2012 Bored of Studies Trial Examinations

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 1 – 4

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 5 – 17

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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1 A particle is travelling in simple harmonic motion such that its velocity, in metres per second, is given by the equation $v^2 = a^2 - b^2x^2$, where $a, b \neq 0$.

What is the period of motion?

(A) a.

(B) b.

(C) $2\pi/a$.

(D) $2\pi/b$.

2 For what positive values of n does the expansion of $\left(x^2 - \frac{1}{x}\right)^n$ have a constant term?

(A) Odd values of n.

(B) Even values of n.

(C) Multiples of 3.

(D) Multiples of 4.

3 Solve the following expression for x, where a and k are positive real numbers.

$$\frac{a-x}{x} \leq k .$$

(A) $0 < x \leq \dfrac{a}{k+1}$.

(B) $0 < x < \dfrac{a}{k+1}$.

(C) $x > \dfrac{a}{k+1}$ or $x < 0$.

(D) $x \geq \dfrac{a}{k+1}$ or $x < 0$.

4 Consider the curve $y = \cos^{-1}\left(\frac{1}{x}\right)$. What is the domain and range?

(A) Domain: $x \le -1$ or $x \ge 1$.

Range: $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

(B) Domain: $-1 \le x \le 1$.

Range: $0 < y < \pi$, $y \ne \frac{\pi}{2}$.

(C) Domain: $x < -1$ or $x > 1$.

Range: $0 < y < \pi$, $y \ne \frac{\pi}{2}$.

(D) Domain: $-1 < x < 1$.

Range: $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

5 The line $y = kx + 4$ makes an acute angle θ with the line $y = -2x + 3$.

For what values of k is $\theta \le 45^\circ$?

(A) $-\frac{1}{3} \le k \le 3$.

(B) $\frac{1}{3} \le k \le 3$.

(C) $k \ge 3$ or $k \le -\frac{1}{3}$.

(D) $k \ge 3$ or $k \le \frac{1}{3}$.

6 At their formal, two twin girls Elle and Daisy are to be seated in a circle with five other friends. However, the two of them insist that they be seated next to each other. They are all assigned seats randomly.

What is the probability that such an arrangement is made?

(A) $1/6$.

(B) $1/3$.

(C) $2/7$.

(D) $1/21$.

7 The two curves $y = x$ and $y = e^{-x}$ intersect near $x = 0.5$.

Use Newton's method to find a better approximation, correct to 3 significant figures.

(A) $x = 0.57$.

(B) $x = 0.570$.

(C) $x = 0.56$

(D) $x = 0.566$.

8 Consider two points $A(a_1, a_2)$ and $B(b_1, b_2)$. A point $C(x, y)$ divides the interval AB internally in the ratio $k:1$. If C lies on the line $a_1x + a_2y = 0$, then

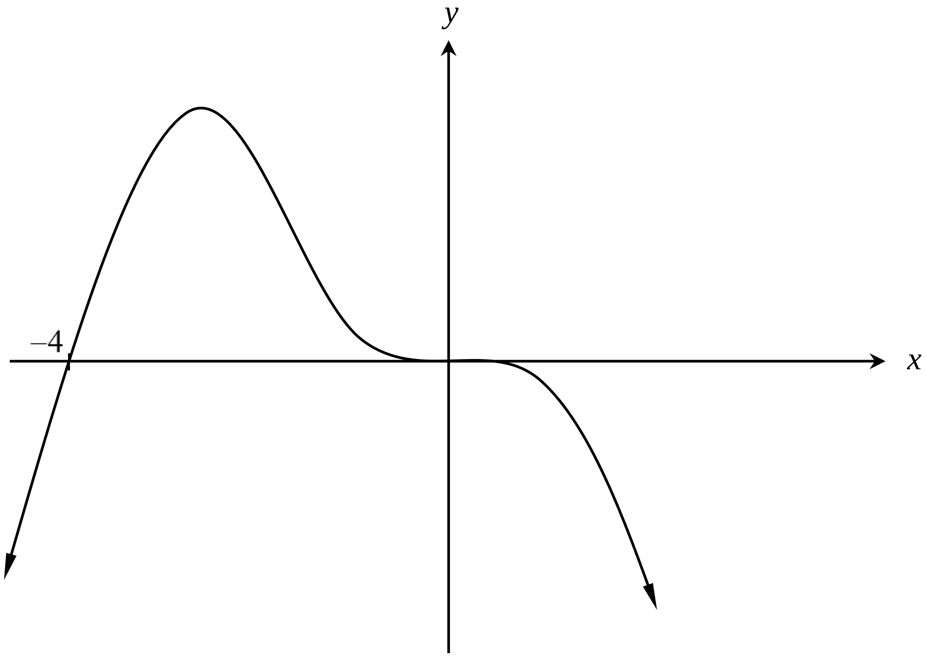
(A) $k = \dfrac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}$.

(B) $k = \dfrac{a_1^2 + a_2^2}{a_1b_1 + a_2b_2}$.

(C) $k = -\dfrac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}$.

(D) $k = -\dfrac{a_1^2 + a_2^2}{a_1b_1 + a_2b_2}$.

9 Which of the following equations best model the curve below?



(A) $y = -x^3(x-4)$.

(B) $y = -x^3(x+4)$.

(C) $y = x^3(x-4)$.

(D) $y = x^3(x+4)$.

10 An n sided 'die' has each of its faces labeled numbers from 1 to n. It is then tossed 3 times, and it is known that the probability of obtaining a particular number exactly 2 times is approximately 2%.

Which of the following is the most likely number of faces that the 'die' has, assuming each number has equal probability?

(A) 6.

(B) 8.

(C) 10.

(D) 12.

Total marks – 60
Attempt Questions 11 – 14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve for $0 \leq \theta \leq 2\pi$. **3**

$$\sin\theta - \cos\theta = \tan\left(\frac{\theta}{2}\right).$$

(b) How many ways can the letters of the word HOLOMORPHIC be arranged if

(i) There are no restrictions? **1**

(ii) The vowels are to be grouped together? **1**

(iii) No two vowels are next to each other? **2**

(c) Evaluate **3**

$$\int_0^1 \frac{\cos^2\left(\tan^{-1} x\right)}{1+x^2}\,dx\ ,$$

using the substitution $u = \tan^{-1} x$.

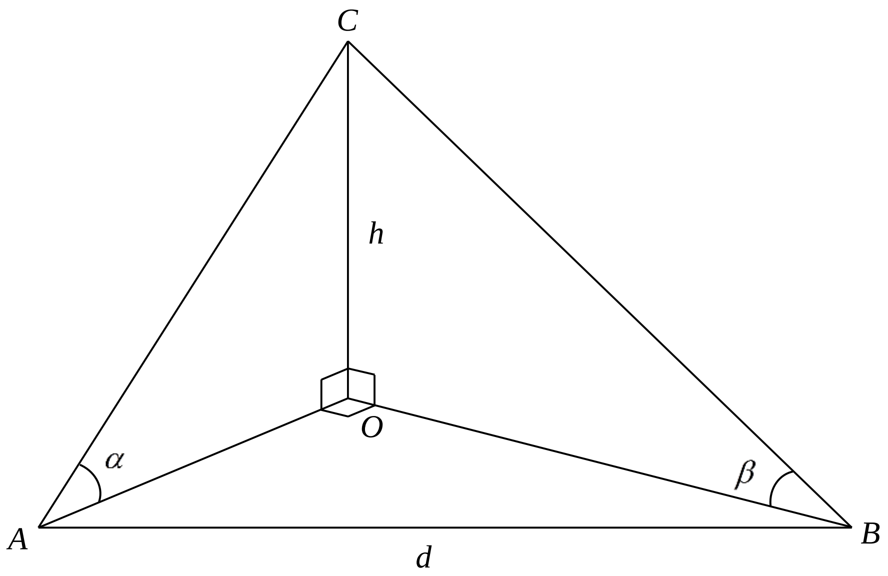
(d) A polynomial $P(x)$ has remainder p when divided by $x - p$, and has remainder q when divided by $x - q$, where p and q are real constants such that $p \neq q$. **2**

Show that the remainder when $P(x)$ is divided by $(x-p)(x-q)$, is exactly 1.

Question 11 continues on page 6

Question 11 (continued)

(e) Two people A and B observe a tower OC, which is h metres high with angles of elevation being α and β respectively. They are standing d metres apart, such that $\angle AOB = 90^{\circ}$, where O is the base of the tower. **3**



Prove that

$$h = \frac{d\tan\alpha\tan\beta}{\sqrt{\tan^2\alpha + \tan^2\beta}}.$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A small aero-plane travels with acceleration $\ddot{x} = \dfrac{k}{x+b}$, for some $b, k > 0$. **3**

when the plane initially starts, it has an acceleration of 3ms^{-2}. It is known that once the jet has travelled 10 metres, its acceleration is now 2ms^{-2} and it travels with a velocity of $10\,\text{ms}^{-1}$.

The pilot Jin lands his plane on an island that has a runway exactly 125 metres long, and now wishes to leave. It is known that $V_L = 17$, where V_L is the minimum speed needed for take-off, in metres per second.

Determine if the runway is long enough for the Jin to leave the island.

(b) Define the expression $S(n) = \sum_{p=2}^{n} \dfrac{1}{p^2 - 1}$.

(i) Use Mathematical Induction to prove that **3**

$$S(n) = \frac{(n-1)(3n+2)}{4n(n+1)}.$$

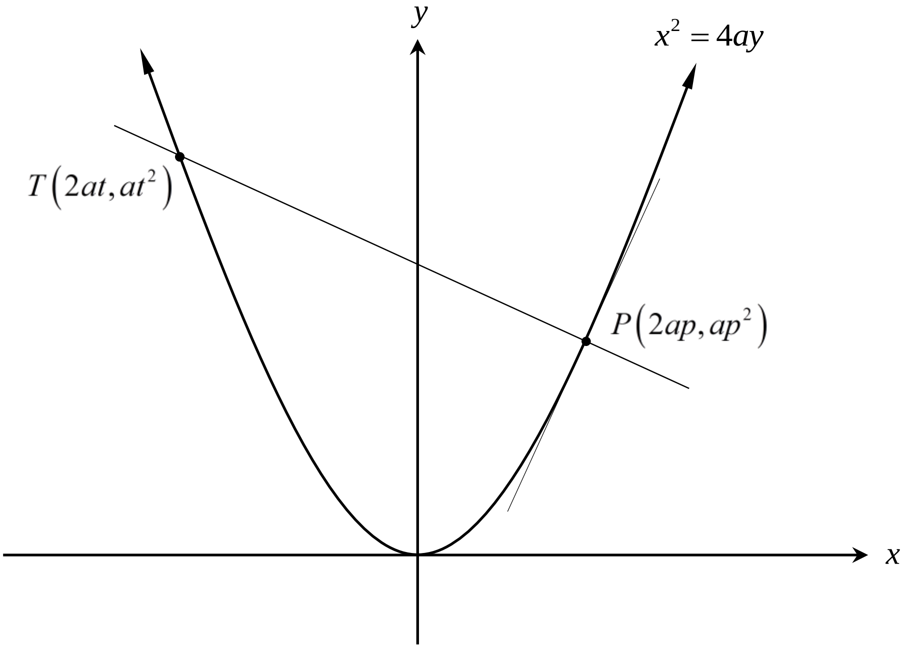
(ii) Hence, or otherwise, find the value of **1**

$$\sum_{p=2}^{\infty} \frac{1}{p^2 - 1}.$$

Question 12 continues on page 8

(c) Two points $P\left(2ap, ap^2\right)$ and $T\left(2at, at^2\right)$ lie on the parabola $x^2 = 4ay$.

A normal is constructed from P such that it meets at T.



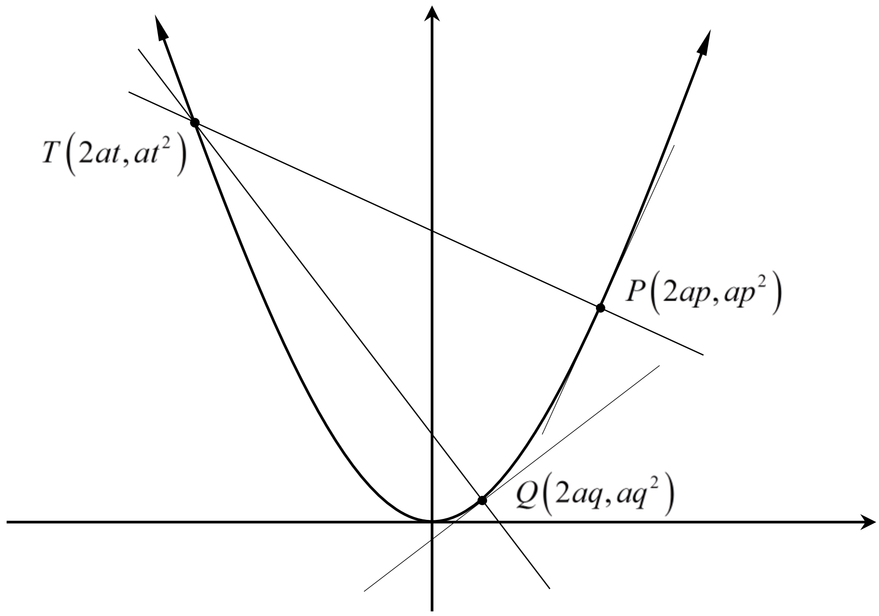
You may assume that the equation of the normal at P is

$$x + py = ap\left(p^2 + 2\right)$$ **(Do NOT prove this).**

(i) Show that $p^2 + pt + 2 = 0$. **2**

Question 12 continues on page 9

(ii) Suppose from another point $Q\left(2aq, aq^2\right)$ on the parabola, a normal is constructed to meet the parabola at T.



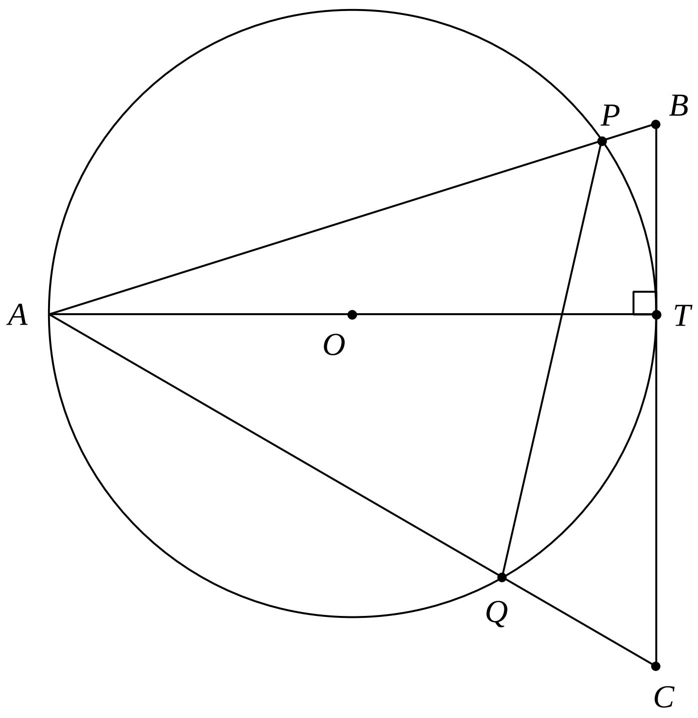
Prove that $p+q+t=0$. **1**

(iii) Deduce that $pq=2$. **1**

Question 12 continues on page 10

Question 12 (continued)

(d) In the diagram below, AT is the diameter of a circle with centre O, and BTC is the tangent to the circle at T. AB and AC intersect the circle at P and Q respectively.



Copy or trace the diagram into your writing booklet.

(i) Prove that $PBCQ$ is a cyclic quadrilateral. **2**

(ii) Hence, or otherwise, prove that $TQ \perp AC$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) An object with initial temperature T_0 is left to cool in a room with some environmental temperature E such that $T_0 > E$. At some time t_n, the temperature is measured to be T_n, where $T_n < T_0$ for all positive integers n. **2**

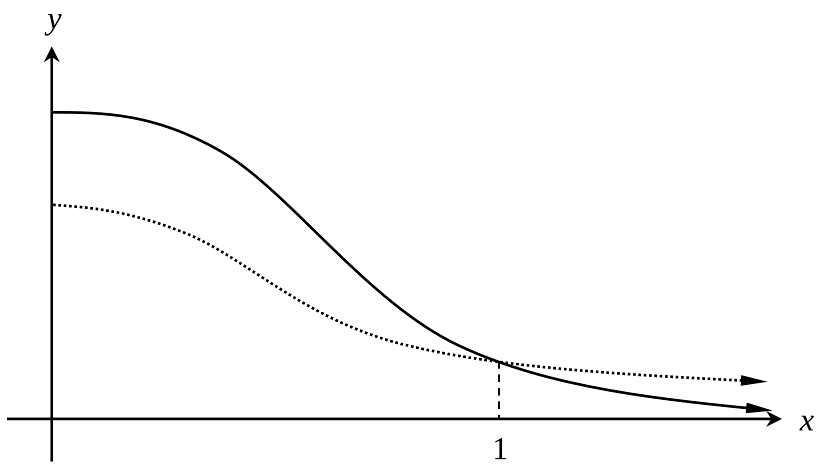
Assuming that the object follows the differential equation

$$\frac{dT}{dt} = k\left(E - T\right),$$

where k is some positive real constant and T is the temperature of the object at some time t, prove that

$$k = \frac{1}{t_n}\ln\left(\frac{T_0 - E}{T_n - E}\right).$$

(b) The diagram below shows the graph of two curves $f\left(x\right) = \dfrac{1}{\sqrt{a^2 + b^2x^2}}$ and $g\left(x\right) = \dfrac{1}{\sqrt{b^2 + a^2x^2}}$, defined for $x \geq 0$, where $a > b > 1$. Let k be some x coordinate such that $k > 1$.



(i) Briefly explain why for $0 \leq x < 1$, $f\left(x\right) < g\left(x\right)$ and for $x \geq 1$, $f\left(x\right) \geq g\left(x\right)$. **1**

Question 13 continues on page 12

(ii) The area bounded by the y axis, the two curves and the line $x=1$, is rotated about the x axis to form a solid. **2**

Prove that the solid has volume

$$V=\frac{\pi}{ab}\tan^{-1}\left(\frac{a^2-b^2}{2ab}\right).$$

You may assume, without proof, that $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$.

(iii) The area bounded by the two curves and the line $x=k$, is rotated about the x axis to form another solid V_k. **2**

Find V_k and hence prove that $\lim\limits_{k\to\infty} V_k=V$.

(c) A particle satisfies the displacement time equation

$$x=A\cos^2\left(\frac{nt}{2}\right)+B\sin^2\left(\frac{nt}{2}\right),$$

where $A<B$.

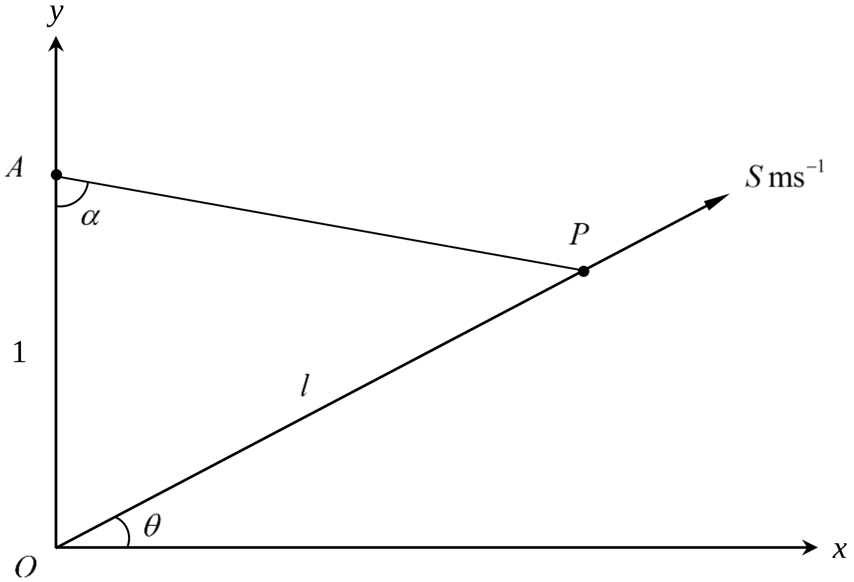
(i) Prove that the motion is simple harmonic. **2**

(ii) Show that **1**

$$A\le x\le B.$$

Question 13 continues on page 13

(d) A point A lies on the y axis, 1 unit away from the origin in the positive direction. Another point P lies on the line $y = x\tan\theta$, where $0^\circ < \theta < 90^\circ$, and travels along the line with some speed $S\text{ ms}^{-1}$. Let l be the distance from P to the origin, and let $\angle OAP$ be α.



(i) Prove that $l = \dfrac{\sin\alpha}{\cos(\alpha - \theta)}$. **1**

(ii) Let $\dot{\alpha} = \dfrac{d\alpha}{dt}$ and $\ddot{\alpha} = \dfrac{d^2\alpha}{dt^2}$. Show that when $\alpha = 2\theta$,

(1) $\dot{\alpha} = S\cos\theta.$ **2**

(2) $\ddot{\alpha} = -\dot{\alpha} \times S \times l.$ **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let k, m and n be positive integers such that $n > m > k$.

(i) Prove that **2**

$$\binom{m}{0}\binom{n-m}{k}+\binom{m}{1}\binom{n-m}{k-1}+\ldots+\binom{m}{k}\binom{n-m}{0}=\binom{n}{k}.$$

(ii) Hence deduce that **1**

$$\sum_{r=0}^{n}\binom{n}{r}^2=\binom{2n}{n}.$$

(iii) Hence, show that **2**

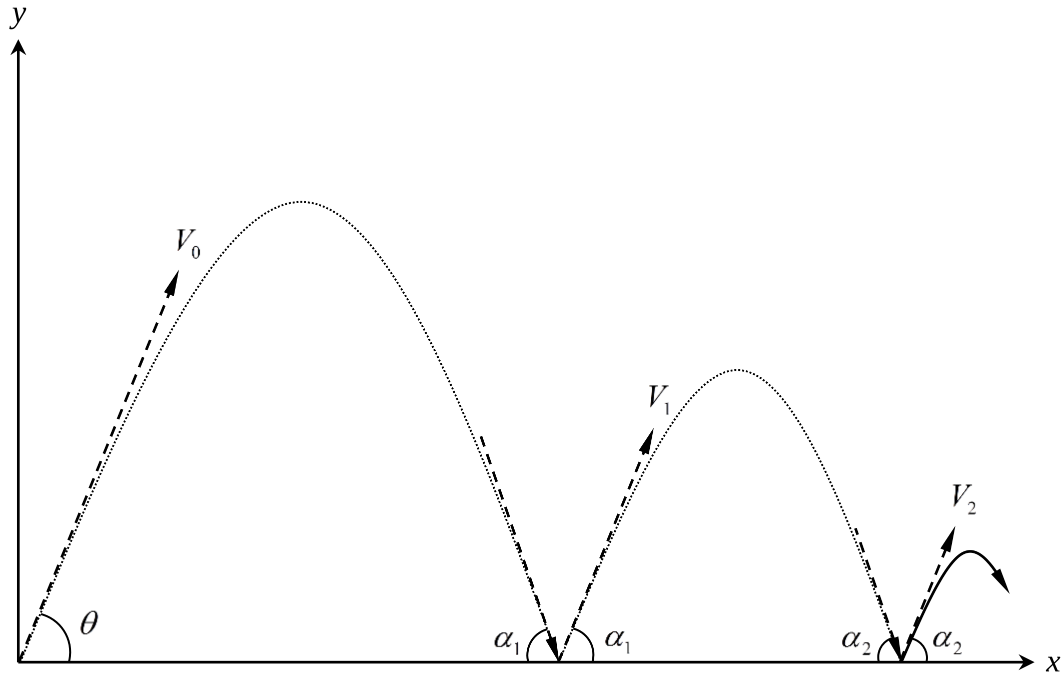
$$\sum_{r=0}^{n}r\binom{n}{r}^2=\frac{n}{2}\binom{2n}{n}.$$

Question 14 continues on page 15

(b) A ball is projected from level ground with some initial velocity V_0 and angle θ. Upon landing, it bounces an infinite number of times.

The n^{th} bounce has some new initial angle α_n being the same acute angle at which it landed, and initial velocity V_n, where $n = 0, 1, 2, \ldots$. However, the velocity V_n is given by the recurrence formula $V_n = kV_{n-1}$, where $0 < k < 1$, and $n \geq 1$.

Let the limiting range be R and let the amount of time it takes to reach that position be T_{V_0} .



You may assume the following equations:

$$x = V_n t \cos\theta,$$

$$y = -\frac{1}{2}gt^2 + V_n t \sin\theta,$$

where g is the acceleration due to gravity, and t is time in seconds.

Question 14 continues on page 16

(i) Briefly explain why $\theta = \alpha_n$, for all $n = 1, 2, 3, \ldots$ **1**

(ii) Show that the amount of time it takes to reach the total limiting range R is **2**

$$T_{V_0} = \frac{2V_0 \sin\theta}{g\left(1-k\right)}.$$

(iii) Prove that the total limiting range after infinite bounces is **2**

$$R = \frac{{V_0}^2 \sin 2\theta}{g\left(1-k^2\right)}.$$

Question 14 continues on page 17

(iv) The ball is now set to be projected with the same initial angle θ, but with some velocity $V_R > V_0$ such that when it lands, it lands with horizontal range R in a single projection of the ball. **3**

Let T_R be the time of flight for the ball to have horizontal range R, with initial velocity V_R.

Show that

$$\frac{T_R}{T_{V_0}} = \sqrt{\frac{1-k}{1+k}} .$$

(v) Deduce that for any $0 < k < 1$, we have **2**

$$T_R < T_{V_0} ,$$

and interpret this result physically.

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2012 Bored of Studies Trial Examinations

Mathematics Extension 1

SOLUTIONS

Disclaimer: These solutions may contain small errors. If any are found, please feel free to contact either Carrotsticks or Trebla on www.boredofstudies.org, regarding them.

Thanks: To Trebla, for his many hours spent verifying solutions and suggesting alternate methods.

Multiple Choice

1. D
2. C
3. D
4. A
5. C
6. B
7. D
8. D
9. B
10. D

Brief Explanations

Question 1 Re-arrange into standard form $v^2 = n^2\left(A^2 - x^2\right)$.

Question 2 Let the exponent of x in the general term be zero to acquire $2n = 3k$, $k \in \mathbb{N}$.

Question 3 Split numerator into two terms and draw a diagram.

Question 4 Observe limit as $x \to \pm\infty$, and that x cannot lie in $-1 < x < 1$.

Question 5 Angle between two lines formula, and let the expression be ≤ 1.

Question 6 Standard permutations problem. Note that they are in a circle, so it's $(n-1)!$.

Question 7 Standard Newton's method of approximation question.

Question 8 Find the coordinates of C, then substitute into the line.

Question 9 Negative quartic, with a triple root at the origin and a single root at $x = -4$.

Question 10 Binomial probability question. Use guess/check to acquire closest solution.

Written Response

Question 11 (a)

We will use the t formula substitutions.

Let $t = \tan\left(\frac{\theta}{2}\right)$

So our expression is:

$$\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = t$$

Re-arrange:

$$\begin{aligned} 2t - 1 + t^2 &= t\left(1+t^2\right) \\ &= t^3 + t \end{aligned}$$

Form a cubic polynomial in t, then solve:

$$\begin{aligned} t^3 - t^2 - t + 1 &= 0 \\ t\left(t^2-1\right) - \left(t^2-1\right) &= 0 \\ \left(t-1\right)\left(t^2-1\right) &= 0 \\ \left(t-1\right)^2\left(t+1\right) &= 0 \\ t &= 1, -1 \\ \tan\left(\frac{\theta}{2}\right) &= 1, -1 \end{aligned}$$

Solve for $0 \le \frac{\theta}{2} \le \pi$:

$$\frac{\theta}{2} = \frac{\pi}{4}, \frac{3\pi}{4}$$

So therefore we have $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Question 11 (b) (i)

There are a total of 11 letters, and we have 2 H's and 3 O's.

So the number of permutations is $\frac{11!}{2!3!}$.

Question 11 (b) (ii)

There are 4 vowels and 7 consonants, and the vowels are to be grouped together.

Example: HCL**(OIOO)**PMHR

Arrange all the consonants and the group (OIOO) to get $\frac{8!}{2!}$. Note we have 2! in the denominator because we have 2 H's.

Arrange the vowels in the group, noting that we have three O', to get $\frac{4!}{3!}$.

So therefore the answer is $\frac{8!}{2!}\times\frac{4!}{3!}$.

Question 11 (b) (iii)

We will count the number of gaps between consonants, and then insert the vowels into these gaps.

We have 7 consonants, so therefore 8 gaps. We insert 4 vowels into these 8 gaps, and thus we have $\binom{8}{4}$.

We must now permute the vowels to acquire $\frac{4!}{3!}$.

Permute the consonants to acquire $\frac{7!}{2!}$.

So therefore the answer is $\binom{8}{4}\times\frac{4!}{3!}\times\frac{7!}{2!}$.

Question 11 (c)

Let $u = \tan^{-1} x$ such that $du = \dfrac{dx}{1+x^2}$.

$u = 1 \Rightarrow x = \dfrac{\pi}{4}$

$u = 0 \Rightarrow x = 0$

$$\begin{aligned}\int_0^1 \frac{\cos^2\left(\tan^{-1} x\right)}{1+x^2}\,dx &= \int_0^{\frac{\pi}{4}} \cos^2 u\,du \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\cos 2u + 1\right)\,du \\ &= \frac{1}{2}\left[\frac{1}{2}\sin 2u + u\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\frac{1}{2}+\frac{\pi}{4}\right) \\ &= \frac{1}{8}\left(2+\pi\right)\end{aligned}$$

Question 11 (d)

We know that $P(p) = p$ and $P(q) = q$.

$P(x) = (x-p)(x-p)Q(x) + Ax + B$

Using the above conditions:

$$\begin{aligned}P(p) &= Ap + B = p \qquad (1) \\ P(q) &= Aq + B = q \qquad (2)\end{aligned}$$

$(1)-(2)$:

$A(p-q) = p-q \Rightarrow A = 1$

Substitute into (1):

$p + B = p \Rightarrow B = 0$

Hence the remainder is exactly *x*. □

Question 11 (e)

From ΔAOC, $OA = \dfrac{h}{\tan\alpha}$ and similarly in ΔBOC, we have $OB = \dfrac{h}{\tan\beta}$.

Using Pythagoras' Theorem, $OA^2 + OB^2 = d^2$.

$$\left(\frac{h}{\tan\alpha}\right)^2 + \left(\frac{h}{\tan\beta}\right)^2 = d^2$$

$$\frac{h^2}{\tan^2\alpha} + \frac{h^2}{\tan^2\beta} = d^2$$

$$h^2\left(\frac{1}{\tan^2\alpha} + \frac{1}{\tan^2\beta}\right) = d^2$$

$$h^2\left(\frac{\tan^2\alpha + \tan^2\beta}{\tan^2\alpha\tan^2\beta}\right) = d^2$$

And therefore:

$$h^2 = \frac{d^2\tan^2\alpha\tan^2\beta}{\tan^2\alpha + \tan^2\beta}$$

Since $\alpha < 90^\circ$, $\beta < 90^\circ$, we have $\tan\alpha > 0,\ \tan\beta > 0$. Also, we must have $h > 0$ and hence:

$$h = \frac{d\tan\alpha\tan\beta}{\sqrt{\tan^2\alpha + \tan^2\beta}} \qquad \square$$

Question 12 (a)

When $x = 0,\ \ddot{x} = 3$, so we have $3 = \dfrac{k}{b}$ and thus $3b = k$.

When $x = 10,\ \ddot{x} = 2$, so we have $2 = \dfrac{k}{10+b}$ and thus $20 + 2b = k$.

Solving simultaneously yields $b = 20$ and thus $k = 60$.

So therefore $\ddot{x} = \dfrac{60}{x+20}$ and thus $\dfrac{d}{dx}\left(\dfrac{1}{2}V^2\right) = \dfrac{60}{x+20}$. Integrating both sides with respect to x yields:

$$\frac{1}{2}V^2 = 60\ln\left(x+20\right) + C$$

We are given that when $x = 10,\ v = 10$, so:

$$\begin{aligned} 50 &= 60\ln\left(30\right) + C \\ C &= 50 - 60\ln\left(30\right) \end{aligned}$$

So our expression is now:

$$\begin{aligned} \frac{1}{2}V^2 &= 60\ln\left(x+20\right) + 50 - 60\ln\left(30\right) \\ V^2 &= 120\ln\left(x+20\right) - 120\ln\left(30\right) + 100 \\ &= 120\ln\left(\frac{x+20}{30}\right) + 100 \end{aligned}$$

Let $V = 17$:

$$\begin{aligned} 120\ln\left(\frac{x+20}{30}\right) + 100 &= 17^2 \\ 120\ln\left(\frac{x+20}{30}\right) &= 189 \\ \ln\left(\frac{x+20}{30}\right) &= 1.575 \\ \frac{x+20}{30} &\approx 4.83 \\ x &\approx 124.92\text{m} \end{aligned}$$

So Jin JUST makes it out.

Alternatively

From

$$V^2 = 120\ln(x+20) - 120\ln(30) + 100$$
$$= 120\ln\left(\frac{x+20}{30}\right) + 100$$

Substitute $x = 125$:

$$V^2 \approx 289.064$$
$$V \approx 17.001$$

And hence, Jin JUST makes it out.

Question 12 (b) (i)

Base Case: $n = 2$.

$$LHS = \sum_{p=2}^{2} \frac{1}{p^2 - 1} = \frac{1}{2^2 - 1} = \frac{1}{3} \qquad RHS = \frac{6+2}{8(3)} = \frac{8}{24} = \frac{1}{3}$$

Therefore true for $n = 2$.

Inductive Hypothesis: $n = k$**.**

$$\sum_{p=2}^{k} \frac{1}{p^2 - 1} = \frac{(k-1)(3k+2)}{4k(k+1)}$$

Inductive Step: $k \Rightarrow k+1$.

Required to prove:

$$\sum_{p=2}^{k+1} \frac{1}{p^2 - 1} = \frac{k(3k+5)}{4(k+1)(k+2)}$$

$$\begin{aligned}
LHS &= \sum_{p=2}^{k+1} \frac{1}{p^2 - 1} \\
&= \sum_{p=2}^{k} \frac{1}{p^2 - 1} + \frac{1}{(k+1)^2 - 1} \\
&= \frac{(k-1)(3k+2)}{4k(k+1)} + \frac{1}{(k+1)^2 - 1} \\
&= \frac{(k-1)(3k+2)}{4k(k+1)} + \frac{1}{k(k+2)} \\
&= \frac{(k-1)(3k+2)(k+2) + 4(k+1)}{4k(k+1)(k+2)} \\
&= \frac{3k^3 + 5k^2 - 4k - 4 + 4k + 4}{4k(k+1)(k+2)} \\
&= \frac{3k^3 + 5k^2}{4k(k+1)(k+2)} \\
&= \frac{k(3k+5)}{4(k+1)(k+2)} \\
&= RHS
\end{aligned}$$

Hence true by induction for all $n \ge 2$.

Question 12 (b) (ii)

$$\begin{aligned}
\lim_{n\to\infty} S(n) &= \lim_{n\to\infty}\sum_{p=2}^{n}\frac{1}{p^2-1}\\
&= \lim_{n\to\infty}\frac{(n-1)(3n+2)}{4n(n+1)}\\
&= \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)\left(3+\frac{2}{n}\right)}{4\left(1+\frac{1}{n}\right)}\\
&= \frac{3}{4}
\end{aligned}$$

Question 12 (c) (i)

There are a couple of ways to do this question.

Method #1:

The equation of the normal is given to be $x+py=ap\left(p^2+2\right)$. But we know that the point T lies on it, so we will substitute in the point $T\left(2at,at^2\right)$.

$$2at+apt^2=ap\left(p^2+2\right)$$
$$2at+apt^2=ap^3+2ap$$

Re-arrange:

$$\begin{aligned}
ap^3-apt^2+2ap-2at&=0\\
ap\left(p^2-t^2\right)+2a(p-t)&=0\\
ap(p-t)(p+t)+2a(p-t)&=0 \qquad \ldots(\text{Noting that } p\neq t)\\
ap(p+t)+2a&=0\\
p(p+t)+2&=0\\
p^2+pt+2&=0 \qquad \square
\end{aligned}$$

Method #2:

The equation of the normal intersects the parabola twice, but we know one of the roots is $x = 2ap$. We could easily do it the other way around, by substituting x into $x^2 = 4ay$, but that would be quite tedious.

Substitute the equation of the normal into the parabola:

$$x + py = ap\left(p^2 + 2\right)$$
$$y = a\left(p^2 + 2\right) - \frac{x}{p}$$

Hence we have:

$$x^2 = 4a\left(a\left(p^2 + 2\right) - \frac{x}{p}\right)$$
$$= 4a^2\left(p^2 + 2\right) - \frac{4a}{p}x$$

Re-arranging:

$$x^2 + \frac{4a}{p}x - 4a^2\left(p^2 + 2\right) = 0$$

Sum of roots is $x_1 + x_2 = -\frac{4a}{p}$. But we already know that one of the roots is $x = 2ap$ and the other is $x = 2at$, so therefore we have:

$$2ap + 2at = -\frac{4a}{p}$$
$$p + t = -\frac{2}{p}$$

And hence the result $p^2 + pt + 2 = 0$. □

Method #3:

The chord PT must be perpendicular to the tangent at P.

$$\begin{aligned}\nabla PT &= \frac{ap^2 - at^2}{2ap - 2at} \\ &= \frac{a(p-t)(p+t)}{2a(p-t)} \\ &= \frac{p+t}{2}\end{aligned}$$

The gradient of the tangent at P is

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dp}{dx/dp} \\ &= \frac{2ap}{2a} \\ &= p\end{aligned}$$

Hence

$$\begin{aligned}p \times \frac{p+t}{2} &= -1 \\ p^2 + pt &= -2 \\ p^2 + pt + 2 &= 0 \qquad \square\end{aligned}$$

Question 12 (c) (ii)

Similarly to (i), we can deduce the same expression, except with q.

So we have:

$$p^2 + pt + 2 = 0$$

$$q^2 + qt + 2 = 0$$

Subtract the two equations:

$$\begin{aligned}p^2 - q^2 + t(p-q) &= 0 \\ (p-q)(p+q) + t(p-q) &= 0 \quad \text{...(note that } p \neq q) \\ p + q + t &= 0 \quad \square\end{aligned}$$

Question 12 (c) (iii)

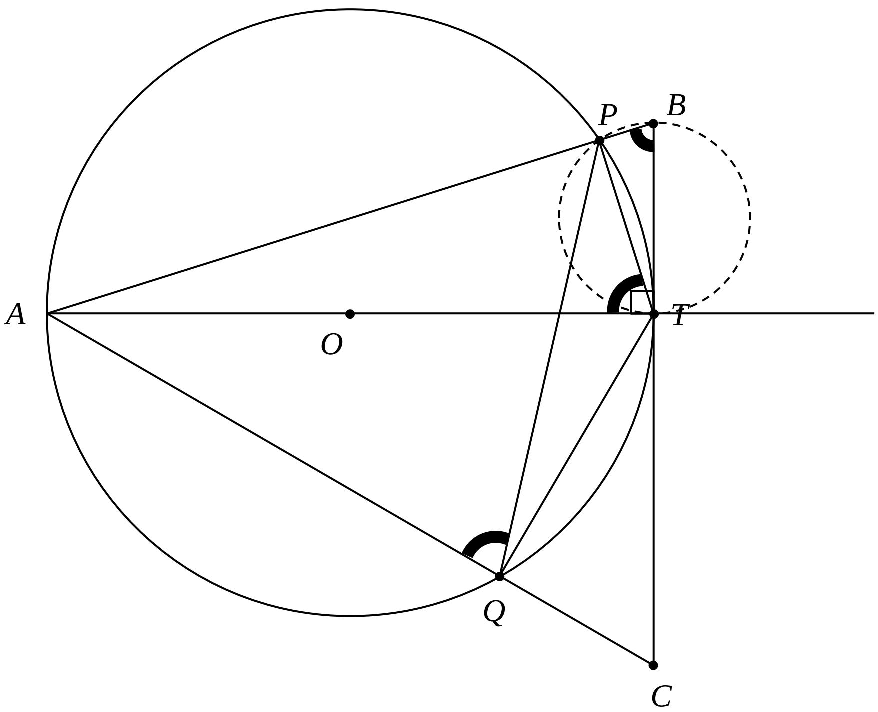
So we now have $p+q+t=0$ and $p^2+pt+2=0$.

Make t the subject to acquire $t=-(p+q)$, then substitute into $p^2+pt+2=0$:

$$\begin{aligned} p^2-p(p+q)+2&=0 \\ p^2-p^2-pq+2&=0 \\ pq&=2 \qquad \square \end{aligned}$$

Question 12 (d) (i)

First, we construct PT and TQ.



Let $\angle PBT=\theta$

$\angle APT=90^\circ$ (Angle subtended from diameter)

Therefore a circle can be constructed through points B, P and T such that BT is a diameter (converse of Thale's Theorem). This implies that AT is tangential to the circle (also since $\angle ATB=90^\circ$).

Hence, $\angle PBT=\theta=\angle PTA$ (Alternate Segment Theorem)

But $\angle PTA=\angle PQA$ (Angle subtended by common chord)

Therefore $\angle PBT = \theta = \angle PQA$.

Hence *PBCQ* is a cyclic quadrilateral (converse of exterior angle from cyclic quadrilateral theorem). □

Alternatively

Let $\angle PTA = \theta$

$\angle PQA = \theta$ (Angle subtended by a common chord)

$\angle APT = 90^{\circ}$ (Angle subtended from diameter)

$\therefore \angle PAT = 90 - \theta$ (Angle sum of ΔPAT)

But ΔBTA is also right-angled, so

$$\begin{aligned}\angle PBT &= 90 - (90 - \theta) \quad \text{(Angle sum of } \Delta BTA\text{)} \\ &= \theta\end{aligned}$$

Hence, by the converse of Exterior Angle = Opposite Interior Angle Theorem:

PBCQ is a cyclic quadrilateral □

Question 12 (d) (ii)

A basic angle chase yields the result immediately.

$\angle BCQ = \angle APQ$ (Exterior angle opposite interior angle of a cyclic quadrilateral)

$\angle APQ = \angle ATQ$ (Angle subtended by common chord)

Hence by the converse of the Alternate Segment Theorem, we have the result. □

Alternatively

We can simply observe that $\angle AQT = 90^{\circ}$, since it is an angle subtended from a diameter. It follows, by supplementary angles, that $\angle TQC = 90^{\circ}$ and hence the result by the converse of Thale's Theorem (Angle subtended from diameter is 90°). □

Question 13 (a)

We begin with the differential equation $\frac{dT}{dt} = k\left(E - T\right)$.

Separating the terms and grouping them appropriately, we have:

$$-\frac{dT}{T - E} = k\,dt$$

Note that we make the arrangement from $E - T$ to $T - E$ since $E < T$.

Integrate both sides with respect to the appropriate variable:

$$-\int_{T_0}^{T_n} \frac{dT}{T - E} = \int_{t_0}^{t_n} k\,dt$$

$$-\ln\left(T - E\right) \Bigg|_{T_0}^{T_n} = kt\Bigg|_{t_0}^{t_n}$$

Substituting and re-arranging, we have:

$$-\ln\left(T_n - E\right) + \ln\left(T_0 - E\right) = k\left(t_n - t_0\right)$$
$$\ln\left(\frac{T_0 - E}{T_n - E}\right) = k\left(t_n - t_0\right)$$

And hence:

$$k = \frac{\ln\left(\frac{T_0 - E}{T_n - E}\right)}{\left(t_n - t_0\right)}$$

But recall that $t_0 = 0$, hence:

$$k = \frac{1}{t_n}\ln\left(\frac{T_0 - E}{T_n - E}\right) \qquad \square$$

Question 13 (b) (i)

We are given the domain $0 \le x < 1$, from which we observe that $x^2 < 1$.

Multiply both sides by $a^2 - b^2$:

$$\left(a^2 - b^2\right)x^2 < a^2 - b^2$$

This is allowed since $a > b > 0$.

Expand and re-arrange:

$$a^2x^2 - b^2x^2 < a^2 - b^2$$
$$b^2 + a^2x^2 < a^2 + b^2x^2$$

We carefully square root both sides, knowing that the inequality is still preserved.

$$\sqrt{b^2 + a^2x^2} < \sqrt{a^2 + b^2x^2}$$

Flip both sides, and thus the inequality:

$$\frac{1}{\sqrt{a^2 + b^2x^2}} < \frac{1}{\sqrt{b^2 + a^2x^2}}$$

Hence $f(x) < g(x)$.

And so the other inequality follows.

Alternatively

Let $f(x) < g(x)$:

$$\frac{1}{\sqrt{a^2 + b^2x^2}} < \frac{1}{\sqrt{b^2 + a^2x^2}}$$

$$\sqrt{a^2 + b^2x^2} > \sqrt{b^2 + a^2x^2}$$

Square both sides carefully, noting that the inequality is preserved.

$$a^2 + b^2x^2 > b^2 + a^2x^2$$
$$x^2\left(a^2 - b^2\right) < a^2 - b^2$$

Hence $x^2 < 1$ and thus $0 \le x < 1$, since $x \ge 0$. The other direction of the inequality follows.

Question 13 (b) (ii)

This is a normal volumes problem now.

Since for $0 \le x < 1$, we have $f(x) < g(x)$, we can compute *V*.

$$\begin{aligned}
V &= \pi \int_0^1 \left(\frac{1}{b^2 + a^2x^2} - \frac{1}{a^2 + b^2x^2} \right) dx \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{ax}{b}\right) - \tan^{-1}\left(\frac{bx}{a}\right) \right]_0^1 \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{b}{a}\right) \right] \\
&= \frac{\pi}{ab} \tan^{-1}\left(\frac{\frac{a}{b} - \frac{b}{a}}{1 + \frac{a}{b} \times \frac{b}{a}} \right) \\
&= \frac{\pi}{ab} \tan^{-1}\left(\frac{a^2 - b^2}{2ab} \right) \quad \square
\end{aligned}$$

Question 13 (b) (iii)

This is essentially the same thing, with different limits and $f(x) \ge g(x)$.

$$\begin{aligned}
V_k &= \pi \int_1^k \frac{1}{a^2 + b^2x^2} - \frac{1}{b^2 + a^2x^2} \, dx \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{bx}{a}\right) - \tan^{-1}\left(\frac{ax}{b}\right) \right]_1^k \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{bk}{a}\right) - \tan^{-1}\left(\frac{ak}{b}\right) - \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{a}{b}\right) \right]
\end{aligned}$$

Note that as $k \to \infty$, $\tan^{-1}\left(\frac{bk}{a}\right) \to \frac{\pi}{2}$ and $\tan^{-1}\left(\frac{ak}{b}\right) \to \frac{\pi}{2}$.

Hence:

$$\begin{aligned}
V_k &\to \frac{\pi}{ab} \left[-\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{a}{b}\right) \right] \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{b}{a}\right) \right]
\end{aligned}$$

And this is the same expression as (i). $\square$

Question 13 (c) (i)

We will use the identity $\sin^2\theta + \cos^2\theta = 1$.

Since $A \le B$, we have:

$$
\begin{aligned}
x &= A\left[\cos^2\left(\frac{nt}{2}\right) + \sin^2\left(\frac{nt}{2}\right)\right] + (B-A)\sin^2\left(\frac{nt}{2}\right) \\
&= A + (B-A)\sin^2\left(\frac{nt}{2}\right) \\
&= A + \frac{B-A}{2}(1-\cos nt) \\
&= \frac{A+B}{2} - \left(\frac{B-A}{2}\right)\cos nt
\end{aligned}
$$

Differentiate once with respect to t:

$$\dot{x} = n\left(\frac{B-A}{2}\right)\sin nt$$

Differentate again with respect to t:

$$
\begin{aligned}
\ddot{x} &= n^2\left(\frac{B-A}{2}\right)\cos nt \\
&= -n^2\left[x - \frac{A+B}{2}\right]
\end{aligned}
$$

Hence, the particle moves in Simple Harmonic Motion, with centre of motion being $x = \dfrac{A+B}{2}$. □

Alternatively

Using the results $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, we have:

$$\begin{aligned} x &= A\cos^2\left(\frac{nt}{2}\right) + B\sin^2\left(\frac{nt}{2}\right) \\ &= \frac{A}{2}(1+\cos nt) + \frac{B}{2}(1-\cos nt) \\ &= \frac{1}{2}(A+B) + \frac{1}{2}(A-B)\cos nt \end{aligned}$$

Differentiate once with respect to t :

$$\dot{x} = -\frac{n}{2}(A-B)\sin nt$$

Differentiate again with respect to t :

$$\begin{aligned} \ddot{x} &= -\frac{n^2}{2}(A-B)\cos nt \\ &= -n^2\left[x - \frac{A+B}{2}\right] \end{aligned}$$

Question 13 (c) (ii)

Observe that the centre of motion is $x=\dfrac{A+B}{2}$ and amplitude is $\left(\dfrac{B-A}{2}\right)$.

One endpoint is $x_1=\dfrac{A+B}{2}+\dfrac{B-A}{2}=B$.

The other endpoint is $x_2=\dfrac{A+B}{2}-\dfrac{B-A}{2}=A$.

Hence $A\le x\le B$.

Question 13 (d) (i)

Using the Sine Rule in ΔAOP, we have:

$$\frac{l}{\sin\alpha}=\frac{1}{\sin\angle APO}$$

But we also have:

$$\begin{aligned}\angle APO &= 180^\circ-\alpha-\angle AOP\\ &=180^\circ-\alpha-\left(90^\circ-\theta\right)\\ &=180^\circ-\alpha-90^\circ+\theta\\ &=90^\circ-\left(\alpha-\theta\right)\end{aligned}$$

So:

$$\begin{aligned}\frac{l}{\sin\alpha}&=\frac{1}{\sin\left(90^\circ-\left(\alpha-\theta\right)\right)}\\ &=\frac{1}{\cos\left(\alpha-\theta\right)}\\ l&=\frac{\sin\alpha}{\cos\left(\alpha-\theta\right)}\qquad\square\end{aligned}$$

Question 13 (d) (ii) (1)

Using the Chain Rule, we have $\frac{d\alpha}{dt} = \frac{d\alpha}{dl} \times \frac{dl}{dt} = \frac{d\alpha}{dl} \times S$.

$$\frac{dl}{d\alpha} = \frac{\cos\alpha\cos(\alpha-\theta) + \sin(\alpha-\theta)\sin\alpha}{\cos^2(\alpha-\theta)}$$

$$= \frac{\cos\theta}{\cos^2(\alpha-\theta)}$$

$$\frac{d\alpha}{dl} = \frac{\cos^2(\alpha-\theta)}{\cos\theta}$$

$$\begin{aligned}\dot{\alpha} &= \frac{d\alpha}{dt}\\ &= \frac{d\alpha}{dl} \times \frac{dl}{dt}\\ &= \frac{\cos^2(\alpha-\theta)}{\cos\theta} \times S\end{aligned}$$

Let $\alpha = 2\theta$:

$$\begin{aligned}\dot{\alpha} &= \frac{\cos^2(2\theta-\theta)}{\cos\theta} \times S\\ &= \frac{\cos^2\theta}{\cos\theta} \times S\\ &= S\cos\theta \qquad \square\end{aligned}$$

Question 13 (d) (ii) (2)

We will use the formula $\ddot{\alpha} = \dot{\alpha} \times \frac{d\dot{\alpha}}{d\alpha}$.

It may seem unrecognisable now, but it is actually more commonly known as $a = v \times \frac{dv}{dx}$, which is much more well-known (as it is taught that way).

$$\begin{aligned}
\ddot{\alpha} &= \dot{\alpha} \times \frac{d\dot{\alpha}}{d\alpha} \\
&= \dot{\alpha} \times \frac{d}{d\alpha}\left(\frac{\cos^2(\alpha - \theta)}{\cos\theta} \times S \right) \\
&= \dot{\alpha} \times \frac{-2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \times S \\
&= -\dot{\alpha} \times S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta}
\end{aligned}$$

Let $\alpha = 2\theta$:

$$\begin{aligned}
\ddot{\alpha} &= -\dot{\alpha} \times S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \\
&= -\dot{\alpha} \times S \times \frac{2\cos\theta\sin\theta}{\cos\theta} \\
&= -\dot{\alpha} \times S \times 2\sin\theta
\end{aligned}$$

But $l = \frac{\sin\alpha}{\cos(\alpha-\theta)}$ and when $\alpha = 2\theta$,

$$\begin{aligned}
l &= \frac{\sin 2\theta}{\cos\theta} \\
&= \frac{2\sin\theta\cos\theta}{\cos\theta} \\
&= 2\sin\theta
\end{aligned}$$

Hence:

$$\begin{aligned}
\ddot{\alpha} &= -\dot{\alpha} \times S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \\
&= -\dot{\alpha} \times S \times \frac{2\cos\theta\sin\theta}{\cos\theta} \\
&= -\dot{\alpha} \times S \times l \qquad \square
\end{aligned}$$

Alternatively

$$
\begin{aligned}
\ddot{\alpha} &= \frac{d\dot{\alpha}}{dt} \\
&= \frac{d\dot{\alpha}}{d\alpha} \times \frac{d\alpha}{dt} \\
&= \frac{d\dot{\alpha}}{d\alpha} \times S\cos\theta
\end{aligned}
$$

But recall that $\dot{\alpha} = \dfrac{\cos^2(\alpha-\theta)}{\cos\theta} \times S$

$$
\begin{aligned}
\frac{d\dot{\alpha}}{d\alpha} &= -S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \\
&= -\frac{S}{\cos\theta}\sin(2\alpha-2\theta)
\end{aligned}
$$

Hence :

$$\ddot{\alpha} = -S^2 \times \sin(2\alpha - 2\theta)$$

Substitute $\alpha = 2\theta$:

$$
\begin{aligned}
\ddot{\alpha} &= -S^2 \times \sin 2\theta \\
&= -2S^2 \sin\theta\cos\theta
\end{aligned}
$$

But recall that $\dot{\alpha} = S\cos\theta$. Also, similarly to the alternative solution above, $l = 2\sin\theta$.

Hence $\ddot{\alpha} = -\dot{\alpha} \times S \times l$ $\square$

Question 14 (a) (i)

Consider the expansion $(1+x)^m (1+x)^{n-m} = (1+x)^n$.

Coefficient of x^k from RHS: $\binom{n}{k}$

Coefficient of x^k from LHS:

$$x^0 \times x^k \Rightarrow \binom{m}{0}\binom{n-m}{k}$$
$$x^1 \times x^{k-1} \Rightarrow \binom{m}{1}\binom{n-m}{k-1}$$
$$x^2 \times x^{k-2} \Rightarrow \binom{m}{2}\binom{n-m}{k-2}$$

...

$$x^k \times x^0 \Rightarrow \binom{m}{k}\binom{n-m}{0}$$

Hence coefficient of x^k from LHS is:

$$\binom{n-m}{0}\binom{n}{k}+\binom{n-m}{1}\binom{n}{k-1}+\binom{n-m}{2}\binom{n}{k-2}+\ldots+\binom{n-m}{k}\binom{n}{0}$$

And hence the result. $\square$

Question 14 (a) (ii)

We make the following substitutions, $n \to 2n$, $m \to n$, $k \to n$.

Then the identity from (i) now becomes:

$$\binom{2n-n}{0}\binom{n}{n}+\binom{2n-n}{1}\binom{n}{n-1}+\binom{2n-n}{2}\binom{n}{n-2}+\ldots+\binom{2n-n}{n}\binom{n}{0}=\binom{2n}{n}$$

Simplifying this:

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\ldots+\binom{n}{n}\binom{n}{0}=\binom{2n}{n}$$

But recall the identity $\binom{n}{k}=\binom{n}{n-k}$:

Hence $\binom{n}{0}=\binom{n}{n}$, $\binom{n}{1}=\binom{n}{n-1}$, … , $\binom{n}{n}=\binom{n}{0}$.

Therefore:

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n} \qquad \square$$

Question 14 (a) (iii)

We now make the substitution $r \to n-r$ in the summation on the right.

This will mean that in a similar fashion to Integration by Substitution, $r=0 \Rightarrow n=r$ and $r=n \Rightarrow n=0$. But also note that if we sum from 0 to *n*, or from *n* to 0, it makes no difference so we can just write the bottom limit as 0 and the top limit as *n*, to follow general convention.

Hence, our new expression is:

$$\sum_{r=0}^{n} r\binom{n}{r}^2 = \sum_{r=1}^{n}(n-r)\binom{n}{n-r}^2$$

Expanding the RHS, we get:

$$\sum_{r=0}^{n} r\binom{n}{r}^2 = \sum_{r=1}^{n} n\binom{n}{n-r}^2 - \sum_{r=1}^{n} r\binom{n}{n-r}^2$$

But recall that $\binom{n}{k} = \binom{n}{n-k}$. Applying it, we have:

$$\begin{aligned}\sum_{r=0}^{n} r\binom{n}{r}^2 &= \sum_{r=1}^{n} n\binom{n}{n-(n-r)}^2 - \sum_{r=1}^{n} r\binom{n}{n-(n-r)}^2 \\ &= \sum_{r=1}^{n} n\binom{n}{r}^2 - \sum_{r=1}^{n} r\binom{n}{r}^2\end{aligned}$$

Re-arranging the terms, we have:

$$2\sum_{r=0}^{n} r\binom{n}{r}^2 = n\sum_{r=1}^{n}\binom{n}{r}^2$$

And this is possible, since $\sum_{r=1}^{n} r\binom{n}{r}^2 = \sum_{r=0}^{n} r\binom{n}{r}^2$.

Substitute the result in (ii):

$$\begin{aligned}2\sum_{r=0}^{n} r\binom{n}{r}^2 &= n\sum_{r=1}^{n}\binom{n}{r}^2 \\ &= n\binom{2n}{n}\end{aligned}$$

Dividing both sides by 2 yields the required result. □

Alternatively

$$\binom{n}{1}^2+2\binom{n}{2}^2+\ldots+n\binom{n}{n}^2=\binom{n}{n-1}^2+2\binom{n}{n-2}^2+\ldots+n\binom{n}{n-n}^2$$

$$=n\binom{n}{0}^2+(n-1)\binom{n}{1}^2+(n-2)\binom{n}{2}^2+\ldots+\binom{n}{n-1}^2$$

$$=n\binom{n}{0}^2+\left[n\binom{n}{1}^2-\binom{n}{1}^2\right]+\left[n\binom{n}{2}^2-2\binom{n}{2}^2\right]+\ldots+\binom{n}{n-1}^2$$

$$=n\left[\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n-1}^2\right]-\left[\binom{n}{1}^2+2\binom{n}{2}^2+\ldots+(n-1)\binom{n}{n-1}^2\right]$$

Or in a more compact form

$$\sum_{k=1}^{n}k\binom{n}{k}^2=n\times\left[\sum_{k=0}^{n}\binom{n}{k}^2\right]-\left[\sum_{k=1}^{n}k\binom{n}{k}^2\right]$$

$$2\times\sum_{k=1}^{n}k\binom{n}{k}^2=n\times\left[\sum_{k=0}^{n}\binom{n}{k}^2\right]$$

$$\sum_{k=1}^{n}k\binom{n}{k}^2=\frac{n}{2}\binom{2n}{n}\qquad\square$$

Question 14 (b) (i)

By symmetry of the parabola, we have $\theta = \alpha_1$. Similarly for other trajectories, $\alpha_1 = \alpha_2$, $\alpha_2 = \alpha_3$, …. and inductively, we have $\theta = \alpha_1 = \alpha_2 = \alpha_3 = ...$ Hence $\theta = \alpha_n$, for all $n \in \mathbb{N}$.

Question 14 (b) (ii)

We will first find the time for the first trajectory.

Let $y = 0$:

$$-\frac{1}{2}gt^2 + V_n t \sin\theta = 0$$

$$-\frac{1}{2}gt + V_n \sin\theta = 0 \quad ...\left(\text{since } t = 0 \text{ is when it is at the origin}\right)$$

$$t = \frac{2V_n \sin\theta}{g}$$

So therefore, we have $t_0 = \dfrac{2V_0 \sin\theta}{g}$.

Similarly:

$t_1 = \dfrac{2V_1 \sin\theta}{g} = \dfrac{2kV_0 \sin\theta}{g}$, using the recurrence $V_n = kV_{n-1}$.

$$t_2 = \frac{2V_2 \sin\theta}{g} = \frac{2kV_1 \sin\theta}{g} = \frac{2k^2V_0 \sin\theta}{g} \dots$$

$$t_n = \frac{2k^nV_0 \sin\theta}{g}$$

So summing up an infinite number of these, we have:

$$\begin{aligned}
T_{V_0} &= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{2k^r V_0 \sin\theta}{g} \\
&= \frac{2V_0\sin\theta}{g}\times\lim_{n\to\infty}\sum_{r=0}^{n}k^r \\
&= \frac{2V_0\sin\theta}{g}\times\frac{1}{1-k} \qquad ...\left(\text{since } |k|<1\right) \\
&= \frac{2V_0\sin\theta}{g\left(1-k\right)} \qquad \square
\end{aligned}$$

Question 14 (b) (iii)

We will first find the distance of the first trajectory.

$$\begin{aligned}
x_0 &= V_0\, t_0 \cos\theta \\
&= V_0\times\frac{2V_0\sin\theta}{g}\times\cos\theta \\
&= \frac{2V_0^{\,2}\sin\theta\cos\theta}{g} \\
&= \frac{V_0^{\,2}\sin 2\theta}{g}
\end{aligned}$$

Similarly,

$$x_1 = \frac{V_1^2\sin 2\theta}{g} = \frac{k^2V_0^{\,2}\sin 2\theta}{g}$$

$$x_2 = \frac{V_2^{\,2}\sin 2\theta}{g} = \frac{k^2V_1^2\sin 2\theta}{g} = \frac{k^4V_0^{\,2}\sin 2\theta}{g}$$

…

$$x_n = \frac{k^{2n}V_0^{\,2}\sin 2\theta}{g}$$

Hence, the total distance is:

$$
\begin{aligned}
R &= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{k^{2r}V_0^{\,2}\sin 2\theta}{g}\\
&= \frac{V_0^{\,2}\sin 2\theta}{g}\lim_{n\to\infty}\sum_{r=0}^{n}k^{2r}\\
&= \frac{V_0^{\,2}\sin 2\theta}{g}\times\frac{1}{1-k^2} \qquad \ldots\left(\text{since } |k|<1\right)\\
&= \frac{V_0^{\,2}\sin 2\theta}{g\left(1-k^2\right)} \qquad \square
\end{aligned}
$$

Question 14 (b) (iv)

We know that $T_R = \dfrac{2V_R\sin\theta}{g}$.

So placing it as a ratio:

$$
\begin{aligned}
\frac{T_R}{T_{V_0}} &= \frac{\dfrac{2V_R\sin\theta}{g}}{\dfrac{2V_0\sin\theta}{g\left(1-k\right)}}\\
&= \frac{2V_R\sin\theta}{g}\times\frac{g\left(1-k\right)}{2V_0\sin\theta}\\
&= \frac{V_R}{V_0}\times\left(1-k\right)
\end{aligned}
$$

So we must now find $\dfrac{V_R}{V_0}$.

Equate R with the range of any normal trajectory:

$$\frac{V_R{}^2 \sin 2\theta}{g} = \frac{V_0{}^2 \sin 2\theta}{g\left(1-k^2\right)}$$

$$V_R{}^2 = \frac{V_0{}^2}{1-k^2}$$

$$\frac{V_R{}^2}{V_0{}^2} = \frac{1}{1-k^2}$$

$$\frac{V_R}{V_0} = \frac{1}{\sqrt{(1-k)(1+k)}}$$

Hence, substituting it in:

$$\frac{T_R}{T_{V_0}} = \frac{V_R}{V_0} \times (1-k)$$

$$= \frac{1-k}{\sqrt{(1-k)(1+k)}}$$

$$= \frac{\sqrt{1-k}}{\sqrt{1+k}}$$

$$= \sqrt{\frac{1-k}{1+k}} \qquad \square$$

Question 14 (b) (v)

We know that $0 < k < 1$.

Hence $1+k > 1$, and $1-k < 1$ and therefore $0 < \dfrac{1-k}{1+k} < 1$.

Square rooting both sides yields the same bounds, so $0 < \sqrt{\dfrac{1-k}{1+k}} < 1$.

But recall that $\dfrac{T_R}{T_{V_0}} = \sqrt{\dfrac{1-k}{1+k}}$, so therefore $0 < \dfrac{T_R}{T_{V_0}} < 1$, and hence $0 < T_R < T_{V_0}$. $\square$

Physically, this means that it will always take less time to get the ball to a location via landing it there in a single larger trajectory, as opposed to bouncing it there with a smaller one.