**2024** **BORED OF STUDIES TRIAL EXAMINATION**

6th October

# Mathematics Extension 2

**General instructions**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

**Total marks: 100**

**Section I – 10 marks** (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks** (pages 5–14)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1−10.

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**1** Suppose that $\hat{\underset{\sim}{a}}$, $\hat{\underset{\sim}{b}}$ and $\hat{\underset{\sim}{c}}$ are unit vectors in three dimensional space.

What is the maximum value of $\hat{\underset{\sim}{a}} \cdot (\hat{\underset{\sim}{b}} + \hat{\underset{\sim}{c}})$?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

**2** The velocity of a 2kg object at displacement $x$ metres from the origin is given by $v = x^2$ metres per second.

What is the force acting on the particle in Newtons when $x = -2$?

(A) $-32$ (B) $-16$ (C) 16 (D) 32

**3** On the complex plane, the points $A, B$ and $C$ are represented by the complex numbers $-5 + 4i$, $1 + i$ and $-1 - 3i$ respectively.

Which of the following best describes $\Delta ABC$?

(A) Equilateral triangle (B) Isosceles triangle

(C) Right-angled triangle (D) Scalene triangle

**4** Which of the following statements is NOT always true?

(A) $\forall x \in \mathbb{N} \quad \exists y \in \mathbb{N} \quad x + y \in \mathbb{N}$ (B) $\forall x \in \mathbb{N} \quad \exists y \in \mathbb{N} \quad \dfrac{x}{y} \in \mathbb{N}$

(C) $\forall x \in \mathbb{C} \quad \exists y \in \mathbb{C} \quad x = y^2$ (D) $\forall x \in \mathbb{C} \quad \exists y \in \mathbb{C} \quad xy = 1$

**5** A quadratic polynomial $P(x) = ax^2 + bx + c$ has the roots $z_1$ and $z_2$, where $a$ is a positive real number, whilst $b$ and $c$ are non-zero complex numbers.

Let $\arg(z_1) = \alpha$, $\arg(z_2) = \beta$ and $\arg(z_1 + z_2) = \gamma$.

Which of the following is equal to $\arg(abc)$?

(A) $\alpha + \beta + \gamma$

(B) $\alpha\beta\gamma$

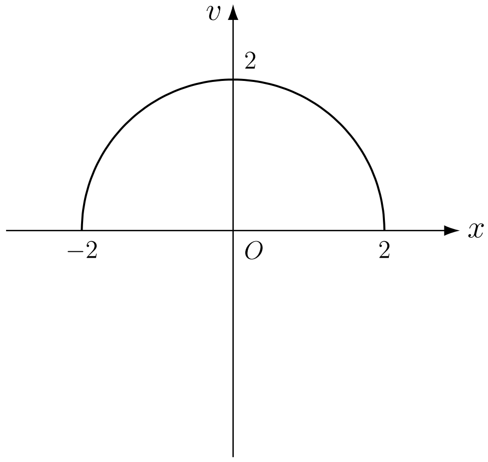
(C) $\alpha + \beta + \gamma + \pi$

(D) $\alpha\beta\gamma + \pi$

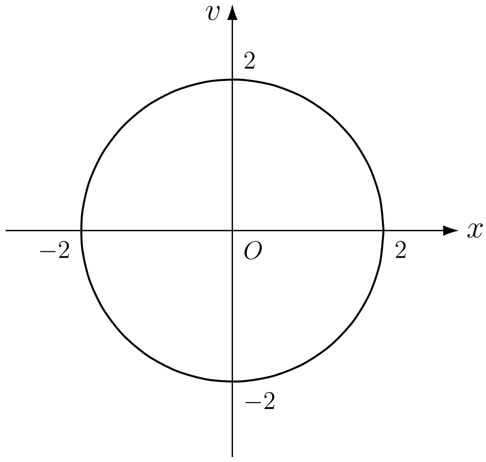
**6** A particle moves in simple harmonic motion between $x = -2$ and $x = 2$ and has an acceleration equation of $\ddot{x} = -x$.

Which of the following best represents the graph of the particle's velocity versus displacement curve?

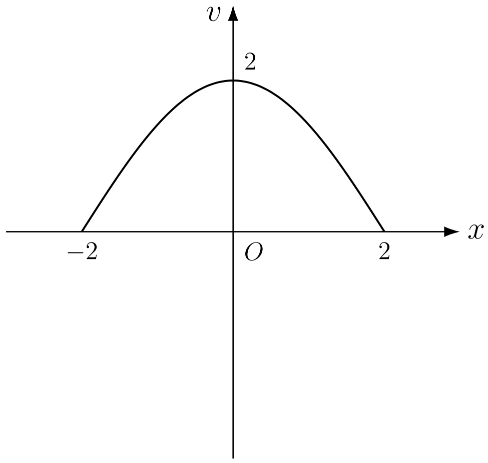
(A)



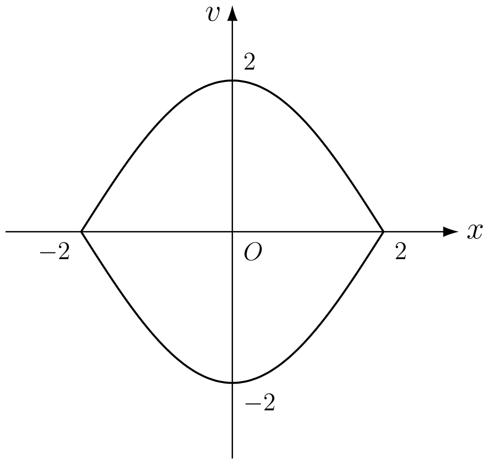

(B)




(C)




(D)



**7** Let $\underset{\sim}{a} = p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$ where $0 < p < q < r$ and let $\underset{\sim}{b}$ be a vector perpendicular to $\underset{\sim}{a}$.

Suppose that $\underset{\sim}{c}$ is a vector such that the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b} - \underset{\sim}{c}$ is obtuse.

Which of the following *could* be $\underset{\sim}{c}$?

(A) $-\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ (B) $\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ (C) $\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ (D) $-\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

**8** Let $\alpha$ and $\beta$ be any two non-real roots of the equation $z^6 = 1$.

Which of the following is always true?

(A) $\text{Re}(\alpha^2 + \beta^2) < 0$ (B) $\text{Re}(\alpha^2 + \beta^2) > 0$

(C) $\text{Re}(\alpha^2 - \beta^2) < 0$ (D) $\text{Re}(\alpha^2 - \beta^2) > 0$

**9** Which of the following is the value of $\displaystyle\int_0^1 \frac{4x - 1}{4x^4 - 4x^3 + x^2 + 1}\, dx$?

(A) $-\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{4} - \ln 2$ (D) $\dfrac{\pi}{4} + \ln 2$

**10** Let $P(z) = z^n \cos\theta_n + z^{n-1}\cos\theta_{n-1} + z^{n-2}\cos\theta_{n-2} + \cdots + z\cos\theta_1 + \cos\theta_0 - 2$, where $\theta \in \mathbb{R}$.

Which of the following is the smallest region on the complex plane that contains all the roots of $P(z)$?

(A) $|z| < \dfrac{1}{3}$ (B) $|z| < \dfrac{1}{2}$ (C) $|z| > \dfrac{1}{3}$ (D) $|z| > \dfrac{1}{2}$

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use the Question 11 Writing Booklet

(a) Evaluate $\displaystyle\int e^x \sec x \tan^2 x \, dx$. **3**

(b) By considering the modulus-argument forms of $1+i$ and $\sqrt{3}-i$, find the exact values of $\sin\dfrac{\pi}{12}$ and $\cos\dfrac{\pi}{12}$. **2**

(c) The line $\ell_1$ passes through the points $A(7,6,2)$ and $B(12,10,5)$. The line $\ell_2$ passes through the points $C(1,-4,-8)$ and $D(11,4,-2)$.

(i) Show that $\ell_1$ and $\ell_2$ are parallel. **1**

(ii) Find the vector equations of the line $\ell_1$ using parameter $\lambda$ and the line $\ell_2$ using parameter $\mu$. **2**

(iii) Let $P$ be a point on $\ell_2$. Find the length of the vector $\overrightarrow{AP}$ in terms of $\mu$. **2**

(iv) Hence, find the shortest distance between $\ell_1$ and $\ell_2$. **2**

(d) The roots of $P(x) = x^4 - 4x^3 + 6x^2 - 4x + 5$ are all non-real and make up the vertices of a square on the complex plane. Find the roots of $P(x)$. **3**

**End of Question 11**

**Question 12** (15 marks) Use the Question 12 Writing Booklet

(a) By considering an appropriate substitution, show that **2**

$$\int_0^{\pi} \frac{\sin^3 x}{\sec x + \tan x + \cos x}\, dx = 0.$$

(b) Let $z$ be a complex number which satisfies **3**

$$|e^z - 1| = 1 \qquad \text{and} \qquad 0 \leq \mathrm{Im}(z) < \frac{\pi}{2}.$$

Sketch the curve which describes the above relation on the complex plane.

(c) Let $f(x)$ be a continuous and differentiable function over the interval $[a, b]$, where $a \neq b$.

(i) Consider the following statement. **1**

If $f(a) < f(b)$ then there exists a $x_0 \in [a, b]$ where $f'(x_0) > 0$

By considering the contrapositive, explain why this statement is true.

(ii) Consider the following statement. **1**

If $f(a) < f(b)$ then $f'(x) > 0$ for all $x \in [a, b]$

Show by counterexample that this statement is not always true.

**Question 12 continues on page 7**

(d) Two particles $P_1$ and $P_2$ are oscillating with equal amplitudes along parallel number lines in simple harmonic motion. Let $x_1$ and $x_2$ be the displacement of the particles $P_1$ and $P_2$ respectively at time $t$. Their acceleration equations are

$$\ddot{x}_1 = -\frac{\pi^2}{4}x_1 \qquad \text{and} \qquad \ddot{x}_2 = -\frac{9\pi^2}{400}x_2.$$

Particle $P_1$ is initially at the midpoint between its centre of motion and its maximum displacement, whilst moving in the positive direction.

Particle $P_2$ is initially at the centre of its motion, whilst moving in the positive direction.

(i) Find the time taken for $P_1$ to reach its maximum displacement for the fifth time. **2**

(ii) When $P_1$ reaches its maximum displacement for the fifth time, particle $P_2$ also reaches its maximum displacement for the $n$th time. **2**

Find the value of $n$.

(e) (i) By considering the complex roots of the equation $z^7 - 1 = 0$, show that the solutions to $x^3 + x^2 - 2x - 1 = 0$ are $2\cos\dfrac{2\pi}{7}$, $2\cos\dfrac{4\pi}{7}$ and $2\cos\dfrac{6\pi}{7}$. **3**

(ii) Hence, show that **1**

$$\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right) = \frac{1}{8}.$$

**End of Question 12**

**Question 13** (15 marks) Use the Question 13 Writing Booklet

(a) Suppose that $z_1$ and $z_2$ both have a modulus of 1. Show that **1**

$$|1 + z_1| + |1 + z_2| + |1 + z_1 z_2| \geq 2.$$

(b) An object of mass $m$ is projected vertically upwards from the ground. At time $t$ the object has a velocity of $v$ and a displacement of $y$ relative to the ground. The object is initially at the origin with a velocity of $u$.

The object is subjected to a gravitational force of magnitude $mg$ and a resistance force of magnitude $mkv^2$ where $g$ is the acceleration due to gravity and $k$ is a positive constant. Let upwards be the positive direction for the equations of motion.

(i) Explain why the object's acceleration equation for both upwards and downwards motion can be written as **1**

$$\ddot{y} = -g - kv|v|.$$

(ii) Let $H$ be the maximum height of the object's motion. Show that **2**

$$H = \frac{1}{2k} \ln\left(\frac{g + ku^2}{g}\right).$$

(iii) Let $T_u$ be the time taken for the object to reach its maximum height. Show that **2**

$$T_u = \frac{1}{\sqrt{kg}} \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right).$$

(iv) After reaching its maximum height, the object then falls back to its initial position on the ground at a velocity of $v_g$. Show that **3**

$$v_g^2 = \frac{gu^2}{g + ku^2}.$$

(v) Let $T_d$ be the time taken for the object to fall from its maximum height back to the ground. Show that **4**

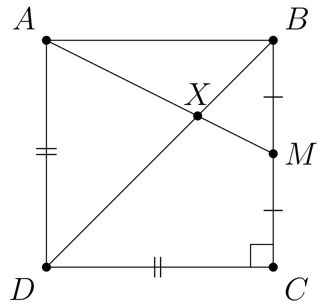
$$T_d = \frac{1}{\sqrt{kg}} \ln\left(\frac{\sqrt{g + ku^2} + u\sqrt{k}}{\sqrt{g}}\right).$$

(vi) Hence, show that $T_d > T_u$. **2**

**End of Question 13**

**Question 14** (15 marks) Use the Question 14 Writing Booklet

(a) Suppose that $ABCD$ is a square. Let $M$ be the midpoint of $BC$ and $X$ be the intersection of $BD$ and $AM$, as shown in the diagram below.

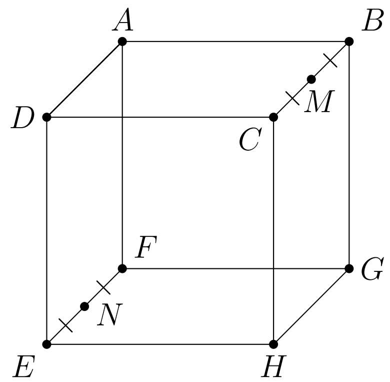



(i) Let $\overrightarrow{AB} = \underset{\sim}{u}$ and $\overrightarrow{AD} = \underset{\sim}{v}$. Show that $\overrightarrow{AX} = \frac{2}{3}\underset{\sim}{u} + \frac{1}{3}\underset{\sim}{v}$. **2**

(ii) Hence, show that $\left|\overrightarrow{XD}\right| = 2\left|\overrightarrow{BX}\right|$. **1**

(iii) Let $P$ be the midpoint of $AB$. By considering the vector equation of the line in the direction of $\overrightarrow{PC}$, show that $X$ lies on $PC$. **2**

(b) A cube with side length of 2 units has vertices $A, B, C, D, E, F, G$ and $H$ as shown in the diagram below. Let $M$ and $N$ be the midpoints of $BC$ and $EF$ respectively.




(i) Let $\overrightarrow{AB} = \underset{\sim}{u}$, $\overrightarrow{AD} = \underset{\sim}{v}$ and $\overrightarrow{AF} = \underset{\sim}{w}$. By expressing the sides in terms of $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$, prove that $AMHN$ is a rhombus. **2**

(ii) Let $P$ and $Q$ be the midpoints of $AB$ and $EH$ respectively. Suppose that $PC$ intersects $AM$ at point $X$ and $NH$ intersects $FQ$ at $Y$. **2**

Using the result(s) in part (a), show that $\left|\overrightarrow{XY}\right| = \frac{2\sqrt{11}}{3}$.

**Question 14 continues on page 10**

Question 14 (continued)

(c) For integer $n \geq 0$ and $a > 0$, let $I_n = \displaystyle\int_0^1 \frac{x^n}{x+a}\, dx$.

(i) Show that for integer $n \geq 1$ **1**

$$I_n = \frac{1}{n} - aI_{n-1}.$$

(ii) Deduce that **2**

$$I_n = \sum_{k=0}^{n-1} \frac{(-1)^k a^k}{n-k} + (-1)^n a^n \ln\left(1 + \frac{1}{a}\right).$$

(iii) Hence, find the value of $\displaystyle\int_0^1 \frac{x^4}{(x+1)(2x+1)}\, dx$. **3**

**End of Question 14**

**Question 15** (15 marks) Use the Question 15 Writing Booklet

(a) Suppose that the point $(0, 0, -1)$ lies on the plane $\mathcal{P}$ in three dimensional space. The position vector $\underset{\sim}{i} - \underset{\sim}{k}$ is perpendicular to this plane.

(i) By considering a general point $P(x, y, z)$ on the plane $\mathcal{P}$ and an appropriate dot product, show that the Cartesian equation of the plane is $x - z = 1$. **1**

(ii) A cylinder with Cartesian equation $x^2 + y^2 = 8$ for all values of $z$ intersects with the plane $\mathcal{P}$. The set of all points of intersection describes the curve $\mathcal{C}$ in three dimensional space. **1**

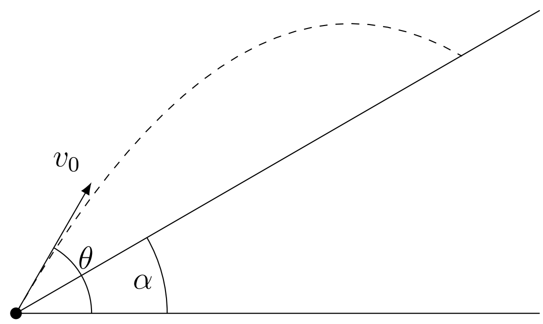
By considering the parametric coordinates of a circle, find the coordinates of a general point $Q(x(\theta), y(\theta), z(\theta))$ on $\mathcal{C}$, in terms of parameter $\theta$.

(iii) It can be shown that at a given point of contact $R(x(\theta_c), y(\theta_c), z(\theta_c))$, the direction vector of the tangent line $\underset{\sim}{r}_d$ to the curve $\mathcal{C}$ is **3**

$$\underset{\sim}{r}_d = x'(\theta_c)\underset{\sim}{i} + y'(\theta_c)\underset{\sim}{j} + z'(\theta_c)\underset{\sim}{k}. \quad \textbf{(Do NOT prove this)}$$

Suppose that the line $\ell$ is tangent to the curve $\mathcal{C}$ with a point of contact of $(2, 2, 1)$. Find the vector equation of the line $\ell$.

(b) An object is projected at the bottom of a frictionless inclined plane with an initial speed of $v_0$ and an acute angle of $\theta$ to the horizontal. The inclined plane has an acute angle of $\alpha$ to the horizontal, where $\alpha < \theta$. **2**




Let $g$ be the acceleration due to gravity. The displacement vector of the object is

$$\underset{\sim}{r} = v_0 t \cos\theta\, \underset{\sim}{i} + \left(v_0 t \sin\theta - \frac{gt^2}{2}\right) \underset{\sim}{j}. \qquad \textbf{(Do NOT prove this)}$$

Let $T$ be the time taken for the object to land on the inclined plane. Show that

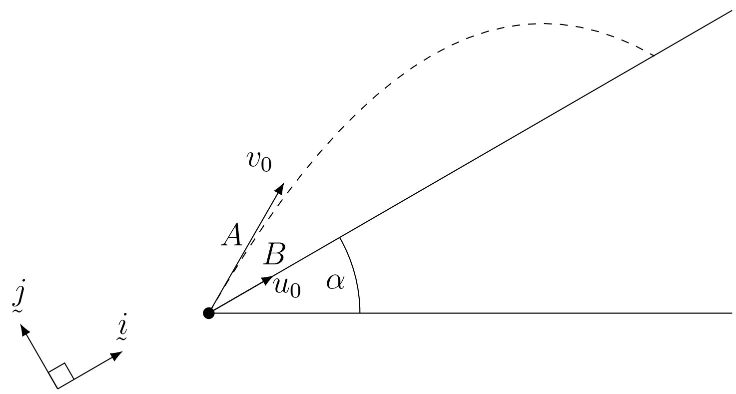
$$T = \frac{2v_0 \cos\theta}{g}(\tan\theta - \tan\alpha).$$

**Question 15 continues on page 12**

(c) Object $A$ is projected at the bottom of a frictionless inclined plane with an initial speed of $v_0$ and an acute angle of $\theta$ to the horizontal. The inclined plane has an acute angle of $\alpha$ to the horizontal, where $\alpha < \theta$.

At the same time, another object $B$ is launched uphill along the plane with an initial speed of $u_0$.

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors parallel and perpendicular to the inclined plane respectively, as shown in the diagram below. The only force action acting on both objects is gravity with acceleration $g$.




Let $\underset{\sim}{r}_A$ and $\underset{\sim}{r}_B$ be the displacement of the objects $A$ and $B$ respectively at time $t$.

(i) By finding the acceleration equation of $A$ in terms of $g$ and $\alpha$, show that **3**

$$\underset{\sim}{r}_A = \left(v_0 t\cos(\theta-\alpha) - \frac{gt^2\sin\alpha}{2}\right)\underset{\sim}{i} + \left(v_0 t\sin(\theta-\alpha) - \frac{gt^2\cos\alpha}{2}\right)\underset{\sim}{j}.$$

(ii) Find $\underset{\sim}{r}_B$ in terms of $g, t, u_0$ and $\alpha$. **2**

(iii) Suppose that the two objects collide when object $B$ reaches its maximum displacement along the plane. **3**

Use the result in part (b) to show that

$$\tan(\theta-\alpha)\tan\alpha = \frac{1}{2}.$$

**End of Question 15**

**Question 16** (15 marks) Use the Question 16 Writing Booklet

(a) For positive integer $n$ and $a_k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with $a_{2n-1} \neq 0$, let

$$u_{2n} = \sum_{k=0}^{2n-1} 10^k a_k \qquad \text{and} \qquad w_{2n} = \sum_{k=0}^{2n-1} (-1)^k a_k.$$

Note that $u_{2n}$ represents a $2n$ digit number with leading digit $a_{2n-1}$ and final digit $a_0$.

(i) Suppose that $w_{2n}$ is divisible by 11 for all positive integer $n$. Prove by mathematical induction that $u_{2n}$ is also divisible by 11. **3**

(ii) A *palindromic* number is a number which is unchanged when it's digits are in reverse order. Using the result in part (i), deduce that all palindromic numbers with an even number of digits is divisible by 11. **1**

(b) Let $f(x)$ be a continuous function where $f(x) > 0$ and $f'(x) > 0$ over the domain $[a, b]$. Let $F(x)$ be a function where $F'(x) = f(x)$. **2**

Show that

$$\int_a^b \frac{f(x)}{f'(x)}\, dx \geq 2[F(b) - F(a)] + \frac{1}{2}[f(a) - f(b)][f(a) + f(b)].$$

**Question 16 continues on page 14**

Question 16 (continued)

(c) Suppose that $\pi = \dfrac{p}{q}$ where $p$ and $q$ are positive integers with no common factors other than 1.

For integer $n > 1$, let

$$f(x) = \frac{x^n(p-qx)^n}{n!}$$

and

$$g(x) = f(x) + \sum_{k=1}^{n}(-1)^k f^{(2k)}(x),$$

where $f^{(2k)}(x)$ is the $2k$th derivative of $f(x)$. For example, $f^{(2)}(x)$ is the second derivative of $f(x)$ and $f^{(4)}(x)$ is the fourth derivative of $f(x)$.

(i) Show that $f(\pi - x) = f(x)$. **1**

(ii) Explain why $g(0) + g(\pi)$ is an integer. **3**

(iii) By evaluating $\dfrac{d}{dx}\left[g'(x)\sin x - g(x)\cos x\right]$ show that $\displaystyle\int_0^{\pi} f(x)\sin x\,dx$ is an integer. **2**

(iv) Show that **1**

$$0 \le \int_0^{\pi} f(x)\sin x\,dx \le \frac{\pi^{n+1}p^n}{n!}.$$

(v) Suppose that $n$ is sufficiently large such that there exists a positive integer $m$ where $\pi p < m < n$. Show that **1**

$$\frac{(\pi p)^n}{n!} < \frac{(\pi p)^m}{(m-1)!}\cdot\frac{1}{n}.$$

(vi) Hence, deduce that $\pi$ is irrational. **1**

**End of paper**

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**2024** BORED OF STUDIES TRIAL EXAMINATION

# Mathematics Extension 2

# Solutions

## Section I

### Answers

| | | | |
|---|---|---|---|
| **1** | D | **6** | B |
| **2** | A | **7** | B |
| **3** | C | **8** | A |
| **4** | D | **9** | B |
| **5** | C | **10** | D |

### Brief explanations

**1** Let $\theta$ be the angle between $\hat{\underset{\sim}{a}}$ and $\hat{\underset{\sim}{b}} + \hat{\underset{\sim}{c}}$, then the dot product gives

$$\begin{aligned}\hat{\underset{\sim}{a}} \cdot (\hat{\underset{\sim}{b}} + \hat{\underset{\sim}{c}}) &= |\hat{\underset{\sim}{a}}|\,|\hat{\underset{\sim}{b}} + \hat{\underset{\sim}{c}}| \cos\theta \quad \text{but } |\hat{\underset{\sim}{a}}| = 1 \\ &= \left|\hat{\underset{\sim}{b}} + \hat{\underset{\sim}{c}}\right| \cos\theta\end{aligned}$$

From the triangle inequality, this is maximised if the unit vectors $\hat{\underset{\sim}{b}}$ and $\hat{\underset{\sim}{c}}$ are parallel, which means the maximum value is $|\hat{\underset{\sim}{b}}| + |\hat{\underset{\sim}{c}}|$, which is 2 units.

Hence, the answer is (D).

**2** When $x = -2$

$$\begin{aligned}F &= ma \\ &= mv\frac{dv}{dx} \\ &= 4x^3 \\ &= 4(-2)^3 \\ &= -32\end{aligned}$$

Hence, the answer is (A).

**3** Let $a = -5+4i, b = 1+i$ and $c = -1-3i$. The complex numbers representing $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CA}$ are $b-a, c-b$ and $a-c$ respectively.

$$\begin{aligned} b-a &= 6-3i \\ c-b &= -2-4i \\ a-c &= -4+7i \end{aligned}$$

No pair of the vectors have equal length, so it remains to check whether there are any right-angles. First note that $b-a = 3(2-i)$ and $c-b = -2(1+2i)$ so

$$\begin{aligned} i(b-a) &= 3(1+2i) \\ &= -\frac{3}{2}(c-b) \end{aligned}$$

This means that there is a rotation of $\dfrac{\pi}{2}$ between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Hence, the answer is (C).

**4** (A) is always true as there will exist any natural number $y$ to ensure that $x+y$ is a natural number.

(B) is always true as there will exist $y = 1$ to ensure that $\dfrac{x}{y}$ is a natural number.

(C) is always true as all complex numbers have a square root.

(D) is not always true, because for the case when $x = 0$ it is not possible to find any complex number $y$ to ensure $xy = 1$.

Hence, the answer is (D).

**5** First note that $z_1 + z_2 = -\dfrac{b}{a}$ and $z_1 z_2 = \dfrac{c}{a}$, so

$$\begin{aligned} \arg(abc) &= \arg a + \arg b + \arg c \quad \text{but since } a \text{ is positive real number then } \arg a = 0 \\ &= \arg b + \arg c \\ &= \arg(-a(z_1+z_2)) + \arg(a z_1 z_2) \\ &= \arg(-a) + \arg(z_1+z_2) + \arg a + \arg z_1 + \arg z_2 \\ &= \alpha + \beta + \gamma + \pi \end{aligned}$$

Hence, the answer is (C).

**6** Since the motion is between $x = -2$ and $x = 2$, then these are the displacement values where $v = 0$. From the acceleration equation

$$\begin{aligned}
\ddot{x} &= -x \\
\frac{d}{dx}\left(\frac{v^2}{2}\right) &= -x \\
\frac{v^2}{2} &= -\frac{x^2}{2} + c \qquad \text{when } x = 2, v = 0 \Rightarrow c = 2 \\
\frac{v^2}{2} &= -\frac{x^2}{2} + 2 \\
x^2 + v^2 &= 4
\end{aligned}$$

On the $v$-$x$ plane this is a circle with the origin as the centre and radius of 2 units. Note that $v$ can be both positive and negative.

Hence, the answer is (B).

**7** If the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b} - \underset{\sim}{c}$ is obtuse then its dot product is negative. Also, since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$. A necessary condition is then

$$\begin{aligned}
\underset{\sim}{a} \cdot (\underset{\sim}{b} - \underset{\sim}{c}) &< 0 \\
\underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{c} &< 0 \\
\underset{\sim}{a} \cdot \underset{\sim}{c} &> 0
\end{aligned}$$

For (A), if $\underset{\sim}{c} = -\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ this requires $-p - q - r > 0$. However, this cannot occur given $0 < p < q < r$.

For (C), if $\underset{\sim}{c} = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ this requires $p - q - r > 0$. However, this cannot occur given $p < q$ and $-r < 0$.

For (D), if $\underset{\sim}{c} = -\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ this requires $-p + q - r > 0$. However, this cannot occur given $q < r$ and $-p < 0$.

For (B), if $\underset{\sim}{c} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ this requires $p - q + r > 0$. This is possible if $r > q - p$ and still follows the condition $0 < p < q < r$.

Hence, the answer is (B).

**8** The possible non-real roots of $z^6 = 1$ are

$$\begin{aligned} z &= e^{-i\frac{2\pi}{3}}, e^{-i\frac{\pi}{3}}, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}} \\ z^2 &= e^{-i\frac{4\pi}{3}}, e^{-i\frac{2\pi}{3}}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}} \\ &= e^{-i\frac{2\pi}{3}}, e^{i\frac{2\pi}{3}} \end{aligned}$$

This means that there are two distinct values for $\alpha^2$ and $\beta^2$. Evaluating their sum and differences gives

$$\begin{aligned} \mathrm{Re}(\alpha^2 + \beta^2) &= \mathrm{Re}\left(e^{-i\frac{2\pi}{3}} + e^{i\frac{2\pi}{3}}\right) \\ &= 2\cos\frac{2\pi}{3} < 0 \end{aligned}$$

$$\mathrm{Re}(\alpha^2 - \beta^2) = \begin{cases} \mathrm{Re}\left(e^{-i\frac{2\pi}{3}} - e^{i\frac{2\pi}{3}}\right) = 0 \\ \mathrm{Re}\left(e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}\right) = 0 \end{cases}$$

Since $\mathrm{Re}(\alpha^2 - \beta^2) = 0$, but $\mathrm{Re}(\alpha^2 + \beta^2) < 0$ always, the answer is (A).

**9** By completing the square

$$\begin{aligned} 4x^4 - 4x^3 + x^2 + 1 &= x^2(4x^2 - 4x + 1) + 1 \\ &= x^2(2x-1)^2 + 1 \\ &= (2x^2 - x)^2 + 1 \\ \int_0^1 \frac{4x-1}{4x^4 - 4x^3 + x^2 + 1}\,dx &= \int_0^1 \frac{4x-1}{(2x^2-x)^2+1}\,dx \\ &= \left[\tan^{-1}(2x^2 - x)\right]_0^1 \\ &= \frac{\pi}{4} \end{aligned}$$

Hence, the answer is (B).

**10** Let $\omega$ be a root of $P(z)$ and let

$$Q(z) = z^n \cos\theta_n + z^{n-1}\cos\theta_{n-1} + z^{n-2}\cos\theta_{n-2} + \cdots + z\cos\theta_1 + \cos\theta_0.$$

From the triangle inequality and using the fact that $|\cos\theta_k| \leq 1$ for $k = 0, 1, 2, \ldots, n$

$$\begin{aligned}
|Q(\omega)| &= |\omega^n \cos\theta_n + \omega^{n-1}\cos\theta_{n-1} + \omega^{n-2}\cos\theta_{n-2} + \cdots + \omega\cos\theta_1 + \cos\theta_0| \\
&\leq |\omega^n \cos\theta_n| + |\omega^{n-1}\cos\theta_{n-1}| + |\omega^{n-2}\cos\theta_{n-2}| + \cdots + |\omega\cos\theta_1| + |\cos\theta_0| \\
&= |\omega|^n|\cos\theta_n| + |\omega|^{n-1}|\cos\theta_{n-1}| + |\omega|^{n-2}|\cos\theta_{n-2}| + \cdots + |\omega||\cos\theta_1| + |\cos\theta_0| \\
&\leq |\omega|^n + |\omega|^{n-1} + |\omega|^{n-2} + \cdots + |\omega| + 1 \\
&= \frac{1-|\omega|^{n+1}}{1-|\omega|} \quad \text{but } |\omega| > 0 \\
&< \frac{1}{1-|\omega|}
\end{aligned}$$

Since $P(\omega) = 0$ then $Q(\omega) = 2$ so a necessary condition is $\dfrac{1}{1-|\omega|} > 2$.

From (A), if $|\omega| < \dfrac{1}{3}$, then $\dfrac{1}{1-|\omega|} < \dfrac{3}{2}$ which contradicts the necessary condition.

From (B), if $|\omega| < \dfrac{1}{2}$, then $\dfrac{1}{1-|\omega|} < 2$ which contradicts the necessary condition.

Also, note that the contradictions also occur at $|\omega| = \dfrac{1}{3}$ and $|\omega| = \dfrac{1}{2}$.

This implies that the roots must sit in a region that satisfies both $|\omega| > \dfrac{1}{3}$ and $|\omega| > \dfrac{1}{2}$.

Therefore, the smallest region from the choices that contains the roots is $|z| > \dfrac{1}{2}$.

Hence, the answer is (D).

## Question 11

(a) Using integration by parts

$u = e^x \tan x \Rightarrow du = e^x(\tan x + \sec^2 x)\,dx$

$dv = \sec x \tan x\,dx \Rightarrow v = \sec x$

$$\begin{aligned}\int e^x \sec x \tan^2 x\,dx &= e^x \sec x \tan x - \int e^x \sec x(\tan x + \sec^2 x)\,dx \\ &= e^x \sec x \tan x - \int e^x \sec x \tan x\,dx - \int e^x \sec^3 x\,dx\end{aligned}$$

Using integration by parts on $\displaystyle\int e^x \sec x \tan x\,dx$

$u = e^x \Rightarrow du = e^x\,dx$

$dv = \sec x \tan x\,dx \Rightarrow v = \sec x$

$$\begin{aligned}\int e^x \sec x \tan^2 x\,dx &= e^x \sec x \tan x - e^x \sec x + \int e^x \sec x\,dx - \int e^x \sec^3 x\,dx \\ &= e^x \sec x \tan x - e^x \sec x - \int e^x \sec x(\sec^2 x - 1)\,dx \\ &= e^x \sec x \tan x - e^x \sec x - \int e^x \sec x \tan^2 x\,dx \\ \int e^x \sec x \tan^2 x\,dx &= \frac{e^x \sec x(\tan x - 1)}{2} + C\end{aligned}$$

(b) In modulus argument form

$$\begin{aligned}1 + i &= \sqrt{2}e^{i\frac{\pi}{4}} \\ \sqrt{3} - i &= 2e^{-i\frac{\pi}{6}}\end{aligned}$$

Since $\dfrac{\pi}{4} - \dfrac{\pi}{6} = \dfrac{\pi}{12}$ then

$$\begin{aligned}(1+i)(\sqrt{3}-i) &= 2\sqrt{2}e^{i\frac{\pi}{12}} \\ \cos\frac{\pi}{12} + i\sin\frac{\pi}{12} &= \frac{(1+i)(\sqrt{3}-i)}{2\sqrt{2}} \\ &= \frac{\sqrt{3} - i + i\sqrt{3} - i^2}{2\sqrt{2}} \\ &= \frac{\sqrt{3} + 1 + i(\sqrt{3} - 1)}{2\sqrt{2}}\end{aligned}$$

Equating real and imaginary parts gives

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad \text{and} \qquad \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(c) (i) Noting that $\overrightarrow{AB}$ is parallel to $\ell_1$ and $\overrightarrow{CD}$ is parallel to $\ell_2$

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= 12\underset{\sim}{i} + 10\underset{\sim}{j} + 5\underset{\sim}{k} - (7\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}) \\
&= 5\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k} \\
\overrightarrow{CD} &= \overrightarrow{OD} - \overrightarrow{OC} \\
&= 11\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k} - (\underset{\sim}{i} - 4\underset{\sim}{j} - 8\underset{\sim}{k}) \\
&= 10\underset{\sim}{i} + 8\underset{\sim}{j} + 6\underset{\sim}{k} \\
&= 2\overrightarrow{AB}
\end{aligned}$$

Hence, $\ell_1$ and $\ell_2$ are parallel.

(ii) The vector equation of the line $\ell_1$ for some parameter $\lambda$ is

$$\begin{aligned}
\underset{\sim}{r} &= \overrightarrow{OA} + \lambda\overrightarrow{AB} \\
&= 7\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(5\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k})
\end{aligned}$$

The vector equation of the line $\ell_2$ for some parameter $\mu$ is

$$\begin{aligned}
\underset{\sim}{r} &= \overrightarrow{OC} + \mu\overrightarrow{CD} \\
&= \underset{\sim}{i} - 4\underset{\sim}{j} - 8\underset{\sim}{k} + \mu(10\underset{\sim}{i} + 8\underset{\sim}{j} + 6\underset{\sim}{k})
\end{aligned}$$

(iii) Using the vector equation of $\ell_2$ from part (ii)

$$\begin{aligned}
\overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA} \\
&= (1 + 10\mu)\underset{\sim}{i} + (-4 + 8\mu)\underset{\sim}{j} + (-8 + 6\mu)\underset{\sim}{k} - (7\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}) \\
&= (-6 + 10\mu)\underset{\sim}{i} + (-10 + 8\mu)\underset{\sim}{j} + (-10 + 6\mu)\underset{\sim}{k} \\
\left|\overrightarrow{AP}\right|^2 &= (-6 + 10\mu)^2 + (-10 + 8\mu)^2 + (-10 + 6\mu)^2 \\
&= 36 - 120\mu + 100\mu^2 + 100 - 160\mu + 64\mu^2 + 100 - 120\mu + 36\mu^2 \\
\left|\overrightarrow{AP}\right| &= \sqrt{200\mu^2 - 400\mu + 236}
\end{aligned}$$

(iv) Minimising distance between $\ell_1$ and $\ell_2$ is equivalent to minimising $|\overrightarrow{AP}|$ or $|\overrightarrow{AP}|^2$.

$$\begin{aligned}
\frac{d}{d\mu}\left|\overrightarrow{AP}\right|^2 &= 400\mu - 400 \\
\frac{d^2}{d\mu^2}\left|\overrightarrow{AP}\right|^2 &= 400 > 0
\end{aligned}$$

Since $\left|\overrightarrow{AP}\right|^2$ is concave up, then it has a minimum value at $\mu = 1$. Since there are no other turning points this is also the global minimum.

Hence, the minimum value of $\left|\overrightarrow{AP}\right|$ is 6 units.

**Remark:** The vector equation could be equivalently written as

$$\underset{\sim}{r} = \underset{\sim}{i} - 4\underset{\sim}{j} - 8\underset{\sim}{k} + \mu(5\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}).$$

This gives $\left|\overrightarrow{AP}\right| = \sqrt{50\mu^2 - 200\mu + 236}$, which is minimised when $\mu = 2$. This gives the same minimum value of $\left|\overrightarrow{AP}\right|$ of 6 units.

(d) Since $P(x)$ has real coefficients and the roots are all non-real then they must occur in conjugate pairs. This means that the square formed by plotting the roots on the complex plane is bisected by the real axis.

Let the roots be $\alpha, \overline{\alpha}, \beta$, and $\overline{\beta}$. Denote $\alpha = p + iq$ where $p$ and $q$ are real numbers, then $\overline{\alpha} = p - iq$. The distance between the roots $\alpha$ and $\overline{\alpha}$ on the complex plane is $2q$. However, since the roots lie on a square, then the side length of the square must also be $2q$.

The other roots can be derived in terms of $p$ and $q$. In this case denote $\beta = (p + 2q) + iq$ and $\overline{\beta} = (p + 2q) - iq$.

Taking the sum of the roots.

$$\begin{aligned} p + iq + p - iq + (p + 2q) + iq + (p + 2q) - iq &= 4 \\ 4p + 4q &= 4 \\ p + q &= 1 \end{aligned}$$

Taking the product of the roots and substitute $q = 1 - p$.

$$\begin{aligned} (p + iq)(p - iq)[(p + 2q) + iq][(p + 2q) - iq] &= 5 \\ (p^2 + q^2)[(p + 2q)^2 + q^2] &= 5 \\ [(1 - q)^2 + q^2][(1 + q)^2 + q^2] &= 5 \\ (1 - q)^2(1 + q)^2 + q^2(1 - q)^2 + q^2(1 + q^2) + q^4 &= 5 \\ (1 - q^2)^2 + q^2[(1 - q)^2 + (1 + q)^2] + q^4 &= 5 \\ 1 - 2q^2 + q^4 + 2q^2(1 + q^2) + q^4 &= 5 \\ 4q^4 &= 4 \\ q &= \pm 1 \end{aligned}$$

When $q = -1 \Rightarrow p = 2$, so $\alpha, \overline{\alpha}, \beta$, and $\overline{\beta}$ are $2 - i, 2 + i, -i$ and $i$ respectively.

When $q = 1 \Rightarrow p = 0$, so $\alpha, \overline{\alpha}, \beta$, and $\overline{\beta}$ are $i, -i, 2 + i$ and $2 - i$ respectively.

Note that both cases give the same set of roots.

Hence, the roots of $P(x)$ are $i, -i, 2 + i$ and $2 - i$.

**Question 12**

(a) Let $u = \dfrac{\pi}{2} - x \Rightarrow du = -dx$. When $x = 0$ then $u = \dfrac{\pi}{2}$ and when $x = \pi$ then $u = -\dfrac{\pi}{2}$.

$$\begin{aligned}\int_0^{\pi} \frac{\sin^3 x}{\sec x + \tan x + \cos x}\,dx &= -\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\sin^3\left(\frac{\pi}{2} - u\right)}{\sec\left(\frac{\pi}{2} - u\right) + \tan\left(\frac{\pi}{2} - u\right) + \cos\left(\frac{\pi}{2} - u\right)}\,du \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3 u}{\mathrm{cosec} u + \cot u + \sin u}\,du\end{aligned}$$

Let $f(u) = \dfrac{\cos^3 u}{\mathrm{cosec} u + \cot u + \sin u}$, then

$$\begin{aligned}f(-u) &= \frac{\cos^3(-u)}{\mathrm{cosec}(-u) + \cot(-u) + \sin(-u)} \\ &= \frac{\cos^3 u}{-\mathrm{cosec} u - \cot u - \sin u} \\ &= -f(u)\end{aligned}$$

Since $f(u)$ is an odd function, then $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3 u}{\mathrm{cosec} u + \cot u + \sin u}\,du = 0$, hence

$$\int_0^{\pi} \frac{\sin^3 x}{\sec x + \tan x + \cos x}\,dx = 0.$$

(b) Let $z = x + iy$ where $x$ and $y$ are real.

$$\begin{aligned}|e^z - 1| &= 1 \\ \left|e^{x+iy} - 1\right| &= 1 \\ |e^x \cos y - 1 + ie^x \sin y| &= 1 \\ (e^x \cos y - 1)^2 + (e^x \sin y)^2 &= 1 \\ e^{2x}\cos^2 y - 2e^x \cos y + 1 + e^{2x}\sin^2 y &= 1 \\ e^{2x} - 2e^x \cos y &= 0 \\ e^x(e^x - 2\cos y) &= 0 \qquad \text{but } e^x > 0 \\ 2\cos y &= e^x\end{aligned}$$

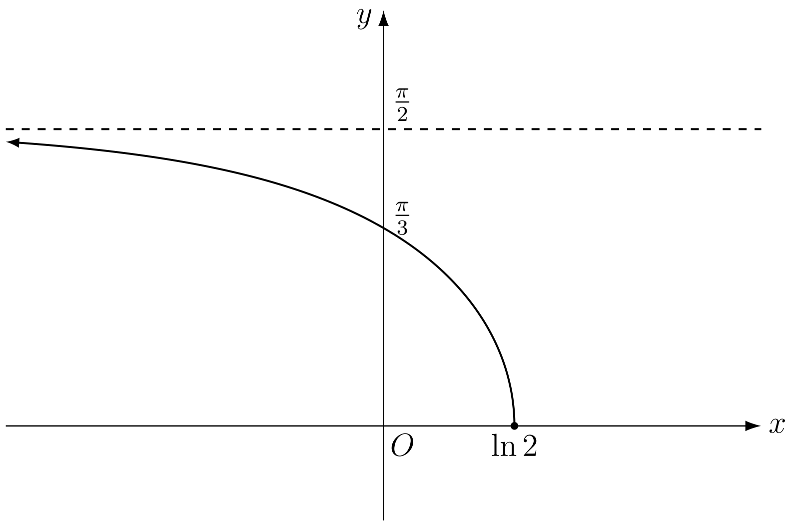
Note that the domain is the solution to $-2 \le e^x \le 2$ which gives $x \le \ln 2$.

Also, the condition $0 \le \mathrm{Im}(z) < \dfrac{\pi}{2}$ defines the range and ensures there is a unique inverse trigonometric function so that the Cartesian equation is

$$y = \cos^{-1}\left(\frac{e^x}{2}\right)$$

When $x = \ln 2, y = 0$ and as $x \to -\infty, y \to \frac{\pi}{2}$.

Putting this information together gives the following sketch.




(c) (i) The contrapositive statement is

If there does not exist an $x_0 \in [a, b]$ where $f'(x_0) > 0$ then $f(a) \geq f(b)$.

Another way to phrase this is

If $f'(x_0) \leq 0$ for all $x_0 \in [a, b]$ then $f(a) \geq f(b)$.

If $f'(x_0) \leq 0$ for all $x_0 \in [a, b]$ then $f(x)$ is non-increasing in that domain. For continuous and differentiable $f(x)$ this means that $f(a)$ cannot be less than $f(b)$ for $a < b$. Since the contrapositive must be true, then so is the original statement.

(ii) Consider $f(x) = x^2$ in the domain $[-2, 3]$. Since $f(-2) = 4$ and $f(3) = 9$, then $f(-2) < f(3)$. However, $f'(x) = 2x$ which is negative in the interval $[-2, 0)$. This means that the statement is not always true.

(d) (i) From particle $P_1$'s acceleration equation, $n = \dfrac{\pi}{2}$.

Since the centre is the origin, the displacement equation for some constants $a$ (defined as maximum displacement) and $\alpha_1$ is

$$x_1 = a\cos\left(\frac{\pi t}{2} + \alpha_1\right)$$

$$\dot{x}_1 = -\frac{a\pi}{2}\sin\left(\frac{\pi t}{2} + \alpha_1\right)$$

When $t = 0, x_1 = \dfrac{a}{2}$ and $\dot{x}_1 > 0$ which means

$$\cos\alpha_1 = \frac{1}{2} \quad \text{and} \quad -\frac{a\pi}{2}\sin\alpha_1 > 0$$

This gives $\alpha_1 = -\dfrac{\pi}{3}$. Hence, the displacement equation of $P_1$ is

$$x_1 = a\cos\left(\frac{\pi t}{2} - \frac{\pi}{3}\right)$$

When $x_1 = a$ then

$$\begin{aligned} \cos\left(\frac{\pi t}{2} - \frac{\pi}{3}\right) &= 1 \\ \frac{\pi t}{2} - \frac{\pi}{3} &= 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, \ldots \end{aligned}$$

The particle is at maximum displacement for the fifth time when

$$\begin{aligned} \frac{\pi t}{2} - \frac{\pi}{3} &= 8\pi \\ t &= \frac{50}{3} \end{aligned}$$

(ii) From particle $P_2$'s acceleration equation, $n = \dfrac{3\pi}{20}$.

Since the centre is the origin, the displacement equation for some constants $a$ (defined as maximum displacement) and $\alpha_2$ is

$$\begin{aligned} x_2 &= a\cos\left(\frac{3\pi t}{20} + \alpha_2\right) \\ \dot{x}_2 &= -\frac{3a\pi}{20}\sin\left(\frac{3\pi t}{20} + \alpha_2\right) \end{aligned}$$

When $t = 0, x_2 = 0$ and $\dot{x}_2 > 0$ which means

$$\cos\alpha_2 = 0 \quad \text{and} \quad -\frac{3a\pi}{20}\sin\alpha_2 > 0$$

This gives $\alpha_2 = -\dfrac{\pi}{2}$. Hence, the displacement equation of $P_2$ is

$$x_2 = a\cos\left(\frac{3\pi t}{20} - \frac{\pi}{2}\right)$$

When $x_2 = a$ then

$$\begin{aligned} \cos\left(\frac{3\pi t}{20} - \frac{\pi}{2}\right) &= 1 \\ \frac{3\pi t}{20} - \frac{\pi}{2} &= 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, \ldots \\ \frac{3\pi t}{20} &= \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}, \frac{21\pi}{2}, \ldots \\ t &= \frac{10}{3}, \frac{50}{3}, \frac{90}{3}, \frac{130}{3}, \frac{170}{3}, \frac{210}{3}, \ldots \end{aligned}$$

Since $P_1$ reaches its maximum displacement for the fifth time when $t = \frac{50}{3}$, this coincides with the second occasion that $P_2$ reaches its maximum displacement. Hence, $n = 2$.

(e) (i) Let $\omega$ be a non-real root of $z^7 = 1$. For integer values of $k = \pm1, \pm2, \pm3$, the root $\omega$ has the general form $\omega = \cos\dfrac{2k\pi}{7} + i\sin\dfrac{2k\pi}{7}$.

Let $P(x) = x^3 + x^2 - 2x - 1$. Using the fact that $\omega + \dfrac{1}{\omega} = 2\cos\dfrac{2k\pi}{7}$

$$\begin{aligned}
P\left(2\cos\frac{2k\pi}{7}\right) &= P\left(\omega + \frac{1}{\omega}\right)\\
&= \left(\omega + \frac{1}{\omega}\right)^3 + \left(\omega + \frac{1}{\omega}\right)^2 - 2\left(\omega + \frac{1}{\omega}\right) - 1\\
&= \omega^3 + 3\omega^2\left(\frac{1}{\omega}\right) + 3\omega\left(\frac{1}{\omega^2}\right) + \frac{1}{\omega^3} + \omega^2 + 2 + \frac{1}{\omega^2} - 2\omega - \frac{2}{\omega} - 1\\
&= \omega^3 + \omega + \frac{1}{\omega} + \frac{1}{\omega^3} + \omega^2 + \frac{1}{\omega^2} + 1\\
&= \frac{\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1}{\omega^3}\\
&= \frac{\omega^7 - 1}{\omega^3(\omega - 1)} \quad \text{but } \omega^7 = 1 \text{ and } \omega \neq 1 \quad \text{for non-real } \omega\\
&= 0
\end{aligned}$$

However, note that since $\cos(-x) = \cos x$ then there are only just three distinct values of $2\cos\dfrac{2k\pi}{7}$.

Hence, the solutions of $x^3 + x^2 - 2x - 1 = 0$ are $2\cos\dfrac{2\pi}{7}, 2\cos\dfrac{4\pi}{7}$ and $2\cos\dfrac{6\pi}{7}$.

(ii) Taking the product of the roots

$$\begin{aligned}
2\cos\frac{2\pi}{7} \times 2\cos\frac{4\pi}{7} \times 2\cos\frac{6\pi}{7} &= 1\\
\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7} &= \frac{1}{8}\\
\sin\left(\frac{\pi}{2} - \frac{2\pi}{7}\right)\sin\left(\frac{\pi}{2} - \frac{4\pi}{7}\right)\sin\left(\frac{\pi}{2} - \frac{6\pi}{7}\right) &= \frac{1}{8}\\
\sin\left(\frac{3\pi}{14}\right)\sin\left(-\frac{\pi}{14}\right)\sin\left(-\frac{5\pi}{14}\right) &= \frac{1}{8}\\
\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right) &= \frac{1}{8}
\end{aligned}$$

**Question 13**

(a) Given that $|z_1| = 1$ and $|z_2| = 1$

$$\begin{aligned}
\text{LHS} &= |1+z_1| + |1+z_2| + |1+z_1z_2| \\
&= |1+z_1| + |1+z_2| + |-(1+z_1z_2)| \\
&\geq |1+z_1| + |1+z_2-(1+z_1z_2)| \quad \text{by triangle inequality} \\
&= |1+z_1| + |z_2||1-z_1| \\
&= |1+z_1| + |1-z_1| \\
&\geq |1+z_1+1-z_1| \quad \text{by triangle inequality} \\
&= 2 \\
&= \text{RHS}
\end{aligned}$$

(b) (i) In the upwards trajectory, the object has resistance in the downwards direction. Given $v \geq 0$, it has the net force equation

$$m\ddot{y} = -mg - mkv^2.$$

In the downwards trajectory, the object has resistance in the upwards direction. Given $v \leq 0$, it has the net force equation

$$m\ddot{y} = -mg + mkv^2.$$

This is equivalent to $\ddot{y} = -g - kv|v|$ which accounts for the different cases where $v \geq 0$ and $v \leq 0$.

(ii) For the upwards trajectory, when $y = 0$, $v = u$ and when $y = H$, $v = 0$. The acceleration equation is

$$\begin{aligned}
\ddot{y} &= -g - kv^2 \\
v\frac{dv}{dy} &= -(g + kv^2) \\
\int_0^H dy &= -\int_u^0 \frac{v}{g + kv^2}\, dv \\
[y]_0^H &= -\frac{1}{2k}\left[\ln(g + kv^2)\right]_u^0 \\
H &= \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)
\end{aligned}$$

(iii) For the upwards trajectory, when $t = 0$, $v = u$ and when $t = T_u$, $v = 0$. The acceleration equation is

$$\begin{aligned}
\ddot{y} &= -g - kv^2 \\
\frac{dv}{dt} &= -(g + kv^2) \\
\int_0^{T_u} dt &= -\int_u^0 \frac{dv}{g + kv^2} \\
[t]_0^{T_u} &= -\frac{1}{k}\int_u^0 \frac{dv}{\frac{g}{k} + v^2} \\
T_u &= -\frac{1}{\sqrt{kg}}\left[\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right)\right]_u^0 \\
&= \frac{1}{\sqrt{kg}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)
\end{aligned}$$

(iv) For the downwards trajectory, when $y = H$, $v = 0$ and when $y = 0$, $v = v_g$. The acceleration equation is

$$\begin{aligned}
\ddot{y} &= -g + kv^2 \\
v\frac{dv}{dy} &= -g + kv^2 \\
\int_H^0 dy &= -\int_0^{v_g} \frac{v}{g - kv^2}\,dv \\
[y]_H^0 &= \frac{1}{2k}\left[\ln(g - kv^2)\right]_0^{v_g} \\
-H &= \frac{1}{2k}\ln\left(\frac{g - k{v_g}^2}{g}\right) \\
H &= \frac{1}{2k}\ln\left(\frac{g}{g - k{v_g}^2}\right)
\end{aligned}$$

Equate this with the result in part (ii)

$$\begin{aligned}
\frac{1}{2k}\ln\left(\frac{g}{g - k{v_g}^2}\right) &= \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right) \\
\frac{g}{g - k{v_g}^2} &= \frac{g + ku^2}{g} \\
g^2 &= g^2 + kgu^2 - k{v_g}^2(g + ku^2) \\
v_g^2 &= \frac{gu^2}{g + ku^2}
\end{aligned}$$

**Remark:** Note that the net acceleration of downward trajectory $\ddot{y} < 0$ which implies $g - kv^2 > 0$.

(v) For the downwards trajectory, when $t = T_u$, $v = 0$ and when $t = T_u + T_d$, $v = v_g$. The acceleration equation is

$$\begin{aligned}
\frac{dv}{dt} &= -g + kv^2 \\
\int_{T_u}^{T_u+T_d} dt &= -\int_0^{v_g} \frac{dv}{g - kv^2} \qquad \text{note that } g - kv^2 > 0 \\
[t]_{T_u}^{T_u+T_d} &= -\int_0^{v_g} \frac{dv}{(\sqrt{g} - v\sqrt{k})(\sqrt{g} + v\sqrt{k})} \\
T_d &= -\frac{1}{2\sqrt{g}} \int_0^{v_g} \frac{\sqrt{g} - v\sqrt{k} + \sqrt{g} + v\sqrt{k}}{(\sqrt{g} - v\sqrt{k})(\sqrt{g} + v\sqrt{k})}\, dv \\
&= -\frac{1}{2\sqrt{g}} \int_0^{v_g} \left( \frac{1}{\sqrt{g} + v\sqrt{k}} + \frac{1}{\sqrt{g} - v\sqrt{k}} \right) dv \\
&= -\frac{1}{2\sqrt{kg}} \left[ \ln\left(\sqrt{g} + v\sqrt{k}\right) - \ln\left(\sqrt{g} - v\sqrt{k}\right) \right]_0^{v_g} \\
&= -\frac{1}{2\sqrt{kg}} \ln \left( \frac{\sqrt{g} + v_g\sqrt{k}}{\sqrt{g} - v_g\sqrt{k}} \times \frac{\sqrt{g} - v_g\sqrt{k}}{\sqrt{g} - v_g\sqrt{k}} \right) \\
&= \frac{1}{2\sqrt{kg}} \ln \left( \frac{\left(\sqrt{g} - v_g\sqrt{k}\right)^2}{g - kv_g^2} \right)
\end{aligned}$$

$$\begin{aligned}
\text{But} \quad g - kv_g^2 &= g - \frac{kgu^2}{g + ku^2} \qquad \text{using the result from part (iv)} \\
&= \frac{g^2}{g + ku^2}
\end{aligned}$$

$$\begin{aligned}
\text{Also} \quad \sqrt{g} - v_g\sqrt{k} &= \sqrt{g} + \frac{u\sqrt{kg}}{\sqrt{g + ku^2}} \qquad \text{since } v_g \leq 0 \text{ in downward trajectory} \\
&= \frac{\sqrt{g}\left(\sqrt{g + ku^2} + u\sqrt{k}\right)}{\sqrt{g + ku^2}} \\
\left(\sqrt{g} - v_g\sqrt{k}\right)^2 &= \frac{g\left(\sqrt{g + ku^2} + u\sqrt{k}\right)^2}{g + ku^2} \\
\Rightarrow \frac{\left(\sqrt{g} - v_g\sqrt{k}\right)^2}{g - kv_g^2} &= \frac{\left(\sqrt{g + ku^2} + u\sqrt{k}\right)^2}{g}
\end{aligned}$$

$$\begin{aligned}
\text{Hence} \quad T_d &= \frac{1}{2\sqrt{kg}} \ln \left( \frac{\left(\sqrt{g + ku^2} + u\sqrt{k}\right)^2}{g} \right) \\
T_d &= \frac{1}{2\sqrt{kg}} \ln \left( \frac{\sqrt{g + ku^2} + u\sqrt{k}}{\sqrt{g}} \right)^2 \\
\therefore T_d &= \frac{1}{\sqrt{kg}} \ln \left( \frac{\sqrt{g + ku^2} + u\sqrt{k}}{\sqrt{g}} \right)
\end{aligned}$$

(vi) Let $\omega = u\sqrt{\dfrac{k}{g}}$ where $k$ and $g$ are positive constants, and $u > 0$

$$
\begin{aligned}
T_d - T_u &= \frac{1}{\sqrt{kg}}\ln\left(\frac{\sqrt{g+ku^2}+u\sqrt{k}}{\sqrt{g}}\right) - \frac{1}{\sqrt{kg}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) \\
&= \frac{1}{\sqrt{kg}}\ln\left(\sqrt{1+\frac{ku^2}{g}}+u\sqrt{\frac{k}{g}}\right) - \frac{1}{\sqrt{kg}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) \\
&= \frac{1}{\sqrt{kg}}\ln\left(\sqrt{1+\omega^2}+\omega\right) - \frac{1}{\sqrt{kg}}\tan^{-1}\omega
\end{aligned}
$$

Let $f(\omega) = \dfrac{1}{\sqrt{kg}}\ln\left(\sqrt{1+\omega^2}+\omega\right) - \dfrac{1}{\sqrt{kg}}\tan^{-1}\omega$

$$
\begin{aligned}
f'(\omega) &= \frac{1}{\sqrt{kg}} \times \frac{\frac{\omega}{\sqrt{1+\omega^2}}+1}{\sqrt{1+\omega^2}+\omega} - \frac{1}{\sqrt{kg}} \times \frac{1}{1+\omega^2} \\
&= \frac{w+\sqrt{1+\omega^2}}{\sqrt{kg(1+\omega^2)}(\sqrt{1+\omega^2}+\omega)} - \frac{1}{\sqrt{kg}(1+\omega^2)} \\
&= \frac{1}{\sqrt{kg}}\left(\frac{1}{\sqrt{1+\omega^2}} - \frac{1}{1+\omega^2}\right) \\
&= \frac{1}{\sqrt{kg}}\left(\frac{\sqrt{1+\omega^2}-1}{1+\omega^2}\right)
\end{aligned}
$$

When $\omega > 0$ then $\omega^2 > 0 \Rightarrow \sqrt{1+\omega^2} > 1$, hence $f'(\omega) > 0$ for $\omega > 0$.

Since $f(0) = 0$ and $f'(\omega) > 0$ for $\omega > 0$, then it must be that $f(\omega) > 0$ for $\omega > 0$. Hence, $T_d - T_u > 0$, or equivalently $T_d > T_u$.

**Question 14**

(a) (i) Note that the defined position vectors are all relative to $A$. The equation of the line through $\overrightarrow{AM}$ for some parameter $\lambda_1$ is

$$\begin{aligned} \underset{\sim}{r} &= \lambda_1\left(\overrightarrow{AB}+\overrightarrow{BM}\right) \\ &= \lambda_1\left(\underset{\sim}{u}+\frac{1}{2}\underset{\sim}{v}\right) \end{aligned}$$

The equation of the line through $\overrightarrow{BD}$ for some parameter $\lambda_2$ is

$$\begin{aligned} \underset{\sim}{r} &= \overrightarrow{AB}+\lambda_2\left(\overrightarrow{BA}+\overrightarrow{AD}\right) \\ &= \underset{\sim}{u}+\lambda_2\left(\underset{\sim}{v}-\underset{\sim}{u}\right) \end{aligned}$$

The position of $X$ is the point of intersection of these two lines, so

$$\begin{aligned} \lambda_1\left(\underset{\sim}{u}+\frac{1}{2}\underset{\sim}{v}\right) &= \underset{\sim}{u}+\lambda_2\left(\underset{\sim}{v}-\underset{\sim}{u}\right) \\ \lambda_1\underset{\sim}{u}+\frac{\lambda_1}{2}\underset{\sim}{v} &= \left(1-\lambda_2\right)\underset{\sim}{u}+\lambda_2\underset{\sim}{v} \end{aligned}$$

Since $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular, the components can be equated.

$$\begin{aligned} \text{Equating } \underset{\sim}{u} \text{ components} \quad & \lambda_1 = 1-\lambda_2 \Rightarrow \lambda_2 = 1-\lambda_1 \\ \text{Equating } \underset{\sim}{v} \text{ components} \quad & \frac{\lambda_1}{2} = \lambda_2 \\ & \quad\;\; = 1-\lambda_1 \\ & \lambda_1 = \frac{2}{3} \quad \text{and} \quad \lambda_2 = \frac{1}{3} \end{aligned}$$

Substitute into equation of line passing through $\overrightarrow{AM}$ to get position vector $\overrightarrow{AX}$

$$\begin{aligned} \underset{\sim}{r} &= \frac{2}{3}\left(\underset{\sim}{u}+\frac{1}{2}\underset{\sim}{v}\right) \\ \overrightarrow{AX} &= \frac{2}{3}\underset{\sim}{u}+\frac{1}{3}\underset{\sim}{v} \end{aligned}$$

(ii) Given that $\overrightarrow{BD} = \underset{\sim}{v}-\underset{\sim}{u}$

$$\begin{aligned} \overrightarrow{BX} &= \overrightarrow{AX}-\overrightarrow{AB} \\ &= \frac{2}{3}\underset{\sim}{u}+\frac{1}{3}\underset{\sim}{v}-\underset{\sim}{u} \\ &= \frac{1}{3}(\underset{\sim}{v}-\underset{\sim}{u}) \\ &= \frac{1}{3}\overrightarrow{BD} \\ \left|\overrightarrow{BX}\right| &= \frac{1}{3}\left|\overrightarrow{BD}\right| \\ &= \frac{1}{3}\left(\left|\overrightarrow{BX}\right|+\left|\overrightarrow{XD}\right|\right) \\ \left|\overrightarrow{XD}\right| &= 2\left|\overrightarrow{BX}\right| \end{aligned}$$

(iii) The equation of the line through $\overrightarrow{PC}$ for some parameter $\lambda_3$ is

$$\begin{aligned}
\underset{\sim}{r} &= \overrightarrow{AP} + \lambda_3\left(\overrightarrow{PB} + \overrightarrow{BC}\right) \\
&= \frac{1}{2}\underset{\sim}{u} + \lambda_3\left(\frac{1}{2}\underset{\sim}{u} + \underset{\sim}{v}\right) \\
&= \frac{\lambda_3 + 1}{2}\,\underset{\sim}{u} + \lambda_3\,\underset{\sim}{v}
\end{aligned}$$

If $X$ lies on this line then there should exist a value of $\lambda_3$ where $\underset{\sim}{r} = \overrightarrow{AX}$.

Note that if $\lambda_3 = \dfrac{1}{3}$ then $\dfrac{\lambda_3 + 1}{2} = \dfrac{2}{3} \Rightarrow \underset{\sim}{r} = \dfrac{2}{3}\underset{\sim}{u} + \dfrac{1}{3}\underset{\sim}{v}$,

Hence, $X$ lies on $PC$.

(b) (i) Find the vectors of each side in terms of $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$

$$\begin{aligned}
\overrightarrow{AM} &= \overrightarrow{AB} + \overrightarrow{BM} \\
&= \underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v} \\
\overrightarrow{NH} &= \overrightarrow{NE} + \overrightarrow{EH} \\
&= \frac{1}{2}\underset{\sim}{v} + \underset{\sim}{u} \\
\overrightarrow{MH} &= \overrightarrow{MC} + \overrightarrow{CH} \\
&= \frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w} \\
\overrightarrow{AN} &= \overrightarrow{AF} + \overrightarrow{FN} \\
&= \underset{\sim}{w} + \frac{1}{2}\underset{\sim}{v}
\end{aligned}$$

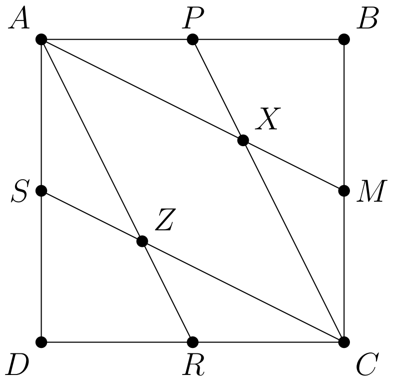
Since, $\overrightarrow{AM} = \overrightarrow{NH}$ and $\overrightarrow{MH} = \overrightarrow{AN}$ then opposite sides are parallel. Also,

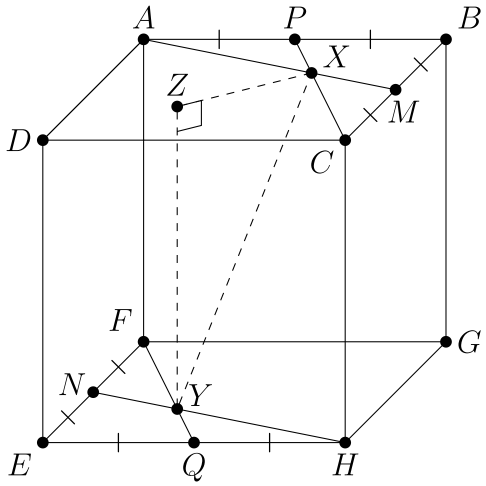
$$\begin{aligned}
\left|\overrightarrow{AM}\right| &= \sqrt{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BM}\right|^2} \\
&= \sqrt{5} \\
\left|\overrightarrow{MH}\right| &= \sqrt{\left|\overrightarrow{CM}\right|^2 + \left|\overrightarrow{CH}\right|^2} \\
&= \sqrt{5}
\end{aligned}$$

Since opposite sides are parallel and adjacent sides are equal then $AMHN$ is a rhombus.

(ii) Let $S$ and $R$ be the midpoints of $AD$ and $DC$ respectively. Let $Z$ be the intersection of $AR$ and $SC$.

By symmetry of the cube, $Z$ is directly positioned above $Y$, so $\left|\overrightarrow{YZ}\right| = 2$.




Using the result from (a)(ii) and symmetry of the square

$$\left|\overrightarrow{BX}\right| = \frac{1}{3}\left|\overrightarrow{BD}\right|$$
$$\left|\overrightarrow{DZ}\right| = \frac{1}{3}\left|\overrightarrow{BD}\right|$$

From (a)(iii), the diagonal $\overrightarrow{BD}$ passes through $X$ and $Z$ so

$$\begin{aligned}
\left|\overrightarrow{XZ}\right| &= \left|\overrightarrow{BD}\right| - \left|\overrightarrow{BX}\right| - \left|\overrightarrow{DZ}\right| \\
&= \frac{1}{3}\left|\overrightarrow{BD}\right| \\
&= \frac{1}{3}\sqrt{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AD}\right|^2} \\
&= \frac{2\sqrt{2}}{3} \\
\left|\overrightarrow{XY}\right|^2 &= \left|\overrightarrow{XZ}\right|^2 + \left|\overrightarrow{YZ}\right|^2 \\
&= \frac{8}{9} + 4 \\
\left|\overrightarrow{XY}\right| &= \frac{2\sqrt{11}}{3}
\end{aligned}$$

(c) (i) Consider

$$\begin{aligned} I_n + aI_{n-1} &= \int_0^1 \frac{x^n}{x+a}\,dx + \int_0^1 \frac{ax^{n-1}}{x+a}\,dx \\ &= \int_0^1 \frac{x^{n-1}(x+a)}{x+a}\,dx \\ &= \int_0^1 x^{n-1}\,dx \\ &= \left[\frac{x^n}{n}\right]_0^1 \\ &= \frac{1}{n} \end{aligned}$$

$$\therefore I_n = \frac{1}{n} - aI_{n-1}$$

(ii) Applying the recurrence relations repeatedly

$$\begin{aligned} I_n &= \frac{1}{n} - aI_{n-1} \\ &= \frac{1}{n} - a\left(\frac{1}{n-1} - aI_{n-2}\right) \\ &= \frac{1}{n} - \frac{a}{n-1} + a^2\left(\frac{1}{n-2} - aI_{n-3}\right) \\ &= \frac{1}{n} - \frac{a}{n-1} + \frac{a^2}{n-2} - a^3\left(\frac{1}{n-3} - aI_{n-4}\right) \\ &= \frac{1}{n} - \frac{a}{n-1} + \frac{a^2}{n-2} - \frac{a^3}{n-3} + a^4\left(\frac{1}{n-4} - aI_{n-5}\right) \\ &\vdots \\ &= \frac{1}{n} - \frac{a}{n-1} + \frac{a^2}{n-2} - \frac{a^3}{n-3} + \cdots + \frac{(-1)^{n-2}a^{n-2}}{2} + (-1)^{n-1}a^{n-1}(1 - aI_0) \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k a^k}{n-k} + (-1)^n a^n \int_0^1 \frac{dx}{x+a} \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k a^k}{n-k} + (-1)^n a^n \left[\ln(x+a)\right]_0^1 \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k a^k}{n-k} + (-1)^n a^n \ln\left(\frac{a+1}{a}\right) \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k a^k}{n-k} + (-1)^n a^n \ln\left(1 + \frac{1}{a}\right) \end{aligned}$$

(iii) For some constants $a$ and $b$, let

$$\frac{1}{(x+1)(2x+1)} = \frac{a}{x+1} + \frac{b}{2x+1}$$
$$a(2x+1) + b(x+1) = 1$$

When $x = -1 \Rightarrow a = -1$ and when $x = -\frac{1}{2} \Rightarrow b = 2$, hence

$$\frac{1}{(x+1)(2x+1)} = -\frac{1}{x+1} + \frac{2}{2x+1}.$$

$$\begin{aligned}\int_0^1 \frac{x^4}{(x+1)(2x+1)}\,dx &= -\int_0^1 \frac{x^4}{x+1}\,dx + \int_0^1 \frac{2x^4}{2x+1}\,dx \\ &= \int_0^1 \frac{x^4}{x+\frac{1}{2}}\,dx - \int_0^1 \frac{x^4}{x+1}\,dx\end{aligned}$$

Using the result in part (ii)

$$\begin{aligned}\int_0^1 \frac{x^4}{x+\frac{1}{2}}\,dx &= \sum_{k=0}^{3} \frac{(-1)^k\left(\frac{1}{2}\right)^k}{4-k} + (-1)^4\left(\frac{1}{2}\right)^4 \ln\left(1+\frac{1}{\frac{1}{2}}\right) \\ &= \sum_{k=0}^{3} \frac{(-1)^k\left(\frac{1}{2}\right)^k}{4-k} + \frac{1}{16}\ln 3 \\ &= \left(\frac{1}{4} - \frac{\frac{1}{2}}{3} + \frac{\frac{1}{2^2}}{2} - \frac{\frac{1}{2^3}}{1}\right) + \frac{1}{16}\ln 3 \\ &= \frac{1}{12} + \frac{1}{16}\ln 3 \\ \int_0^1 \frac{x^4}{x+1}\,dx &= \sum_{k=0}^{3} \frac{(-1)^k 1^k}{4-k} + (-1)^4 1^4 \ln\left(1+\frac{1}{1}\right) \\ &= \sum_{k=0}^{3} \frac{(-1)^k}{4-k} + \ln 2 \\ &= \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{2} - \frac{1}{1}\right) + \ln 2 \\ &= -\frac{7}{12} + \ln 2\end{aligned}$$

Hence

$$\begin{aligned}\int_0^1 \frac{x^4}{(x+1)(2x+1)}\,dx &= \frac{1}{12} + \frac{1}{16}\ln 3 + \frac{7}{12} - \ln 2 \\ &= \frac{2}{3} - \ln 2 + \frac{1}{16}\ln 3\end{aligned}$$

**Question 15**

(a) (i) Let $(x, y, z)$ be a general point on the plane. Since $(0, 0, -1)$ lies on the plane and the position vector $\underset{\sim}{i} - \underset{\sim}{k}$ is perpendicular to the plane then

$$\begin{aligned}(\underset{\sim}{i} - \underset{\sim}{k}) \cdot ((x-0)\underset{\sim}{i} + (y-0)\underset{\sim}{j} + (z+1)\underset{\sim}{k}) &= 0 \\ x - (z+1) &= 0 \\ x - z &= 1\end{aligned}$$

(ii) The parametric coordinates of $x^2 + y^2 = 8$ are

$$x = 2\sqrt{2}\cos\theta \quad \text{and} \quad y = 2\sqrt{2}\sin\theta$$

Where the cylinder intersects with $x - z = 1$, gives the parametric coordinates of $z$ for $\mathcal{C}$, so $z = 2\sqrt{2}\sin\theta - 1$. Hence, the parametric coordinates of $\mathcal{C}$ are

$$\left(2\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta, 2\sqrt{2}\sin\theta - 1\right)$$

(iii) At the point $(2, 2, 1)$ on $\mathcal{C}$, a value of $\theta$ can be derived

$$\begin{aligned}2\sqrt{2}\cos\theta &= 2 \\ \cos\theta &= \frac{1}{\sqrt{2}} \\ \theta &= \frac{\pi}{4}\end{aligned}$$

The direction vector at $\theta = \dfrac{\pi}{4}$ is

$$\begin{aligned}\underset{\sim}{r}_d &= -2\sqrt{2}\sin\theta\underset{\sim}{i} + 2\sqrt{2}\cos\theta\underset{\sim}{j} + 2\sqrt{2}\cos\theta\underset{\sim}{k} \\ &= -2\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} \\ &= 2(-\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})\end{aligned}$$

Hence, the vector equation of the tangent is

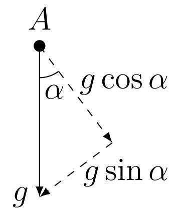
$$\begin{aligned}\underset{\sim}{r} &= 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} + \mu(-2\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}) \\ \text{or equivalently} \quad \underset{\sim}{r} &= 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} + \lambda(-\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})\end{aligned}$$

for some parameters $\mu$ and $\lambda$.

(b) Let $(x_c, y_c)$ be the point that the object lands on the inclined plane, which occurs when $t = T$, where $T \neq 0$. Substitute this into the displacement components.

$$\begin{aligned}
x_c &= v_0 T \cos\theta \\
y_c &= v_0 T \sin\theta - \frac{gT^2}{2} \\
\tan\alpha &= \frac{y_c}{x_c} \\
&= \frac{v_0 T \sin\theta - \frac{gT^2}{2}}{v_0 T \cos\theta} \\
v_0 \cos\theta \tan\alpha &= v_0 \sin\theta - \frac{gT}{2} \\
T &= \frac{2v_0}{g}(\sin\theta - \cos\theta\tan\alpha) \\
&= \frac{2v_0 \cos\theta}{g}(\tan\theta - \tan\alpha)
\end{aligned}$$

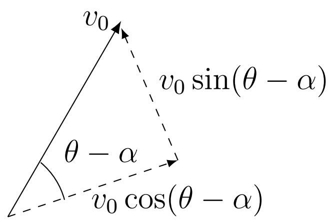
(c) (i) Resolving the gravitational acceleration vector in the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ directions.




The net acceleration on $A$ is

$$\ddot{\underset{\sim}{r}}_A = -g\sin\alpha\,\underset{\sim}{i} - g\cos\alpha\,\underset{\sim}{j}.$$

Resolving the initial velocity vector into the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components.




So when $t = 0$ the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of the velocity vector are $v_0\cos(\theta-\alpha)$ and $v_0\sin(\theta-\alpha)$ respectively. Hence

$$\dot{\underset{\sim}{r}}_A = (-gt\sin\alpha + v_0\cos(\theta-\alpha))\,\underset{\sim}{i} + (-gt\cos\alpha + v_0\sin(\theta-\alpha))\,\underset{\sim}{j}$$

When $t = 0$, the displacement vector is $0\underset{\sim}{i} + 0\underset{\sim}{j}$ so

$$\underset{\sim}{r}_A = \left(v_0 t\cos(\theta-\alpha) - \frac{gt^2\sin\alpha}{2}\right)\underset{\sim}{i} + \left(v_0 t\sin(\theta-\alpha) - \frac{gt^2\cos\alpha}{2}\right)\underset{\sim}{j}.$$

(ii) Since object $B$ moves parallel to the direction of $\underset{\sim}{i}$, then its $\underset{\sim}{j}$ component is always zero. By similarly resolving the gravitational force, its acceleration vector is

$$\ddot{\underset{\sim}{r}}_B = -g\sin\alpha\,\underset{\sim}{i}$$

When $t = 0$, its initial velocity in the $\underset{\sim}{i}$ direction has magnitude $u_0$, so

$$\dot{\underset{\sim}{r}}_B = (-gt\sin\alpha + u_0)\,\underset{\sim}{i}$$

When $t = 0$, the displacement vector is $0\underset{\sim}{i} + 0\underset{\sim}{j}$ so

$$\underset{\sim}{r}_B = \left(u_0 t - \frac{gt^2\sin\alpha}{2}\right)\underset{\sim}{i}$$

(iii) The maximum of $\underset{\sim}{r}_B$ occurs when $\dot{\underset{\sim}{r}}_B = 0\underset{\sim}{i}$.

$$\begin{aligned} -gt\sin\alpha + u_0 &= 0 \\ t &= \frac{u_0}{g\sin\alpha} \end{aligned}$$

Equate this with $T$ from part (b) when object $A$ and object $B$ collide.

$$\begin{aligned} \frac{u_0}{g\sin\alpha} &= \frac{2v_0\cos\theta}{g}(\tan\theta - \tan\alpha) \\ u_0 &= 2v_0\cos\theta\sin\alpha(\tan\theta - \tan\alpha) \\ &= 2v_0\cos\theta\sin\alpha\left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right) \\ &= 2v_0\cos\theta\sin\alpha\left(\frac{\sin\theta\cos\alpha - \sin\alpha\cos\theta}{\cos\alpha\cos\theta}\right) \\ &= 2v_0\tan\alpha\sin(\theta - \alpha) \end{aligned}$$

When objects $A$ and $B$ collide, their displacements must be equal. So equating the $\underset{\sim}{i}$ components of $\underset{\sim}{r}_A$ and $\underset{\sim}{r}_B$ gives

$$\begin{aligned} v_0 t\cos(\theta - \alpha) - \frac{gt^2\sin\alpha}{2} &= u_0 t - \frac{gt^2\sin\alpha}{2} \qquad \text{noting } t \neq 0 \\ u_0 &= v_0\cos(\theta - \alpha) \end{aligned}$$

Equating this with the earlier expression for $u_0$ gives

$$\begin{aligned} v_0\cos(\theta - \alpha) &= 2v_0\tan\alpha\sin(\theta - \alpha) \\ \tan(\theta - \alpha)\tan\alpha &= \frac{1}{2} \end{aligned}$$

**Remark:** Note that when object $A$ hits the inclined plane, its $\underset{\sim}{j}$ component is zero. When applying this to solve for $t$, the same time of flight result as part (b) is derived. The equations lead to similar results regardless of whether the reference coordinates are horizontal/vertical or parallel/perpendicular to the inclined plane.

**Question 16**

(a) (i) When $n = 1$, it is given that $w_2$ is divisible by 11 so $a_0 - a_1 = 11M$ where $M \in \mathbb{Z}$.

$$\begin{aligned}\text{Consider} \quad u_2 &= a_0 + 10a_1 \\ &= 11M + a_1 + 10a_1 \\ &= 11(M + a_1) \\ &= 11N \quad \text{where } N = M + a_1 \in \mathbb{Z}\end{aligned}$$

The statement is true for $n = 1$.

Assume that the statement is true for $n = k$, that is for integer $Q$,

$$u_{2k} = 11Q.$$

Required to prove statement is true for $n = k + 1$, that is for integer $S$,

$$u_{2k+2} = 11S.$$

It is given that $w_{2k} = 11P$ and $w_{2k+2} = 11R$ for integers $P$ and $R$, so

$$\begin{aligned} w_{2k+2} &= \sum_{k=0}^{2k+1} (-1)^j a_j \\ 11R &= \sum_{k=0}^{2k-1} (-1)^j a_j + a_{2k} - a_{2k+1} \\ &= 11P + a_{2k} - a_{2k+1} \\ a_{2k} - a_{2k+1} &= 11(R - P) \end{aligned}$$

Now consider

$$\begin{aligned} u_{2k+2} &= \sum_{k=0}^{2k+1} 10^j a_j \\ &= \sum_{k=0}^{2k-1} 10^j a_j + 10^{2k} a_{2k} + 10^{2k+1} a_{2k+1} \\ &= 11Q + 10^{2k} a_{2k} + 10^{2k+1} a_{2k+1} \quad \text{by assumption} \\ &= 11Q + 10^{2k}(a_{2k} + 10a_{2k+1}) \\ &= 11Q + 10^{2k}(a_{2k+1} + 11(R - P) + 10a_{2k+1}) \\ &= 11Q + 11.10^{2k}(a_{2k+1} + R - P) \\ &= 11[Q + 10^{2k}(a_{2k+1} + R - P)] \\ &= 11S \quad \text{where } S = Q + 10^{2k}(a_{2k+1} + R - P) \in \mathbb{Z} \end{aligned}$$

Since the statement is true for $n = 1$, then by induction it is true for all positive integers $n$.

(ii) If $u_{2n}$ is palindromic number then $a_0 = a_{2n-1}, a_1 = a_{2n-2}, \ldots, a_{n-1} = a_n$. This implies that

$$\begin{aligned} w_{2n} &= (a_0 - a_{2n-1}) + (-a_1 + a^{2n-2}) + (a_2 - a_{2n-3}) + \cdots + ((-1)^{n-1}a_{n-1} + (-1)^n a_n) \\ &= 0 \end{aligned}$$

Since $w_{2n}$ is divisible by 11 then from part (i), $u_{2n}$ is also divisible by 11. Hence, all palindromic numbers with an even number of digits is divisible by 11.

(b) For $x \in [a, b]$, noting that $f(x) > 0$ and $f'(x) > 0$

$$\begin{aligned} \left(\sqrt{f'(x)} - \frac{1}{\sqrt{f'(x)}}\right)^2 &\geq 0 \\ f'(x) + \frac{1}{f'(x)} &\geq 2 \\ f(x)f'(x) + \frac{f(x)}{f'(x)} &\geq 2f(x) \\ \int_a^b f(x)f'(x)\,dx + \int_a^b \frac{f(x)}{f'(x)}\,dx &\geq 2\int_a^b f(x)\,dx \\ \int_a^b \frac{f(x)}{f'(x)}\,dx &\geq 2[F(x)]_a^b - \frac{1}{2}\left[(f(x))^2\right]_a^b \\ &= 2[F(b) - F(a)] + \frac{1}{2}[(f(a))^2 - (f(b))^2] \\ &= 2[F(b) - F(a)] + \frac{1}{2}[f(a) - f(b)][f(a) + f(b)] \end{aligned}$$

(c) (i) Given $\pi = \dfrac{p}{q}$

$$\begin{aligned} \text{LHS} &= f(\pi - x) \\ &= f\left(\frac{p}{q} - x\right) \\ &= \frac{\left(\frac{p}{q} - x\right)^n \left(p - q\left(\frac{p}{q} - x\right)\right)^n}{n!} \\ &= \frac{1}{n!}\left(\frac{p - qx}{q}\right)^n (p - p + qx)^n \\ &= \frac{x^n(p - qx)^n}{n!} \\ &= f(x) \\ &= \text{RHS} \end{aligned}$$

(ii) Firstly, let $q(x) = x^k$ for sufficiently large positive integer $k$. By considering the multiple derivatives

$$
\begin{aligned}
q^{(1)}(x) &= kx^{k-1} \\
q^{(2)}(x) &= k(k-1)x^{k-2} \\
q^{(3)}(x) &= k(k-1)(k-2)x^{k-3} \\
&\vdots \\
q^{(k-1)}(x) &= k(k-1)(k-2)\dots 3.2x^1 \\
q^{(k)}(x) &= k(k-1)(k-2)\dots 3.2.1 \\
q^{(k+1)}(x) &= 0.
\end{aligned}
$$

This means that $q^{(1)}(0), q^{(2)}(0), q^{(3)}(0), \dots, q^{(k-1)}(0), q^{(k+1)}(0)$ all equal zero, which implies that for some positive integer $m$, $q^{(m)}(0) = 0$ if $m \neq k$. The only non-zero derivative at $x = 0$ is when $m = k$, which gives $q^{(k)}(0) = k!$.

Now the above result will be applied on the derivatives of $f(x)$, which are in $g(x)$. First note that with $f(0) = 0$ then

$$g(0) = \sum_{k=1}^{n} (-1)^k f^{(2k)}(0).$$

Now $x^n(p - qx)^n$ can be written generally as a polynomial of degree $2n$ when the binomial is expanded. Let the coefficient of $x^k$ be $a_k$, which are integers (since $p$ and $q$ are integers), then

$$
\begin{aligned}
f(x) &= \frac{x^n(p-qx)^n}{n!} \\
&= \frac{1}{n!}(a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + a_{2n-2}x^{2n-2} + \dots + a_n x^n)
\end{aligned}
$$

First consider the individual $f^{(2k)}(0)$ terms in $g(x)$ where $2k < n$. Since all terms in $f(x)$ have integer powers of $x$ of at least $n$, then $f^{(2k)}(0) = 0$.

Secondly, consider the individual $f^{(2k)}(0)$ terms in $g(x)$ where $n \leq 2k \leq 2n$. Each of the individual terms of $f^{(2k)}(0)$ will vanish at $x = 0$, except for one term which will be constant and has a factor of $n!$. For example,

$$
\begin{aligned}
f^{(2n-2)}(0) &= \frac{1}{n!}(a_{2n-2}(2n-2)!) \\
&= \frac{a_{2n-2}(2n-2)(2n-3)(2n-4)\dots(n+2)(n+1)n!}{n!} \\
&= a_{2n-2}(2n-2)(2n-3)\dots(n+2)(n+1) \quad \text{which is an integer}
\end{aligned}
$$

Applying this similarly to each $f^{(2k)}(0)$ term for $n \leq 2k \leq 2n$ will return an integer. Hence, it must be that $g(0)$ is an integer.

Now consider $g(\pi)$. From part (i),

$$\begin{aligned} f(\pi - x) &= f(x) \\ -f^{(1)}(\pi - x) &= f^{(1)}(x) \\ f^{(2)}(\pi - x) &= f^{(2)}(x) \\ -f^{(3)}(\pi - x) &= f^{(3)}(x) \\ &\vdots \\ (-1)^{2k} f^{(2k)}(\pi - x) &= f^{(2k)}(x) \end{aligned}$$

Substituting $x = \pi$, gives $f(0) = f(\pi)$ and $f^{(2k)}(0) = f^{(2k)}(\pi)$ so

$$\begin{aligned} g(\pi) &= f(\pi) + \sum_{k=1}^{n} (-1)^k f^{(2k)}(\pi) \\ &= f(0) + \sum_{k=1}^{n} (-1)^k f^{(2k)}(0) \\ &= g(0) \quad \text{which was shown to be an integer} \end{aligned}$$

Hence, $g(0) + g(\pi)$ is an integer.

(iii)

$$\begin{aligned} \frac{d}{dx}\left[g'(x)\sin x - g(x)\cos x\right] &= g''(x)\sin x + g'(x)\cos x - g'(x)\cos x + g(x)\sin x \\ &= (g''(x) + g(x))\sin x \end{aligned}$$

But

$$\begin{aligned} g(x) &= f(x) + \sum_{k=1}^{n} (-1)^k f^{(2k)}(x) \\ &= f(x) - f^{(2)}(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots + (-1)^{n-1} f^{(2n-2)}(x) + (-1)^n f^{(2n)}(x) \\ g''(x) &= f^{(2)}(x) - f^{(4)}(x) + f^{(6)}(x) - f^{(8)}(x) + \cdots + (-1)^{n-1} f^{(2n)}(x) + (-1)^n f^{(2n+2)}(x) \end{aligned}$$

Noting that $f(x)$ has $x^{2n}$ as the term with the largest power

$$\begin{aligned} g''(x) + g(x) &= f(x) + (-1)^n f^{(2n+2)}(x) \quad \text{but } f^{(2n+2)}(x) = 0 \\ &= f(x) \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dx}\left[g'(x)\sin x - g(x)\cos x\right] &= f(x)\sin x \\ \int_0^{\pi} f(x)\sin x\,dx &= \int_0^{\pi} \frac{d}{dx}\left[g'(x)\sin x - g(x)\cos x\right]\,dx \\ &= \left[g'(x)\sin x - g(x)\cos x\right]_0^{\pi} \\ &= g(\pi) + g(0) \quad \text{which is an integer from part (ii)} \end{aligned}$$

(iv) For $0 \leq x \leq \pi$ then

$$\begin{aligned} x &\leq \frac{p}{q} \\ p - qx &\geq 0 \\ \Rightarrow f(x) &\geq 0 \\ f(x)\sin x &\geq 0 \quad \text{since } \sin x \geq 0 \quad \text{for } 0 \leq x \leq \pi \\ \int_0^\pi f(x)\sin x\, dx &\geq 0 \end{aligned}$$

Also $\sin x \leq 1$ for $0 \leq x \leq \pi$, so

$$\begin{aligned} f(x)\sin x &\leq f(x) \\ &= \frac{x^n(p-qx)^n}{n!} \\ &\leq \frac{\pi^n p^n}{n!} \\ \int_0^\pi f(x)\sin x\, dx &\leq \frac{\pi^n p^n}{n!}\int_0^\pi dx \\ &= \frac{\pi^{n+1}p^n}{n!} \end{aligned}$$

$$\therefore 0 \leq \int_0^\pi f(x)\sin x\, dx \leq \frac{\pi^{n+1}p^n}{n!}$$

(v) Given that $\pi p < m$ then

$$\begin{aligned} \pi p &< m \\ \pi p &< m+1 \\ \pi p &< m+2 \\ &\vdots \\ \pi p &< m + (n-m-1) \\ \Rightarrow (\pi p)^{n-m} &< m(m+1)(m+2)\ldots(m+(n-m-1)) \\ \frac{(\pi p)^{n-m}}{n!} &< \frac{m(m+1)(m+2)\ldots(n-1)}{n!} \times \frac{(m-1)}{(m-1)!} \\ &= \frac{(n-1)!}{n!(m-1)!} \\ &= \frac{1}{n(m-1)!} \end{aligned}$$

$$\therefore \frac{(\pi p)^n}{n!} < \frac{(\pi p)^m}{(m-1)!}\cdot\frac{1}{n}$$

(vi) From the results in part (iv) and part (v)

$$0 \leq \int_0^\pi f(x)\sin x\, dx < \frac{(\pi p)^m}{(m-1)!} \cdot \frac{1}{n}$$

This inequality holds true for **any** integer $n > 1$. This means that it should hold for large values of $n$.

However, note that as $n \to \infty$ then $\dfrac{(\pi p)^m}{(m-1)!} \cdot \dfrac{1}{n} \to 0$. This means that is possible for $n$ to be sufficiently large such that $\dfrac{(\pi p)^m}{(m-1)!} \cdot \dfrac{1}{n} < 1$.

This means that there exists some values of $n$ such that

$$0 \leq \int_0^\pi f(x)\sin x\, dx < 1.$$

This is a contradiction because $\displaystyle\int_0^\pi f(x)\sin x\, dx$ is an integer from (iii).

The initial assumption that $\pi$ is rational is false. Hence, $\pi$ is irrational.

**Remark:** Since $f(x)\sin x \geq 0$, it is only possible for $\displaystyle\int_0^\pi f(x)\sin x\, dx = 0$ if $f(x)\sin x = 0$ for all $0 \leq x \leq \pi$, which is obviously false so the left inequality can be written as a strict inequality.