sim}{r}||\underset{\sim}{p}|\right)}{|\underset{\sim}{q}||\underset{\sim}{r}| + |\underset{\sim}{s}||\underset{\sim}{p}|}$$

Hence

$$\begin{aligned} |\underset{\sim}{u}|^2|\underset{\sim}{v}|^2 &= \frac{\left(|\underset{\sim}{p}||\underset{\sim}{s}| + |\underset{\sim}{q}||\underset{\sim}{r}|\right)\left(|\underset{\sim}{p}||\underset{\sim}{r}| + |\underset{\sim}{q}||\underset{\sim}{s}|\right)}{|\underset{\sim}{p}||\underset{\sim}{q}| + |\underset{\sim}{r}||\underset{\sim}{s}|} \times \frac{\left(|\underset{\sim}{p}||\underset{\sim}{q}| + |\underset{\sim}{r}||\underset{\sim}{s}|\right)\left(|\underset{\sim}{q}||\underset{\sim}{s}| + |\underset{\sim}{p}||\underset{\sim}{r}|\right)}{|\underset{\sim}{q}||\underset{\sim}{r}| + |\underset{\sim}{p}||\underset{\sim}{s}|} \\ &= \left(|\underset{\sim}{p}||\underset{\sim}{r}| + |\underset{\sim}{q}||\underset{\sim}{s}|\right)^2 \end{aligned}$$

Noting that $|\underset{\sim}{p}||\underset{\sim}{r}| + |\underset{\sim}{q}||\underset{\sim}{s}|$, $|\underset{\sim}{u}|$ and $|\underset{\sim}{v}|$ are all positive then

$$|\underset{\sim}{p}||\underset{\sim}{r}| + |\underset{\sim}{q}||\underset{\sim}{s}| = |\underset{\sim}{u}||\underset{\sim}{v}|$$

From the position vector definitions

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} \quad \overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c} \quad \overrightarrow{CD} = \underset{\sim}{d} - \underset{\sim}{c} \quad \overrightarrow{AD} = \underset{\sim}{d} - \underset{\sim}{a} \quad \overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a} \quad \overrightarrow{BD} = \underset{\sim}{d} - \underset{\sim}{b}$$

Hence

$$|\underset{\sim}{b} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{c}| + |\underset{\sim}{d} - \underset{\sim}{a}||\underset{\sim}{c} - \underset{\sim}{b}| = |\underset{\sim}{c} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{b}|.$$

(ii) Since three distinct non-collinear points will always uniquely define a plane, consider the case (without loss of generality) where A, B and C form a plane and D does not lie on the plane.

There exists a point D^* (represented by position vector $\underset{\sim}{d}^*$) on the plane of A, B and C such that $AD = AD^*$ and $CD = CD^*$ and $\angle ADC = \angle AD^*C$. This forms a quadrilateral $ABCD^*$ and the following holds as shown in part (i).

$$|\underset{\sim}{b} - \underset{\sim}{a}||\underset{\sim}{d}^* - \underset{\sim}{c}| + |\underset{\sim}{d}^* - \underset{\sim}{a}||\underset{\sim}{c} - \underset{\sim}{b}| = |\underset{\sim}{c} - \underset{\sim}{a}||\underset{\sim}{d}^* - \underset{\sim}{b}|$$

But $AD = AD^*$ and $CD = CD^*$ so $|\underset{\sim}{d}^* - \underset{\sim}{c}| = |\underset{\sim}{d} - \underset{\sim}{c}|$ and $|\underset{\sim}{d}^* - \underset{\sim}{a}| = |\underset{\sim}{d} - \underset{\sim}{a}|$

$$|\underset{\sim}{b} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{c}| + |\underset{\sim}{d} - \underset{\sim}{a}||\underset{\sim}{c} - \underset{\sim}{b}| = |\underset{\sim}{c} - \underset{\sim}{a}||\underset{\sim}{d}^* - \underset{\sim}{b}|$$

However, $BD^* > BD$ (see remark) which means that $|\underset{\sim}{d}^* - \underset{\sim}{b}| > |\underset{\sim}{d} - \underset{\sim}{b}|$ so

$$|\underset{\sim}{b} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{c}| + |\underset{\sim}{d} - \underset{\sim}{a}||\underset{\sim}{c} - \underset{\sim}{b}| > |\underset{\sim}{c} - \underset{\sim}{a}||\underset{\sim}{d} - \underset{\sim}{b}|$$

Remark: An intuitive way to visualise this is to consider a flat rectangular piece of paper labelled $ABCD$. If one of the vertices of the paper, say D, is lifted up but the other vertices stay where they are, then notice that the direct distance from vertex B to the vertex D has shortened, compared to the original positioning on the plane of the paper.

Question 15

(a) (i) Using integration by parts for $n \geq 2$

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n x\,dx \\
&= \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\,dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x(1-\sin^2 x)\,dx \\
&= (n-1)(I_{n-2} - I_n) \\
I_n &= \frac{n-1}{n} I_{n-2}
\end{aligned}$$

(ii) From part (i)

$$\begin{aligned}
I_{2n} &= \frac{2n-1}{2n} I_{2n-2} \\
&= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} I_{2n-4} \\
&\vdots \\
&= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{3}{4} I_2 \\
&= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{3}{4} \times \frac{1}{2} I_0 \\
&= \frac{(2n-1)\times(2n-3)\times\cdots\times 3\times 1}{2^n n!} \int_0^{\frac{\pi}{2}} dx \times \frac{2n\times(2n-2)\times\cdots\times 4\times 2}{2n\times(2n-2)\times\cdots\times 4\times 2} \\
&= \frac{\pi}{2} \times \frac{(2n)!}{2^{2n}(n!)^2}
\end{aligned}$$

(iii) For the domain $\left[0, \frac{\pi}{2}\right]$ note that

$$\begin{aligned}
0 \leq \sin x &\leq 1 \\
\sin^{2n+1} x &\leq \sin^{2n} x \\
\sin^{2n+2} &\leq \sin^{2n+1} x \\
\Rightarrow \sin^{2n+2} x \leq \sin^{2n+1} x &\leq \sin^{2n} x \\
\int_0^{\frac{\pi}{2}} \sin^{2n+2} x\,dx \leq \int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx &\leq \int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx \\
I_{2n+2} \leq I_{2n+1} &\leq I_{2n} \\
\frac{2n+1}{2n+2} I_{2n} \leq I_{2n+1} &\leq I_{2n} \\
\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} &\leq 1
\end{aligned}$$

Noting that since $0 \leq \sin x \leq 1$ in the domain $\left[0, \frac{\pi}{2}\right]$ then $I_{2n} > 0$.

(iv) Consider the ratio

$$\begin{aligned}\frac{I_{2n+1}}{I_{2n}} &= \frac{2^{2n}(n!)^2}{(2n+1)!} \times \frac{2}{\pi} \times \frac{2^{2n}(n!)^2}{(2n)!} \\ &= \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} \times \frac{2}{\pi}\end{aligned}$$

From part (iii), when $n \to \infty$ then $\dfrac{2n+1}{2n+2} \to 1$ so $\dfrac{I_{2n+1}}{I_{2n}} \to 1$, hence

$$\lim_{n\to\infty} \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} = \frac{\pi}{2}$$

(v) Consider the ratio

$$\begin{aligned}\frac{(u_n)^4}{(u_{2n})^2} &= \frac{e^{4n}(n!)^4}{n^{4n+2}} \times \frac{(2n)^{4n+1}}{e^{4n}[(2n)!]^2} \\ &= \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} \times \frac{2(2n+1)}{n} \\ &= \frac{2^{4n}(n!)^4}{(2n)!(2n+1)!} \times \left(4 + \frac{2}{n}\right)\end{aligned}$$

As $n \to \infty$ then $4 + \dfrac{2}{n} \to 4$ and $\dfrac{2^{4n}(n!)^4}{(2n)!(2n+1)!} \to \dfrac{\pi}{2}$ from part (iv). Hence

$$\lim_{n\to\infty} \frac{(u_n)^4}{(u_{2n})^2} = 2\pi$$

However, when $n \to \infty$ then $u_n \to L$ and similarly when $2n \to \infty$ then $u_{2n} \to L$. This means that

$$\begin{aligned}\lim_{n\to\infty} \frac{(u_n)^4}{(u_{2n})^2} &= \frac{L^4}{L^2} \\ L^2 &= 2\pi \\ L &= \sqrt{2\pi} \quad \text{noting that } L > 0\end{aligned}$$

(b) **Method 1:**

For $t > 0$

$$\begin{aligned}
0 < \frac{1}{t+1} &< 1 \\
0 < \frac{t^2}{t+1} &< t^2 \\
0 < \frac{t^2 - 1 + 1}{t+1} &< t^2 \\
0 < t - 1 + \frac{1}{t+1} &< t^2 \\
0 < \int_0^x \left(t - 1 + \frac{1}{t+1}\right) dt &< \int_0^x t^2\, dt \\
0 < \ln(1+x) - x + \frac{x^2}{2} &< \frac{x^3}{3}
\end{aligned}$$

Method 2:

Let $f(x) = \ln(1+x) - x + \dfrac{x^2}{2}$ for $x > 0$

$$\begin{aligned}
f'(x) &= \frac{1}{x+1} - 1 + x \\
&= \frac{x^2}{x+1} \\
&> 0 \quad \text{for } x > 0
\end{aligned}$$

Since $f(0) = 0$ and $f(x)$ is increasing for $x > 0$ then $f(x) > 0$ in that domain. Hence

$$\ln(1+x) - x + \frac{x^2}{2} > 0$$

Let $g(x) = \ln(1+x) - x + \dfrac{x^2}{2} - \dfrac{x^3}{3}$ for $x > 0$

$$\begin{aligned}
g'(x) &= \frac{x^2}{x+1} - x^2 \\
&= \frac{x^2(1 - (x+1))}{x+1} \\
&= -\frac{x^3}{x+1} \\
&< 0 \quad \text{for } x > 0
\end{aligned}$$

Since $g(0) = 0$ and $g(x)$ is decreasing for $x > 0$ then $g(x) < 0$ in that domain. Hence

$$\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} < 0$$

Putting these inequalities together gives

$$0 < \ln(1+x) - x + \frac{x^2}{2} < \frac{x^3}{3}$$

(c) (i) Let $F(x)$ and $G(z)$ be the cumulative distribution functions of random variables X and Z respectively

$$\begin{aligned} G(z) &= P(Z \leq z) \\ &= P\left(\frac{X-n}{\sqrt{n}} \leq z\right) \\ &= P(X \leq n + z\sqrt{n}) \\ &= F(n + z\sqrt{n}) \end{aligned}$$

Differentiate both sides with respect to z to get the probability density functions

$$g(z) = \sqrt{n}\, f(n + z\sqrt{n})$$

(ii) Substituting part (i) into the probability density function for X

$$\begin{aligned} g(z) &= \sqrt{n} \times \frac{(n+z\sqrt{n})^n}{n!} e^{-(n+z\sqrt{n})} \\ &= \frac{n^{n+\frac{1}{2}}}{e^n n!}\left(1 + \frac{z}{\sqrt{n}}\right)^n e^{-z\sqrt{n}} \end{aligned}$$

From part (b), let $x = \frac{z}{\sqrt{n}}$ where $z > 0$

$$\begin{aligned} 0 < \ln\left(1 + \frac{z}{\sqrt{n}}\right) - \frac{z}{\sqrt{n}} + \frac{z^2}{2n} &< \frac{z^3}{3n\sqrt{n}} \\ -\frac{z^2}{2} < n\ln\left(1 + \frac{z}{\sqrt{n}}\right) - z\sqrt{n} &< \frac{z^3}{3\sqrt{n}} - \frac{z^2}{2} \\ e^{-\frac{z^2}{2}} < \left(1 + \frac{z}{\sqrt{n}}\right)^n e^{-z\sqrt{n}} &< e^{-\frac{z^2}{2} + \frac{z^3}{3\sqrt{n}}} \\ \frac{n^{n+\frac{1}{2}}}{e^n n!} e^{-\frac{z^2}{2}} < g(z) &< \frac{n^{n+\frac{1}{2}}}{e^n n!} e^{-\frac{z^2}{2} + \frac{z^3}{3\sqrt{n}}} \end{aligned}$$

As $n \to \infty$ then $\dfrac{z^3}{3\sqrt{n}} \to 0$ and $\dfrac{n^{n+\frac{1}{2}}}{e^n n!} \to \dfrac{1}{\sqrt{2\pi}}$ from part (a)(v). Hence

$$\lim_{n\to\infty} g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

This means that as the sample size n gets large, the distribution of Z approaches a standard normal distribution (in the positive domain).

Question 16

(a) (i) From the horizontal displacement equation

$$\begin{aligned} x &= \frac{u^2}{g}\cos\theta\left(1-e^{-\frac{gt}{u}}\right) \\ x\sec\theta &= \frac{u^2}{g}\left(1-e^{-\frac{gt}{u}}\right) \\ e^{-\frac{gt}{u}} &= 1-\frac{gx\sec\theta}{u^2} \\ t &= -\frac{u}{g}\ln\left(1-\frac{gx\sec\theta}{u^2}\right) \end{aligned}$$

Substitute into the vertical displacement equation

$$\begin{aligned} y &= \frac{u^2}{g}(1+\sin\theta)\left(1-e^{-\frac{gt}{u}}\right)-ut \\ y &= x\sec\theta\,(1+\sin\theta)+\frac{u^2}{g}\ln\left(1-\frac{gx\sec\theta}{u^2}\right) \end{aligned}$$

At the range when $x = R$ then $y = 0$ hence

$$\begin{aligned} R\sec\theta\,(1+\sin\theta)+\frac{u^2}{g}\ln\left(1-\frac{gR\sec\theta}{u^2}\right) &= 0 \\ \ln\left(1-\frac{gR\sec\theta}{u^2}\right) &= -\frac{gR}{u^2}(\sec\theta+\tan\theta) \end{aligned}$$

Let $f(\theta) = \dfrac{gR}{u^2}(\sec\theta+\tan\theta)$ which means that

$$\begin{aligned} \frac{gR\sec\theta}{u^2} &= 1-e^{-f(\theta)} \\ R &= \frac{u^2}{g}\cos\theta\left(1-e^{-f(\theta)}\right) \end{aligned}$$

(ii) Using the expression for the range in part (i)

$$\begin{aligned} R(\theta) &= \frac{u^2}{g}\cos\theta\left(1-e^{-f(\theta)}\right) \\ R'(\theta) &= -\frac{u^2}{g}\sin\theta\left(1-e^{-f(\theta)}\right)+\frac{u^2}{g}\cos\theta f'(\theta)e^{-f(\theta)} \end{aligned}$$

To remove the exponential terms, notice that by rearranging the equation of $R(\theta)$

$$\frac{u^2}{g}\cos\theta e^{-f(\theta)} = \frac{u^2}{g}\cos\theta - R(\theta)$$

Hence

$$R'(\theta) = -R(\theta)\tan\theta + f'(\theta)\left(\frac{u^2}{g}\cos\theta - R(\theta)\right)$$

Looking into $f'(\theta)$

$$\begin{aligned}
f(\theta) &= \frac{gR(\theta)}{u^2}(\sec\theta + \tan\theta) \\
f'(\theta) &= \frac{g}{u^2}\left(R'(\theta)(\sec\theta + \tan\theta) + R(\theta)(\sec\theta\tan\theta + \sec^2\theta)\right) \\
&= \frac{g}{u^2}(\sec\theta + \tan\theta)(R'(\theta) + R(\theta)\sec\theta)
\end{aligned}$$

Substitute this into the derivative equation

$$R'(\theta) = -R(\theta)\tan\theta + \frac{g}{u^2}(\sec\theta + \tan\theta)(R'(\theta) + R(\theta)\sec\theta)\left(\frac{u^2}{g}\cos\theta - R(\theta)\right)$$

This simplifies to

$$R'(\theta)\left(\frac{gR(\theta)}{u^2}(\sec\theta + \tan\theta) - \sin\theta\right) = R(\theta)\sec\theta - \frac{g\sec\theta}{u^2}(\sec\theta + \tan\theta)[R(\theta)]^2$$

Hence

$$R'(\theta) = \frac{R(\theta)\sec\theta\left(1 - \dfrac{gR(\theta)}{u^2}(\sec\theta + \tan\theta)\right)}{\dfrac{gR(\theta)}{u^2}(\sec\theta + \tan\theta) - \sin\theta}$$

A necessary condition for the maximum range is $R'(\theta) = 0$ so either

$$R(\theta) = 0 \quad \text{or} \quad R(\theta) = \frac{u^2}{g(\sec\theta + \tan\theta)}$$

Take the positive solution and substitute it back into the original equation

$$\begin{aligned}
R(\theta) &= \frac{u^2}{g}\cos\theta\left(1 - e^{-f(\theta)}\right) \\
\frac{1}{\sec\theta + \tan\theta} &= \cos\theta(1 - e^{-f(\theta)}) \\
1 - e^{-f(\theta)} &= \frac{\sec\theta}{\sec\theta + \tan\theta} \\
\frac{\sec\theta + \tan\theta}{\tan\theta} &= e^{f(\theta)} \\
\sin\theta &= \frac{1}{e^{f(\theta)} - 1}
\end{aligned}$$

But when $R(\theta) = \dfrac{u^2}{g(\sec\theta + \tan\theta)}$ then $f(\theta) = 1$, hence $\sin\theta = \dfrac{1}{e - 1}$.

(iii) Using the expression for the first derivative in terms of $f(\theta)$

$$\begin{aligned} R'(\theta) &= \frac{R(\theta)\sec\theta\,(1-f(\theta))}{f(\theta)-\sin\theta} \\ R''(\theta) &= \frac{g(\theta)-h(\theta)}{(f(\theta)-\sin\theta)^2} \end{aligned}$$

Where

$$\begin{aligned} g(\theta) &= \left\{ R'(\theta)[\sec\theta(1-f(\theta))] + R(\theta)\frac{d}{d\theta}[\sec\theta(1-f(\theta))] \right\}(f(\theta)-\sin\theta) \\ h(\theta) &= R(\theta)\sec\theta\,(1-f(\theta))\,[f'(\theta)-\cos\theta] \end{aligned}$$

When $\sin\theta = \dfrac{1}{e-1}$ then $R(\theta) = \dfrac{u^2}{g(\sec\theta+\tan\theta)}$, $R'(\theta) = 0$ and $f(\theta) = 1$ as shown in the previous part, so these simplify to

$$\begin{aligned} g(\theta) &= R(\theta)(1-\sin\theta)\frac{d}{d\theta}[\sec\theta(1-f(\theta))]\bigg|_{f(\theta)=1} \\ &= R(\theta)(1-\sin\theta)[\sec\theta\tan\theta(1-f(\theta)) - f'(\theta)\sec\theta]\bigg|_{f(\theta)=1} \\ &= -R(\theta)(1-\sin\theta)f'(\theta)\sec\theta \\ h(\theta) &= 0 \end{aligned}$$

Hence at the stationary point

$$R''(\theta) = -\frac{R(\theta)f'(\theta)\sec\theta}{1-\sin\theta}$$

At the stationary point $\sin\theta < 1$, $\sec\theta > 0$ and $R(\theta) > 0$. Also

$$\begin{aligned} f'(\theta) &= \frac{g}{u^2}(\sec\theta+\tan\theta)(R'(\theta)+R(\theta)\sec\theta) \\ &= \frac{gR(\theta)}{u^2}\sec\theta(\sec\theta+\tan\theta) \qquad \text{as } R'(\theta) = 0 \\ &> 0 \end{aligned}$$

This means that $R''(\theta) < 0$ at the stationary point. Therefore, the range is maximised at $\theta = \sin^{-1}\left(\dfrac{1}{e-1}\right)$.

(b) (i) Suppose that $\sqrt[3]{2}$ is rational, that is $\sqrt[3]{2} = \frac{p}{q}$ for some integers p and q, where p and q have no common factors and $q \neq 0$. Since $\sqrt[3]{4} = \frac{p^2}{q^2}$ is also a ratio of two integers with no common factors, then $\sqrt[3]{4}$ must also be rational. Hence, if $\sqrt[3]{2}$ rational then $\sqrt[3]{4}$ is rational.

For the converse of the above, suppose that $\sqrt[3]{4}$ is rational, that is $\sqrt[3]{4} = \frac{a}{b}$ for some integers a and b, where a and b have no common factors and $b \neq 0$. Squaring both sides gives

$$2\sqrt[3]{2} = \frac{a^2}{b^2}$$

$$\sqrt[3]{2} = \frac{a^2}{2b^2} \quad \text{which is rational}$$

Hence, if $\sqrt[3]{4}$ rational then $\sqrt[3]{2}$ is rational.

$\therefore$ $\sqrt[3]{2}$ is rational if and only if $\sqrt[3]{4}$ is rational.

(ii) If m is an integer where $m \in S_2$ then

$$m\sqrt[3]{2} = M \quad \text{where } M \text{ is an integer}$$

$$m^2\sqrt[3]{2} = Mm$$

Since Mm is an integer then $m^2 \in S_2$.

Squaring both sides of the first expression gives $m^2\sqrt[3]{4} = M^2$.
Since M^2 is an integer then $m^2 \in S_4$.

Since $m^2 \in S_2$ and $m^2 \in S_4$ then $m^2 \in S$, so it follows that there is at least one positive integer in S.

(iii) If k is an integer where $k \in S$, then

$$k\sqrt[3]{2} = P \quad \text{and} \quad k\sqrt[3]{4} = Q$$

where P and Q are integers.

First note that $[\sqrt[3]{4} - 1]k = Q - k$ which is an integer
Also $[\sqrt[3]{4} - 1]k \times \sqrt[3]{2} = 2k - P$ which is an integer so $[\sqrt[3]{4} - 1]k \in S_2$
Similarly $[\sqrt[3]{4} - 1]k \times \sqrt[3]{4} = 2P - Q$ which is an integer so $[\sqrt[3]{4} - 1]k \in S_4$

Hence, if $k \in S$, then $[\sqrt[3]{4} - 1]k \in S$.

(iv) Since there can only be positive integers in S then there must exist a particular $k \in S$, which is the smallest positive integer in the set S. From the previous part $[\sqrt[3]{4}-1]k \in S$ as well. However since

$$\begin{aligned} 1 &< 4 < 8 \\ \sqrt[3]{1}-1 &< \sqrt[3]{4}-1 < \sqrt[3]{8}-1 \\ 0 &< [\sqrt[3]{4}-1]k < k \end{aligned}$$

This implies that the set S contains a smaller integer than the minimum k. This is a contradiction.

Hence, it must be the case that the original statement, that $\sqrt[3]{2}$ is rational if and only if $\sqrt[3]{4}$ is rational, is not true.

Therefore, both $\sqrt[3]{2}$ and $\sqrt[3]{4}$ must be irrational.

Alternate Method: Using the same inequality it can be seen that

$$\begin{aligned} [\sqrt[3]{4}-1]k &< k \\ [\sqrt[3]{4}-1]^2k &< [\sqrt[3]{4}-1]k \\ [\sqrt[3]{4}-1]^3k &< [\sqrt[3]{4}-1]^2k \\ \text{and so on}\ldots \end{aligned}$$

This leads to an infinite sequence of strictly decreasing positive integers in S given by $\{k, [\sqrt[3]{4}-1]k, [\sqrt[3]{4}-1]^2k, [\sqrt[3]{4}-1]^3k, \ldots\}$.

However, there exists only finitely many positive integers below any given positive integer. By contradiction, it must be the case that the original statement, that $\sqrt[3]{2}$ is rational if and only if $\sqrt[3]{4}$ is rational, is not true.

Therefore, both $\sqrt[3]{2}$ and $\sqrt[3]{4}$ must be irrational.