2019 Bored of Studies Trial Examinations

Mathematics Extension 1

10th October 2019

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- NESA-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 2 – 3

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 4 – 9

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

Section I

10 marks
Attempt Questions 1—10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1—10.

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1 Consider three points on a line from left to right A, B and C where AB is m units in length and BC is n units in length. Which of the following statements is correct?

(A) B divides AC internally in the ratio $n : m$

(B) B divides CA internally in the ratio $m : n$

(C) A divides BC externally in the ratio $(m + n) : n$

(D) C divides AB externally in the ratio $(m + n) : n$

2 A population is modelled based on the differential equation

$$\frac{dN}{dt} = k(N - b).$$

where k and b are constants, and N is the size of the population at time t. Which of the following is NOT a possible solution to the differential equation, given some constant a?

(A) $N = ae^{kt} + b$ (B) $N = ae^{kt} - b$ (C) $N = b - ae^{kt}$ (D) $N = b$

3 Consider the trigonometric equation

$$\frac{2\tan x}{1 + \tan^2 x} = \cos 2x.$$

Which of the following best represents the general solution to the equation for some integer n?

(A) $\dfrac{n\pi}{2} + \dfrac{\pi}{4}$ (B) $\dfrac{n\pi}{4} + \dfrac{\pi}{4}$ (C) $\dfrac{n\pi}{2} + \dfrac{\pi}{8}$ (D) $\dfrac{n\pi}{4} + \dfrac{\pi}{8}$

4 Which of the following best describes the set of solutions to the inequality below?

$$|x - 1| < \frac{1}{x - 1}$$

(A) $1 < x < 2$ (B) $0 < x < 2$ (C) $x < 0$ or $x > 1$ (D) $x < 0$ or $1 < x < 2$

5 A polynomial $P(x)$ has remainder $x+3$ when divided by x^2+2x+1 and satisfies $P(-1)=2$. Which of the following is NOT a possible remainder when $P(x)$ is divided by $(x+1)^3$?

(A) $5x+2$ (B) $x+3$ (C) $3x^2+7x+6$ (D) $4x^2+9x+7$

6 Given that x^2-2x+2 is a factor of x^4+4, what is $\displaystyle\int_{-1}^{1} \frac{(x+1)^2+1}{x^4+4}\,dx$?

(A) $\tan^{-1}\left(\frac{1}{2}\right)$ (B) $\tan^{-1}(2)$ (C) $\dfrac{\pi}{4}$ (D) $\dfrac{\pi}{2}$

7 Which of the following integrals is equivalent to

$$\int_{-2}^{-1} \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx$$

after applying the substitution $x=\sec\theta$?

(A) $\displaystyle\int_{\frac{\pi}{3}}^{\pi} \tan^4\theta\,d\theta$ (B) $\displaystyle\int_{\frac{2\pi}{3}}^{\pi} \tan^4\theta\,d\theta$ (C) $\displaystyle\int_{\frac{4\pi}{3}}^{\pi} \tan^4\theta\,d\theta$ (D) $\displaystyle\int_{\frac{5\pi}{3}}^{\pi} \tan^4\theta\,d\theta$

8 The coefficient of x^k in the binomial expansion of $(1-x)^n$ for positive odd integers n is

$$1+2+3+\ldots+(n-1).$$

What is the value of k?

(A) 1 (B) 2 (C) $n-1$ (D) $n-2$

9 Suppose that $2n$ students sitting the BoS Trial exam can completely fill two rows in the exam room. Each row contains n students. There are two different versions of the exam. The exam papers are to be distributed such that no adjacent student (left/right or front/back) has the same version of the exam.

How many ways can be the students and exams be arranged in this way?

(A) $(n!)^2$ (B) $2\times(n!)^2$ (C) $(2n)!$ (D) $2\times(2n)!$

10 The graph of $y=\sin^{-1}\sqrt{x}$ is increasing in the domain $0<x<1$. What is the value of the following definite integral?

$$\int_0^1 \sin^{-1}\sqrt{x}\,dx$$

(A) $\dfrac{\pi}{8}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{2}$ (D) π

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Suppose that α, β and γ are angles such that **2**

$$\frac{\sin\alpha+\sin\beta+\sin\gamma}{\sin(\alpha+\beta+\gamma)}=\frac{\cos\alpha+\cos\beta+\cos\gamma}{\cos(\alpha+\beta+\gamma)}.$$

Show that

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\alpha+\gamma)=0.$$

(b) Two distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ where $a > 0$. The tangents at P and Q intersect at the point $T(a(p+q), apq)$ (Do NOT prove this).

Suppose that $R(2ar, ar^2)$ is a point between P and Q which also lies on the parabola. The tangent at R intersects the tangents at P and Q at the points P' and Q' respectively.

(i) Show that $PT = a|p-q|\sqrt{1+p^2}$. **1**

(ii) Show that $P'T = a|q-r|\sqrt{1+p^2}$. **1**

(iii) Hence show that for any position of P and Q on the parabola **3**

$$\frac{P'T}{PT}+\frac{Q'T}{QT}=1.$$

(c) The polynomial $P(x) = x^3 + 2x^2 - 3x - 1$ has roots α, β and γ. Find the value of **2**

$$\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta).$$

Question 11 continues on page 5

(d) Newton's method is used to find an approximation of a root of $f(x)$. Let x_0 be the initial approximation and x_k be the approximation of the root after k applications of Newton's method. After $(n+1)$ applications, $x_{n+1} = x_0$. Assume that none of the approximations are stationary points. Show that **1**

$$\sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)} = 0.$$

(e) Matthew tosses a fair coin n times. Jonathan independently tosses another fair coin n times.

(i) Explain why the probability that Matthew and Jonathan each toss exactly k heads is **1**

$$\binom{n}{k}^2 \frac{1}{2^{2n}}.$$

(ii) Suppose that Matthew tosses k heads and Jonathan tosses $n-k$ heads. Explain why the probability of this occurring is equal to the result in part (i). **1**

(iii) Hence, show that the probability that Matthew and Jonathan will toss exactly the same number of heads is **2**

$$\binom{2n}{n} \frac{1}{2^{2n}}.$$

(iv) Deduce that **1**

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) It can be shown that the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ has a minimum turning point at $(0, 1)$ **2**
(Do NOT prove this).

Use this information to sketch the graph of $y = \dfrac{1}{\sqrt{1-x^2}}$.

(b) Consider the function $f(x) = \sin^{-1}(2x^2 - 1)$.

(i) Find the domain of $f(x)$. **1**

(ii) Using the result in part (a), or otherwise, sketch the graph of $y = f'(x)$. **3**

(iii) Find the range of $f(x)$. **1**

(iv) Hence sketch the graph of $y = f(x)$. **2**

(c) A particle in a system of springs has the following acceleration equation

$$\frac{d^2x}{dt^2} = \begin{cases} -x\,, & \text{for } x < 2 \\ 4 - x\,, & \text{for } x \geq 2 \end{cases}$$

where x is the particle's displacement at time t. The particle is initially at the origin with velocity v_0.

(i) Show that if $|v_0| < 2$ then the particle will exhibit simple harmonic motion. **2**

(ii) If $|v_0| < 2$, find the amplitude of the particle's motion in terms of v_0. **1**

(iii) Now suppose that the particle's initial velocity is given by $v_0 = 2$. Find the maximum displacement of the particle. **2**

(iv) Hence, describe the particle's displacement over time if $v_0 = 2$. **1**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Suppose that the graphs of $y = a^x$ and its inverse $y = \log_a x$ are tangent to each other at the point (p, p) for some constant $0 < a < 1$. Let m be the slope of the tangent of the graphs at (p, p).

(i) Explain why $m^2 = 1$. **1**

(ii) Hence, find the values of a and p. **3**

(iii) On the same set of axes, sketch the graphs of $y = a^x$ and its inverse $y = \log_a x$ for the value of a found in part (ii). **2**

(b) A particle is projected at an initial speed of V at an angle of θ to the horizontal. Let g be the acceleration due to gravity. The displacement equations of motion at time t are

$$x = Vt\cos\theta \qquad y = Vt\sin\theta - \frac{gt^2}{2} \qquad \text{(Do NOT prove this.)}$$

The particle has to clear two walls. One wall has a horizontal displacement of d_1 from the point of projection and vertical height of d_2 where $d_1 \neq d_2$. Another wall has a horizontal displacement of d_2 from the point of projection and vertical height of d_1.

(i) Show that the angle of projection θ needed for the particle to just clear the two walls satisfies **3**

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}.$$

(ii) Suppose that the angle of projection is chosen such that $\tan\theta \leq 3$. Show that the particle cannot clear the two walls for any values of d_1 and d_2. **2**

(c) Let n be a positive integer.

(i) Show that $\left(1 + \dfrac{1}{n}\right)^n \geq 2$. **1**

(ii) Hence, or otherwise, prove by mathematical induction **3**

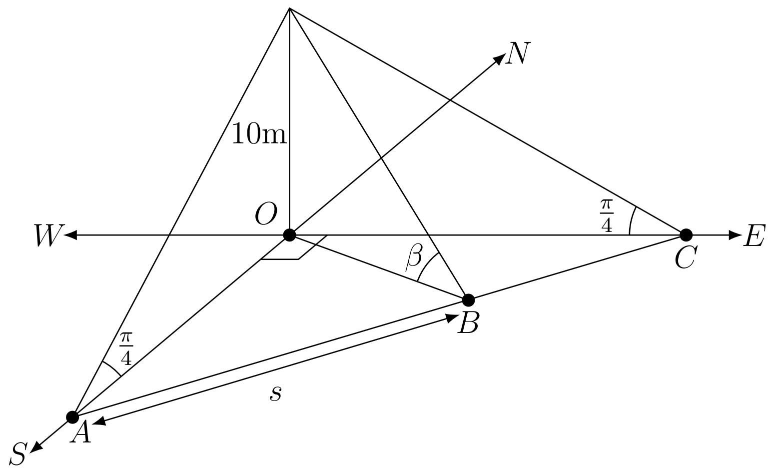
$$(2n-1)! \leq n^{2n-1}.$$

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) A monument of height 10 metres stands on point O on the ground. A visitor's centre A is located due south of the monument with an angle of elevation of $\frac{\pi}{4}$ to the top of the monument. Another visitor's centre C is located due east of the monument also with an angle of elevation of $\frac{\pi}{4}$ to the top of the monument.

Visitor B cycles directly from centre A to centre C in a straight line. Let β be the angle of elevation of the cycling visitor to the top of the monument at time t.



Let s be the displacement of the visitor from centre A at time t in metres. Let $\angle ABO = \theta$ where $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

(i) Show that $s = \dfrac{10\sin\left(\frac{3\pi}{4} - \theta\right)}{\sin\theta}$. **2**

(ii) Suppose that the visitor cycles from A to C at a fixed speed of 5 metres per second. Show that **2**

$$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{\sqrt{2}}.$$

(iii) Hence, show that the rate of change of the visitor's angle of elevation is given by **3**

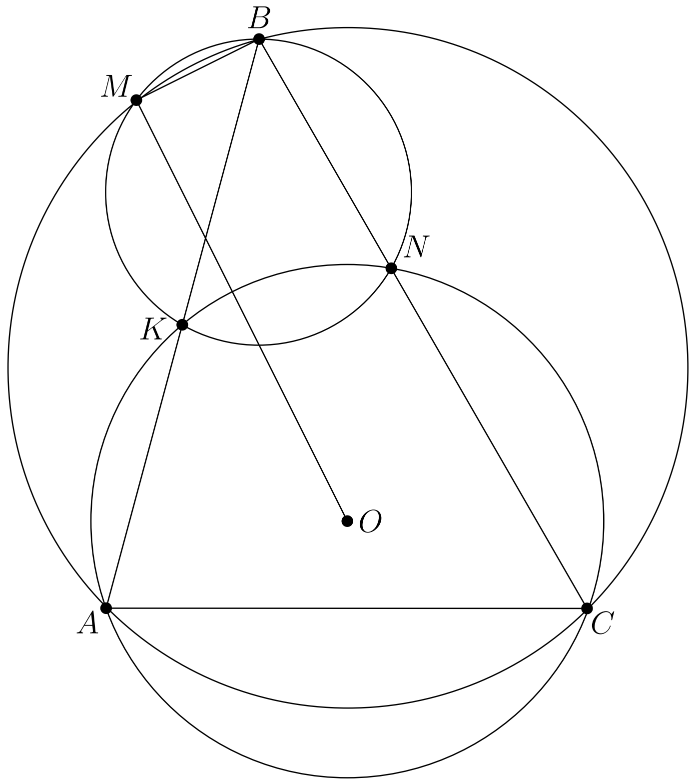
$$\frac{d\beta}{dt} = -\frac{\cos\theta}{2 + \text{cosec}^2\theta}.$$

(iv) What is the largest possible angle of elevation that the visitor can have cycling from A to C? Explain your answer. **1**

Question 14 continues on page 9

(b) Let O be the centre of a circle $\mathscr{C}_1$ which passes through the vertices A and C of $\triangle ABC$. Let K and N be the points where this circle intersects AB and BC respectively.

Another circle $\mathscr{C}_2$ is drawn to pass through the points B, K and N. A third circle $\mathscr{C}_3$ is also drawn to pass through the points A, B and C. Let M be the point of intersection of circles $\mathscr{C}_2$ and $\mathscr{C}_3$.



(i) Prove that $\triangle MKA$ and $\triangle MNC$ are similar. **2**

(ii) Let S and T be the midpoints of AK and CN respectively. Prove that $MSTB$ is a cyclic quadrilateral. **3**

(iii) Explain why $SBTO$ is a cyclic quadrilateral. **1**

(iv) Hence, deduce that $OM \perp BM$. **1**

End of Question 14

End of paper

2019 Bored of Studies Trial Examinations

Mathematics Extension 1

Solutions

Section I

Answers

1	D	**6**	B
2	B	**7**	C
3	C	**8**	B
4	A	**9**	D
5	A	**10**	B

Brief explanations

1 $AB = m$ and $BC = n$, so B must divide AC internally with ratio $m : n$. Also, A divides BC externally in the ratio $m : (m+n)$ or C divides AB externally in the ratio $(m+n) : n$. The only option which matches these is (D).

2 Differentiating each of the options, it can be seen that the only answer which cannot satisfy $\dfrac{dN}{dt} = k(N-b)$ is option (B).

3 From t-formula results $\sin 2x = \dfrac{2t}{1+t^2}$ where $t = \tan x$. The equation then simplifies to $\tan 2x = 1$, which has general solution $\dfrac{n\pi}{2} + \dfrac{\pi}{8}$. Hence, the answer is (C).

4 Since $|x-1| \geq 0$ then the inequality implies $\dfrac{1}{x-1} > 0$. This means that the only valid domain is $x > 1$, which implies $|x-1| = x-1$ and the inequality reduces to $(x-1)^2 < 1$. Solving this leads to $0 < x < 2$ but since $x > 1$ then the final solution is $1 < x < 2$, which is (A).

5 By the remainder theorem, $P(x) = (x+1)^3 Q(x) + R(x)$ where $Q(x)$ and $R(x)$ are polynomials. The remainder $R(x)$ must be a quadratic, otherwise it can be further divided by the cubic polynomial $(x+1)^3$. Hence, $R(x) = ax^2 + bx + c$ for some coefficients a, b and c. Since $P(-1) = 2$ then $a - b + c = 2$. The coefficients of all the options satisfy this equality except for (A).

6 Applying long division or inspection, the other factor of x^4+4 is x^2+2x+2 so

$$\begin{aligned}\int_{-1}^{1}\frac{(x+1)^2+1}{x^4+4}\,dx &= \int_{-1}^{1}\frac{x^2+2x+2}{(x^2+2x+2)(x^2-2x+2)}\,dx\\ &= \int_{-1}^{1}\frac{1}{x^2-2x+2}\,dx\\ &= \int_{-1}^{1}\frac{1}{(x-1)^2+1}\,dx\\ &= \left[\tan^{-1}(x-1)\right]_{-1}^{1}\\ &= -\tan^{-1}(-2)\\ &= \tan^{-1}2\end{aligned}$$

Hence, the answer is (B).

7 Let $x=\sec\theta$. When $x=-1$ then $\theta=\pi$. When $x=-2$ then $\theta=\frac{2\pi}{3}$ or $\theta=\frac{4\pi}{3}$. This means the answer cannot be (A) or (D).

Note that $\frac{(x^2-1)^{\frac{3}{2}}}{x}\le 0$ in the domain $-2\le x\le -1$ so the value of the definite integral must be negative. Since $\tan^4\theta$ is non-negative, then the only way to get a negative definite integral is for the lower limit to be greater than the upper limit. Hence, the answer is (C).

8 The general term of the binomial expansion of $(1-x)^n$ is $\binom{n}{k}(-1)^k x^k$. Note that

$$\begin{aligned}1+2+3+...+(n-1) &= \frac{n(n-1)}{2}\\ &= \frac{n(n-1)(n-2)!}{(n-2)!2!}\\ &= \frac{n!}{2!(n-2)!}\\ &= \binom{n}{2} \quad \text{or} \quad \binom{n}{n-2}\end{aligned}$$

This means $k=2$ or $k=n-2$ are possible candidate choices. The coefficient of x^{n-2} is $\binom{n}{n-2}(-1)^{n-2}$, but since n is odd then the coefficient simplifies to $-\binom{n}{n-2}$. On the other hand, the coefficient of x^2 is $\binom{n}{2}(-1)^2$, which is $\binom{n}{2}$. Hence, the answer is (B).

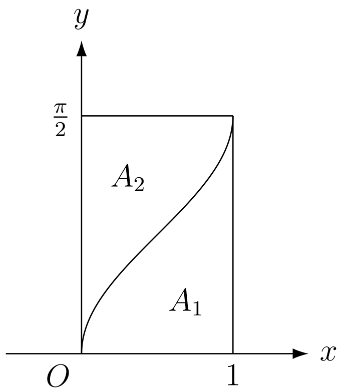
9 Since there are no restrictions on the seating arrangement of the students, there are $(2n)!$ ways of arranging them anywhere in the two rows. Each student has two possible exams they could be allocated with. Therefore, the total number of arrangements is $2\times(2n)!$, so the answer is (D).

10 Since $y = \sin^{-1}\sqrt{x}$ is increasing in the domain $0 < x < 1$ (noting its actual domain is $0 \leq x \leq 1$), the range must be $0 \leq y \leq \frac{\pi}{2}$.

Let A_1 be the area bounded by the curve $y = \sin^{-1}\sqrt{x}$, the x-axis and $x = 1$. Let A_2 be the area bounded by the curve $y = \sin^{-1}\sqrt{x}$, the y-axis and $y = \frac{\pi}{2}$.

$$\begin{aligned} A_1 + A_2 &= \int_0^1 y\,dx + \int_0^{\frac{\pi}{2}} x\,dy \\ &= \int_0^1 \sin^{-1}\sqrt{x}\,dx + \int_0^{\frac{\pi}{2}} \sin^2 y\,dy \end{aligned}$$

However, the sum of the areas A_1 and A_2 form a rectangle of side lengths 1 and $\frac{\pi}{2}$.



By inspection and deduction of the available options, the most likely answer is (B). This can be formally shown as follows

$$\begin{aligned} \int_0^1 \sin^{-1}\sqrt{x}\,dx + \int_0^{\frac{\pi}{2}} \sin^2 y\,dy &= \frac{\pi}{2} \times 1 \\ \Rightarrow \int_0^1 \sin^{-1}\sqrt{x}\,dx &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 y\,dy \\ &= \frac{\pi}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2y - 1)\,dy \\ &= \frac{\pi}{2} + \frac{1}{2}\left[\frac{1}{2}\sin 2y - y\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} \end{aligned}$$

Hence, the answer is (B).

Question 11

(a) Rearranging the given result $\dfrac{\sin\alpha+\sin\beta+\sin\gamma}{\sin(\alpha+\beta+\gamma)} = \dfrac{\cos\alpha+\cos\beta+\cos\gamma}{\cos(\alpha+\beta+\gamma)}$ gives

$$(\cos\alpha+\cos\beta+\cos\gamma)\sin(\alpha+\beta+\gamma) - (\sin\alpha+\sin\beta+\sin\gamma)\cos(\alpha+\beta+\gamma) = 0$$

Note that

$$\begin{aligned}\sin(\alpha+\beta+\gamma)\cos\alpha - \cos(\alpha+\beta+\gamma)\sin\alpha &= \sin(\alpha+\beta+\gamma-\alpha)\\ &= \sin(\beta+\gamma)\end{aligned}$$

Similarly

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha+\beta+\gamma)\cos\gamma - \cos(\alpha+\beta+\gamma)\sin\gamma\\ \sin(\alpha+\gamma) &= \sin(\alpha+\beta+\gamma)\cos\beta - \cos(\alpha+\beta+\gamma)\sin\beta\end{aligned}$$

Adding all the equations gives

$$\begin{aligned}&\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\alpha+\gamma)\\ &= (\cos\alpha+\cos\beta+\cos\gamma)\sin(\alpha+\beta+\gamma) - (\sin\alpha+\sin\beta+\sin\gamma)\cos(\alpha+\beta+\gamma)\\ &= 0\end{aligned}$$

(b) (i) Using the distance formula noting that $a > 0$ and $1+p^2 > 0$

$$\begin{aligned}PT^2 &= (2ap - a(p+q))^2 + (ap^2 - apq)^2\\ &= a^2(2p-p-q)^2 + a^2p^2(p-q)^2\\ &= a^2(p-q)^2(1+p^2)\\ PT &= a|p-q|\sqrt{1+p^2}\end{aligned}$$

(ii) The point T is the point of intersection of the tangents at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$. The coordinates of T are $(a(p+q), apq)$.

Similarly, the point P' is the point of intersection of the tangents at $R(2ar, ar^2)$ and $P(2ap, ap^2)$. The coordinates of P' are $(a(p+r), apr)$.

Using the distance formula noting that $a > 0$ and $1+p^2 > 0$

$$\begin{aligned}P'T^2 &= (a(p+r) - a(p+q))^2 + (apr - apq)^2\\ &= a^2(r-q)^2 + a^2p^2(r-q)^2\\ &= a^2(r-q)^2(1+p^2)\\ P'T &= a|q-r|\sqrt{1+p^2}\end{aligned}$$

(iii) Using the results in (i) and (iii), and interchanging the parameter p with q.

$$QT = a|p-q|\sqrt{1+q^2}$$

$$Q'T = a|p-r|\sqrt{1+q^2}$$

$$\begin{aligned}
\frac{P'T}{PT} + \frac{Q'T}{QT} &= \frac{a|q-r|\sqrt{1+p^2}}{a|p-q|\sqrt{1+p^2}} + \frac{a|p-r|\sqrt{1+q^2}}{a|p-q|\sqrt{1+q^2}} \\
&= \frac{|q-r|+|p-r|}{|p-q|}
\end{aligned}$$

Since R lies between P and Q on the parabola, then either $p < r < q$ or $q < r < p$.

If $p < r < q$ then $|q-r| = q-r$, $|p-r| = -(p-r)$ and $|p-q| = -(p-q)$ so

$$\begin{aligned}
\frac{|q-r|+|p-r|}{|p-q|} &= \frac{q-r-p+r}{q-p} \\
&= 1
\end{aligned}$$

If $q < r < p$ then $|q-r| = -(q-r)$, $|p-r| = p-r$ and $|p-q| = p-q$ so

$$\begin{aligned}
\frac{|q-r|+|p-r|}{|p-q|} &= \frac{-q+r+p-r}{p-q} \\
&= 1
\end{aligned}$$

Hence, in all cases

$$\frac{P'T}{PT} + \frac{Q'T}{QT} = 1$$

(c) From the sum of roots

$$\alpha + \beta + \gamma = -2$$

Using this result

$$\begin{aligned}
\alpha^2(\beta+\gamma) + \beta^2(\alpha+\gamma) + \gamma^2(\alpha+\beta) &= \alpha^2(-2-\alpha) + \beta^2(-2-\beta) + \gamma^2(-2-\gamma) \\
&= -(\alpha^3+\beta^3+\gamma^3) - 2(\alpha^2+\beta^2+\gamma^2)
\end{aligned}$$

Since α, β and γ are the roots of $P(x)$ then

$$\begin{aligned}
\alpha^3 + 2\alpha^2 - 3\alpha - 1 &= 0 \\
\beta^3 + 2\beta^2 - 3\beta - 1 &= 0 \\
\gamma^3 + 2\gamma^2 - 3\gamma - 1 &= 0 \\
-(\alpha^3+\beta^3+\gamma^3) - 2(\alpha^2+\beta^2+\gamma^2) &= -3(\alpha+\beta+\gamma) - 3 \\
&= 3 \\
\therefore \alpha^2(\beta+\gamma) + \beta^2(\alpha+\gamma) + \gamma^2(\alpha+\beta) &= 3
\end{aligned}$$

(d) For the $(k+1)$th application of Newton's method

$$\begin{aligned}
x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\
\frac{f(x_k)}{f'(x_k)} &= x_k - x_{k+1} \\
\sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)} &= \sum_{k=0}^{n} (x_k - x_{k+1}) \\
&= (x_0 - x_1) + (x_1 - x_2) + (x_2 - x_3) + ... + (x_{n-1} - x_n) + (x_n - x_{n+1}) \\
&= x_0 - x_{n+1} \quad \text{but } x_{n+1} = x_0 \\
&= 0
\end{aligned}$$

(e) Let X_m be the number of heads Matthew tosses and let X_j be the number of heads Jonathan tosses.

(i) Since Matthew and Jonathan toss the coins independently then the probability that both toss exactly k heads is the product of the probabilities that each of them toss k heads.

$$\begin{aligned}
P(X_m = k, X_j = k) &= P(X_m = k) \times P(X_j = k) \\
&= \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \times \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \\
&= \binom{n}{k}^2 \frac{1}{2^{2n}}
\end{aligned}$$

(ii) Computing the probability that Matthew tosses k heads and Jonathan tosses $n-k$ heads

$$\begin{aligned}
P(X_m = k, X_j = n-k) &= P(X_m = k) \times P(X_j = n-k) \\
&= \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \times \binom{n}{n-k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2}\right)^k \\
&= \binom{n}{k}^2 \frac{1}{2^{2n}} \quad \text{since } \binom{n}{k} = \binom{n}{n-k}
\end{aligned}$$

Hence, the probability that Matthew tosses k heads and Jonathan tosses $n-k$ heads is equal to the probability that Matthew tosses k heads and Jonathan tosses k heads computed in part (i).

Alternative solution:

The probability that Jonathan tosses $n-k$ heads is equal to the probability that Jonathan tosses $n-k$ tails by symmetry of the coin toss. Tossing $n-k$ tails is equivalent to tossing k heads. Hence, the probability that Matthew tosses k heads and Jonathan tosses $n-k$ heads is equal to the probability that both of them each toss k heads, as computed in part (i).

(iii) From part (ii), it was shown that

$$P(X_m = k, X_j = k) = P(X_m = k, X_j = n - k)$$

For Matthew and Jonathan to toss the same number of heads then

$$\begin{aligned}
&P(X_m = X_j) \\
&= P(X_m = 0, X_j = 0) + P(X_m = 1, X_j = 1) + ... + P(X_m = n, X_j = n) \\
&= P(X_m = 0, X_j = n) + P(X_m = 1, X_j = n - 1) + ... + P(X_m = n, X_j = 0) \\
&= P(X_m + X_j = n)
\end{aligned}$$

If Matthew tosses k heads and Jonathan tosses $n - k$ heads, then out of the entire $2n$ tosses there are n heads. This is true for all possible integer values of $0 \leq k \leq n$. The probability that in $2n$ tosses there are n heads is given by

$$\begin{aligned}
P(X_m + X_j = n) &= \binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n \\
\therefore P(X_m = X_j) &= \binom{2n}{n}\frac{1}{2^{2n}}
\end{aligned}$$

(iv) Using the result in part (i)

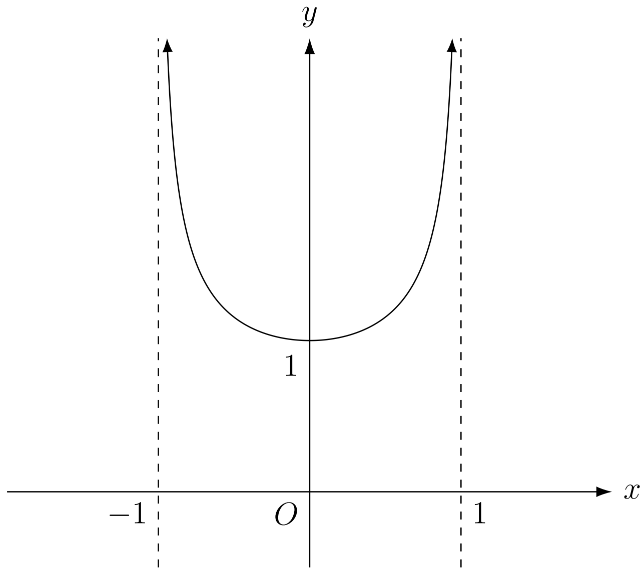
$$\begin{aligned}
P(X_m = X_j) &= P(X_m = 0, X_j = 0) + P(X_m = 1, X_j = 1) + ... + P(X_m = n, X_j = n) \\
&= \binom{n}{0}^2\frac{1}{2^{2n}} + \binom{n}{1}^2\frac{1}{2^{2n}} + ... + \binom{n}{n}^2\frac{1}{2^{2n}} \\
&= \sum_{k=0}^{n}\binom{n}{k}^2\frac{1}{2^{2n}}
\end{aligned}$$

Equating this to the result in (iii)

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}^2\frac{1}{2^{2n}} &= \binom{2n}{n}\frac{1}{2^{2n}} \\
\therefore \sum_{k=0}^{n}\binom{n}{k}^2 &= \binom{2n}{n}
\end{aligned}$$

Question 12

(a) The domain of the curve is $-1 < x < 1$ with asymptotes at $x = -1$ and $x = 1$. The function is also even so there is symmetry about the y-axis. Using this and the given minimum point of $(0, 1)$, the following sketch can be obtained.



(b) (i) The domain is the solution to $-1 \le 2x^2 - 1 \le 1$

$$
\begin{aligned}
0 &\le 2x^2 \le 2 \\
0 &\le x^2 \le 1 \\
\therefore \text{Domain is } &-1 \le x \le 1
\end{aligned}
$$

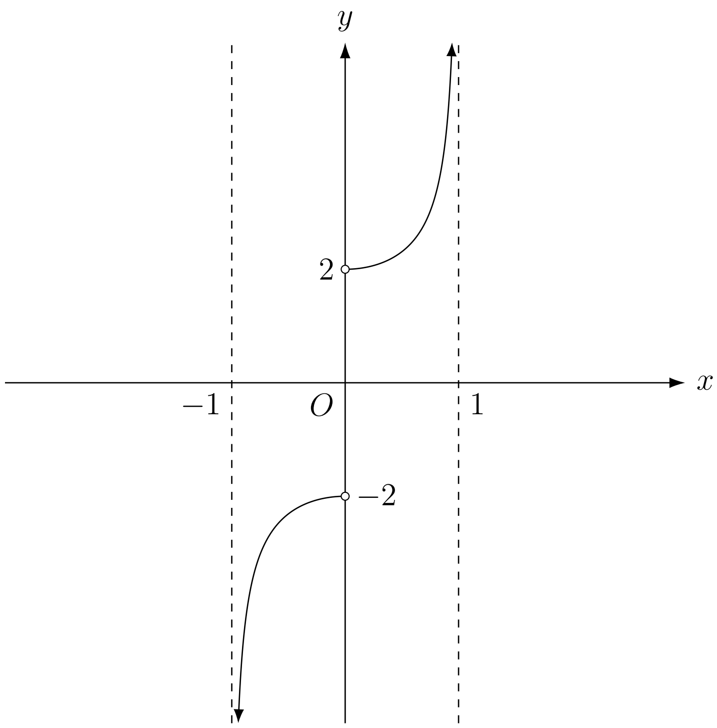
(ii) Finding the derivative of $f(x) = \sin^{-1}(2x^2 - 1)$

$$
\begin{aligned}
f'(x) &= \frac{4x}{\sqrt{1 - (2x^2 - 1)^2}} \\
&= \frac{4x}{\sqrt{1 - 4x^4 + 4x^2 - 1}} \\
&= \frac{4x}{\sqrt{4x^2(1 - x^2)}} \\
&= \frac{2x}{|x|\sqrt{1 - x^2}}
\end{aligned}
$$

Note that $f'(x)$ is not defined at $x = 0$ and so

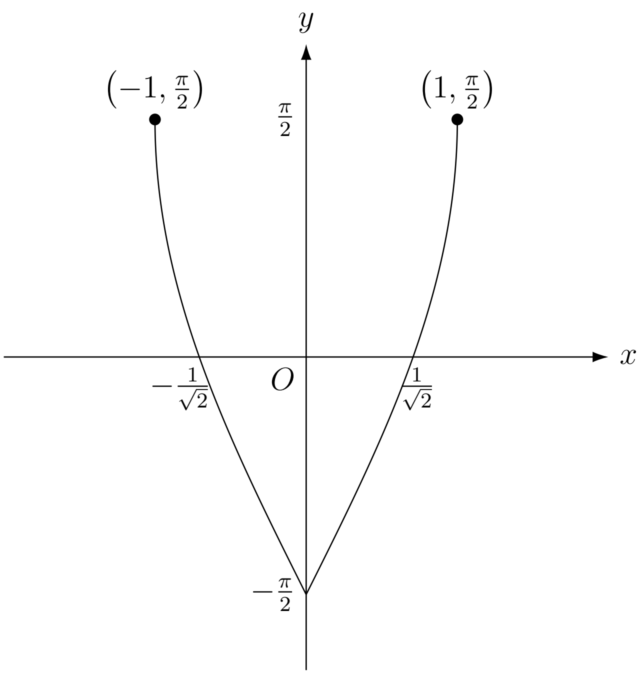
$$
f'(x) = \begin{cases} \dfrac{2}{\sqrt{1 - x^2}}, & \text{for } x > 0 \\[2ex] -\dfrac{2}{\sqrt{1 - x^2}}, & \text{for } x < 0 \end{cases}
$$

Using the result in (a), the following sketch of $y = f'(x)$ can be obtained



(iii) From $f'(x)$ obtained in (ii), $f(x)$ is increasing in the domain $0 < x < 1$ and decreasing in the domain $-1 < x < 0$. Also, observe that $f(x)$ is an even function. Hence, the minimum value of $f(x)$ occurs at $f(0) = -\frac{\pi}{2}$. The maximum value of $f(x)$ occurs at $f(1)$ or $f(-1)$ which is $\frac{\pi}{2}$. Hence, the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

(iv) Using all the information above to sketch $y = f(x)$.



(c) (i) Let v be the velocity of the particle at time t. When the particle moves from its initial displacement at the origin

$$\begin{aligned}\frac{d}{dx}\left(\frac{v^2}{2}\right) &= -x \\ \frac{v^2}{2} &= -\frac{x^2}{2} + c \quad \text{when } x = 0, v = v_0 \Rightarrow c = \frac{v_0^2}{2} \\ v^2 &= v_0^2 - x^2\end{aligned}$$

If $|v_0| < 2$ or equivalently $v_0^2 < 4$ then

$$\begin{aligned}4 - x^2 &> v^2 \quad \text{but } v^2 \geq 0 \\ \Rightarrow 4 - x^2 &> 0 \\ \Rightarrow -2 < x &< 2\end{aligned}$$

Since the displacement equation satisfies $x < 2$ then the acceleration equation will always be $\dfrac{d^2x}{dt^2} = -x$. Hence, provided $|v_0| < 2$ the particle will exhibit simple harmonic motion.

(ii) From the equation $v^2 = v_0^2 - x^2$, the extreme points occur when $v = 0$

$$\begin{aligned}v_0^2 - x^2 &= 0 \\ x^2 &= v_0^2 \\ x &= \pm v_0\end{aligned}$$

The particle exhibits simple harmonic motion and from the acceleration equation, it can be seen that the centre is the origin. Hence, the amplitude of the particle's motion is $|v_0|$.

(iii) If $v_0 = 2$ then the velocity equation becomes $v^2 = 4 - x^2$ for $x < 2$. However, since the particle will reach the displacement $x = 2$ after moving initially from the origin, the other piece of the acceleration equation must be considered.

$$\begin{aligned}\frac{d}{dx}\left(\frac{v^2}{2}\right) &= 4 - x \\ \frac{v^2}{2} &= 4x - \frac{x^2}{2} + c \quad \text{when } x \to 2^-, v \to 0^+ \Rightarrow c = -6 \\ v^2 &= -12 + 8x - x^2 \\ &= -(x-6)(x-2)\end{aligned}$$

The extreme points of the displacement occur when $v = 0$, which are $x = 2$ and $x = 6$. Thus, the maximum displacement of the particle is $x = 6$.

(iv) The particle is initially at the origin before moving in simple harmonic motion (described by $\ddot{x} = -x$) up to $x = 2$. Once it reaches $x = 2$, it transitions to a different simple harmonic motion (described by $\ddot{x} = 4 - x$). From that point, it then continually oscillates between $x = 2$ and $x = 6$.

Question 13

(a) (i) For $y = a^x$

$$\begin{aligned}\frac{dy}{dx} &= a^x \ln a \\ m &= a^p \ln a\end{aligned}$$

For $y = \log_a x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x \ln a} \\ m &= \frac{1}{p \ln a}\end{aligned}$$

Multiplying the two expressions for m

$$\begin{aligned}m^2 &= \frac{a^p \ln a}{p \ln a} \\ &= \frac{a^p}{p} \\ &= 1\end{aligned}$$

since (p, p) lies on $y = a^x$ so $p = a^p$

Alternative solution:

For a function $y = f(x)$ at (p, p)

$$\begin{aligned}\frac{dy}{dx} &= f'(x) \\ m &= f'(p)\end{aligned}$$

For its inverse $y = f^{-1}(x)$ or equivalently $x = f(y)$ at (p, p)

$$\begin{aligned}\frac{dx}{dy} &= f'(y) \\ \frac{dy}{dx} &= \frac{1}{f'(y)} \\ m &= \frac{1}{f'(p)}\end{aligned}$$

Multiplying the two expressions for m gives the result $m^2 = 1$

(ii) From part (i), $m^2 = 1$ gives two cases for m

When $m = 1$, then $\dfrac{1}{p\ln a} = 1$ and $a^p \ln a = 1 \Rightarrow a^p = p$

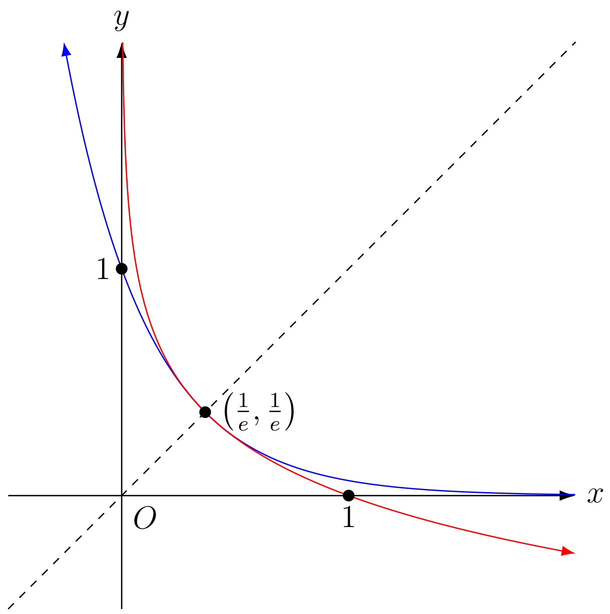
$$\begin{aligned}
\frac{1}{p\ln a} &= 1 \\
\ln a^p &= 1 \quad \text{but } p = a^p \\
\ln p &= 1 \\
\Rightarrow p &= e \quad \text{substitute into } p = a^p \\
a^e &= e \\
\Rightarrow a &= e^{\frac{1}{e}}
\end{aligned}$$

But $0 < a < 1$ so this cannot be the value for a, so consider the other case.
When $m = -1$, then $\dfrac{1}{p\ln a} = -1$ and $a^p \ln a = -1 \Rightarrow a^p = p$

$$\begin{aligned}
\frac{1}{p\ln a} &= -1 \\
\ln a^p &= -1 \quad \text{but } a^p = p \\
\ln p &= -1 \\
\Rightarrow p &= \frac{1}{e} \quad \text{substitute into } p = a^p \\
a^{\frac{1}{e}} &= \frac{1}{e} \\
\Rightarrow a &= e^{-e}
\end{aligned}$$

Here a satisfies the condition $0 < a < 1$.

(iii) Since $0 < a < 1$, the exponential curve is decreasing as x increases. Hence, sketching the graphs of $y = e^{-ex}$ (blue) and $y = \log_{e^{-e}} x$ (red)



(b) (i) Using the given displacement equations

$$\begin{aligned} t &= \frac{x}{V\cos\theta} \quad \text{substitute into } y \\ y &= V\sin\theta \times \frac{x}{V\cos\theta} - \frac{g}{2} \times \frac{x^2}{V^2\cos^2\theta} \\ &= -\frac{gx^2}{2V^2\cos^2\theta} + x\tan\theta \end{aligned}$$

The projectile just clears a wall on the point (d_1, d_2). This point satisfies the Cartesian equation of the projectile motion so

$$\begin{aligned} d_2 &= -\frac{gd_1^2}{2V^2\cos^2\theta} + d_1\tan\theta \\ -\frac{g}{2V^2\cos^2\theta} &= \frac{d_2 - d_1\tan\theta}{d_1^2} \end{aligned}$$

The projectile just clears another wall on the point (d_2, d_1). This point satisfies the Cartesian equation of the projectile motion so

$$\begin{aligned} d_1 &= -\frac{gd_2^2}{2V^2\cos^2\theta} + d_2\tan\theta \\ -\frac{g}{2V^2\cos^2\theta} &= \frac{d_1 - d_2\tan\theta}{d_2^2} \end{aligned}$$

Equating the two expressions

$$\begin{aligned} \frac{d_2 - d_1\tan\theta}{d_1^2} &= \frac{d_1 - d_2\tan\theta}{d_2^2} \\ d_2^3 - d_1d_2^2\tan\theta &= d_1^3 - d_1^2d_2\tan\theta \\ \tan\theta(d_1^2d_2 - d_1d_2^2) &= d_1^3 - d_2^3 \\ d_1d_2\tan\theta(d_1 - d_2) &= (d_1 - d_2)(d_1^2 + d_1d_2 + d_2^2) \\ \therefore \tan\theta &= \frac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2} \quad \text{where } d_1 \neq d_2 \end{aligned}$$

(ii) Suppose that the particle clears the walls when $\tan\theta \leq 3$. From part (i), this implies that

$$\begin{aligned} \frac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2} &\leq 3 \\ d_1^2 + d_1d_2 + d_2^2 &\leq 3d_1d_2 \\ d_1^2 - 2d_1d_2 + d_2^2 &\leq 0 \\ (d_1 - d_2)^2 &\leq 0 \end{aligned}$$

Since $d_1 \neq d_2$, there are no real values of d_1 and d_2 which could satisfy this inequality. Hence, the initial assumption must be false. In other words, the particle cannot clear the walls when the angle of projection θ is low enough such that $\tan\theta \leq 3$.

(c) (i) Since n is a positive integer

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \ldots + \binom{n}{n}\frac{1}{n^n} \\ &\geq \binom{n}{0} + \binom{n}{1}\frac{1}{n} \\ &= 1 + n \times \frac{1}{n} \\ &= 2\end{aligned}$$

(ii) When $n = 1$, LHS $= 1!$ and RHS $= 1^{2\times 1-1}$. The statement is true for $n = 1$ because both sides equal 1.

Assume the statement is true for some $n = k$

$$(2k-1)! \leq k^{2k-1}$$

Required to prove the statement is true for $n = k+1$

$$(2k+1)! \leq (k+1)^{2k+1}$$

$$\begin{aligned}\text{LHS} &= (2k+1)! \\ &= (2k+1)(2k)(2k-1)! \\ &\leq (2k+1)(2k)k^{2k-1} \quad \text{by assumption} \\ &= 2(2k+1)k^{2k} \\ &= 2(2k+1)\left(\frac{k}{k+1}\right)^{2k}(k+1)^{2k} \\ &\leq 2(2k+1)\left(\frac{1}{2}\right)^2(k+1)^{2k} \quad \text{using the result in (i)*} \\ &= \frac{1}{2}(2k+1)(k+1)^{2k} \\ &\leq \frac{1}{2}(2k+2)(k+1)^{2k} \\ &= (k+1)^{2k+1} \\ &= \text{RHS}\end{aligned}$$

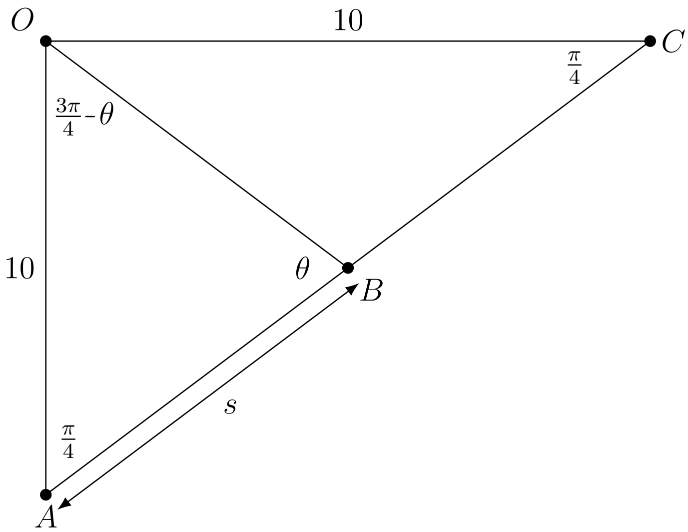
Hence, the statement is true by mathematical induction

*Rearranging the result in (i):

$$\begin{aligned} &\left(1+\frac{1}{n}\right)^n \geq 2 \\ \Rightarrow &\left(\frac{n+1}{n}\right)^n \geq 2 \\ \Rightarrow &\left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}\end{aligned}$$

Question 14

(a) (i) Viewing from above



Since the angle of elevation to the top of the monument for both visitor centres A and C is $\frac{\pi}{4}$, the associated right-angled triangles are isosceles so $OC = 10$ and $OA = 10$.

This means that $\triangle AOC$ is right-angled isosceles, so $\angle OAB = \frac{\pi}{4}$. Since $\angle ABO = \theta$ then by angle sum of triangle $\angle AOB = \frac{3\pi}{4} - \theta$.

Using the sine rule on $\triangle AOB$

$$\begin{aligned}
\frac{AB}{\sin \angle AOB} &= \frac{OA}{\sin \angle ABO} \\
\frac{s}{\sin\left(\frac{3\pi}{4} - \theta\right)} &= \frac{10}{\sin\theta} \\
\therefore s &= \frac{10\sin\left(\frac{3\pi}{4} - \theta\right)}{\sin\theta}
\end{aligned}$$

(ii) Using the result in (i)

$$\begin{aligned}
s &= \frac{10\sin\left(\frac{3\pi}{4} - \theta\right)}{\sin\theta} \\
\frac{ds}{d\theta} &= \frac{10(-\cos\left(\frac{3\pi}{4} - \theta\right)\sin\theta - \sin\left(\frac{3\pi}{4} - \theta\right)\cos\theta)}{\sin^2\theta} \\
&= -\frac{10\sin\left(\frac{3\pi}{4} - \theta + \theta\right)}{\sin^2\theta} \\
&= -\frac{10\sin\frac{3\pi}{4}}{\sin^2\theta} \\
&= -\frac{10}{\sqrt{2}\sin^2\theta}
\end{aligned}$$

Since the visitor cycles from A to C at 5 metres per second then $\dfrac{ds}{dt} = 5$, so

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d\theta}{ds} \times \frac{ds}{dt} \\ &= -\frac{\sqrt{2}\sin^2\theta}{10} \times 5 \\ &= -\frac{\sin^2\theta}{\sqrt{2}}\end{aligned}$$

(iii) From the visitor's perspective

$$\begin{aligned}\tan\beta &= \frac{10}{OB} \\ OB &= 10\cot\beta\end{aligned}$$

Using the sine rule on $\triangle AOB$

$$\begin{aligned}\frac{OB}{\sin\angle OAB} &= \frac{10}{\sin\theta} \\ \frac{10\cot\beta}{\sin\frac{\pi}{4}} &= \frac{10}{\sin\theta} \\ \tan\beta &= \sqrt{2}\sin\theta \\ \beta &= \tan^{-1}(\sqrt{2}\sin\theta) \\ \frac{d\beta}{d\theta} &= \frac{\sqrt{2}\cos\theta}{1+2\sin^2\theta}\end{aligned}$$

Using the result in (ii)

$$\begin{aligned}\frac{d\beta}{dt} &= \frac{d\beta}{d\theta} \times \frac{d\theta}{dt} \\ &= \frac{\sqrt{2}\cos\theta}{1+2\sin^2\theta} \times -\frac{\sin^2\theta}{\sqrt{2}} \\ &= -\frac{\cos\theta\sin^2\theta}{1+2\sin^2\theta} \times \frac{\operatorname{cosec}^2\theta}{\operatorname{cosec}^2\theta} \\ &= -\frac{\cos\theta}{2+\operatorname{cosec}^2\theta}\end{aligned}$$

(iv) When the visitor moves from A to B, $\angle AOB$ increases from 0 to $\frac{\pi}{2}$. This means that $\angle ABO$ or θ decreases from $\frac{3\pi}{4}$ to $\frac{\pi}{4}$ over time.

When $\theta = \dfrac{3\pi}{4}$ then $\dfrac{d\beta}{dt} = \dfrac{1}{4\sqrt{2}}$ and when $\theta = \dfrac{\pi}{4}$ then $\dfrac{d\beta}{dt} = -\dfrac{1}{4\sqrt{2}}$.

This means there is a maximum point where $\frac{d\beta}{dt} = 0$. This occurs when $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$. Substitute this into the relationship $\beta = \tan^{-1}(\sqrt{2}\sin\theta)$ and the maximum angle of elevation the visitor can have is $\tan^{-1}\sqrt{2}$.

(b) (i)

$$\begin{aligned} \angle AMC &= \angle ABC \quad \text{(angles in the same segment in } \mathscr{C}_1\text{)} \\ \angle KMN &= \angle ABC \quad \text{(angles in the same segment in } \mathscr{C}_2\text{)} \end{aligned}$$

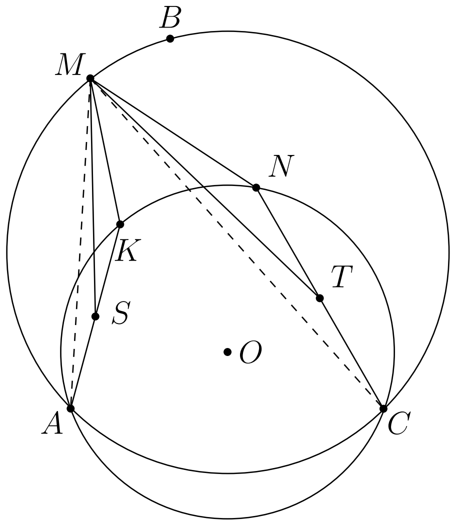
But

$$\begin{aligned} \angle AMC &= \angle AMK + \angle KMC \\ \angle KMN &= \angle KMC + \angle CMN \\ \Rightarrow \angle AMK &= \angle CMN \end{aligned}$$

Also

$$\begin{aligned} \angle MAK &= \angle MCN \quad \text{(angles in the same segment in } \mathscr{C}_3\text{)} \\ \therefore \triangle MKA \,|||\, \triangle MNC & \quad \text{(equiangular)} \end{aligned}$$

(ii) The points S and T are shown in the diagram below.



Since $\triangle MKA$ and $\triangle MNC$ are similar

$$\frac{NC}{KA} = \frac{MN}{MK} \quad \text{(corresponding sides of similar triangles in proportion)}$$

But $NT = \frac{1}{2}NC$ and $KS = \frac{1}{2}KA \Rightarrow \dfrac{NC}{KA} = \dfrac{NT}{KS}$ so

$$\frac{MN}{MK} = \frac{NT}{KS}$$

$$\begin{aligned} \text{Also} \quad \angle MNT &= \angle MKS \quad \text{(corresponding angles of similar triangles)} \\ \therefore \triangle MNT \,|||\, \triangle MKS & \end{aligned}$$

as corresponding sides are in proportion and included angles are equal. This means that

$$\begin{aligned} \angle MSB &= \angle MTB \quad \text{(corresponding angles of similar triangles)} \\ \therefore MSTB & \text{ is a cyclic quadrilateral (angles in same segment)} \end{aligned}$$

(iii) Since S bisects chord KA and similarly T bisects chord NC

$$OS \perp KA \quad \text{and} \quad OT \perp NC$$

This is because the line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Since $\angle BSO = \frac{\pi}{2}$ and $\angle BTO = \frac{\pi}{2}$ then $SBTO$ is a cyclic quadrilateral as opposite angles $\angle BSO$ and $\angle BTO$ are supplementary.

(iv) Since S, T and B are three non-collinear points, they uniquely define a circle. In part (ii), it was shown that M lies on this circle and in part (iii), it was shown that O also lies on this circle. This means that S, T, B, M and O are concyclic, so

$$\begin{aligned} & \angle OSB = \angle OMB \quad \text{(angles in the same segment)} \\ \text{but} \quad & \angle OSB = \frac{\pi}{2} \quad (OS \text{ bisects chord } KA \text{ as explained in part (iii))} \\ & \therefore OM \perp BM \end{aligned}$$