2023 **BORED OF STUDIES TRIAL EXAMINATION**

3rd October

Mathematics Extension 2

General instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5–16)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 A sphere with equation $(x-a)^2+(y-2a)^2+(z-2a)^2=81$ passes through the origin.

Which of the following is a possible value of a?

(A) $a=0$

(B) $a=3$

(C) $a=6$

(D) $a=9$

2 Which is the remainder when $x^3-ix^2+(1+i)x-2-i$ is divided by x^2+1?

(A) -3

(B) -1

(C) $ix-2$

(D) $2ix-1$

3 Let $\dfrac{dy}{dx}=\dfrac{1}{ax^2-2abx+c}$ be a differential equation for non-zero constants a, b and c.

Which of the following restrictions ensures that the general solution to the differential equation is $y=A\tan^{-1}(Bx+C)+D$ for some non-zero constants A, B, C and D?

(A) $a<0$, all real $b\neq 0$ and $c<0$

(B) $a>0$, all real $b\neq 0$ and $c<0$

(C) $a>0$, $b>0$ and all real $c\neq 0$

(D) $a<0$, $b>0$ and all real $c\neq 0$

4 At time t seconds, a particle is at the position vector $\underset{\sim}{r} = 5\underset{\sim}{i} + (3-t)\underset{\sim}{j} - 6\sqrt{t}\underset{\sim}{k}$.

Which of the below vectors describes the particle's direction of motion at $t = 9$ seconds?

(A) $-\underset{\sim}{j} - \underset{\sim}{k}$

(B) $5\underset{\sim}{i} - 6\underset{\sim}{j} - 18\underset{\sim}{k}$

(C) $-3\underset{\sim}{j} - \underset{\sim}{k}$

(D) $5\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

5 Let x, y and z be real numbers where $x < y$. Suppose that x, y and z satisfy the inequality

$$|x-y| < |x-z| + |z-y|.$$

Which of the following statements is correct?

(A) The inequality is true for all real z.

(B) The inequality is only true if $x < z < y$.

(C) The inequality is only true if $z < x$ or $z > y$.

(D) The inequality is only true for $z = x$ and $z = y$.

6 Let z and w be complex numbers such that $\dfrac{\pi}{2} < \text{Arg}(z) - \text{Arg}(w) < \pi$.

On the complex plane, the origin and the points representing the complex numbers w, z and $z + w$ are the vertices of a rhombus. Which of the following is correct?

(A) $\dfrac{z-w}{z+w} = ki$ where $0 < k < 1$

(B) $\dfrac{z-w}{z+w} = ki$ where $k > 1$

(C) $\dfrac{z+w}{z-w} = ki$ where $0 < k < 1$

(D) $\dfrac{z+w}{z-w} = ki$ where $k > 1$

7 Suppose that a and b are non-zero constants. Which of the following could *possibly* be a unit vector?

(A) $\dfrac{1}{a}\underset{\sim}{i} + b\underset{\sim}{j} + \underset{\sim}{k}$

(B) $a^2b^2\underset{\sim}{i} + 2ab\underset{\sim}{j} + 2\underset{\sim}{k}$

(C) $a\underset{\sim}{i} + \dfrac{1}{b}\underset{\sim}{j} + \dfrac{1}{a}\underset{\sim}{k}$

(D) $a^2\underset{\sim}{i} + 2ab\underset{\sim}{j} + 2b^2\underset{\sim}{k}$

8 A spring is vertically fixed to the ground. A light object is pushed on top of the spring such that it first compresses the spring downward and is then released such that the object ordinarily performs simple harmonic motion with amplitude A and period ω. Let the centre of this simple harmonic motion be the origin and upwards as the positive direction. The forces acting on the object are gravity and an opposing constant normal force N.

However, on this occasion the object is pushed excessively onto the spring such that when released it loses contact with the spring at its maximum vertical extension.

Let g be the acceleration due to gravity. What is the value of A?

(A) $\dfrac{g}{\omega^2}$ (B) $\dfrac{g}{\omega}$

(C) $\dfrac{g\omega^2}{4\pi^2}$ (D) $\dfrac{g\omega}{2\pi}$

9 Which of the following statements is true?

(A) $\dfrac{1}{b} > \dfrac{1}{a}$ if and only if $a > b$.

(B) $\Phi^{-1}(a) > \Phi^{-1}(b)$ if and only if $a > b$, where $\Phi(x)$ is the standard cumulative normal distribution function.

(C) $f'(a) > f'(b)$ if and only if $a > b$ where $f(x)$ is some differentiable function at $x = a$ and $x = b$.

(D) $\displaystyle\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$ if and only if $f(x) > g(x)$ for all $a \le x \le b$, where $f(x)$ and $g(x)$ are some continuous functions over all real x.

10 Which of the following inequalities is FALSE?

(A) $\displaystyle\int_0^1 \sqrt{x}\cos^{-1} x\,dx < \frac{\pi}{2}$ (B) $\displaystyle\int_1^e \frac{\tan^{-1} x}{x}\,dx < \frac{\pi}{2}$

(C) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^4 \sin x\,dx < \frac{\pi}{15}$ (D) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{x}}\,dx < \frac{\pi}{15}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Evaluate **3**

$$\int_0^1 \frac{x^2 \ln\left(1+x+x^9+x^{10}\right)}{1+x^3}\,dx.$$

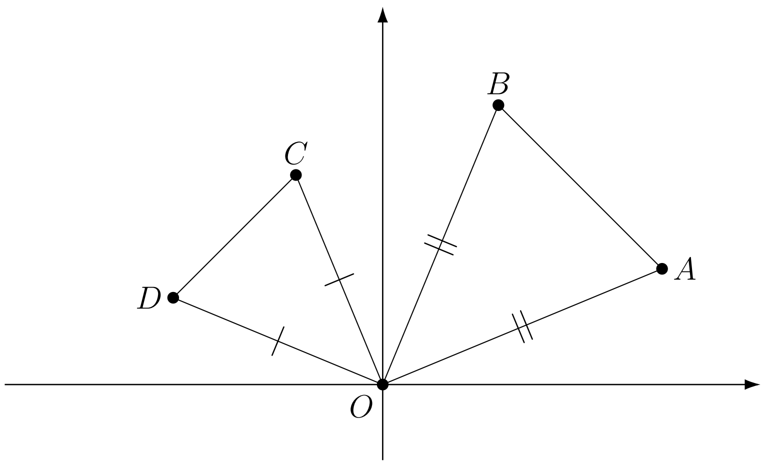
(b) The coordinates of the centre of a circular plane in space is $(4,-1,3)$. **2**
Suppose that $(1,-1,7)$ lies on the circular edge of the plane.

Find a parametric vector equation of the plane's circular edge. Express your answer in the form

$$f(t)\underset{\sim}{i} + g(t)\underset{\sim}{j} + h(t)\underset{\sim}{k}$$

where t is a parameter, and $f(t), g(t)$ and $h(t)$ are some functions of t.

(c) Let $\triangle OAB$ and $\triangle OCD$ be two similar but non-overlapping isosceles triangles on the complex plane with $OA = OB$ and $OC = OD$ as shown in the diagram below. **1**



Let the complex number representing the points A, B, C and D be a, b, c and d respectively.

Show that $|a-c| = |b-d|$.

Question 11 continues on page 6

(d) A line ℓ_1 has the vector equation $\underset{\sim}{r} = 10\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$. Another line ℓ_2 intersects ℓ_1 at point M and passes through the points $A(4,1,3)$ and $B(6,5,-3)$.

(i) Show that ℓ_1 and ℓ_2 are perpendicular. **1**

(ii) Let C and D be points on ℓ_1 such that M is the midpoint of CD and the distance from A to C is $\sqrt{62}$. Find the coordinates of C and D. **3**

(e) Suppose that z is a complex number with modulus r and argument θ.
Let R_1 be the region on the complex plane defined by the inequality

$$0 \leq \arg(z-1) \leq \frac{\pi}{3}.$$

(i) Sketch the region R_1. **1**

(ii) Suppose that $\omega = r(\sin\theta + i\cos\theta)$ where r and θ are specific values such that z lies in the region R_1. **1**

Let R_2 be the region on the complex plane that represents the locus of ω. Sketch the region R_2.

(iii) The region on the complex plane which represents the set of complex numbers which lie in both R_1 and R_2 can be described by the inequality **3**

$$\theta_1 \leq \arg(z - z_0) \leq \theta_2.$$

Find z_0, θ_1 and θ_2.

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Let $P(z) = z^4 + 3z^2 + z - 1$. How many roots of $P(z)$ have a positive real part? Justify your answer. **3**

(b) Let $I_n = \lim_{a \to \infty} \left(\int_{-a}^{a} \frac{x^n e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx \right)$ for some non-negative integer n.

(i) Show that $I_{n+2} = (n+1)I_n$. **2**

You may assume without proof that $\lim_{x \to \infty} \left(x^n e^{-\frac{x^2}{2}} \right) = 0$.

(ii) Hence, show that $I_{2n} = \frac{(2n)!}{2^n n!}$. **1**

(c) Consider the polynomial $P(z) = z^{2mn} - az^{mn} + 1$ for some positive integers m, n and real value $-2 < a < 2$. Suppose that $Q(z) = z^{2n} - az^n + 1$ is a factor of $P(z)$.

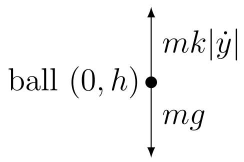
(i) Show that the roots of both $P(z)$ and $Q(z)$ must satisfy $|z| = 1$. **2**

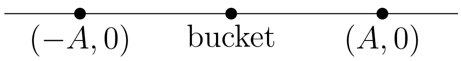
(ii) Hence, find all possible values of a in terms of m and n. **2**

Question 12 continues on page 8

(d) On the x-y plane, a ball of mass m is dropped vertically from rest at a sufficiently large height from the point $(0, h)$ above the origin. Whilst falling, the ball has a vertical displacement of y relative to the ground at time t. It is subjected to air resistance of magnitude $mk|\dot{y}|$, where k is a positive constant.

On the ground, a bucket is mounted on a machine which moves horizontally in simple harmonic motion with displacement x at time t relative to its centre of motion and amplitude A. The bucket is initially at its centre of motion $(0, 0)$.





(i) By considering the relevant acceleration equation, show that **2**

$$\dot{y} = \frac{g}{k}(e^{-kt} - 1)$$

where g is the acceleration due to gravity.

(ii) The machine holding the bucket moves according to the acceleration equation $\ddot{x} = -\omega^2 x$ for some positive constant ω. Suppose that it is timed such that the bucket catches the ball at exactly the nth time it crosses its centre of motion (excluding its initial position). **3**

Show that

$$\frac{\omega k h}{\pi g} < n < \frac{\omega}{\pi}\left(\frac{kh}{g} + \frac{1}{k}\right).$$

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ where $a_0, a_1, a_2, \ldots, a_n$ are integers with $a_0 \neq 0$ and $a_n \neq 0$. Suppose that $x = \frac{p}{q}$ is a root of $P(x)$ where p and q are integers with no common factors, other than 1.

(i) Show that a_0 is divisible by p and a_n is divisible by q. **2**

(ii) Hence, prove that $\cos\dfrac{\pi}{9}$ is irrational. **2**

You may use the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ without proof.

(b) The points $P(p^\alpha, p^{1-\alpha})$ and $Q(q^\alpha, q^{1-\alpha})$ lie on the curve $y = x^{\frac{1-\alpha}{\alpha}}$ for some constants $p > 0$, $q > 0$ and $0 < \alpha < 1$.

(i) By considering areas, or otherwise, show that $p^\alpha q^{1-\alpha} \leq \alpha p + (1-\alpha)q$. **2**

(ii) Hence, show that for positive values x, y and z **1**

$$\frac{x+y+z}{3} \geq (xyz)^{\frac{1}{3}}.$$

(iii) Let x_i be a positive real number for $i = 1, 2, \ldots, n$. Let α_i be a positive real number such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and $0 < \alpha_i < 1$ for $i = 1, 2, \ldots, n$. Use the result in part (i) to prove by mathematical induction that for integer $n \geq 2$ **3**

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \geq x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

(iv) Let X be a discrete random variable which takes the values $x_1, x_2, \ldots, x_n$ with probabilities $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. Use the result in part (iii) to show that **1**

$$\mathrm{Var}\left(\sqrt{X}\right) \leq \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n - x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

(c) A particle moves in simple harmonic motion along the x-axis with its centre of motion at $x = 1$. The square of the particle's velocity v^2 at time t has the relation

$$v^2 = 4 - 4\sin 4t.$$

(i) Find the time that the particle is first at rest. **1**

(ii) Hence, find the displacement equations of the particle in terms of t. **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) (i) Express $\dfrac{5}{(x-2)(2x+1)}$ as a sum of partial fractions. **1**

(ii) Use the substitution $t = \tan\frac{x}{2}$ to show that for some constant c_1 **3**

$$\int \frac{5}{4\cos x + 3\sin x}\,dx = \ln\left(\frac{\sin x - 2\cos x + 2}{2\sin x + \cos x - 1}\right) + c_1$$

where $0 \leq x \leq \dfrac{\pi}{2}$.

(iii) By writing $4\cos x + 3\sin x$ in the form $R\cos(x-\alpha)$ for some real numbers R and α, show that for some constant c_2 **3**

$$\int \frac{5}{4\cos x + 3\sin x}\,dx = \ln\left(\frac{4\sin x - 3\cos x + 5}{3\sin x + 4\cos x}\right) + c_2$$

where $0 \leq x \leq \dfrac{\pi}{2}$.

(iv) Hence, carefully explain why for $0 \leq x \leq \dfrac{\pi}{2}$ **1**

$$\frac{\sin x - 2\cos x + 2}{2\sin x + \cos x - 1} \equiv \frac{4\sin x - 3\cos x + 5}{3\sin x + 4\cos x}.$$

(b) Let $P(x) = x^{2n} + x^{2n-1} + x^{2n-2} + \cdots + x + 1$.

(i) Find all complex roots of $P(x)$. **1**

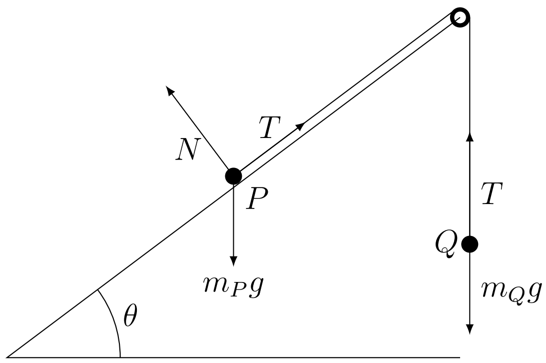
(ii) Hence, show that **2**

$$\tan\left(\frac{\pi}{2n+1}\right)\tan\left(\frac{2\pi}{2n+1}\right)\tan\left(\frac{3\pi}{2n+1}\right)\ldots\tan\left(\frac{2n\pi}{2n+1}\right) = (-1)^n(2n+1).$$

Question 14 continues on page 11

Question 14 (continued)

(c) An object P of mass m_P sits on a smooth inclined slope at a fixed acute angle of θ to the horizontal. It is held by a light string which extends up to the edge of the slope into a smooth pulley as shown in the diagram below. Another object Q of mass m_Q is held vertically by the same light string from that pulley over the edge of the inclined slope.



The forces acting on object P are gravity (with acceleration g), the tension on the string of magnitude T and a normal force of magnitude N perpendicular to the slope. The forces acting on object Q are gravity and tension on the string T.

The net forces on each object are such that they each maintain a constant speed at any point in time.

(i) By resolving the forces on both objects, show that $m_P > m_Q$. **1**

(ii) Suppose that the inclined slope is rough instead and object P is subjected to a friction force F opposing its direction of motion. As long as the ratio $F : N$ is no more than $1 : \sqrt{3}$ then neither object will accelerate in any direction. **3**

Show that if the ratio of $m_P : m_Q$ is $\sqrt{3} : \sqrt{2}$ and neither object accelerates in any direction, then $\dfrac{\pi}{12} \leq \theta \leq \dfrac{5\pi}{12}$.

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) Suppose that $\triangle ABC$ is an acute angled triangle. **4**

Let P lie on AB such that the ratio $|\overrightarrow{AP}| : |\overrightarrow{PB}|$ is $1 : 2$ and let Q lie on AC such that the ratio $|\overrightarrow{AQ}| : |\overrightarrow{QC}|$ is $1 : 3$.

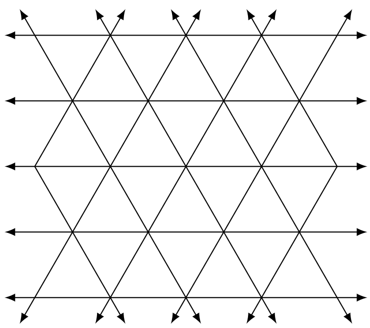
Let O be intersection of the intervals BQ and CP. The interval AO is produced until it intersects with BC at R.

By considering the relevant vector additions, find the ratio of $|\overrightarrow{BR}| : |\overrightarrow{RC}|$.

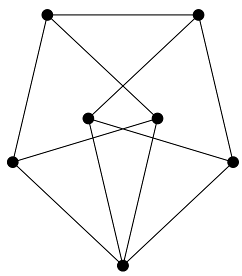
(b) Suppose that the x-y plane is made up of a set of individual points. Each point in the plane is assigned a colour. Suppose that the assignment is such that any two points 1 unit apart on the plane will *always* have different colours. There are N distinct colours in total.

(i) A triangular lattice on a plane is provided below where all triangles are congruent and equilateral with each side having a length of 1 unit. **1**

By considering the intersection points in this triangular lattice, explain why $N \geq 3$.



(ii) Now consider seven points on the plane arranged as shown below. **2**
Each interval drawn between two points in the diagram has unit length.



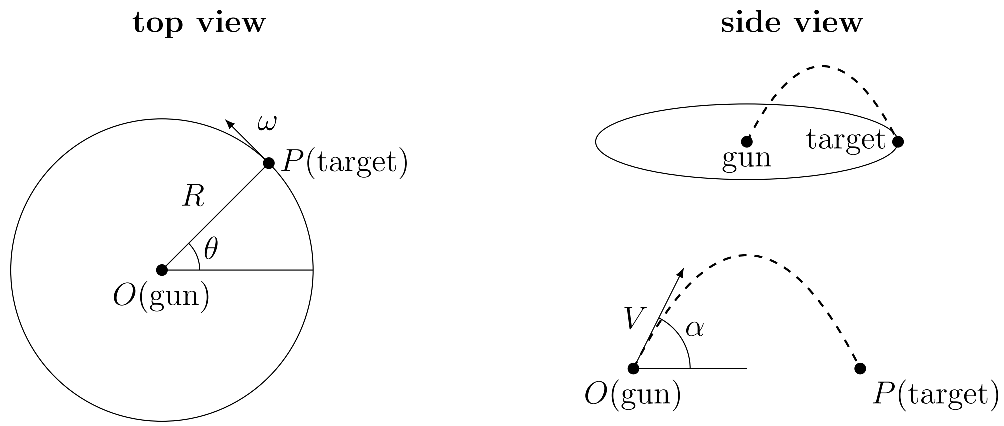
By considering the colour of each of these points, prove that $N \geq 4$.

Question 15 continues on page 13

Question 15 (continued)

(c) An automated gun is fixed at position O on the ground pointing at an angle of elevation of α to the horizontal. A target P circles the gun in a horizontal plane on the ground with centre O and radius R.

The gun fires a bullet from O in the air at an initial speed of V. At time t seconds after the gun was fired, the target P has displaced an angle of θ at the centre O, where $\theta > 0$. The target P moves along the circle such that the rate of change of θ is a constant positive value ω. Assume that the gun is orientated such that it will always aim for the target after firing.



Let g be the acceleration due to gravity. The displacement vector of the bullet is

$$\underset{\sim}{r} = Vt\cos\alpha\,\underset{\sim}{i} + \left(Vt\sin\alpha - \frac{gt^2}{2}\right)\underset{\sim}{j}. \qquad \textbf{(Do NOT prove this)}$$

(i) Show that if $V < \sqrt{gR}$, the bullet will never hit the target P. **1**

(ii) Suppose that $V \geq \sqrt{gR}$ and the bullet hits the target P. Show that the corresponding value(s) of θ must satisfy **3**

$$g^2\theta^4 - 4\omega^2V^2\theta^2 + 4R^2\omega^4 = 0.$$

(iii) Explain why if $V > \sqrt{gR}$ then there are two distinct values of θ where the bullet hits the target. **1**

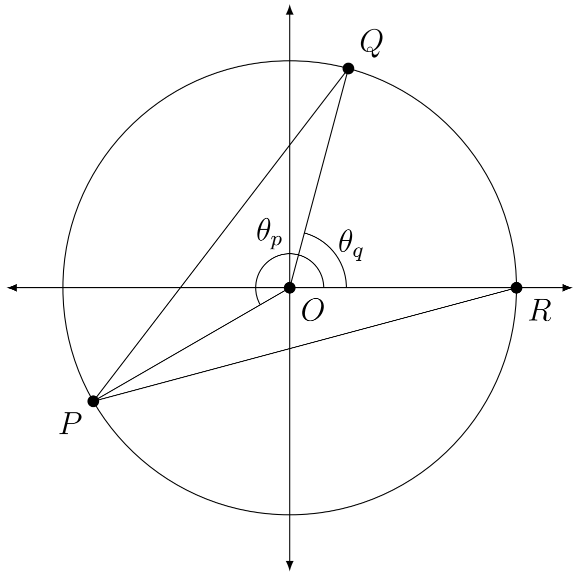
(iv) Let θ_1 and θ_2 be the distinct values of θ corresponding to the points P_1 and P_2 respectively, where the bullet hits the target. **3**

Show that the length of the interval P_1P_2 is $2R\left|\sin\left(\dfrac{\omega\sqrt{V^2 - gR}}{g}\right)\right|$.

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) On the complex plane, let P and Q be points on a circle of radius r with the centre as the origin, as shown on the diagram below. Let R be the point on the circle that intersects the positive real axis.



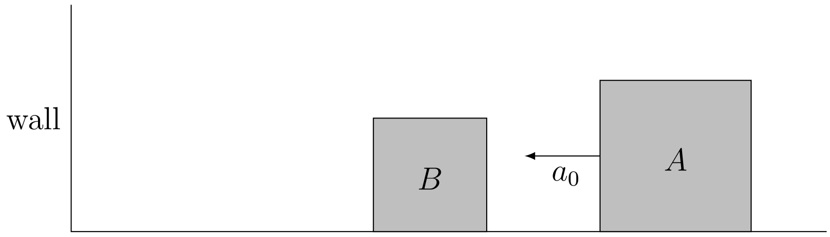
Let the complex numbers representing P and Q be p and q respectively.
Suppose that $\arg p = \theta_p$ and $\arg q = \theta_q$ where $\theta_p \in [0, 2\pi)$, $\theta_q \in [0, 2\pi)$ and $\theta_p > \theta_q$.

(i) Show that $\angle QPR = \arg\left(\dfrac{e^{i\theta_q} - e^{i\theta_p}}{1 - e^{i\theta_p}}\right)$. **1**

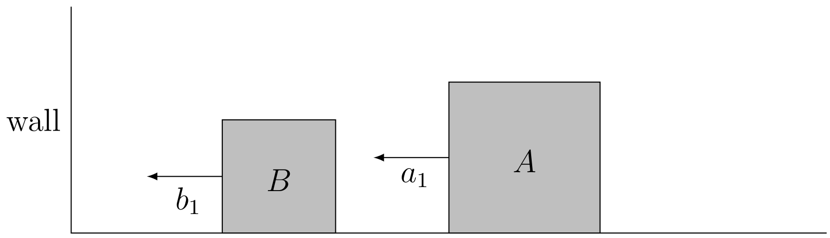
(ii) Hence, prove that $\angle QOR = 2\angle QPR$. **2**

Question 16 continues on page 15

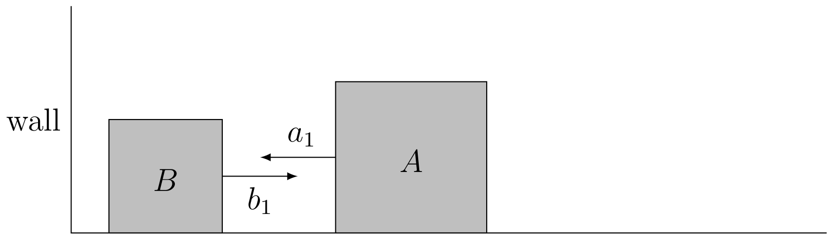
(b) An object A of mass m_A moves in a horizontal line on a friction-less surface at a constant speed a_0 towards a second object B of mass m_B, initially at rest.



Object A collides with object B, which subsequently moves object B towards a stationary wall with speed b_1 and object A decelerates to a speed of a_1 in the same direction.



Object B then collides with the wall and moves back towards object A at the same speed b_1 but opposite direction, before colliding with it. The wall remains stationary at all times.



This process repeats until both objects are moving away from the wall and the speed of object A object exceeds the speed of object B so they do not collide again.

Let v_A and F_A be the velocity and force of object A respectively at time t, where the positive direction is to the right. Let v_B and F_B be similarly defined.

Question 16 continues on page 16

(i) Explain why during the collision between object A and object B that **1**

$$F_A + F_B = 0.$$

(ii) Let $p = v_A\sqrt{m_A}$ and $q = v_B\sqrt{m_B}$. By solving the differential equations associated with the force equation in part (i), show that at the first collision **3**

$$p^2 + q^2 = a_0^2 m_A \qquad \text{and} \qquad p\sqrt{m_A} + q\sqrt{m_B} = -a_0 m_A.$$

(iii) Hence, on the p-q axes (with p being the horizontal axis), plot the two points corresponding to the velocities of both objects just before and just after the first collision with each other. Justify your answer. **2**

(iv) On the same diagram using the p-q axes, plot the next two points corresponding to the velocities of both objects just before and just after their second collision with each other. Justify your answer. You do not need to identify the exact coordinates of the two points. **2**

(v) Let N be the total number of collisions that occur before both objects move away from the wall and never collide again. This includes collisions between object A and object B as well as collisions between object B and the wall. **4**

Using the result in part (a)(ii), show that for $N > 2$

$$\tan^2\left(\frac{\pi}{N+1}\right) < \frac{m_B}{m_A} < \tan^2\left(\frac{\pi}{N}\right).$$

End of paper

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2023 **BORED OF STUDIES TRIAL EXAMINATION**

Mathematics Extension 2

Solutions

Section I

Answers

1	B	**6**	B
2	C	**7**	D
3	A	**8**	C
4	A	**9**	B
5	C	**10**	D

Brief explanations

1 To pass through the origin, the point $(0,0,0)$ must satisfy the equation so

$$\begin{aligned}(0-a)^2+(0-2a)^2+(0-2a)^2&=81\\ 9a^2&=81\\ a&=\pm 3\end{aligned}$$

Hence, the answer is (B).

2 For some constants a and b, the polynomial can also be written as

$$x^3-ix^2+(1+i)x-2-i=(x^2+1)Q(x)+ax+b$$

Substitute $x=i$

$$\begin{aligned}i^3-i^3+(1+i)i-2-i&=(i^2+1)Q(i)+ai+b\\ ai+b&=-3\end{aligned}$$

Substitute $x=-i$

$$\begin{aligned}-i^3-i^3-(1+i)i-2-i&=(i^2+1)Q(i)-ai+b\\ -ai+b&=-1\end{aligned}$$

Solving simultaneously gives $a=i, b=-2$, hence the answer is (C)

3 In order to get an inverse tangent primitive function, the quadratic polynomial in the denominator must have non-real roots. This implies that

$$\begin{aligned} 4a^2b^2 - 4ac &< 0 \\ a(ab^2 - c) &< 0 \end{aligned}$$

The only choice which always satisfies this condition is (A).

4 The vector describes a curve in space as parameter t varies. The direction at time t is defined by the rate of change as the particle moves along the curve.

$$\begin{aligned} \dot{\underset{\sim}{r}} &= -\underset{\sim}{j} - \frac{3}{\sqrt{t}}\underset{\sim}{k} \\ &= -\underset{\sim}{j} - \underset{\sim}{k} \quad \text{when } t = 9 \end{aligned}$$

Hence, the answer is (A).

5 (D) cannot be the answer as this does not satisfy the given condition that $x < y$.

Consider the case where $x < z < y$ then the inequality reduces to

$$\begin{aligned} -(x-y) &< -(x-z) - (z-y) \\ y - x &< y - x \end{aligned}$$

This is false as the two sides are in fact equal, so this means the answer cannot be (B) and consequently cannot be (A).

For the case $z < x$ (which means $z < x < y$) then the inequality reduces to

$$\begin{aligned} -(x-y) &< (x-z) - (z-y) \\ y - x &< x - 2z + y \\ z &< x \end{aligned}$$

This is true for all z values under this case.

For the case $z > y$ (which means $x < y < z$) then the inequality reduces to

$$\begin{aligned} -(x-y) &< -(x-z) + (z-y) \\ y - x &< -x + 2z - y \\ z &> y \end{aligned}$$

This is true for all z values under this case.

Hence, the answer is (C)

6 Let Z be the point on the complex plane represented by complex number z and similarly define this for W and w. The given restriction on the arguments suggests that Z is anti-clockwise relative to W and $\angle ZOW$ is obtuse.

The vector diagonals of the rhombus are represented by the complex numbers $z+w$ and $z-w$. Their lengths are not necessarily equal but they are perpendicular to each other, so for some positive constant k

$$\begin{aligned}(z-w) &= ki(z+w)\\ \frac{z-w}{z+w} &= ki\end{aligned}$$

Since $\angle ZOW$ is obtuse then the diagonal vector represented by $z-w$ must be longer than the other diagonal vector represented by $z+w$. This means the scaling factor k on $z+w$ must be greater than 1. Hence, the answer is (B).

7 For option (A)

$$\begin{aligned}\left|\frac{1}{a}\underset{\sim}{i}+b\underset{\sim}{j}+\underset{\sim}{k}\right|^2 &= \frac{1}{a^2}+b^2+1 \qquad \text{but } a>0, b>0\\ &> 1\end{aligned}$$

For option (B)

$$\begin{aligned}\left|a^2b^2\underset{\sim}{i}+2ab\underset{\sim}{j}+2\underset{\sim}{k}\right|^2 &= a^4b^4+4a^2b^2+4\\ &= (a^2b^2+2)^2 \qquad \text{but } a>0, b>0\\ &> 4\end{aligned}$$

For option (C)

$$\begin{aligned}\left|a\underset{\sim}{i}+\frac{1}{b}\underset{\sim}{j}+\frac{1}{a}\underset{\sim}{k}\right|^2 &= a^2+\frac{1}{b^2}+\frac{1}{a^2} \qquad \text{but } b>0, a^2+\frac{1}{a^2}\geq 2\\ &> 2\end{aligned}$$

For option (D)

$$\begin{aligned}\left|a^2\underset{\sim}{i}+2ab\underset{\sim}{j}+2b^2\underset{\sim}{k}\right|^2 &= a^4+4a^2b^2+4b^4\\ &= (a^2+2b^2)^2 \qquad \text{but } a>0, b>0\\ &> 0\end{aligned}$$

Only option (D) could potentially give a unit vector as the other options have lower bounds greater than 1. For example, the values $a=\frac{1}{\sqrt{2}}$ and $b=\frac{1}{2}$ produce a unit vector for option (D).

8 The two forces acting on the object are gravity and the normal force. The net force is the simple harmonic motion.

Let m be the mass of the object and g be the acceleration due to gravity. Let n be such that the period is $\omega = \frac{2\pi}{n}$ and y be the vertical displacement,

Resolving the forces gives

$$\begin{aligned}-mn^2y &= N - mg \\ y &= \frac{mg - N}{mn^2} \\ &= \frac{\omega^2(mg - N)}{4m\pi^2}\end{aligned}$$

When the object loses contact $N = 0$. This occurs at full vertical extension which is when $y = A$. Hence, $A = \dfrac{g\omega^2}{4\pi^2}$ so the answer is (C).

9 For option (A), if $a > b$ then dividing both sides of the inequality by ab will give $\dfrac{1}{b} > \dfrac{1}{a}$, with the sign only preserved if $ab > 0$. This is not true for all a and b.

For option (C), it is possible to find a counterexample function such as $f(x) = \ln x$ where the gradient gets smaller with larger x-values.

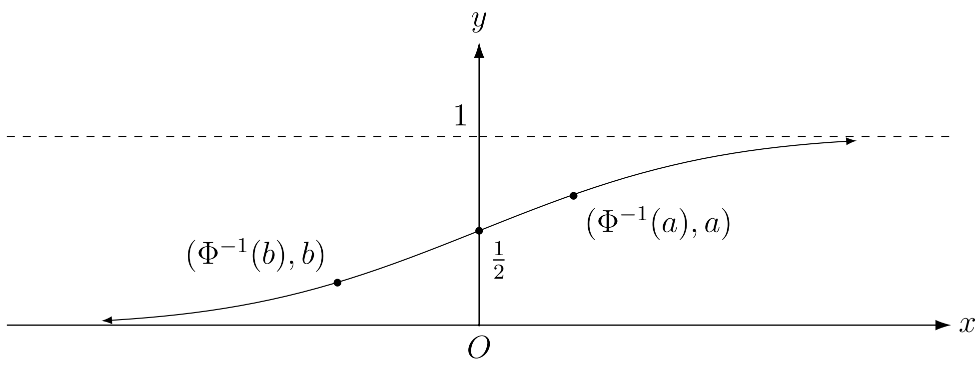
For option (D), if $f(x) > g(x)$ for all $x \in [a, b]$ then it is true that $\displaystyle\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$.

However, the converse is not necessarily true, particularly when there is a mix of positive and negative function values involved. A counterexample would be $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in [0, \pi]$.

For option (B), a standard cumulative normal distribution function is defined as

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx.$$

As x increases, so does $\Phi(x)$ (as the area gets larger). This means that $\Phi(x)$ is increasing for all real x and subsequently it has a unique inverse $\Phi^{-1}(x)$. Hence, the answer is (B).



10 Since all the integrals evaluate into a unique numeric value, the question can be re-framed as finding which option has a negation that is true.

For option (A), consider the negation statement $\displaystyle\int_0^1 \sqrt{x}\cos^{-1} x\,dx \geq \frac{\pi}{2}$.

Note that $0 \leq \sqrt{x} \leq 1$ and $0 \leq \cos^{-1} x \leq \frac{\pi}{2}$ for $0 \leq x \leq 1$. This means that the area under the curve $y = \sqrt{x}\cos^{-1} x$ must be less than the area of the rectangle with vertices at $(0,0)$, $(0,\frac{\pi}{2})$, $(1,\frac{\pi}{2})$ and $(1,0)$ which contradicts the negation and shows that option (A) is true.

For option (B), consider the negation statement $\displaystyle\int_1^e \frac{\tan^{-1} x}{x}\,dx \geq \frac{\pi}{2}$.

Note that if $1 \leq x \leq e$ then $\dfrac{\pi}{4} \leq \tan^{-1} x \leq \tan^{-1} e$ so

$$\begin{aligned}
\frac{\pi}{4}\int_1^e \frac{dx}{x} &\leq \int_1^e \frac{\tan^{-1} x}{x}\,dx \leq \tan^{-1} e\int_1^e \frac{dx}{x} \\
\frac{\pi}{4}[\ln x]_1^e &\leq \int_1^e \frac{\tan^{-1} x}{x}\,dx \leq \tan^{-1} e[\ln x]_1^e \\
\frac{\pi}{4} &\leq \int_1^e \frac{\tan^{-1} x}{x}\,dx \leq \tan^{-1} e
\end{aligned}$$

This contradicts the negation and shows that option (B) is true.

The negation statements of (C) and (D) are $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^4\sin x\,dx \geq \frac{\pi}{15}$ and $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{x}}\,dx \geq \frac{\pi}{15}$.

Note that if $\frac{\pi}{6} \leq x \leq \frac{\pi}{4}$, then $\frac{1}{2} \leq \sin x \leq \frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} \leq \cos x \leq \frac{\sqrt{3}}{2}$ so

$$\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{x^4}{2}\,dx &\leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^4\sin x\,dx \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{x^4}{\sqrt{2}}\,dx \\
\left[\frac{x^5}{10}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} &\leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^4\sin x\,dx \leq \left[\frac{x^5}{5\sqrt{2}}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
\frac{211\pi^5}{2488320} &\leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^4\sin x\,dx \leq \frac{211\pi^5}{1244160\sqrt{2}}
\end{aligned}$$

This contradicts the negation and shows that option (C) is true. Similarly,

$$\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sqrt{2x}}\,dx &\leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{x}}\,dx \leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{3}}{2\sqrt{x}}\,dx \\
\sqrt{2}[\sqrt{x}]_{\frac{\pi}{6}}^{\frac{\pi}{4}} &\leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{x}}\,dx \leq \sqrt{3}[\sqrt{x}]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
\frac{\sqrt{\pi}(3-\sqrt{6})}{3\sqrt{2}} &\leq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{x}}\,dx \leq \frac{\sqrt{\pi}(3-\sqrt{6})}{2\sqrt{3}}
\end{aligned}$$

The lower bound shows that the negation is true as it exceeds $\dfrac{\pi}{15}$. Hence, the original inequality is false and the answer is (D).

Remark: This is a question where it is quicker to deduce the answer than to prove it!

Question 11

(a) First split the integral as follows.

$$\int_0^1 \frac{x^2 \ln\left(1+x+x^9+x^{10}\right)}{1+x^3}\,dx$$
$$\int_0^1 \frac{x^2 \ln\left((1+x)+x^9(1+x)\right)}{1+x^3}\,dx$$
$$= \int_0^1 \frac{x^2 \ln\left(1+x\right)}{1+x^3}\,dx + \int_0^1 \frac{x^2 \ln\left(1+x^9\right)}{1+x^3}\,dx$$

For the first term, perform integration by parts where

$$u = \ln(1+x) \Rightarrow du = \frac{dx}{1+x}$$
$$dv = \frac{x^2}{1+x^3}\,dx \Rightarrow v = \frac{1}{3}\ln(1+x^3)$$

$$\begin{aligned}\int_0^1 \frac{x^2 \ln\left(1+x\right)}{1+x^3}\,dx &= \frac{1}{3}\left[\ln(1+x)\ln(1+x^3)\right]_0^1 - \frac{1}{3}\int_0^1 \frac{\ln\left(1+x^3\right)}{1+x}\,dx \\ &= \frac{(\ln 2)^2}{3} - \frac{1}{3}\int_0^1 \frac{\ln\left(1+x^3\right)}{1+x}\,dx\end{aligned}$$

For the second term, let $u = x^3 \Rightarrow du = 3x^2\,dx$. When $x = 0, u = 1$ and when $x = 1, u = 1$ so

$$\int_0^1 \frac{x^2 \ln\left(1+x^9\right)}{1+x^3}\,dx = \frac{1}{3}\int_0^1 \frac{\ln\left(1+u^3\right)}{1+u}\,du$$

Hence, noting that u is a dummy variable

$$\begin{aligned}\int_0^1 \frac{x^2 \ln\left(1+x+x^9+x^{10}\right)}{1+x^3}\,dx &= \frac{(\ln 2)^2}{3} - \frac{1}{3}\int_0^1 \frac{\ln\left(1+x^3\right)}{1+x}\,dx + \frac{1}{3}\int_0^1 \frac{\ln\left(1+u^3\right)}{1+u}\,du \\ &= \frac{(\ln 2)^2}{3} - \frac{1}{3}\int_0^1 \frac{\ln\left(1+x^3\right)}{1+x}\,dx + \frac{1}{3}\int_0^1 \frac{\ln\left(1+x^3\right)}{1+x}\,dx \\ &= \frac{(\ln 2)^2}{3}\end{aligned}$$

(b) The distance between $(4,-1,3)$ and $(1,-1,7)$ is the radius of the circular plane. Note that both points have the same y-coordinate which suggests that the two points lie on the plane $y=-1$.

This means the Cartesian equation is

$$\begin{aligned}(x-4)^2 + (z-3)^2 &= \sqrt{(4-1)^2 + (-1+1)^2 + (3-7)^2} \\ &= 25 \quad \text{and } y = -1\end{aligned}$$

Hence, a parametric vector equation is $(4+5\sin t)\underset{\sim}{i} - \underset{\sim}{j} + (3+5\cos t)\underset{\sim}{k}$ (or alternatively $(4+5\cos t)\underset{\sim}{i} - \underset{\sim}{j} + (3+5\sin t)\underset{\sim}{k}$).

(c) Let $\angle AOB = \theta$. Since the isosceles triangles are similar then $\angle COD = \theta$. Hence, the following rotational relationships hold.

$$\begin{aligned}
b &= ae^{i\theta} \\
d &= ce^{i\theta} \\
\text{LHS} &= |a - c| \\
&= |be^{-i\theta} - de^{-i\theta}| \\
&= |e^{-i\theta}||b - d| \\
&= |b - d| \\
&= \text{RHS}
\end{aligned}$$

(d) (i) A direction vector of ℓ_1 is $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= (6 - 4)\underset{\sim}{i} + (5 - 1)\underset{\sim}{j} + (-3 - 3)\underset{\sim}{k} \\
&= 2\underset{\sim}{i} + 4\underset{\sim}{j} - 6\underset{\sim}{k}
\end{aligned}$$

Consider the dot product of the two vectors.

$$\begin{aligned}
(2\underset{\sim}{i} + 4\underset{\sim}{j} - 6\underset{\sim}{k}) \cdot (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) &= 2 + 4 - 6 \\
&= 0
\end{aligned}$$

Since $\overrightarrow{AB}$ lies on ℓ_2 then ℓ_2 is perpendicular to ℓ_1.

(ii) With $\overrightarrow{AB} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 6\underset{\sim}{k}$ then a convenient direction vector for ℓ_2 is $\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ which leads to the equation

$$\ell_2 : \quad 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k} + \mu(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}) \quad \text{for some parameter } \mu.$$

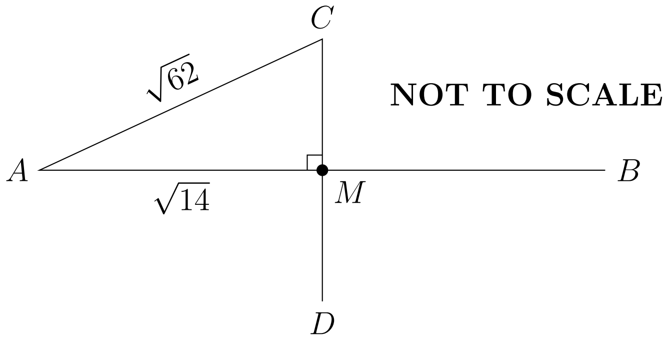
Solve this simultaneously with $\ell_1 : 10\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$ by components.

$$\begin{aligned}
4 + \mu &= 10 + \lambda \\
1 + 2\mu &= 8 + \lambda \\
3 - 3\mu &= 5 + \lambda
\end{aligned}$$

Taking the second equation less the first equation gives $-3 + \mu = -2$ so $\mu = 1$. Substitute this into the second equation gives $\lambda = -5$ which also satisfies the third equation. This means that the coordinates of the intersection point M are $(5, 3, 0)$. Now

$$\begin{aligned}
|\overrightarrow{AM}| &= \sqrt{(5 - 4)^2 + (3 - 1)^2 + (3 - 0)^2} \\
&= \sqrt{14}
\end{aligned}$$

$\overrightarrow{AM}$ is perpendicular to $\overrightarrow{MC}$ with $|\overrightarrow{AC}| = \sqrt{62}$ and $|\overrightarrow{AM}| = \sqrt{14}$.



Hence,

$$\begin{aligned} |\overrightarrow{MC}| &= \sqrt{|\overrightarrow{AC}|^2 - |\overrightarrow{AM}|^2} \\ &= 4\sqrt{3} \end{aligned}$$

This means that

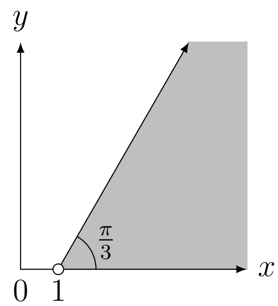
$$\begin{aligned} \overrightarrow{MC} &= \overrightarrow{OC} - \overrightarrow{OM} \\ &= 10\underset{\sim}{i} + 8\underset{\sim}{j} + 5\underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) - 5\underset{\sim}{i} - 3\underset{\sim}{j} \\ &= (5+\lambda)(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) \\ |\overrightarrow{MC}| &= \sqrt{(5+\lambda)^2 + (5+\lambda)^2 + (5+\lambda)^2} \\ &= \sqrt{3}|5+\lambda| \qquad \text{but } |\overrightarrow{MC}| = 4\sqrt{3} \\ \lambda &= -1, -9 \end{aligned}$$

When $\lambda = -1$ then $\overrightarrow{OC} = 9\underset{\sim}{i} + 7\underset{\sim}{j} + 4\underset{\sim}{k}$.

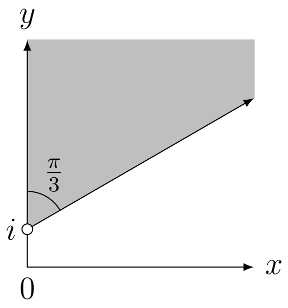
When $\lambda = -9$ then $\overrightarrow{OC} = \underset{\sim}{i} - \underset{\sim}{j} - 4\underset{\sim}{k}$.

The two solutions occur because C and D are interchangeable without loss of generality. Hence, their coordinates are $(9, 7, 4)$ and $(1, -1, -4)$.

(e) (i) Sketching the region $0 \le \arg(z-1) \le \dfrac{\pi}{3}$ gives the below.



(ii) R_2 is the reflection of R_1 along the line $y = x$.



(iii) The upper ray in the region of part (i) can be described as a line which has an x-intercept of $(1, 0)$ and gradient of $\tan\frac{\pi}{3}$. This has the linear equation

$$(y - 0) = \tan\frac{\pi}{3}(x - 1)$$
$$y = \sqrt{3}x - \sqrt{3}$$

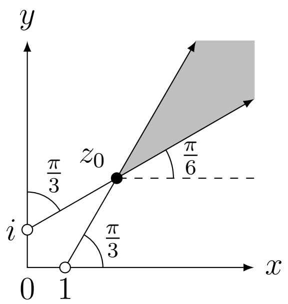
The lower ray of the region of part (ii) can be described as a line which has a y-intercept of $(0, 1)$ and a gradient of $\tan\frac{\pi}{6}$. This has the linear equation

$$(y - 1) = \tan\frac{\pi}{6}(x - 0)$$
$$y = \frac{1}{\sqrt{3}}x + 1$$

Solving simultaneously to find the point of intersection

$$\begin{aligned}
\sqrt{3}x - \sqrt{3} &= \frac{1}{\sqrt{3}}x + 1 \\
\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)x &= 1 + \sqrt{3} \\
x &= \frac{\sqrt{3}+3}{2} \quad \text{substitute into } y = \sqrt{3}x - \sqrt{3} \\
y &= \frac{\sqrt{3}+3}{2}
\end{aligned}$$

The point of intersection of the two rays is represented by $z_0 = \dfrac{\sqrt{3}+3}{2}(1 + i)$.



The boundaries of the intersection are simply the angles the rays make with the positive horizontal. This gives $\theta_1 = \dfrac{\pi}{6}$ and $\theta_2 = \dfrac{\pi}{3}$.

Question 12

(a) $P(z) = z^4 + 3z^2 + z - 1$ so $P'(z) = 4z^3 + 6z + 1$ and $P''(z) = 12z^2 + 6 > 0$. Since $P(z)$ is concave up then it must either have two real roots or no real roots. However, note that $P(0) = -1 < 0$ and $P(1) = 4 > 0$. Since $P(z)$ is continuous over all z, then it must it must have two x-intercepts given it is concave up.

This means that $P(z)$ has two real roots and two non-real roots in conjugate pairs (given it has real coefficients). Let these roots be $a + ib, a - ib, p$ and q. By product of roots

$$\begin{aligned}(a + ib)(a - ib)pq &= -1\\ pq &= -\frac{1}{a^2 + b^2}\end{aligned}$$

This means that $pq < 0$ so the real roots must be opposite in sign and therefore one of them is positive.

The sum of roots gives $p + q = -2a$ and the product of roots in triples gives

$$\begin{aligned}(a + ib)(a - ib)p + (a + ib)(a - ib)q + (a - ib)pq + (a + ib)pq &= -1\\ (a^2 + b^2)(p + q) + 2apq &= -1\\ -2a(a^2 + b^2) + 2apq &= -1\\ 2a(pq - (a^2 + b^2)) &= -1\end{aligned}$$

Since $pq < 0$ then $pq - (a^2 + b^2) < 0$ so $a > 0$ in order for a negative value to be possible on the RHS. Hence, there are 3 roots with positive real parts.

(b) (i) Using integration by parts on $\displaystyle\int_{-a}^{a} \frac{x^{n+2}e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx$.

$u = x^{n+1} \Rightarrow du = (n + 1)x^n dx$

$dv = xe^{-\frac{x^2}{2}} \Rightarrow v = -e^{-\frac{x^2}{2}}$

$$\begin{aligned}\int_{-a}^{a} \frac{x^{n+2}e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx &= -\frac{1}{\sqrt{2\pi}}\left[x^{n-1}e^{-\frac{x^2}{2}}\right]_{-a}^{a} + \frac{n+1}{\sqrt{2\pi}}\int_{-a}^{a} x^n e^{-\frac{x^2}{2}}\,dx\\ &= -\frac{1}{\sqrt{2\pi}}\left(a^{n-1}e^{-\frac{a^2}{2}} - (-1)^{n-1}a^{n-1}e^{-\frac{a^2}{2}}\right) + \frac{n+1}{\sqrt{2\pi}}\int_{-a}^{a} x^n e^{-\frac{x^2}{2}}\,dx\end{aligned}$$

$$\begin{aligned}I_{n+2} &= \lim_{a\to\infty}\left(\int_{-a}^{a} \frac{x^{n+2}e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx\right)\\ &= \lim_{a\to\infty}\left(-\frac{1}{\sqrt{2\pi}}\left(a^{n-1}e^{-\frac{a^2}{2}} - (-1)^{n-1}a^{n-1}e^{-\frac{a^2}{2}}\right) + \frac{n+1}{\sqrt{2\pi}}\int_{-a}^{a} x^n e^{-\frac{x^2}{2}}\,dx\right)\\ &= (n + 1)I_n \quad \text{since } \lim_{a\to\infty}\left(a^{n-1}e^{-\frac{a^2}{2}}\right) = 0\end{aligned}$$

(ii) Note that $I_0 = \lim_{a\to\infty}\left(\int_{-a}^{a} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx\right) = 1$ since the integrand is a standard normal probability density function. Using the result in part (i)

$$\begin{aligned}
I_{2n} &= (2n-1)I_{2n-2} \\
&= (2n-1)(2n-3)I_{2n-4} \\
&= (2n-1)(2n-3)(2n-5)I_{2n-6} \\
&\dots \\
&= (2n-1)(2n-3)(2n-5)\dots 3.1.I_0 \\
&= (2n-1)(2n-3)(2n-5)\dots 3.1 \times \frac{(2n)(2n-2)(2n-4)\dots 4.2}{(2n)(2n-2)(2n-4)\dots 4.2} \\
&= \frac{(2n)!}{2^n n!}
\end{aligned}$$

Remark: The value of I_n is actually $E(X^n)$ where $X \sim N(0,1)$. When n is odd $E(X^n) = 0$ (e.g. $E(X) = 0$ and $E(X^3) = 0$) and when n is even $E(X^n)$ is a product of odd numbers (e.g. $E(X^2) = 1$ and $E(X^4) = 3$).

(c) (i) $P(z) = z^{2mn} - az^{mn} + 1$ is a quadratic polynomial in z^{mn} and its discriminant is $a^2 - 4 < 0$ since $-2 < a < 2$. This means that the two roots are complex conjugates, say α and $\overline{\alpha}$, since the coefficients of $P(z)$ are real. From the product of the two roots $\alpha\overline{\alpha} = 1$ and hence $|\alpha|^2 = 1$, or equivalently $|\alpha| = 1$.

This similarly applies for $Q(z)$ which is a special case of $P(z)$ where $m = 1$.

(ii) Since the roots have a modulus of 1 then they can be written in general form as $z = e^{i\theta}$. Since $Q(z)$ is a factor of $P(z)$ then the roots also have this general form.

$$\begin{aligned}
e^{2in\theta} - ae^{in\theta} + 1 &= 0 \\
a &= \frac{\cos 2n\theta + i\sin 2n\theta + 1}{\cos n\theta + i\sin n\theta} \\
&= \frac{2\cos^2 n\theta + 2i\sin n\theta\cos n\theta}{\cos n\theta + i\sin n\theta} \\
&= 2\cos n\theta
\end{aligned}$$

Similarly, substituting the general form of the root into $P(z)$ gives $a = 2\cos mn\theta$. To eliminate θ as an introduced parameter

$$\begin{aligned}
2\cos n\theta &= 2\cos mn\theta \\
mn\theta &= 2k\pi \pm n\theta \quad \text{for some integer } k \\
\theta &= \frac{2k\pi}{n(m-1)}, \frac{2k\pi}{n(m+1)} \\
a &= 2\cos\left(\frac{2k\pi}{n(m-1)}\right), 2\cos\left(\frac{2k\pi}{n(m+1)}\right) \quad \text{for some integer } k
\end{aligned}$$

(d) (i) Upwards is the positive direction but the object is falling so $\dot{y} < 0$. Resolving the forces vertically

$$\begin{aligned} m\ddot{y} &= mk|\dot{y}| - mg \\ \ddot{y} &= -k\left(\dot{y} + \frac{g}{k}\right) \end{aligned}$$

This is a shifted exponential decay differential equation with general solution $\dot{y} = -\frac{g}{k} + Be^{-kt}$. When $t = 0$, $\dot{y} = 0 \Rightarrow B = \frac{g}{k}$, hence

$$\dot{y} = \frac{g}{k}(e^{-kt} - 1).$$

(ii) Deriving the displacement equation

$$\begin{aligned} \dot{y} &= \frac{g}{k}(e^{-kt} - 1) \\ y &= \frac{g}{k}\left(-\frac{1}{k}e^{-kt} - t\right) + c \qquad \text{when } t = 0, y = h \Rightarrow c = h + \frac{g}{k^2} \\ y &= \frac{g}{k}\left(-\frac{1}{k}e^{-kt} - t + \frac{1}{k}\right) + h \end{aligned}$$

The bucket crosses the origin twice per period. Since it is initially at the origin, the next time it is at the origin is at $t = \frac{\pi}{\omega}$. Hence, the nth time it is at the origin after initial position is $t = \frac{n\pi}{\omega}$. The ball must hit the origin at this time in order for it land in the bucket. So require $y = 0$ at $t = \frac{n\pi}{\omega}$ hence

$$\begin{aligned} 0 &= \frac{g}{k}\left(-\frac{1}{k}e^{-k\frac{n\pi}{\omega}} - \frac{n\pi}{\omega} + \frac{1}{k}\right) + h \\ \frac{g}{k^2}e^{-k\frac{n\pi}{\omega}} &= -\frac{gn\pi}{k\omega} + \frac{g}{k^2} + h \\ e^{-k\frac{n\pi}{\omega}} &= 1 - \frac{kn\pi}{\omega} + \frac{k^2h}{g} \end{aligned}$$

Since $t > 0$ when hitting the bucket then $0 < e^{-k\frac{n\pi}{\omega}} < 1$ so

$$\begin{aligned} 0 < 1 - \frac{kn\pi}{\omega} &+ \frac{k^2h}{g} < 1 \\ \frac{k^2h}{g} < \frac{kn\pi}{\omega} &< \frac{k^2h}{g} + 1 \\ \frac{\omega kh}{\pi g} < n &< \frac{\omega}{\pi}\left(\frac{kh}{g} + \frac{1}{k}\right) \end{aligned}$$

Question 13

(a) (i) If $x = \dfrac{p}{q}$ is a root of $P(x)$ then

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + a_{n-2}\left(\frac{p}{q}\right)^{n-2} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$
$$p(a_np^{n-1} + a_{n-1}p^{n-2}q + a_{n-2}p^{n-3}q^2 + \cdots + a_1q^{n-1}) = -a_0q^n$$

All terms on either side are integers. Since p and q have no common factors other than 1 then it must be that a_0 is divisible by p.

Similarly, the expression can rearranged as

$$q(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_0q^{n-1}) = -p^na_n$$

Since p and q have no common factors other than 1 then it must be that a_n is divisible by q.

(ii) Note that $\cos\dfrac{\pi}{9}$ is a solution to $\cos 3\theta = \dfrac{1}{2}$ so it must be a solution to

$$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$$
$$8\cos^3\theta - 6\cos\theta - 1 = 0$$

This is a cubic polynomial equation in $\cos\theta$.

Suppose that $\cos\dfrac{\pi}{9} = \dfrac{p}{q}$ (i.e. it is rational root) where p and q are integers with no common factors, other than 1.

From part (i), this implies that the constant term -1 is divisible by p and the leading coefficient 8 is divisible by q. This suggests that the only possible roots of this cubic polynomial are $\cos\theta = \pm\dfrac{1}{2}, \pm\dfrac{1}{4}, \pm\dfrac{1}{8}$.

However, note that since $\cos x$ is decreasing when $0 < x < \dfrac{\pi}{2}$ then

$$\frac{\pi}{9} < \frac{\pi}{3}$$
$$\cos\frac{\pi}{9} > \cos\frac{\pi}{3}$$
$$\cos\frac{\pi}{9} > \frac{1}{2}$$

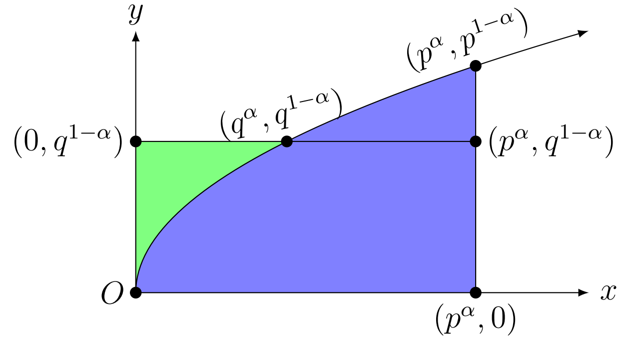
This means that none of the candidate roots can be $\cos\dfrac{\pi}{9}$ which is a contradiction. Hence, it must be that $\cos\dfrac{\pi}{9}$ is irrational.

(b) (i) Without loss of generality, suppose $p > q$. Given $y = x^{\frac{1-\alpha}{\alpha}}$ then $x = y^{\frac{\alpha}{1-\alpha}}$.

Let A_p be the area bounded by the curve $y = x^{\frac{1-\alpha}{\alpha}}$, the x-axis and the line $x = p^\alpha$ (the blue area below).

Let A_q be the area bounded by the curve $y = x^{\frac{1-\alpha}{\alpha}}$, the y-axis and the line $y = q^{1-\alpha}$ (the green area below).

Consider the rectangle with vertices $(0,0), (p^\alpha, 0), (p^\alpha, q^{1-\alpha}), (0, q^{1-\alpha})$.
This rectangle has area $p^\alpha q^{1-\alpha}$.

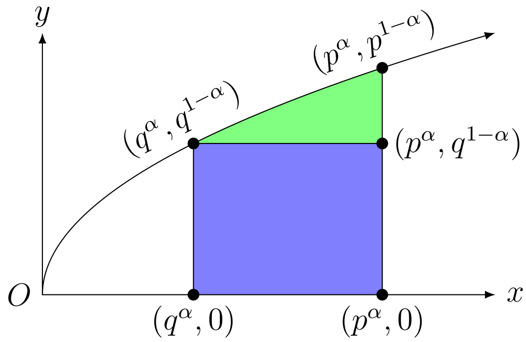


Since $A_p + A_q \geq p^\alpha q^{1-\alpha}$ then

$$\int_0^{p^\alpha} x^{\frac{1-\alpha}{\alpha}}\, dx + \int_0^{q^{1-\alpha}} y^{\frac{\alpha}{1-\alpha}}\, dy \geq p^\alpha q^{1-\alpha}$$
$$\left[\alpha x^{\frac{1}{\alpha}}\right]_0^{p^\alpha} + \left[(1-\alpha) y^{\frac{1}{1-\alpha}}\right]_0^{q^{1-\alpha}} \geq p^\alpha q^{1-\alpha}$$
$$p^\alpha q^{1-\alpha} \leq \alpha p + (1-\alpha) q$$

Alternate Method:

Consider the rectangle with vertices $(q^\alpha, 0), (p^\alpha, 0), (p^\alpha, q^{1-\alpha}), (q^\alpha, q^{1-\alpha})$.
This rectangle has area $q^{1-\alpha}(p^\alpha - q^\alpha)$, or equivalently, $p^\alpha q^{1-\alpha} - q$.



This area is no greater than the area under the curve $y = x^{\frac{1-\alpha}{\alpha}}$ for $p^\alpha \leq x \leq q^\alpha$.

$$p^\alpha q^{1-\alpha} - q \leq \int_{q^\alpha}^{p^\alpha} x^{\frac{1-\alpha}{\alpha}}\, dx$$
$$p^\alpha q^{1-\alpha} \leq q + \left[\alpha x^{\frac{1}{\alpha}}\right]_{q^\alpha}^{p^\alpha}$$
$$p^\alpha q^{1-\alpha} \leq \alpha p + (1-\alpha) q$$

(ii) Let $\alpha = \dfrac{2}{3}, p = \sqrt{xy}$ and $q = z$ for the inequality in part (i).

$$\begin{aligned}(\sqrt{xy})^{\frac{2}{3}} z^{\frac{1}{3}} &\leq \frac{2}{3}\sqrt{xy} + \frac{1}{3}z \\ (xyz)^{\frac{1}{3}} &\leq \frac{2}{3}\sqrt{xy} + \frac{1}{3}z \qquad \text{but } x + y \geq 2\sqrt{xy} \\ \frac{x+y+z}{3} &\geq (xyz)^{\frac{1}{3}}\end{aligned}$$

(iii) When $n = 2$ then $\alpha_1 + \alpha_2 = 1$

$$\begin{aligned}\text{LHS} &= \alpha_1 x_1 + \alpha_2 x_2 \\ &= \alpha_1 x_1 + (1-\alpha_1)x_2 \\ &\geq x_1^{\alpha_1} x_2^{1-\alpha_1} \qquad \text{from (i)} \\ &= \text{RHS}\end{aligned}$$

The statement is true for $n = 2$.

Assume the statement is true for $n = k$, that for some α_i

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \geq x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k} \qquad \text{where } \alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$$

Required to prove that the statement is true for $n = k + 1$, that for some β_i

$$\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k+1} x_{k+1} \geq x_1^{\beta_1} x_2^{\beta_2} \ldots x_{k+1}^{\beta_{k+1}} \qquad \text{where } \beta_1 + \beta_2 + \cdots + \beta_{k+1} = 1$$

Note that α_i and β_i are just **arbitrary** constants which satisfy the same set of conditions each between 0 and 1 and all values summing to 1. So given

$$\begin{aligned}\beta_1 + \beta_2 + \cdots + \beta_{k+1} &= 1 \\ \beta_1 + \beta_2 + \cdots + \beta_k &= 1 - \beta_{k+1} \\ \frac{\beta_1}{1-\beta_{k+1}} + \frac{\beta_2}{1-\beta_{k+1}} + \cdots + \frac{\beta_k}{1-\beta_{k+1}} &= 1\end{aligned}$$

Let $\alpha_i = \dfrac{\beta_i}{1-\beta_{k+1}}$ noting that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$, so now

$$\begin{aligned}\text{LHS} &= \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{k+1} x_{k+1} \\ &= (1-\beta_{k+1})\left(\frac{\beta_1}{1-\beta_{k+1}}x_1 + \frac{\beta_2}{1-\beta_{k+1}}x_2 + \cdots + \frac{\beta_k}{1-\beta_{k+1}}x_k\right) + \beta_{k+1}x_{k+1} \\ &= (1-\beta_{k+1})(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k) + \beta_{k+1}x_{k+1} \\ &\geq (1-\beta_{k+1})(x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}) + \beta_{k+1}x_{k+1} \qquad \text{by assumption} \\ &\geq (x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k})^{1-\beta_{k+1}} x_{k+1}^{\beta_{k+1}} \qquad \text{using result in part (i)} \\ &= x_1^{\alpha_1(1-\beta_{k+1})} x_2^{\alpha_2(1-\beta_{k+1})} \ldots x_k^{\alpha_k(1-\beta_{k+1})} x_{k+1}^{\beta_{k+1}} \\ &= x_1^{\beta_1} x_2^{\beta_2} \ldots x_{k+1}^{\beta_{k+1}} \\ &= \text{RHS}\end{aligned}$$

Hence, since the statement is true for $n = 2$ then by induction it is true for all integers $n \geq 2$.

(iv) Using the result in part (iii)

$$\begin{aligned}
\text{LHS} &= \operatorname{Var}\left(\sqrt{X}\right) \\
&= E(X) - \left[E(\sqrt{X})\right]^2 \\
&= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n - (\alpha_1\sqrt{x_1} + \alpha_2\sqrt{x_2} + \cdots + \alpha_n\sqrt{x_n})^2 \\
&\le \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n - \left(x_1^{\frac{\alpha_1}{2}} x_2^{\frac{\alpha_2}{2}} \dots x_n^{\frac{\alpha_n}{2}}\right)^2 \\
&= \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n - x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \\
&= \text{RHS}
\end{aligned}$$

Remark: Note that this result shows that the variance of $\sqrt{X}$ can never exceed the gap between the arithmetic mean and the geometric mean.

(c) (i) The particle is at rest when $v = 0$ so noting that $t > 0$

$$\begin{aligned}
4 - 4\sin 4t &= 0 \\
\sin 4t &= 1 \\
t &= \frac{\pi}{8}, \frac{5\pi}{8}, \dots
\end{aligned}$$

The particle is first at rest when $t = \dfrac{\pi}{8}$.

(ii) The particle is at rest when $t = \dfrac{\pi}{8}$ and then next at rest when $t = \dfrac{5\pi}{8}$.

Given the motion is simple harmonic, the particle takes half its period to move between the two rest points so the period is π.

Since the centre is $x = 1$ then the general form of the displacement equation for some constants α and A is

$$\begin{aligned}
x &= 1 + A\cos(2t + \alpha) \\
v &= -2A\sin(2t + \alpha)
\end{aligned}$$

When $t = \dfrac{\pi}{8}$ then $v = 0$ so

$$\begin{aligned}
-2A\sin\left(\frac{\pi}{4} + \alpha\right) &= 0 \\
\alpha &= -\frac{\pi}{4}
\end{aligned}$$

Now $v^2 = 4A^2 \sin^2\left(2t - \frac{\pi}{4}\right)$ so

$$\begin{aligned}
4A^2 \sin^2\left(2t - \frac{\pi}{4}\right) &\equiv 4 - 4\sin 4t \\
4A^2 \left(\frac{1}{\sqrt{2}}\sin 2t - \frac{1}{\sqrt{2}}\cos 2t\right)^2 &\equiv 4 - 4\sin 4t \\
\frac{A^2}{2}\left(\sin 2t - \cos 2t\right)^2 &\equiv 1 - \sin 4t \\
\frac{A^2}{2}(\sin^2 2t - 2\sin 2t\cos 2t + \cos^2 2t) &\equiv 1 - \sin 4t \\
\frac{A^2}{2}(1 - \sin 4t) &\equiv 1 - \sin 4t \\
\frac{A^2}{2} &= 1 \\
A &= \pm\sqrt{2}
\end{aligned}$$

Hence, the displacement equations are $x = 1 \pm \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$.

Question 14

(a) (i) For some constants a and b, let

$$\frac{5}{(x-2)(2x+1)} \equiv \frac{a}{x-2} + \frac{b}{2x+1}$$
$$a(2x+1) + b(x-2) = 5$$

When $x = 2 \Rightarrow a = 1$ and when $x = -\dfrac{1}{2} \Rightarrow b = -2$, hence

$$\frac{5}{(x-2)(2x+1)} = \frac{1}{x-2} - \frac{2}{2x+1}.$$

(ii) Let $t = \tan\frac{x}{2} \Rightarrow x = 2\tan^{-1} t \Rightarrow dx = \dfrac{2}{1+t^2}\,dt$ so

$$\begin{aligned}
\int \frac{5}{4\cos x + 3\sin x}\,dx &= \int \frac{5}{4\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt \\
&= \int \frac{5}{2+3t-2t^2}\,dt \\
&= -\int \frac{5}{(2t+1)(t-2)}\,dt \\
&= \int \left(\frac{2}{2t+1} - \frac{1}{t-2}\right) dt \qquad \text{from part (i)} \\
&= \int \left(\frac{2}{2t+1} + \frac{1}{2-t}\right) dt \\
&= \ln\left(\frac{2t+1}{2-t}\right) + c_1 \qquad \text{given } 0 \le x \le \frac{\pi}{2} \text{ then } 2 - \tan\frac{x}{2} > 0 \\
&= \ln\left(\frac{2\tan\frac{x}{2} + 1}{2 - \tan\frac{x}{2}}\right) + c_1 \\
&= \ln\left(\frac{\frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}} + 1}{2 - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} \times \frac{\sin\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}\right) + c_1 \\
&= \ln\left(\frac{2\sin^2\frac{x}{2} + \sin\frac{x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2} - \sin^2\frac{x}{2}}\right) + c_1 \\
&= \ln\left(\frac{(1-\cos x) + \frac{1}{2}\sin x}{\sin x - \frac{1}{2}(1-\cos x)}\right) + c_1 \\
&= \ln\left(\frac{\sin x - 2\cos x + 2}{2\sin x + \cos x - 1}\right) + c_1
\end{aligned}$$

(iii) Let $4\cos x + 3\sin x = R\cos(x-\alpha)$ for some $0 < \alpha < \dfrac{\pi}{2}$ so given $0 \le x \le \dfrac{\pi}{2}$

$$\begin{aligned}
\int \frac{5}{4\cos x + 3\sin x}\,dx &= \int \frac{5}{R\cos(x-\alpha)}\,dx \\
&= \frac{5}{R}\int \sec(x-\alpha)\,dx \\
&= \frac{5}{R}\ln\left(\sec(x-\alpha)+\tan(x-\alpha)\right) + c_2 \\
&= \frac{5}{R}\ln\left(\frac{1+\sin(x-\alpha)}{\cos(x-\alpha)}\right) + c_2
\end{aligned}$$

But

$$\begin{aligned}
R &= \sqrt{3^2+4^2} \\
&= 5 \\
4\cos x + 3\sin x &\equiv 5\cos x\cos\alpha + 5\sin x\sin\alpha \\
\cos\alpha = \frac{4}{5} &\quad \text{and} \quad \sin\alpha = \frac{3}{5}
\end{aligned}$$

Hence

$$\begin{aligned}
\int \frac{5}{4\cos x + 3\sin x}\,dx &= \ln\left(\frac{1+\sin x\cos\alpha - \cos x\sin\alpha}{\cos x\cos\alpha + \sin x\sin\alpha}\right) + c_2 \\
&= \ln\left(\frac{1+\frac{4}{5}\sin x - \frac{3}{5}\cos x}{\frac{4}{5}\cos x + \frac{3}{5}\sin x}\right) + c_2 \\
&= \ln\left(\frac{4\sin x - 3\cos x + 5}{3\sin x + 4\cos x}\right) + c_2
\end{aligned}$$

(iv) Since the answers in part (ii) and part (iii) are evaluations of the same integral then they must only differ by a constant.

This means that for some constant k

$$\ln\left(\frac{\sin x - 2\cos x + 2}{2\sin x + \cos x - 1}\right) - \ln\left(\frac{4\sin x - 3\cos x + 5}{3\sin x + 4\cos x}\right) = k$$

Since this is true for all x (except where $4\cos x + 3\sin x = 0$) then k can be found with any substituted value of x, say $x = \frac{\pi}{2}$

$$\begin{aligned}
k &= \ln\left(\frac{\sin\frac{\pi}{2} - 2\cos\frac{\pi}{2} + 2}{2\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - 1}\right) - \ln\left(\frac{4\sin\frac{\pi}{2} - 3\cos\frac{\pi}{2} + 5}{3\sin\frac{\pi}{2} + 4\cos\frac{\pi}{2}}\right) \\
&= \ln 3 - \ln 3 \\
&= 0
\end{aligned}$$

They are exactly equivalent, hence $\dfrac{\sin x - 2\cos x + 2}{2\sin x + \cos x - 1} \equiv \dfrac{4\sin x - 3\cos x + 5}{3\sin x + 4\cos x}$

(b) (i) Note that $P(x)$ is a geometric series so

$$\begin{aligned} P(x) &= x^{2n} + x^{2n-1} + x^{2n-2} + \cdots + x + 1 \\ &= \frac{x^{2n+1} - 1}{x - 1} \quad \text{for } x \neq 1 \end{aligned}$$

The roots of $P(x)$ are the roots of unity for $x^{2n+1} = 1$ where $x \neq 1$

$$x = e^{i\frac{2k\pi}{2n+1}} \quad \text{where } k = 1, 2, 3, \ldots, 2n$$

(ii) Using the fact that $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ then

$$\begin{aligned} \tan\theta &= \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} \\ &= \frac{e^{2i\theta} - 1}{i(e^{2i\theta} + 1)} \\ &= \frac{1 - e^{2i\theta}}{i(-1 - e^{2i\theta})} \end{aligned}$$

Let $\theta = \dfrac{k\pi}{2n+1}$ then for $k = 1, 2, 3, \ldots, 2n$

$$\begin{aligned} &\tan\left(\frac{\pi}{2n+1}\right)\tan\left(\frac{2\pi}{2n+1}\right)\tan\left(\frac{3\pi}{2n+1}\right)\ldots\tan\left(\frac{2n\pi}{2n+1}\right) \\ &= \frac{\left(1 - e^{i\frac{2\pi}{2n+1}}\right)\left(1 - e^{i\frac{4\pi}{2n+1}}\right)\left(1 - e^{i\frac{6\pi}{2n+1}}\right)\ldots\left(1 - e^{i\frac{4n\pi}{2n+1}}\right)}{i^{2n}\left(-1 - e^{i\frac{2\pi}{2n+1}}\right)\left(-1 - e^{i\frac{4\pi}{2n+1}}\right)\left(-1 - e^{i\frac{6\pi}{2n+1}}\right)\ldots\left(-1 - e^{i\frac{4n\pi}{2n+1}}\right)} \end{aligned}$$

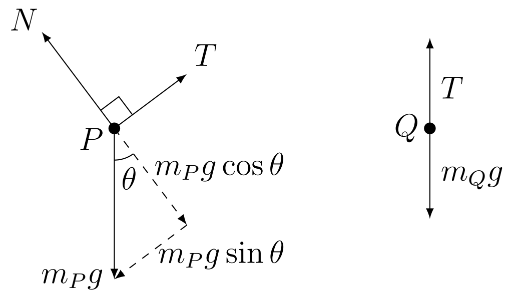
However, from part (i) the polynomial $P(x)$ can be factorised as

$$P(x) = \left(x - e^{i\frac{2\pi}{2n+1}}\right)\left(x - e^{i\frac{4\pi}{2n+1}}\right)\left(x - e^{i\frac{6\pi}{2n+1}}\right)\ldots\left(x - e^{i\frac{4n\pi}{2n+1}}\right)$$

This means that

$$\begin{aligned} &\tan\left(\frac{\pi}{2n+1}\right)\tan\left(\frac{2\pi}{2n+1}\right)\tan\left(\frac{3\pi}{2n+1}\right)\ldots\tan\left(\frac{2n\pi}{2n+1}\right) \\ &= \frac{P(1)}{i^{2n}P(-1)} \times \frac{i^{2n}}{i^{2n}} \quad \text{but } P(1) = 2n + 1 \text{ and } P(-1) = 1 \\ &= (-1)^n(2n + 1) \end{aligned}$$

(c) (i) Resolving the forces for both objects parallel and perpendicular to the slope:



The net force on object P is zero so

$$T - m_P g \sin\theta = 0$$
$$N - m_P g \cos\theta = 0$$

Resolving the forces on object Q gives $T - m_Q g = 0$ or equivalently $T = m_Q g$. Substitute this into the net force equation for object P along the slope gives

$$m_Q g - m_P g \sin\theta = 0$$
$$\sin\theta = \frac{m_Q}{m_P}$$

Since θ is acute then $\sin\theta < 1$, hence $m_P > m_Q$.

(ii) Consider the case that the direction of F is directed up the slope, then the new force equations are

$$T + F - m_P g \sin\theta = 0 \Rightarrow F = m_P g \sin\theta - m_Q g$$
$$N - m_P g \cos\theta = 0 \Rightarrow N = m_P g \cos\theta$$
$$\frac{F}{N} = \frac{m_P \sin\theta - m_Q}{m_P \cos\theta} \quad \text{but } \frac{m_P}{m_Q} = \frac{\sqrt{3}}{\sqrt{2}}$$
$$= \frac{\sqrt{3}\sin\theta - \sqrt{2}}{\sqrt{3}\cos\theta}$$

For neither object to accelerate in any direction, $\dfrac{F}{N} \leq \dfrac{1}{\sqrt{3}}$ so noting that θ is acute it follows that

$$\frac{\sqrt{3}\sin\theta - \sqrt{2}}{\sqrt{3}\cos\theta} \leq \frac{1}{\sqrt{3}}$$
$$\sqrt{3}\sin\theta - \cos\theta \leq \sqrt{2}$$
$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta \leq \frac{1}{\sqrt{2}}$$
$$\sin\left(\theta - \frac{\pi}{6}\right) \leq \frac{1}{\sqrt{2}}$$
$$\theta \leq \frac{5\pi}{12}$$

Noting that θ is acute and $\sin x$ is always increasing for $0 < x < \frac{\pi}{2}$.

Now consider the case where the direction of F is directed down the slope, which can be solved similarly

$$\begin{aligned}
T - F - m_P g \sin\theta = 0 \Rightarrow F = m_Q g - m_P g \sin\theta \\
N - m_P g \cos\theta = 0 \Rightarrow N = m_P g \cos\theta \\
\frac{F}{N} &= \frac{m_Q - m_P \sin\theta}{m_P \cos\theta} \qquad \text{but } \frac{m_P}{m_Q} = \frac{\sqrt{3}}{\sqrt{2}} \\
&= \frac{\sqrt{2} - \sqrt{3}\sin\theta}{\sqrt{3}\cos\theta}
\end{aligned}$$

Require $\dfrac{F}{N} \leq \dfrac{1}{\sqrt{3}}$ so

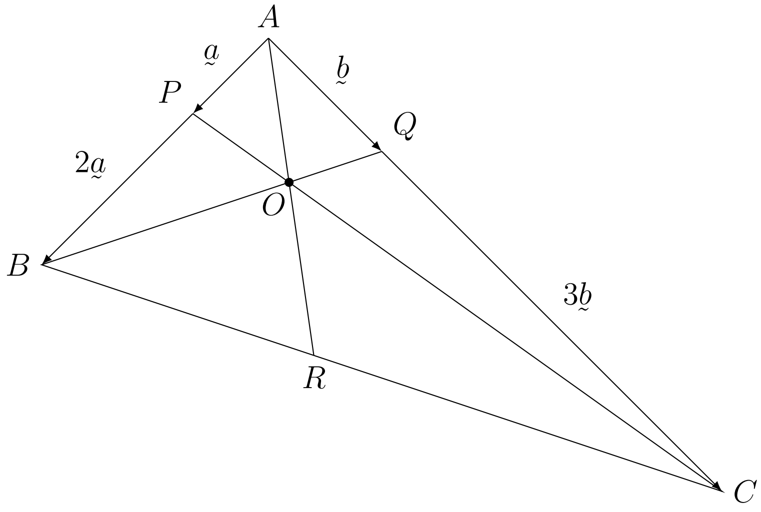
$$\begin{aligned}
\frac{\sqrt{2} - \sqrt{3}\sin\theta}{\sqrt{3}\cos\theta} &\leq \frac{1}{\sqrt{3}} \\
\sqrt{3}\sin\theta + \cos\theta &\geq \sqrt{2} \\
\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta &\geq \frac{1}{\sqrt{2}} \\
\sin\left(\theta + \frac{\pi}{6}\right) &\geq \frac{1}{\sqrt{2}} \\
\theta &\geq \frac{\pi}{12}
\end{aligned}$$

Noting that θ is acute and $\sin x$ is always increasing for $0 < x < \frac{\pi}{2}$.

Hence, $\dfrac{\pi}{12} \leq \theta \leq \dfrac{5\pi}{12}$.

Question 15

(a) Let $\overrightarrow{AP} = \underset{\sim}{a}$ then $\overrightarrow{PB} = 2\underset{\sim}{a}$. Similarly, let $\overrightarrow{AQ} = \underset{\sim}{b}$ then $\overrightarrow{QC} = 3\underset{\sim}{b}$.



The vector $\overrightarrow{AO}$ can be written in two different ways for some constants $\lambda \in (0,1)$ and $\mu \in (0,1)$:

$$\overrightarrow{AO} = \overrightarrow{AB} + \lambda\overrightarrow{BQ}$$
$$\overrightarrow{AO} = \overrightarrow{AC} + \mu\overrightarrow{CP}$$

Equating these gives

$$\begin{aligned}
\overrightarrow{AB} + \lambda\overrightarrow{BQ} &= \overrightarrow{AC} + \mu\overrightarrow{CP} \\
\overrightarrow{AB} + \lambda(\overrightarrow{AQ} - \overrightarrow{AB}) &= \overrightarrow{AC} + \mu(\overrightarrow{AP} - \overrightarrow{AC}) \\
3\underset{\sim}{a} + \lambda(\underset{\sim}{b} - 3\underset{\sim}{a}) &= 4\underset{\sim}{b} + \mu(\underset{\sim}{a} - 4\underset{\sim}{b}) \\
(3 - 3\lambda - \mu)\underset{\sim}{a} + (-4 + \lambda + 4\mu)\underset{\sim}{b} &= \underset{\sim}{0} \\
3\lambda + \mu = 3 \quad \text{and} \quad \lambda + 4\mu &= 4
\end{aligned}$$

From the first equation, substitute $\mu = 3 - 3\lambda$ into the second equation which gives

$$\begin{aligned}
\lambda + 4(3 - 3\lambda) &= 4 \\
\lambda &= \frac{8}{11} \quad \text{substitute into } \overrightarrow{AO} = \overrightarrow{AB} + \lambda\overrightarrow{BQ} \\
\overrightarrow{AO} &= 3\underset{\sim}{a} + \frac{8}{11}(\underset{\sim}{b} - 3\underset{\sim}{a}) \\
&= \frac{9}{11}\underset{\sim}{a} + \frac{8}{11}\underset{\sim}{b}
\end{aligned}$$

Also, notice that for $k > 1$ and $r \in (0, 1)$

$$\begin{aligned}
\overrightarrow{AR} &= k\overrightarrow{AO} \\
\overrightarrow{BR} &= r\overrightarrow{BC} \\
\overrightarrow{AR} &= \overrightarrow{AB} + \overrightarrow{BR} \\
k\overrightarrow{AO} &= \overrightarrow{AB} + r\overrightarrow{BC} \\
&= \overrightarrow{AB} + r(\overrightarrow{AC} - \overrightarrow{AB}) \\
k\left(\frac{9}{11}\underset{\sim}{a} + \frac{8}{11}\underset{\sim}{b}\right) &= 3\underset{\sim}{a} + r(4\underset{\sim}{b} - 3\underset{\sim}{a}) \\
\left(\frac{9k}{11} - 3 + 3r\right)\underset{\sim}{a} + \left(\frac{8k}{11} - 4r\right)\underset{\sim}{b} &= \underset{\sim}{0} \\
\frac{9k}{11} - 3 + 3r = 0 \quad \text{and} \quad \frac{8k}{11} - 4r &= 0
\end{aligned}$$

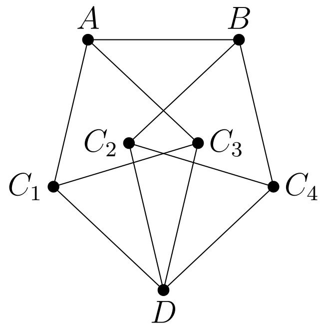
From the first equation substitute $\dfrac{k}{11} = \dfrac{1-r}{3}$ into the second equation.

$$\begin{aligned}
\frac{8(1-r)}{3} - 4r &= 0 \\
8 - 8r &= 12r \\
r &= \frac{2}{5} \\
\frac{|\overrightarrow{BR}|}{|\overrightarrow{BC}|} &= \frac{2}{5} \\
\frac{|\overrightarrow{BR}|}{|\overrightarrow{BR}| + |\overrightarrow{RC}|} &= \frac{2}{5} \\
\frac{|\overrightarrow{BR}| + |\overrightarrow{RC}|}{|\overrightarrow{BR}|} &= \frac{5}{2} \\
\frac{|\overrightarrow{RC}|}{|\overrightarrow{BR}|} &= \frac{3}{2}
\end{aligned}$$

Hence, the ratio of $|\overrightarrow{BR}| : |\overrightarrow{RC}|$ is $2 : 3$.

(b) (i) $N > 1$ since there needs to be at least two colours. If $N = 2$ (i.e. only two colours exist), then the triangular lattice will always have two adjacent vertices with the same colour 1 unit apart. Therefore, $N \neq 2$ so it must be that $N \geq 3$.

(ii) Suppose that $N = 3$ and label the vertices as shown below.



Method 1:

Suppose vertex D is a given colour. Vertices C_1 and C_3 must be the other two colours, by part (i).

It follows that vertex A must be the same colour as D given there are only three distinct colours. Applying the same reasoning to vertex B shows that B and D must also be the same colour.

This means that A and B are the same colour. However, they are 1 unit apart, which contradicts the requirement that any two points 1 unit apart on the plane will *always* have different colours.

Hence, $N \neq 3$ and so it must be that $N \geq 4$.

Method 2:

Vertices A and B must be different colours because they are 1 unit apart. Vertices C_1 and C_3 must be the other two colours because they are 1 unit apart from vertex A. A similar argument can be made for vertices C_2, C_4, and B.

The pairs (C_1, C_3), and (C_2, C_4) have exactly one colour in common.
Hence, all three colours show up in the 4 vertices.

However, this leaves D with no possible colour to satisfy the requirement that any two points 1 unit apart on the plane will *always* have different colours.

Hence, $N \neq 3$ and so it must be that $N \geq 4$.

(c) (i) The particle lands when $y = 0$ for $t > 0$ so

$$\begin{aligned}
Vt\sin\alpha - \frac{gt^2}{2} &= 0 \\
t &= \frac{2V\sin\alpha}{g} \qquad \text{substitute into } x \\
x &= \frac{2V^2\sin\alpha\cos\alpha}{g} \\
V^2 &= \frac{gx}{\sin 2\alpha}
\end{aligned}$$

If $V < \sqrt{gR}$ then

$$\begin{aligned}
\frac{gx}{\sin 2\alpha} &< gR \\
x &< R\sin 2\alpha
\end{aligned}$$

Since $\sin 2\alpha \leq 1$ then this suggests that $x < R$ (i.e. shorter than the circle's radius). This means that the range of the bullet will never reach the circumference of the circle, hence it will never hit P.

(ii) The time of flight of the projectile motion must equal exactly the time it takes for P to displace θ. Since the rate of change of θ is constant at ω then using the working from part (i)

$$\begin{aligned}
\frac{2V\sin\alpha}{g} &= \frac{\theta}{\omega} \\
\sin\alpha &= \frac{g\theta}{2V\omega}
\end{aligned}$$

The range is exactly $x = R$ so substitute the above result into x, giving

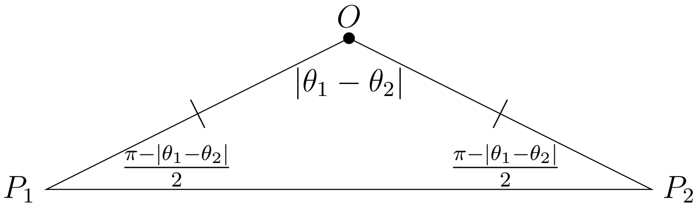
$$\begin{aligned}
R &= \frac{2V^2\sin\alpha\cos\alpha}{g} \\
R &= \frac{2V^2}{g}\left(\frac{g\theta}{2V\omega}\right)\sqrt{1 - \frac{g^2\theta^2}{4V^2\omega^2}} \\
R^2\omega^2 &= V^2\theta^2\left(1 - \frac{g^2\theta^2}{4V^2\omega^2}\right) \\
4R^2\omega^4 &= 4\omega^2V^2\theta^2 - g^2\theta^4 \\
g^2\theta^4 - 4\omega^2V^2\theta^2 + 4R^2\omega^4 &= 0
\end{aligned}$$

(iii) The equation in part (ii) is a quadratic in θ^2 with a discriminant of

$$\begin{aligned}
\Delta &= (4\omega^2V^2)^2 - 4(g^2)(4R^2\omega^4) \\
&= 16\omega^4(V^4 - g^2R^2) \\
&> 0 \quad \text{since } V > \sqrt{gR}
\end{aligned}$$

Hence, there are two distinct roots of the quadratic equation in θ^2. Since $\theta > 0$ this implies only two distinct values for θ.

(iv) The chord P_1P_2 is the base of an isosceles triangle of equal sides R with the angle subtended at the centre as $|\theta_1 - \theta_2|$.



By the sine rule

$$\frac{P_1P_2}{\sin|\theta_1 - \theta_2|} = \frac{OP_1}{\sin\left(\frac{\pi}{2} - \frac{|\theta_1-\theta_2|}{2}\right)} \quad \text{but } OP_1 = R$$

$$P_1P_2 = \frac{2R\sin\frac{|\theta_1-\theta_2|}{2}\cos\frac{|\theta_1-\theta_2|}{2}}{\cos\frac{|\theta_1-\theta_2|}{2}}$$

$$= 2R\sin\frac{|\theta_1 - \theta_2|}{2}$$

However, θ_1^2 and θ_2^2 are the roots of the quadratic in part (ii) so using the sum and product of roots

$$\begin{aligned}
|\theta_1 - \theta_2|^2 &= \theta_1^2 - 2\theta_1\theta_2 + \theta_2^2 \\
&= \theta_1^2 + \theta_2^2 - 2\sqrt{\theta_1^2\theta_2^2} \quad \text{noting that } \theta_1 > 0, \theta_2 > 0 \\
&= \frac{4\omega^2V^2}{g^2} - 2\sqrt{\frac{4R^2\omega^4}{g^2}} \quad \text{noting that } R > 0, \omega > 0, V > 0, g > 0 \\
&= \frac{4\omega^2V^2}{g^2} - \frac{4R\omega^2}{g} \\
&= \frac{4\omega^2(V^2 - gR)}{g^2} \\
|\theta_1 - \theta_2| &= \frac{2\omega\sqrt{V^2 - gR}}{g}
\end{aligned}$$

Hence, $P_1P_2 = 2R\sin\left(\dfrac{\omega\sqrt{V^2 - gR}}{g}\right)$ noting that $V > \sqrt{gR}$.

However, the sine can potentially take negative values so to take absolute values to ensure the length of the P_1P_2 is positive, giving $2R\left|\sin\left(\dfrac{\omega\sqrt{V^2 - gR}}{g}\right)\right|$.

Question 16

(a) (i) Note that

$$\begin{aligned}
\text{LHS} &= \angle QPR \\
&= \arg(\overrightarrow{PQ}) - \arg(\overrightarrow{PR}) \\
&= \arg\left(\overrightarrow{OQ} - \overrightarrow{OP}\right) - \arg\left(\overrightarrow{OR} - \overrightarrow{OP}\right) \\
&= \arg\left(re^{i\theta_q} - re^{i\theta_p}\right) - \arg\left(r - re^{i\theta_p}\right) \\
&= \arg\left(\frac{re^{i\theta_q} - re^{i\theta_p}}{r - re^{i\theta_p}}\right) \\
&= \arg\left(\frac{e^{i\theta_q} - e^{i\theta_p}}{1 - e^{i\theta_p}}\right) \\
&= \text{RHS}
\end{aligned}$$

(ii) Use the fact that for some complex number z

$$\begin{aligned}
z^2 &= \frac{z \times z \times \overline{z}}{\overline{z}} \\
&= |z|^2 \frac{z}{\overline{z}} \qquad \text{but } |z|^2 \text{ is a real number} \\
\arg(z^2) &= \arg\left(\frac{z}{\overline{z}}\right)
\end{aligned}$$

Let $z = \dfrac{e^{i\theta_q} - e^{i\theta_p}}{1 - e^{i\theta_p}}$ so

$$\begin{aligned}
\frac{z}{\overline{z}} &= \frac{e^{i\theta_q} - e^{i\theta_p}}{1 - e^{i\theta_p}} \times \frac{1 - e^{-i\theta_p}}{e^{-i\theta_q} - e^{-i\theta_p}} \times \frac{e^{i(\theta_p+\theta_q)}}{e^{i(\theta_p+\theta_q)}} \\
&= \frac{e^{i\theta_q} - e^{i\theta_p}}{1 - e^{i\theta_p}} \times \frac{e^{i\theta_q}(e^{i\theta_p} - 1)}{e^{i\theta_p} - e^{i\theta_q}} \\
&= e^{i\theta_q} \\
\text{RHS} &= 2\angle QPR \\
&= 2\arg\left(\frac{e^{i\theta_q} - e^{i\theta_p}}{1 - e^{i\theta_p}}\right) \qquad \text{from part (i)} \\
&= \arg\left(\frac{e^{i\theta_q} - e^{i\theta_p}}{1 - e^{i\theta_p}}\right)^2 \\
&= \arg(e^{i\theta_q}) \\
&= \angle QOR \\
&= \text{LHS}
\end{aligned}$$

Hence, $\angle QOR = 2\angle QPR$.

(b) (i) There are no external forces, so the force on object B can only come from object A in equal and opposite direction when they collide (Newton's third law), which means $F_A = -F_B$, or equivalently, $F_A + F_B = 0$.

(ii) From the result in part (i) consider the velocity-displacement relationship and note that the positive direction is to the right.

$$\begin{aligned} F_A + F_B &= 0 \\ m_A \frac{d}{dx}\left(\frac{v_A^2}{2}\right) + m_B \frac{d}{dx}\left(\frac{v_B^2}{2}\right) &= 0 \\ m_A v_A^2 + m_B v_B^2 &= c_1 \quad \text{when } v_A = -a_0, v_B = 0 \Rightarrow c_1 = a_0^2 m_A \\ m_A v_A^2 + m_B v_B^2 &= a_0^2 m_A \quad \text{but } p = v_A\sqrt{m_A}, q = v_B\sqrt{m_B} \\ p^2 + q^2 &= a_0^2 m_A \end{aligned}$$

Now consider the velocity-time relationship.

$$\begin{aligned} F_A + F_B &= 0 \\ m_A \frac{dv_A}{dt} + m_B \frac{dv_B}{dt} &= 0 \\ m_A v_A + m_B v_B &= c_2 \quad \text{when } v_A = -a_0, v_B = 0 \Rightarrow c_2 = -a_0 m_A \\ m_A v_A + m_B v_B &= -a_0 m_A \quad \text{but } p = v_A\sqrt{m_A}, q = v_B\sqrt{m_B} \\ p\sqrt{m_A} + q\sqrt{m_B} &= -a_0 m_A \end{aligned}$$

(iii) Let P_0 be the point (using the p-q axes) before the first collision and P_1 be the point after the first collision. Note that the positive direction is to the right.

Just before the first collision, $v_A = -a_0$ and $v_B = 0$ so $P_0\left(-a_0\sqrt{m_A}, 0\right)$.

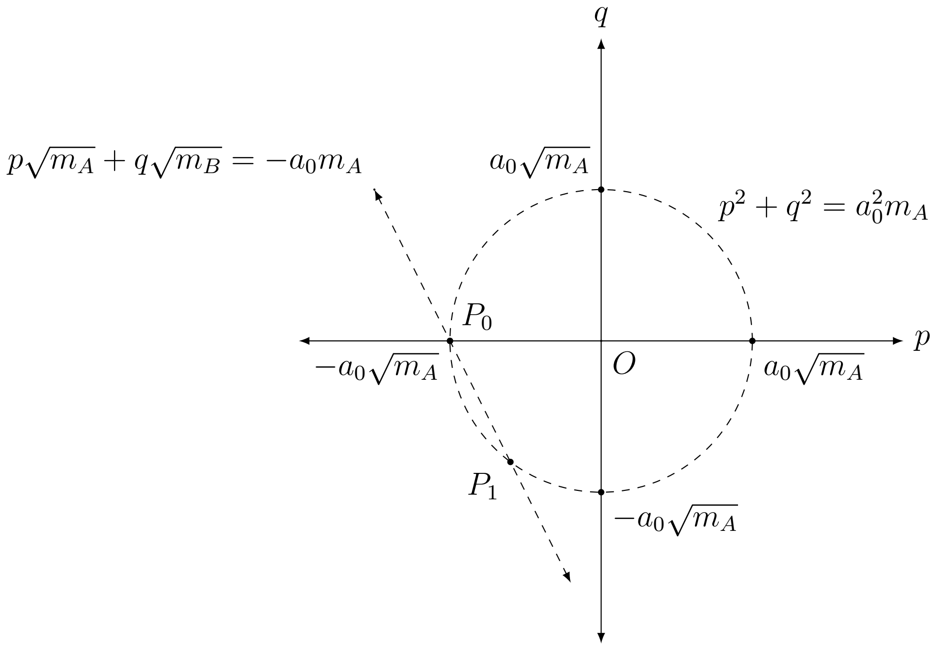
After the first collision, $v_A = -a_1$ and $v_B = -b_1$ so $P_1\left(-a_1\sqrt{m_A}, -b_1\sqrt{m_B}\right)$.

Since the motion before and after collision is governed by the same forces of F_A and F_B, these two points must satisfy the two equations derived in part (ii).

$$p^2 + q^2 = a_0^2 m_A \qquad \text{and} \qquad p\sqrt{m_A} + q\sqrt{m_B} = -a_0 m_A.$$

The first equation is a circle of radius $a_0\sqrt{m_A}$ and the second equation is a line of gradient $-\sqrt{\dfrac{m_A}{m_B}}$ and y-intercept $-\dfrac{a_0 m_A}{\sqrt{m_B}}$.

Plotting the points P_0 and P_1 below gives



(iv) First note that the equation of the circle for the first collision in part (iii) was

$$\begin{aligned} p^2 + q^2 &= a_0^2 m_A \quad \text{but } P_1\left(-a_1\sqrt{m_A}, -b_1\sqrt{m_B}\right) \text{ lies on this circle} \\ a_1^2 m_A + b_1^2 m_B &= a_0^2 m_A \end{aligned}$$

Similarly

$$\begin{aligned} p\sqrt{m_A} + q\sqrt{m_B} &= -a_0 m_A \quad \text{but } P_1\left(-a_1\sqrt{m_A}, -b_1\sqrt{m_B}\right) \text{ lies on this circle} \\ -a_1 m_A - b_1 m_B &= -a_0 m_A \end{aligned}$$

Let P_2 be the point before the second collision between objects A and B and P_3 be the point after that collision.

Just before the collision, $v_A = -a_1$ and $v_B = b_1$ (noting that the positive direction is to the right) so $P_2\left(-a_1\sqrt{m_A}, b_1\sqrt{m_B}\right)$.

A new set of equations of the velocities need to be derived as the initial conditions for the collision have changed. From part (ii) it can be seen that

$$\begin{aligned} m_A v_A^2 + m_B v_B^2 &= c_3 \quad \text{when } v_A = -a_1, v_B = b_1 \Rightarrow c_3 = m_A a_1^2 + m_B b_1^2 \\ m_A v_A^2 + m_B v_B^2 &= m_A a_1^2 + m_B b_1^2 \quad \text{but as shown above } a_1^2 m_A + b_1^2 m_B = a_0^2 m_A \\ p^2 + q^2 &= a_0^2 m_A \end{aligned}$$

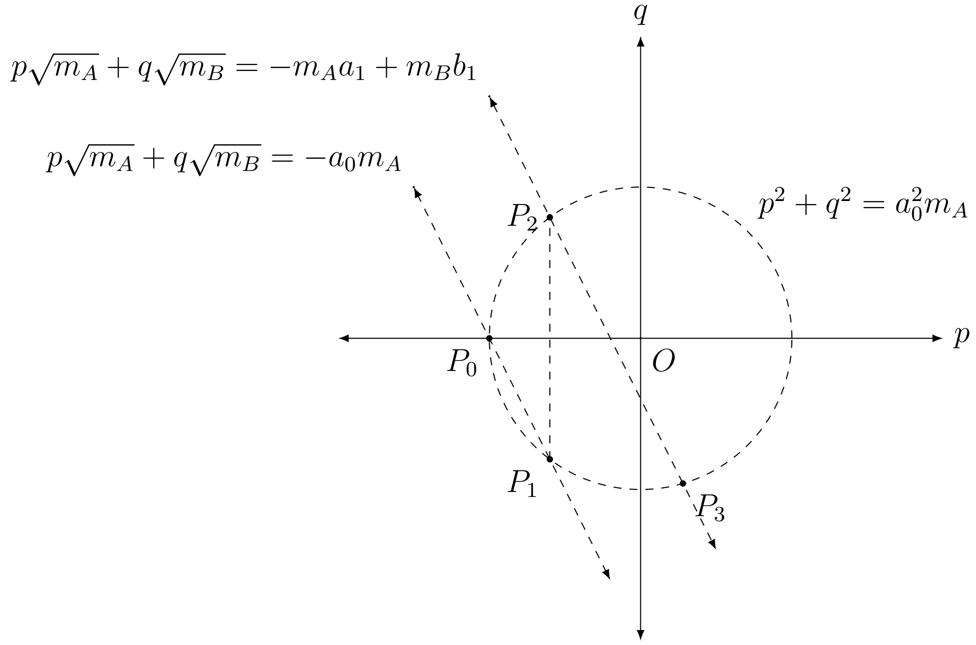
This means that the velocities v_A and v_B associated with the second collision between object A and B lie on the **same** circle $p^2 + q^2 = a_0^2 m_A$ which was derived for the first collision.

For the other velocity equation in part (ii) it can be seen that

$$
\begin{aligned}
m_A v_A + m_B v_B &= c_4 \quad \text{when } v_A = -a_1, v_B = b_1 \Rightarrow c_4 = -m_A a_1 + m_B b_1 \\
m_A v_A + m_B v_B &= -m_A a_1 + m_B b_1 \\
p\sqrt{m_A} + q\sqrt{m_B} &= -m_A a_1 + m_B b_1
\end{aligned}
$$

The constant term is **different** to what it was previously in the first collision, which means the equation of the line is not the same as $p\sqrt{m_A}+q\sqrt{m_B} = -a_0 m_A$. However, note that the gradient is still the same value $-\sqrt{\dfrac{m_A}{m_B}}$.

The velocities of both objects A and B after the collision (call this P_3) must also satisfy the circle and the line simultaneously, so plotting these below



(v) Note that points of (p, q) always lie on the same circle even when initial conditions change because the sign change of the velocity from object B after hitting the wall does not change the equation due to the v_B^2 (or q^2) term.

When going from collision k to collision $k+1$ between object A and B, the initial condition just prior to the $(k+1)$th collision is that:

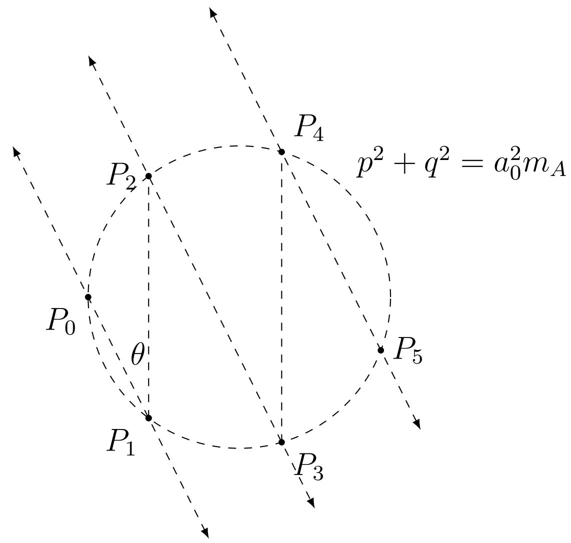
- Object A's velocity is the same as after the kth collision
- Object B's speed is the same as after the kth collision but its velocity has changed in sign.

Graphically (using the p-q axes), this translates to a point that is vertically above the previous point after the kth collision but still lying on the circle (e.g. P_2 is vertically above P_1).

After collision, the velocities change but they must still satisfy the equation of the line and the circle for $(k+1)$th collision. This translates to locating the second intersection point of the line and the circle for the $(k+1)$th collision (e.g. P_3 is found by locating the intersection point other than P_2).

Note that this is process continues where the equation of the line constantly changes after each collision. It will eventually terminate when the equation of the new line is such that it cannot intersect with the circle.

Let $\angle P_0P_1P_2 = \theta$ as shown in the diagram below.



Since the lines have the same gradient then $P_0P_1 \parallel P_2P_3$ so by alternating angles $\angle P_1P_2P_3 = \theta$. Also, $P_1P_2 \parallel P_3P_4$ given they are vertical lines so by alternating angles $\angle P_2P_3P_4 = \theta$. These arguments can be repeated for each successive angle $\angle P_{i-1}P_iP_{i+1}$.

Now consider counting the **total** number of collisions, which includes collisions between objects A and B as well as collisions between object B and the wall.

Notice that there is only one collision associated with P_1 but two collisions (one between object B and the wall plus one between objects A and B) associated with P_2 and P_3. These three collisions are associated with three arc lengths of the circle P_0P_1, P_0P_2 and P_1P_3.

When extending this to P_4 and P_5 there are another two collisions (one between object B and the wall plus one between objects A and B) adding up to five collisions in total. This corresponds to five arc lengths of the circle $P_0P_1, P_0P_2, P_1P_3, P_2P_4$ and P_3P_5.

To find these arc lengths, first consider arc P_0P_1. Since P_1 and P_2 are reflections of each other along the circle's diameter (which P_0 lies on) then

$$\begin{aligned} &P_0P_2 = P_0P_1 \quad \text{(the chords)} \\ &\therefore \triangle P_0P_1P_2 \text{ is isosceles} \\ &\Rightarrow \angle P_0P_2P_1 = \theta \end{aligned}$$

This means that arc P_0P_1 subtends an angle of θ at the circumference. However, from part (a)(ii) this means that the arc P_0P_1 must subtend an angle of 2θ at the centre of the circle. Hence, given the radius is $a_0\sqrt{m_A}$, the arc length of P_0P_1 is $2\theta a_0\sqrt{m_A}$.

Similar arguments can be made about the arc lengths $P_0P_2, P_1P_3, \ldots$ and so on because they all subtend an angle of θ at the circumference and therefore 2θ at the centre, so they all have equal arc lengths.

In order to get N collisions in total there must be N arc lengths associated with it. If this is the terminal point before no more collisions are observed then it must be that N is the largest integer possible such that the sum of the N arc lengths is less than the the total circumference of the circle (i.e. $N+1$ exceeds this circumference). This translates to

$$\begin{aligned} N \times 2\theta a_0\sqrt{m_A} < 2\pi a_0\sqrt{m_A} &< (N+1) \times 2\theta a_0\sqrt{m_A} \\ N\theta < \pi &< (N+1)\theta \\ \frac{\pi}{N+1} < \theta &< \frac{\pi}{N} \end{aligned}$$

Note that θ is the angle between the line and the vertical so the angle the line makes to the horizontal is $\frac{\pi}{2} - \theta$. The gradient (in absolute terms) of the line is therefore $\tan\left(\frac{\pi}{2} - \theta\right)$, or equivalently, $\cot\theta$ so

$$\begin{aligned} \cot\theta &= \sqrt{\frac{m_A}{m_B}} \\ \tan^2\theta &= \frac{m_B}{m_A} \end{aligned}$$

Hence, noting that $\tan^2 x$ is an increasing function

$$\begin{aligned} \tan^2\left(\frac{\pi}{N+1}\right) < \tan^2\theta &< \tan^2\left(\frac{\pi}{N}\right) \\ \tan^2\left(\frac{\pi}{N+1}\right) < \frac{m_B}{m_A} &< \tan^2\left(\frac{\pi}{N}\right) \end{aligned}$$

Remark: This answer is written with accompanying explanations of the concept and thought process, but the actual solution need not be this detailed when written out.