2024 **BORED OF STUDIES TRIAL EXAMINATION**

6th October

Mathematics Extension 1

General instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–10)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 What is the value of $\int_0^{\frac{\pi}{4}} (2\cos^2 x + 1)(4\cos^2 x - 5)\, dx$?

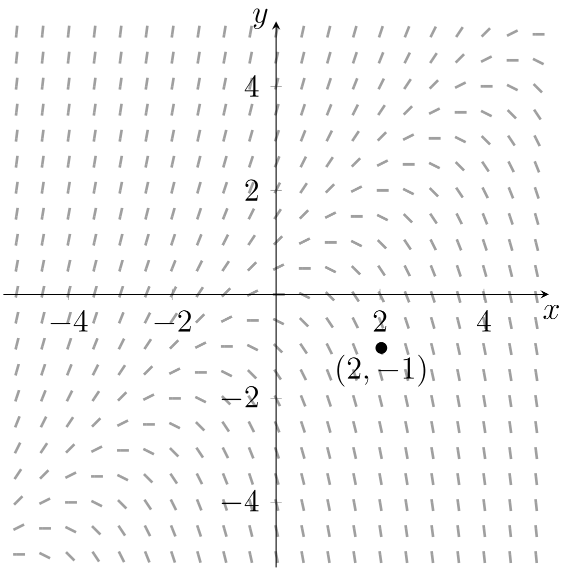
(A) $-\dfrac{5\pi}{4}$ (B) $-\dfrac{3\pi}{4}$ (C) $-\dfrac{1+5\pi}{4}$ (D) $\dfrac{2-5\pi}{4}$

2 Which of the following is NOT equivalent to $\sin(2\cos^{-1} x)$?

(A) $\sin(2\sin^{-1} x)$ (B) $\sin(\pi + 2\sin^{-1}(-x))$

(C) $2x\sqrt{1-x^2}$ (D) $2\sqrt{x^2(1-x^2)}$

3 The following is the direction field of a function $y = f(x)$ for the domain $x \in [-5, 5]$.



Suppose that $f(2) = -1$. Which of the following statements is NOT true about $y = f(x)$?

(A) $f'(2) < 0$ (B) $f''(2) < 0$ (C) $f'(-2) < 0$ (D) $f''(-2) < 0$

4 The displacement vector of a particle at time t is given by $\underset{\sim}{r} = 4\sin 2t\,\underset{\sim}{i} + 3\cos 2t\,\underset{\sim}{j}$.

What is the maximum magnitude of the particle's acceleration?

(A) 8 (B) 16 (C) 20 (D) 28

5 The gradient of the normal to any point P on a particular curve is twice the gradient of the line joining P and the origin.

Which of the following differential equations does this curve satisfy?

(A) $\dfrac{dy}{dx} + \dfrac{x}{2y} = 0$ (B) $\dfrac{dy}{dx} + \dfrac{2x}{y} = 0$

(C) $\dfrac{dy}{dx} - \dfrac{x}{2y} = 0$ (D) $\dfrac{dy}{dx} - \dfrac{2x}{y} = 0$

6 Let $P(x)$ be a quartic polynomial. Suppose that $P'(x) = 0$ has three distinct real solutions x_1, x_2 and x_3, where $x_1 < x_2 < x_3$.

Which of the following inequalities CANNOT be true for $P(x)$?

(A) $P(x_1) < P(x_2) < P(x_3)$ (B) $P(x_1) < P(x_3) < P(x_2)$

(C) $P(x_3) < P(x_1) < P(x_2)$ (D) $P(x_2) < P(x_3) < P(x_1)$

7 Let p be the expected proportion of n students from a school that achieved a band E4 in Mathematics Extension 1. The probability that up to 30% of the school's cohort scores a band E4 in Mathematics Extension 1 is estimated to be 40%.

The following approximate probabilities are provided for $Z \sim N(0, 1)$

$$P(Z < 0.53) \approx 0.7 \quad \text{and} \quad P(Z < 0.26) \approx 0.6.$$

Which of the following values of n and p best support this estimate?

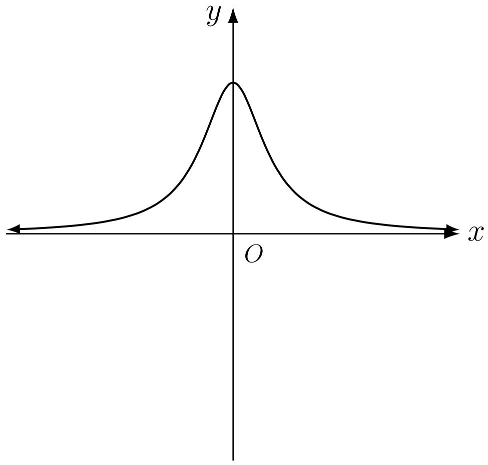
(A) $n = 64$ and $p = 0.315$ (B) $n = 79$ and $p = 0.328$

(C) $n = 70$ and $p = 0.286$ (D) $n = 66$ and $p = 0.271$

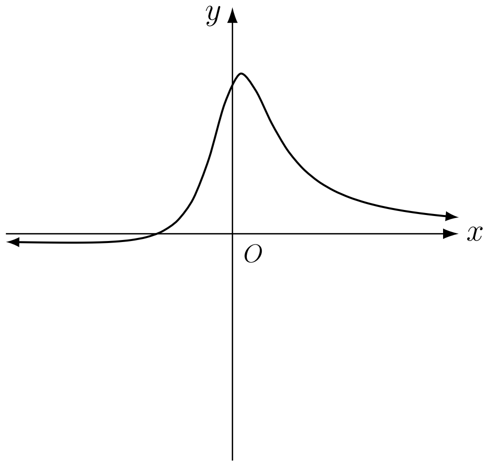
8 A quartic polynomial is divided by $A(x)$ with a remainder of $R(x)$. $A(x)$ is a monic quadratic polynomial with no real roots and $R(x)$ is such that $R(0) = 4$ and $R(1) = 4$.

Which of the following best represents the graph of $y = \dfrac{R(x)}{A(x)}$?

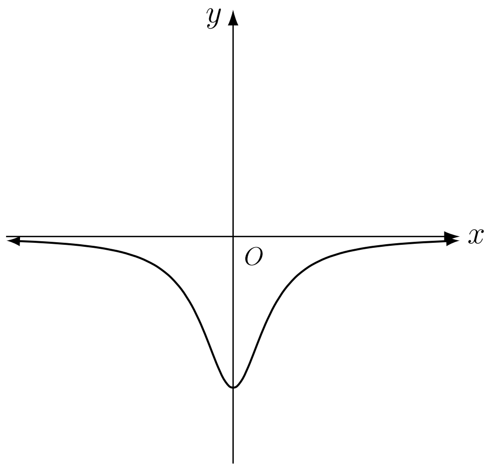
(A)



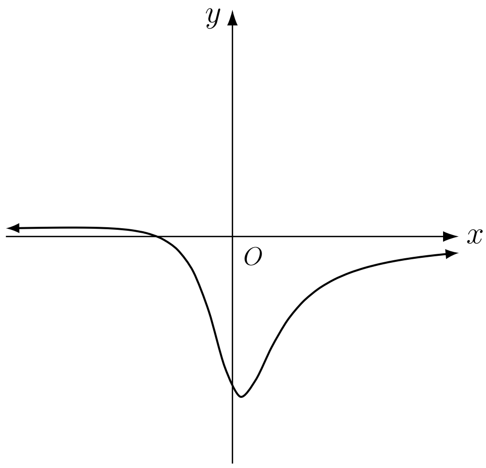
(B)



(C)



(D)



9 A bag contains 4 red, 7 blue, 10 yellow, 12 black and 15 white balls. What is the minimum number of balls that must be taken out of the bag at random, such that there will always be 9 balls of the same colour?

(A) 25 (B) 36 (C) 44 (D) 48

10 What is the range of $y = \sec\left(\tan^{-1}\dfrac{x}{x-1}\right)$?

(A) $y \leq -1$ or $y > \sqrt{2}$ (B) $y \neq \sqrt{2}$

(C) $y \geq 1$ (D) $y \neq 1$

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Let $f(x) = \tan^{-1}\dfrac{x}{x+1} + \tan^{-1}\dfrac{1}{2x+1}$.

(i) Find $f'(x)$. **2**

(ii) Hence, sketch $y = f(x)$. **2**

(b) Suppose that the solutions to the inequality $\dfrac{3x^2-1}{2x^2+x} > k$ are given by $x \in \left(-\frac{1}{2}, 0\right)$. Find the range of values of k. **2**

(c) A monic cubic polynomial $P(x)$ has the roots α, β and γ. Suppose that **3**

$$P'(\alpha) = 0, \quad P'(\beta) = 0, \quad P'(\gamma) = 25,$$
$$\alpha + \beta + \gamma = -1, \text{ and } \alpha\beta\gamma = 12.$$

Find the roots of $P(x)$.

(d) Let A be the area bounded by the curve $y = \sin^{-1} x$, the line $y = \dfrac{\pi}{4}$ and the y-axis.

(i) Show that $A = 1 - \dfrac{1}{\sqrt{2}}$. **1**

(ii) The area bounded by the curves $y = \sin^{-1} x$, $y = \cos^{-1} x$ and the y-axis is rotated about the x-axis to form a solid of revolution. Use the result of A in part (i) to find the volume of this solid. **3**

(e) Suppose that $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ with $\alpha + \beta + \gamma = \dfrac{\pi}{2}$. Show that **2**

$$\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} + \frac{\cos(\beta+\gamma)}{\cos\beta\cos\gamma} + \frac{\cos(\alpha+\gamma)}{\cos\alpha\cos\gamma} = 2.$$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) A beaker containing a chemical is partly submerged in a water bath. Over time, the chemical cools down whilst the water bath warms up. The chemical and the water bath have initial temperatures of 160°C and 20°C respectively. **3**

Let T_c and T_w be the temperatures of the chemical and water bath respectively at time t. The rate of change of the temperatures is given by

$$\frac{dT_c}{dt} = -k(T_c - T_w)$$
$$\frac{dT_w}{dt} = \frac{3}{4}k(T_c - T_w)$$

where k is a positive constant.

Show that the chemical will cool down towards a temperature of 80°C over time.

(b) Prove by mathematical induction that $(2n+1)7^n - 1$ is divisible by 4 for positive integer n. **3**

(c) Show that for any constant $0 < \theta < \frac{\pi}{2}$, **3**

$$\int_0^1 \frac{\cos\theta}{x^2 - 2x\sin\theta + 1}\,dx = \frac{\pi}{4} + \frac{\theta}{2}.$$

(d) A particle has the position $P(x_t, y_t)$ on the x-y plane at time t. Let d be the shortest distance between the particle and the line ℓ, which has the Cartesian equation $3x - 4y + 4 = 0$.

(i) Let Y be the y-intercept of the line ℓ and Q be a point on the line ℓ where $\overrightarrow{QY} \cdot \overrightarrow{QP} = 0$. By considering the projection of $\overrightarrow{YP}$ onto $\overrightarrow{YQ}$, show that **3**

$$d^2 = \frac{(3x_t - 4y_t + 4)^2}{25}.$$

(ii) Suppose that the particle itself moves according to the displacement vector **2**

$$\underset{\sim}{r} = t\underset{\sim}{i} + \frac{t^2}{4}\underset{\sim}{j}.$$

Sketch the graph of d over time, showing any intercepts and turning points.

(iii) How many times is the particle exactly 1 unit away (in terms of shortest distance) from the line ℓ, throughout its movement? **1**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

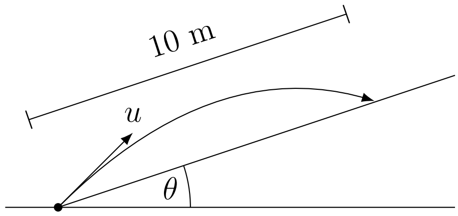
(a) A particle is projected from the ground at an angle of 45° to the horizontal with an initial speed of u ms^{-1}. Let the point of projection be the origin. It can be shown that the velocity vector of the particle relative to the ground is

$$\dot{\underset{\sim}{r}} = \frac{u}{\sqrt{2}}\underset{\sim}{i} + \left(\frac{u}{\sqrt{2}} - gt\right)\underset{\sim}{j} \qquad \textbf{(Do NOT prove this)}$$

where g is the acceleration due to gravity in ms^{-2}.

(i) Show that the Cartesian equation of the particle's path is $y = x - \dfrac{gx^2}{u^2}$. **2**

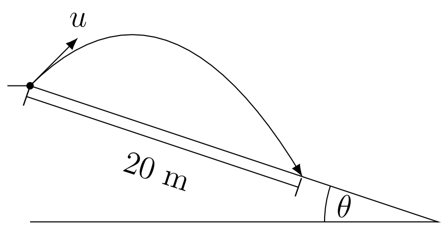
(ii) Suppose that there is an uphill slope with an angle of θ to the horizontal. **1**



When the particle is launched, it lands at a distance of 10 metres up along the slope as shown in the diagram above. Show that

$$\tan\theta = 1 - \frac{10g\cos\theta}{u^2}.$$

(iii) Now suppose instead that there is a downhill slope also with an angle of θ to the horizontal. **1**



When the particle is launched, it lands at a distance of 20 metres down along the slope as shown in the diagram above. Show that

$$\tan\theta = \frac{20g\cos\theta}{u^2} - 1.$$

(iv) Hence, show that $\tan\theta = \dfrac{1}{3}$. **1**

Question 13 continues on page 8

Question 13 (continued)

(b) Use the substitution $t = \tan\left(\frac{\pi}{4} - x\right)$ to show (without induction) that for some integer $n \geq 2$ **3**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x + \cos x)^{2n}} = \frac{\binom{n-1}{0}}{1 \times 2^{n-1}} + \frac{\binom{n-1}{1}}{3 \times 2^{n-1}} + \frac{\binom{n-1}{2}}{5 \times 2^{n-1}} + \cdots + \frac{\binom{n-1}{n-1}}{(2n-1) \times 2^{n-1}}.$$

(c) Let $P(x) = 8x^3 + 4x^2 - 6x - 2$. It can be shown that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \qquad \text{and} \qquad \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

(Do NOT prove these)

(i) Use the above results to find the solutions to **4**

$$\cos\theta P(\cos\theta) = 0$$

for $0 \leq \theta \leq 2\pi$.

(ii) Find the roots of $P(x)$ in the form $a + \sqrt{b}$ for rational values of a and b, given that $P(-1) = 0$. **2**

(iii) Hence, find the exact value of $\cos\dfrac{2\pi}{5}$. **1**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) On a street there are 20 identical vacant houses in a straight line. There are 16 interested tenants who apply to live in these vacant houses, but only 12 can be successful. The successful tenants are then given one of houses to live in at random.

(i) Suppose that each tenant has an equal chance of being successful in their application. How many ways can the 16 interested tenants be arranged amongst the houses, if they were successful? **1**

(ii) Find the probability that two particular tenants are successful and are also neighbours (i.e. living in adjacent houses). **2**

(b) Water from a contaminated lake is being transported into a treatment plant via a tank. Let t be the number of minutes that have passed since commencing the water transportation,

Suppose that the contaminated lake contains $e^{-0.2t}$ grams of radioactive chemicals per litre after t minutes. The tank initially contains 100 litres of pure water (i.e. initially has zero grams of radioactive chemicals).

Water from the lake flows into the tank at a rate of 200 litres per minute and is uniformly mixed. At the same time, water flows out of the tank into the treatment plant at a rate of 100 litres per minute.

Let m be the mass of radioactive chemicals inside the tank (in grams) after t minutes.

(i) By considering the mass of radioactive chemicals per litre in the tank, explain why **2**

$$\frac{dm}{dt} = 200e^{-0.2t} - \frac{m}{1+t}.$$

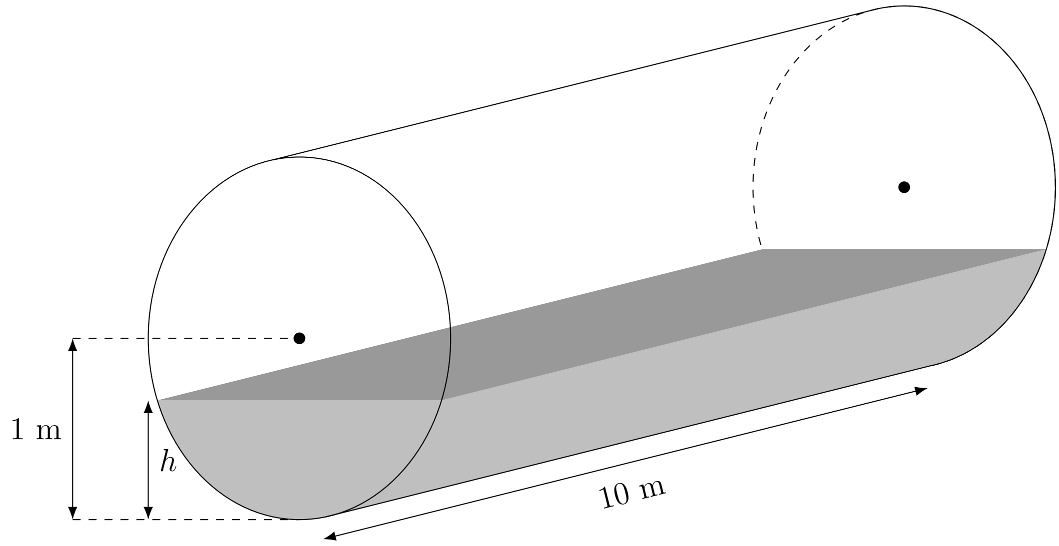
(ii) By considering $\frac{d}{dt}\left[m(1+t)\right]$, find the solution to the differential equation in part (i). **2**

You may assume without proof that $\int xe^{-kx}\,dx = -\frac{1}{k^2}e^{-kx}(1+kx) + C.$

(iii) Calculate the mass of the radioactive chemicals which have flowed *out* of the tank in the first 4 minutes. Provide your answer to the nearest gram. **2**

Question 14 continues on page 10

(c) A horizontal cylindrical enclosed tank has a radius of 1 metre and a length of 10 metres. The tank contains oil at a height of h metres at time t, where $h < 1$ as shown in the diagram below. **4**



Suppose that the oil is evaporating from the tank at a rate of 5 cubic metres per week. Let S be the area of the tank's cylindrical surface that is in contact with the oil after t weeks.

By first finding the volume of the oil after t weeks, show that

$$\frac{dS}{dt} = \frac{5}{h(h-2)} - 1.$$

(d) A fair coin is tossed n times to create a sequence of heads (H) and tails (T). **2**

Define a 'streak' to be a sequence of consecutive coin tosses that have the same outcome. For example, if the coin is tossed 10 times with the sequence of outcomes as $\{HTTTTHHHTH\}$, then the streaks are $\{H\}$, $\{TTTT\}$, $\{HHH\}$, $\{T\}$ and $\{H\}$, In this case, the 10 tosses gave rise to 5 streaks.

Let S be the random variable that represents the total number of streaks from n consecutive coin tosses,

Find $E(S)$ in terms of n.

End of paper

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2024 BORED OF STUDIES TRIAL EXAMINATION

Mathematics Extension 1

Solutions

Section I

Answers

1	D	**6**	A
2	D	**7**	A
3	C	**8**	A
4	B	**9**	B
5	A	**10**	C

Brief explanations

1 Given $\cos 2x = 2\cos^2 x - 1$ and $\cos 4x = 2\cos^2 2x - 1$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} (2\cos^2 x + 1)(4\cos^2 x - 5)\,dx &= \int_0^{\frac{\pi}{4}} (\cos 2x + 2)(2\cos 2x - 3)\,dx \\
&= \int_0^{\frac{\pi}{4}} (2\cos^2 2x + \cos 2x - 6)\,dx \\
&= \int_0^{\frac{\pi}{4}} (\cos 4x + \cos 2x - 5)\,dx \\
&= \left[\frac{\sin 4x}{4} + \frac{\sin 2x}{2} - 5x\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2} - \frac{5\pi}{4} \\
&= \frac{2 - 5\pi}{4}
\end{aligned}$$

Hence, the answer is (D).

2 Assessing each of the options

$$\begin{aligned}
\text{For option (A)} \quad \sin(2\sin^{-1} x) &= \sin\left(2\left(\frac{\pi}{2} - \cos^{-1} x\right)\right) \\
&= \sin(\pi - 2\cos^{-1} x) \\
&= \sin(2\cos^{-1} x)
\end{aligned}$$

$$\begin{aligned}
\text{For option (B)} \quad \sin(\pi + 2\sin^{-1}(-x)) &= \sin(\pi - 2\sin^{-1} x) \\
&= \sin(\pi - 2\sin^{-1} x) \\
&= \sin(2\sin^{-1} x) \\
&= \sin(2\cos^{-1} x) \quad \text{from option (A)}
\end{aligned}$$

For option (C), let $u = \cos^{-1} x \Rightarrow x = \cos u$ noting that $0 \le u \le \pi$.

$$\begin{aligned}\sin 2u &= 2\sin u \cos u\\ &= 2x\sqrt{1-x^2}\end{aligned}$$

For option (D), note that

$$\begin{aligned}2\sqrt{x^2(1-x^2)} &= 2|x|\sqrt{1-x^2}\\ &\neq 2x\sqrt{1-x^2}\end{aligned}$$

Hence, the answer is (D).

3 The curve looks concave down with a maximum turning point in $-2 < x < 2$. This means that $f'(2) < 0, f''(2) < 0$ and $f'(-2) > 0, f''(-2) < 0$.

Hence, the answer is (C).

4 Deriving the acceleration vector

$$\begin{aligned}\underset{\sim}{r} &= 4\sin 2t\,\underset{\sim}{i} + 3\cos 2t\,\underset{\sim}{j}\\ \dot{\underset{\sim}{r}} &= 8\cos 2t\,\underset{\sim}{i} - 6\sin 2t\,\underset{\sim}{j}\\ \ddot{\underset{\sim}{r}} &= -16\sin 2t\,\underset{\sim}{i} - 12\cos 2t\,\underset{\sim}{j}\\ |\ddot{\underset{\sim}{r}}| &= \sqrt{(-16\sin 2t)^2 + (-12\cos 2t)^2}\\ &= \sqrt{256\sin^2 2t + 144\cos^2 2t}\\ &= \sqrt{144 + 112\sin^2 2t}\end{aligned}$$

This is maximised when $\sin^2 2t = 1$, which gives $|\ddot{\underset{\sim}{r}}| = 16$.

Hence, the answer is (B).

5 For a point $P(x, y)$, the gradient of the normal is $-\dfrac{dx}{dy}$.

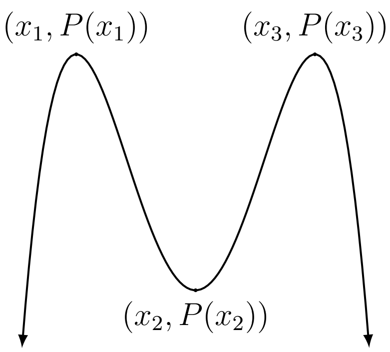
The gradient of the line between $P(x, y)$ and the origin is $\dfrac{y}{x}$.

The gradient of the normal to any point P on a particular curve is twice the gradient of the line joining P and the origin so

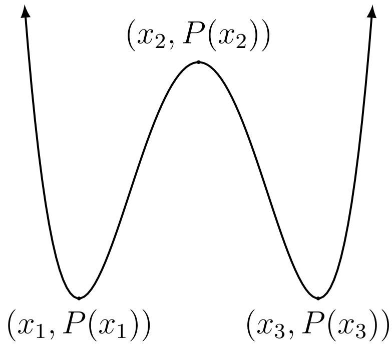
$$\begin{aligned}-\frac{dx}{dy} &= 2 \times \frac{y}{x}\\ \frac{dy}{dx} + \frac{x}{2y} &= 0\end{aligned}$$

Hence, the answer is (A).

6 Given there are distinct stationary points, the shape of $P(x)$ could be



It is possible for $P(x_2) < P(x_3) < P(x_1)$, which is option (D).



It is possible for $P(x_1) < P(x_3) < P(x_2)$ or $P(x_3) < P(x_1) < P(x_2)$ which are options (B) and (C) respectively.

However, it is not possible for $P(x_1) < P(x_2) < P(x_3)$. Hence, the answer is (A).

7 Let $\hat{p}$ be the sample proportion of students in the school that score band E4 in Mathematics Extension 1. The estimate is that

$$P\left(\hat{p} \leq 0.3\right) = 0.4$$

$$P\left(\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \leq \frac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}}\right) = 0.4 \quad \text{for large } n \text{ note that } \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$$P\left(Z \leq \frac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}}\right) \approx 0.4 \quad \text{where } Z \sim N(0,1)$$

Given $P(Z < 0.26) \approx 0.6$ then by symmetry of the normal curve

$$P(Z \geq 0.26) \approx 0.4$$
$$\Rightarrow P(Z \leq -0.26) \approx 0.4$$

Hence, the choices of n and p should have $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} \approx -0.26$.

For option (B), when $p = 0.328$ and $n = 79$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = -0.53$

For option (C), when $p = 0.286$ and $n = 70$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = 0.26$

For option (D), when $p = 0.271$ and $n = 66$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = 0.53$

For option (A), when $p = 0.315$ and $n = 64$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = -0.26$

Hence, the answer is (A).

8 Since $A(x)$ is monic quadratic with no real roots then $A(x)$ is positive definite.

Also, $R(x)$ is a remainder term it must be a linear polynomial or a constant. Since, $R(0) = R(1)$ then it must be a constant (namely equal to 4).

Since $R(x) = 4$ and $A(x) > 0$ for all real x then $\dfrac{A(x)}{R(x)} > 0$ for all real x.

Hence, the answer is (A).

9 First note that it is not possible to get 9 balls of the same colour that are red or blue.

By the pigeonhole principle, to guarantee 9 balls are the same colour, the red and blue need to be exhausted first and then 8 balls of yellow, black and white need to be drawn out.

This means that $4+7+8\times 3$, or equivalently, 35 balls need to be drawn first and the 36th ball will guarantee there will 9 balls of the same colour.

Hence, the answer is (B).

10 First derive the range of $\dfrac{x}{x-1}$. Noting that

$$\begin{aligned}\frac{x}{x-1} &= \frac{x-1+1}{x-1} \\ &= 1 + \frac{1}{x-1}\end{aligned}$$

The range of $\dfrac{x}{x-1}$ is simply $\dfrac{x}{x-1} \neq 1$. This implies that the range of $\tan^{-1}\dfrac{x}{x-1}$ is

$$-\frac{\pi}{2} < \tan^{-1}\frac{x}{x-1} < \frac{\pi}{2} \quad \text{but} \;\; \tan^{-1}\frac{x}{x-1} \neq \frac{\pi}{4}$$

This means that

$$\begin{aligned}0 < \cos\left(\tan^{-1}\frac{x}{x-1}\right) &\leq 1 \\ \sec\left(\tan^{-1}\frac{x}{x-1}\right) &\geq 1\end{aligned}$$

Hence, the answer is (C).

Remark: Although $\tan^{-1}\dfrac{x}{x-1} \neq \dfrac{\pi}{4}$, it is possible for $\tan^{-1}\dfrac{x}{x-1} = -\dfrac{\pi}{4}$. Since $\sec x$ is an even function then it possible to get $\sec\left(\dfrac{\pi}{4}\right)$ because

$$\sec\left(-\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right)$$

This means that this restriction has no impact on the final range.

Question 11

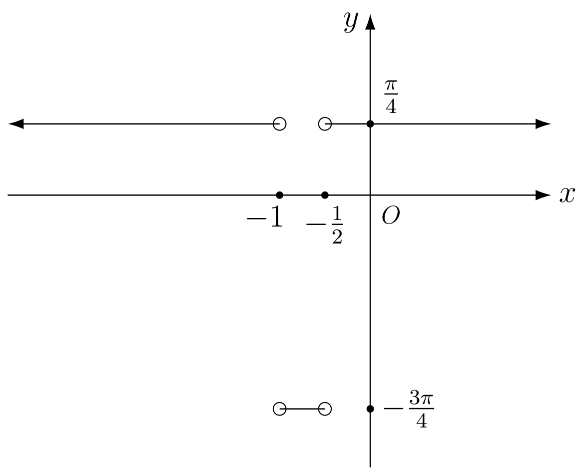
(a) (i) Using the quotient rule

$$\begin{aligned}
f(x) &= \tan^{-1}\frac{x}{x+1} + \tan^{-1}\frac{1}{2x+1} \\
&= \frac{1}{1+\frac{x^2}{(x+1)^2}} \times \frac{(1)(x+1)-x(1)}{(x+1)^2} + \frac{1}{1+\frac{1}{(2x+1)^2}} \times -\frac{2}{(2x+1)^2} \\
&= \frac{1}{(x+1)^2+x^2} - \frac{2}{(2x+1)^2+1} \\
&= \frac{(2x+1)^2+1-2(x+1)^2-2x^2}{[(x+1)^2+x^2][(2x+1)^2+1]} \\
&= \frac{4x^2+4x+1+1-2x^2-4x-2-2x^2}{[(x+1)^2+x^2][(2x+1)^2+1]} \\
&= 0
\end{aligned}$$

(ii) From part (i), $f'(x) = 0$ for all defined values of x. However, $f(x)$ is undefined at $x = -1, -\frac{1}{2}$. Consider the regions $x < -1, -1 < x < -\frac{1}{2}$ and $x > -\frac{1}{2}$. Testing different values by calculator in those regions, say $x = -2, -\frac{3}{4}$ and 0 gives

$$\begin{aligned}
f(-2) &= \frac{\pi}{4} \\
f\left(-\frac{3}{4}\right) &= -\frac{3\pi}{4} \\
f(0) &= \frac{\pi}{4}
\end{aligned}$$

This gives the following sketch.



(b) From the inequality

$$\begin{aligned}
\frac{3x^2-1}{2x^2+x} &> k \\
\frac{3x^2-1-2kx^2-kx}{x(2x+1)} &> 0 \\
x(2x+1)((3-2k)x^2-kx-1) &> 0
\end{aligned}$$

It is given that the solution to this inequality is $x \in \left(-\frac{1}{2}, 0\right)$.

This implies that $x < 0$ and $2x+1 > 0$, hence $x(2x+1) < 0$. This means that the quadratic factor is required to be negative definite in order for the inequality to hold. A necessary condition for this is

$$\begin{aligned}
\Delta &< 0 \\
k^2 - 4(3-2k)(-1) &< 0 \\
k^2 - 8k + 12 &< 0 \\
(k-2)(k-6) &< 0 \\
2 < k &< 6
\end{aligned}$$

The other condition required for the quadratic to be negative definition is $3-2k<0$, or equivalently, $k > \frac{3}{2}$.

This is a superset of the above solution, so the range of values of k is just $2 < k < 6$.

(c) Since $P'(\alpha) = 0$ and $P'(\beta) = 0$ then there is a double root where $\alpha = \beta$. Since $P(x)$ is a monic cubic polynomial then

$$\begin{aligned}
P(x) &= (x-\alpha)^2(x-\gamma) \\
P'(x) &= 2(x-\alpha)(x-\gamma) + (x-\alpha)^2 \\
P'(\gamma) &= (\gamma-\alpha)^2 \\
(\gamma-\alpha)^2 &= 25 \\
\gamma &= \alpha \pm 5
\end{aligned}$$

Consider the case where $\gamma = \alpha - 5$, then from the sum of the roots

$$\begin{aligned}
\alpha + \beta + \gamma &= -1 \\
3\alpha - 5 &= -1 \\
\alpha &= \frac{4}{3} \Rightarrow \beta = \frac{4}{3}, \gamma = -\frac{11}{3}
\end{aligned}$$

However, it is given that $\alpha\beta\gamma = 12$ which does not hold from the above solution set.

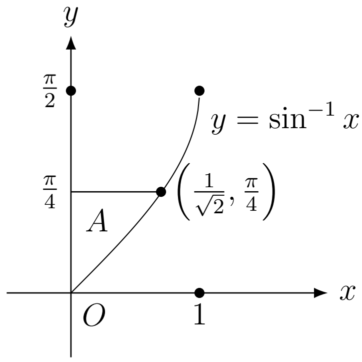
Now consider the case where $\gamma = \alpha + 5$, then from the sum of the roots

$$\begin{aligned}
\alpha + \beta + \gamma &= -1 \\
3\alpha + 5 &= -1 \\
\alpha &= -2 \Rightarrow \beta = -2, \gamma = 3
\end{aligned}$$

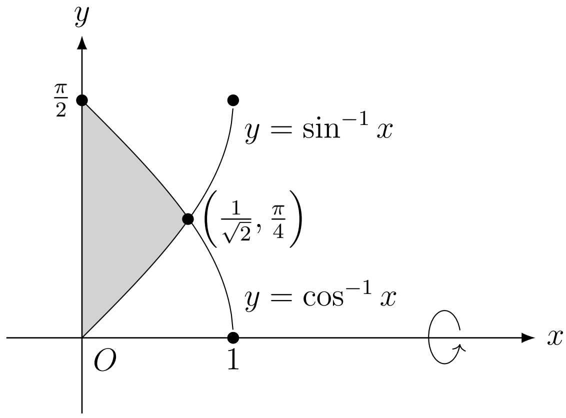
This satisfies the given result $\alpha\beta\gamma = 12$. Hence, the roots are -2 as a double root and 3.

(d) (i) Given $y = \sin^{-1} x$, then $x = \sin y$ so

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} x\,dy \\ &= \int_0^{\frac{\pi}{4}} \sin y\,dy \\ &= [-\cos y]_0^{\frac{\pi}{4}} \\ &= 1 - \frac{1}{\sqrt{2}} \end{aligned}$$



(ii) Note the point of intersection between $y = \sin^{-1} x$ and $y = \cos^{-1} x$ is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$.



The area between the curves is rotated about the x-axis so the volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^{\frac{1}{\sqrt{2}}} \left(\cos^{-1} x\right)^2\,dx - \pi \int_0^{\frac{1}{\sqrt{2}}} \left(\sin^{-1} x\right)^2\,dx \\ &= \pi \int_0^{\frac{1}{\sqrt{2}}} \left(\cos^{-1} x - \sin^{-1} x\right)\left(\cos^{-1} x + \sin^{-1} x\right)\,dx \quad \text{but } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \\ &= \frac{\pi^2}{2} \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{\pi}{2} - 2\sin^{-1} x\right)\,dx \\ &= \frac{\pi^3}{4} \int_0^{\frac{1}{\sqrt{2}}} dx - \pi^2 \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx \\ &= \frac{\pi^3}{4}[x]_0^{\frac{1}{\sqrt{2}}} - \pi^2 \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx \quad \text{but from part (i)} \quad A + \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx = \frac{\pi}{4\sqrt{2}} \\ &= \frac{\pi^3}{4\sqrt{2}} - \pi^2\left(\frac{\pi}{4\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}}\right)\right) \\ &= \pi^2\left(1 - \frac{1}{\sqrt{2}}\right) \end{aligned}$$

(e) Given $\alpha + \beta + \gamma = \dfrac{\pi}{2}$, note that

$$\begin{aligned}
\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\tan\left(\frac{\pi}{2} - \gamma\right) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\frac{1}{\tan\gamma} &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
1 - \tan\alpha\tan\beta &= \tan\alpha\tan\gamma + \tan\beta\tan\gamma \\
\tan\alpha\tan\beta + \tan\alpha\tan\gamma + \tan\beta\tan\gamma &= 1
\end{aligned}$$

Using this result

$$\begin{aligned}
\text{LHS} &= \frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} + \frac{\cos(\beta+\gamma)}{\cos\beta\cos\gamma} + \frac{\cos(\alpha+\gamma)}{\cos\alpha\cos\gamma} \\
&= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta} + \frac{\cos\beta\cos\gamma - \sin\beta\sin\gamma}{\cos\beta\cos\gamma} + \frac{\cos\alpha\cos\gamma - \sin\alpha\sin\gamma}{\cos\alpha\cos\gamma} \\
&= 1 - \tan\alpha\tan\beta + 1 - \tan\beta\tan\gamma + 1 - \tan\alpha\tan\gamma \\
&= 3 - (\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\alpha\tan\gamma) \\
&= 2 \\
&= \text{RHS}
\end{aligned}$$

Question 12

(a) Given the differential equations and for some constant C

$$\begin{aligned}\frac{3}{4}\cdot\frac{dT_c}{dt}+\frac{dT_w}{dt}&=\frac{3}{4}\times -k(T_c-T_w)+\frac{3}{4}k(T_c-T_w)\\&=0\\ \Rightarrow \frac{3}{4}T_c+T_w&=C\end{aligned}$$

When $t=0, T_c=160$ and $T_w=20 \Rightarrow C=140$. This implies that $\frac{3}{4}T_c+T_w=140$, so

$$\begin{aligned}\frac{dT_c}{dt}&=-k(T_c-T_w)\\&=-k\left(T_c-140+\frac{3}{4}T_c\right)\\&=-k\left(\frac{7}{4}T_c-140\right)\\&=-\frac{7}{4}k\left(T_c-80\right)\end{aligned}$$

The general solution is $T_c=80+Ae^{-\frac{7}{4}kt}$ for some constant A. When $t\to\infty$, then $T_c\to 80$. Hence, the chemical will cool down towards a temperature of 80°C over time.

(b) When $n=1$

$$\begin{aligned}(2n+1)7^n-1&=(2(1)+1)7^1-1\\&=20\\&=4\times 5\end{aligned}$$

The statement is true for $n=1$.

Assume the statement is true for $n=k$, that is for some integer M

$$(2k+1)7^k-1=4M.$$

Required to prove the statement is true for $n=k+1$, that is for some integer N

$$(2k+3)7^{k+1}-1=4N.$$

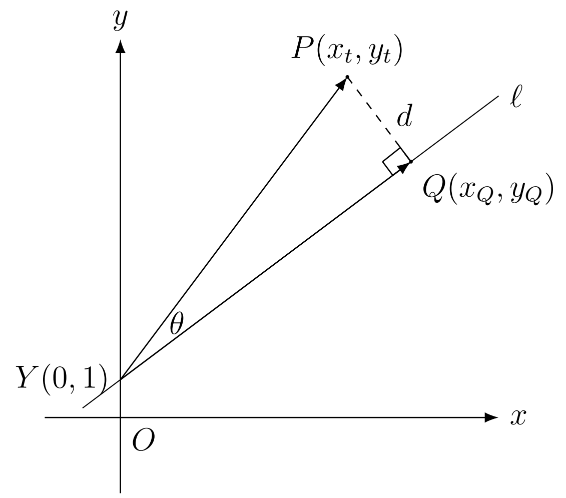
$$\begin{aligned}\text{LHS}&=(2k+3)7^{k+1}-1\\&=(2k+1+2)7^{k+1}-1\\&=(2k+1)7^{k+1}+14\cdot 7^k-1\\&=7(4M+1)+14\cdot(4M+1-2k\cdot 7^k)-1 \quad \text{by assumption}\\&=84M-28k\cdot 7^k+20\\&=4(21M-7k\cdot 7^k+5)\\&=4N \quad \text{where } N=21M-7k\cdot 7^k+5\\&=\text{RHS}\end{aligned}$$

Since statement is true for $n=1$, then by induction it is true for all positive integers n.

(c)

$$
\begin{aligned}
\text{LHS} &= \int_0^1 \frac{\cos\theta}{x^2 - 2x\sin\theta + 1}\,dx \\
&= \int_0^1 \frac{\cos\theta}{x^2 - 2x\sin\theta + \sin^2\theta + \cos^2\theta}\,dx \\
&= \int_0^1 \frac{\cos\theta}{(x - \sin\theta)^2 + \cos^2\theta}\,dx \\
&= \left[\tan^{-1}\left(\frac{x - \sin\theta}{\cos\theta}\right)\right]_0^1 \\
&= \tan^{-1}\left(\frac{1 - \sin\theta}{\cos\theta}\right) - \tan^{-1}\left(\frac{-\sin\theta}{\cos\theta}\right) \\
&= \tan^{-1}\left(\frac{1 - \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}{\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}\right) - \tan^{-1}(-\tan\theta) \\
&= \tan^{-1}\left(\frac{1 - 2\tan\frac{\theta}{2} + \tan^2\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}\right) + \theta \\
&= \tan^{-1}\left(\frac{\left(1 - \tan\frac{\theta}{2}\right)^2}{\left(1 - \tan\frac{\theta}{2}\right)\left(1 + \tan\frac{\theta}{2}\right)}\right) + \theta \\
&= \tan^{-1}\left(\frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}\right) + \theta \quad \text{noting } 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \tan\frac{\theta}{2} < 1 \\
&= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right) + \theta \\
&= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right) + \theta \\
&= \frac{\pi}{4} - \frac{\theta}{2} + \theta \\
&= \frac{\pi}{4} + \frac{\theta}{2} \\
&= \text{RHS}
\end{aligned}
$$

(d) (i) Note that the coordinates of Y are $(0, 1)$. Let θ be the angle between $\overrightarrow{YP}$ and $\overrightarrow{YQ}$ as shown in the diagram below. Let (x_Q, y_Q) be the coordinates of Q.



Consider the projection of $\overrightarrow{YP}$ onto $\overrightarrow{YQ}$ in terms of magnitude

$$\begin{aligned}\left|\overrightarrow{YQ}\right| &= \left|\overrightarrow{YP}\right| \left|\cos\theta\right| \\ &= \left|\overrightarrow{YP}\right| \frac{\left|\overrightarrow{YP}\cdot\overrightarrow{YQ}\right|}{\left|\overrightarrow{YP}\right|\left|\overrightarrow{YQ}\right|} \\ &= \frac{\left|\overrightarrow{YP}\cdot\overrightarrow{YQ}\right|}{\left|\overrightarrow{YQ}\right|} \\ &= \frac{\left|(x_t\underset{\sim}{i} + (y_t-1)\underset{\sim}{j})\cdot(x_Q\underset{\sim}{i} + (y_Q-1)\underset{\sim}{j})\right|}{\sqrt{x_Q^2 + (y_Q-1)^2}}\end{aligned}$$

But Q lies on the line $3x - 4y + 4 = 0$ so

$$\begin{aligned}3x_Q - 4y_Q + 4 &= 0 \\ y_Q - 1 &= \frac{3}{4}x_Q\end{aligned}$$

Substitute this into $\left|\overrightarrow{YQ}\right|$

$$\begin{aligned}\left|\overrightarrow{YQ}\right| &= \frac{\left|(x_t\underset{\sim}{i} + (y_t-1)\underset{\sim}{j})\cdot(x_Q\underset{\sim}{i} + \frac{3}{4}x_Q\underset{\sim}{j})\right|}{\sqrt{x_Q^2 + \frac{9}{16}x_Q^2}} \\ &= \frac{|x_Q||x_t + \frac{3}{4}(y_t-1)|}{|x_Q|\sqrt{\frac{25}{16}}} \\ &= \frac{|4x_t + 3y_t - 3|}{5}\end{aligned}$$

Hence

$$\begin{aligned}
d^2 &= \left|\overrightarrow{YP}\right|^2 - \left|\overrightarrow{YQ}\right|^2 \\
&= x_t^2 + (y_t - 1)^2 - \frac{(4x_t + 3y_t - 3)^2}{25} \\
&= \frac{1}{25}[25x_t^2 + 25y_t^2 - 50y_t + 25 \\
&\quad - (16x_t^2 + 9y_t^2 + 9 + 2(4x_t)(3y_t) + 2(3y_t)(-3) + 2(4x_t)(-3))] \\
&= \frac{1}{25}[9x_t^2 + 16y_t^2 + 16 - 24x_t y_t - 32y_t + 24x_t] \\
&= \frac{1}{25}[9x_t^2 + 16y_t^2 + 16 + 2(3x_t)(-4y_t) + 2(4)(-4y_t) + 2(3x_t)(4)] \\
&= \frac{(3x_t - 4y_t + 4)^2}{25}
\end{aligned}$$

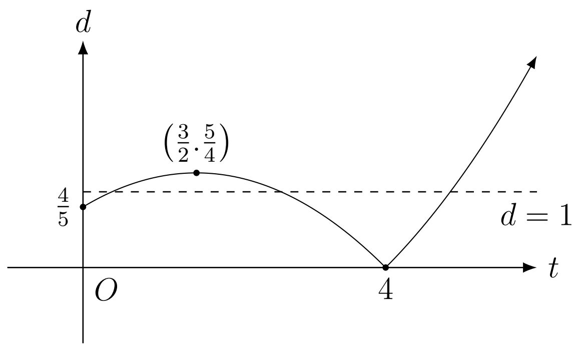
(ii) Substitute $x_t = t$ and $y_t = \dfrac{t^2}{4}$ into d^2

$$\begin{aligned}
d^2 &= \frac{(3t - t^2 + 4)^2}{25} \\
d^2 &= \frac{(t^2 - 3t - 4)^2}{25} \quad \text{but } d \geq 0 \\
d &= \frac{|(t+1)(t-4)|}{5}
\end{aligned}$$

Consider the parabola $y = \dfrac{(x+1)(x-4)}{5}$. The stationary points occur at the midpoint of the x-intercepts which is $x = \dfrac{3}{2} \Rightarrow y = -\dfrac{5}{4}$.

The graph of $y = \left|\dfrac{(x+1)(x-4)}{5}\right|$ will reflect the negative y-values about the x-axis.

Now apply this to the equation for d as a function of t, noting that $t \geq 0$ and $d \geq 0$, with the turning point $\left(\frac{3}{2}.\frac{5}{4}\right)$.



(iii) From part (ii), the local maximum is at $\left(\frac{3}{2}.\frac{5}{4}\right)$ and the d-intercept is at $d = \frac{5}{4}$. Hence, the particle is exactly 1 unit away from the line (represented by the horizontal line $d = 1$ above) exactly three times.

Question 13

(a) (i) Given the velocity vector

$$\dot{\underset{\sim}{r}} = \frac{u}{\sqrt{2}}\underset{\sim}{i} + \left(\frac{u}{\sqrt{2}} - gt\right)\underset{\sim}{j}$$
$$\underset{\sim}{r} = \left(\frac{ut}{\sqrt{2}} + C_x\right)\underset{\sim}{i} + \left(\frac{ut}{\sqrt{2}} - \frac{gt^2}{2} + C_y\right)\underset{\sim}{j}$$

When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} + 0\underset{\sim}{j} \Rightarrow C_x = 0, C_y = 0$, so

$$\begin{aligned} x &= \frac{ut}{\sqrt{2}} \\ t &= \frac{x\sqrt{2}}{u} \quad \text{substitute into } y \\ y &= \frac{u}{\sqrt{2}} \times \frac{x\sqrt{2}}{u} - \frac{g}{2} \times \frac{2x^2}{u^2} \\ &= x - \frac{gx^2}{u^2} \end{aligned}$$

(ii) On the x-y plane, the particle lands on the point $(10\cos\theta, 10\sin\theta)$. Substitute this into the Cartesian equation to get

$$\begin{aligned} 10\sin\theta &= 10\cos\theta - \frac{100g\cos^2\theta}{u^2} \quad \text{noting } 0 < \theta < \frac{\pi}{2} \\ \tan\theta &= 1 - \frac{10g\cos\theta}{u^2} \end{aligned}$$

(iii) On the x-y plane, the particle lands on the point $(20\cos\theta, -20\sin\theta)$. Substitute this into the Cartesian equation to get

$$\begin{aligned} -20\sin\theta &= 20\cos\theta - \frac{400g\cos^2\theta}{u^2} \quad \text{noting } 0 < \theta < \frac{\pi}{2} \\ \tan\theta &= \frac{20g\cos\theta}{u^2} - 1 \end{aligned}$$

(iv) From the result in part (ii)

$$\begin{aligned} \tan\theta &= 1 - \frac{10g\cos\theta}{u^2} \\ \frac{10g\cos\theta}{u^2} &= 1 - \tan\theta \end{aligned}$$

Substitute this into the result from part (iii)

$$\begin{aligned} \tan\theta &= \frac{20g\cos\theta}{u^2} - 1 \\ &= 2(1 - \tan\theta) - 1 \\ \tan\theta &= \frac{1}{3} \end{aligned}$$

(b) First note that

$$\begin{aligned}
\sin x + \cos x &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) \\
&= \sqrt{2}\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right) \\
&= \sqrt{2}\cos\left(\frac{\pi}{4} - x\right) \\
(\sin x + \cos x)^{2n} &= \left[\sqrt{2}\cos\left(\frac{\pi}{4} - x\right)\right]^{2n} \\
&= 2^n\left[\cos^2\left(\frac{\pi}{4} - x\right)\right]^n \\
\frac{1}{(\sin x + \cos x)^{2n}} &= \frac{1}{2^n}\left[\sec^2\left(\frac{\pi}{4} - x\right)\right]^n \\
&= \frac{1}{2^n}\left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]^n
\end{aligned}$$

Using the given substitution

$$\begin{aligned}
t &= \tan\left(\frac{\pi}{4} - x\right) \\
dt &= -\sec^2\left(\frac{\pi}{4} - x\right)\, dx \\
&= -\left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]\, dx
\end{aligned}$$

When $x = 0 \Rightarrow t = 1$ and when $x = \frac{\pi}{2} \Rightarrow -1$ so

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x + \cos x)^{2n}} &= \frac{1}{2^n}\int_0^{\frac{\pi}{2}} \left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]^n\, dx \\
&= \frac{1}{2^n}\int_0^{\frac{\pi}{2}} \left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]^{n-1}\left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]\, dx \\
&= -\frac{1}{2^n}\int_1^{-1} (1 + t^2)^{n-1}\, dt \\
&= \frac{1}{2^n}\int_{-1}^{1} (1 + t^2)^{n-1}\, dt \quad \text{but } (1 - t^2)^{n-1} \text{ is an even function} \\
&= \frac{2}{2^n}\int_0^1 \left[\binom{n-1}{0} + \binom{n-1}{1}t^2 + \binom{n-1}{2}t^4 + \cdots + \binom{n-1}{n-1}t^{2n-2}\right]\, dt \\
&= \frac{1}{2^{n-1}}\left[\frac{\binom{n-1}{0}t}{1} + \frac{\binom{n-1}{1}t^3}{3} + \frac{\binom{n-1}{2}t^5}{5} + \cdots + \frac{\binom{n-1}{n-1}t^{2n-1}}{2n-1}\right]_0^1 \\
&= \frac{\binom{n-1}{0}}{1 \times 2^{n-1}} + \frac{\binom{n-1}{1}}{3 \times 2^{n-1}} + \frac{\binom{n-1}{2}}{5 \times 2^{n-1}} + \cdots + \frac{\binom{n-1}{n-1}}{(2n-1) \times 2^{n-1}}
\end{aligned}$$

(c) (i) Using the given results

$$\cos\theta P(\theta) = 0$$
$$8\cos^4\theta + 4\cos^3\theta - 6\cos^2\theta - 2\cos\theta = 0$$
$$\cos 4\theta + 8\cos^2\theta - 1 + \cos 3\theta + 3\cos\theta - 6\cos^2\theta - 2\cos\theta = 0$$
$$\cos 4\theta + \cos 3\theta + 2\cos^2\theta + \cos\theta - 1 = 0$$
$$(\cos\theta + \cos 4\theta) + (\cos 2\theta + \cos 3\theta) = 0$$
$$\left[\cos\left(\frac{5\theta}{2} - \frac{3\theta}{2}\right) + \cos\left(\frac{5\theta}{2} + \frac{3\theta}{2}\right)\right] + \left[\cos\left(\frac{5\theta}{2} - \frac{\theta}{2}\right) + \cos\left(\frac{5\theta}{2} + \frac{\theta}{2}\right)\right] = 0$$
$$2\cos\frac{5\theta}{2}\cos\frac{3\theta}{2} + 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2} = 0$$
$$\cos\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right) = 0$$
$$\cos\frac{5\theta}{2}\left[\cos\left(\theta + \frac{\theta}{2}\right) + \cos\left(\theta - \frac{\theta}{2}\right)\right] = 0$$
$$\cos\frac{5\theta}{2}\cos\theta\cos\frac{\theta}{2} = 0$$

If $\cos\frac{\theta}{2} = 0$ then for $0 \le \theta \le 2\pi$, or equivalently, $0 \le \frac{\theta}{2} \le \pi$

$$\frac{\theta}{2} = \frac{\pi}{2} \qquad \Rightarrow \theta = \pi$$

If $\cos\theta = 0$ then for $0 \le \theta \le 2\pi$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

If $\cos\frac{5\theta}{2} = 0$ then for $0 \le \theta \le 2\pi$, or equivalently, $0 \le \frac{5\theta}{2} \le 5\pi$

$$\frac{5\theta}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2} \qquad \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$$

Hence, the full solution set is

$$\theta = \frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{3\pi}{2}, \frac{9\pi}{5}$$

(ii) Let the other roots be α and β. By the sum and product of roots

$$\alpha + \beta - 1 = -\frac{1}{2} \qquad \Rightarrow \alpha + \beta = \frac{1}{2}$$
$$\alpha\beta(-1) = \frac{1}{4} \qquad \Rightarrow \alpha\beta = -\frac{1}{4}$$

This implies that α and β are the roots of $4x^2 - 2x - 1 = 0$ which solves to

$$x = \frac{2 \pm \sqrt{4 - 4(4)(-1)}}{8}$$
$$= \frac{1 \pm \sqrt{5}}{4}$$

Hence, the roots of $P(x)$ are -1, $\frac{1-\sqrt{5}}{4}$ and $\frac{1+\sqrt{5}}{4}$.

(iii) The solutions of $P(\cos\theta) = 0$ are derived from $\cos\dfrac{\theta}{2}\cos\dfrac{5\theta}{2} = 0$, which are

$$\begin{aligned}
\theta &= \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5} \\
\cos\theta &= \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}
\end{aligned}$$

However, note that $\cos\pi = -1$ and

$$\begin{aligned}
\cos\frac{9\pi}{5} &= \cos\left(2\pi - \frac{\pi}{5}\right) \\
&= \cos\frac{\pi}{5} \\
\cos\frac{7\pi}{5} &= \cos\left(2\pi - \frac{3\pi}{5}\right) \\
&= \cos\frac{3\pi}{5}
\end{aligned}$$

Hence, there are three distinct roots of $P(x)$, which are $\cos\dfrac{\pi}{5}, \cos\dfrac{3\pi}{5}$ and -1.

Since $\dfrac{\pi}{2} < \dfrac{3\pi}{5} < \pi$ then $\cos\dfrac{3\pi}{5} < 0$, so from part (ii)

$$\begin{aligned}
\cos\frac{3\pi}{5} &= \frac{1-\sqrt{5}}{2} \\
\cos\left(\pi - \frac{2\pi}{5}\right) &= \frac{1-\sqrt{5}}{2} \\
-\cos\frac{2\pi}{5} &= \frac{1-\sqrt{5}}{4} \\
\cos\frac{2\pi}{5} &= \frac{\sqrt{5}-1}{4}
\end{aligned}$$

Question 14

(a) (i) There are ${}^{16}C_{12}$ possible combinations of successful tenants.

There are ${}^{20}P_{12}$ possible arrangements of 12 successful tenants amongst the 20 vacant homes.

Hence, there are ${}^{16}C_{12} \times {}^{20} P_{12}$ ways the interested tenants be arranged amongst the houses, if they were successful.

(ii) If there are two particular tenants that are successful then there are ${}^{14}C_{10}$ combinations of the other successful tenants.

There are ${}^{18}P_{10}$ possible arrangements of the other 10 successful tenants amongst the 18 remaining houses.

There are 19 possible pairs of houses where the two tenants can be neighbours.

There are 2! ways to arrange the two particular tenants between each of the neighbouring houses.

The number of arrangements where two particular tenants are successful and are also neighbours is ${}^{14}C_{10} \times {}^{18} P_{10} \times 19 \times 2!$.

Hence, the probability this occurs is $\dfrac{{}^{14}C_{10} \times {}^{18} P_{10} \times 19 \times 2!}{{}^{16}C_{12} \times {}^{20} P_{12}}$, or equivalently $\dfrac{11}{200}$.

(b) (i) After each minute there will be 200 litres of water flowing into the tank. Since the water coming in will have $e^{-0.2t}$ grams of radioactive chemical per litre, then there will be $200e^{-0.2t}$ grams of radioactive chemical coming into the tank.

If m is the mass of radioactive chemicals inside the tank after t minutes, then there are $\frac{m}{V}$ grams of it per litre in the tank, where V is the volume of the tank after time t minutes.

After each minute there will be 100 litres of water flowing out of the tank. This carries $\frac{100m}{V}$ grams of radioactive chemical out of the tank. Therefore, the rate of change of m is

$$\frac{dm}{dt} = 200e^{-0.2t} - \frac{100m}{V}$$

To find V, note that after time t minutes, there are $200t$ litres of water flowing in and $100t$ litres water flowing out. Since the tank is initially holding 100 litres of water then the volume of water in the tank after t minutes is $V = 100+200t-100t$, or equivalently $V = 100(1 + t)$. Hence,

$$\frac{dm}{dt} = 200e^{-0.2t} - \frac{m}{1+t}$$

(ii) Noting that m is a function of t and using the given result

$$\begin{aligned}\frac{d}{dt}\left[m(1+t)\right] &= \frac{dm}{dt}(1+t)+m\\ &= 200e^{-0.2t}(1+t)-m+m\\ &= 200e^{-0.2t}(1+t)\\ m(1+t) &= 200\int e^{-0.2t}\,dt + 200\int te^{-0.2t}\,dt\\ &= -1000e^{-0.2t} + 200\left(-\frac{1}{(0.2)^2}e^{-0.2t}(1+0.2t)\right)+C\\ &= -1000e^{-0.2t} - 5000e^{-0.2t}(1+0.2t)+C\end{aligned}$$

When $t=0, m=0 \Rightarrow C=6000$

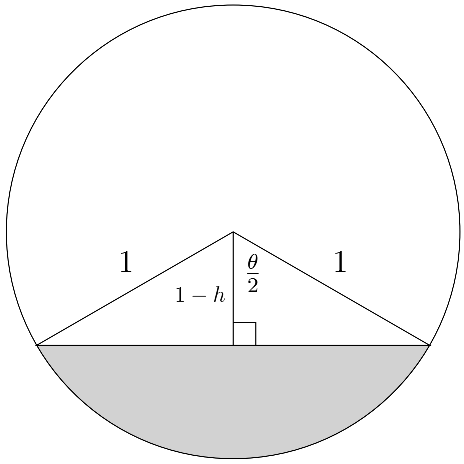
$$\begin{aligned}m &= \frac{-6000e^{-0.2t} - 1000te^{-0.2t} + 6000}{1+t}\\ &= \frac{6000-1000e^{-0.2t}(t+6)}{1+t}\end{aligned}$$

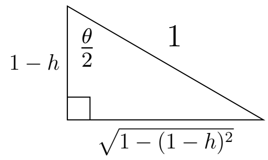
(iii) Let M be the mass of radioactive chemical which flowed out of the tank.

From the derivation in part (i), it was shown that

$$\begin{aligned}\frac{dM}{dt} &= \frac{m}{1+t}\\ &= 200e^{-0.2t} - \frac{dm}{dt}\\ M &= \int_0^4 \left(200e^{-0.2t} - \frac{dm}{dt}\right)\,dt\\ &= -1000[e^{-0.2t}]_0^4 - \left[\frac{6000-1000e^{-0.2t}(t+6)}{1+t}\right]_0^4\\ &= 1000(1-e^{-0.8}) - \left(\frac{6000-10000e^{-0.8}}{5}\right)\\ &\approx 249 \text{ grams}\end{aligned}$$

(c) Let θ be the angle subtended by the chord at the centre.





From the right-angled triangle

$$\begin{aligned}\sin\frac{\theta}{2} &= \sqrt{1-(1-h)^2} \\ &= \sqrt{2h-h^2} \\ \cos\frac{\theta}{2} &= 1-h \\ \frac{\theta}{2} &= \cos^{-1}(1-h)\end{aligned}$$

Consider the area of the segment A

$$\begin{aligned}A &= \frac{1}{2}\times 1^2\times(\theta-\sin\theta) \\ &= \frac{\theta}{2}-\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= \cos^{-1}(1-h)-(1-h)\sqrt{2h-h^2}\end{aligned}$$

Let V be the volume of the oil at time t.

$$\begin{aligned}V &= 10A \\ &= 10\cos^{-1}(1-h)-10(1-h)\sqrt{2h-h^2} \\ \frac{dV}{dh} &= \frac{-10(-1)}{\sqrt{1-(1-h)^2}}-10(-1)\sqrt{2h-h^2}-\frac{10(1-h)(2-2h)\times\frac{1}{2}}{\sqrt{2h-h^2}} \\ &= \frac{10[1+(2h-h^2)-(1-h)^2]}{\sqrt{2h-h^2}} \\ &= \frac{10(1+2h-h^2-1+2h-h^2)}{\sqrt{2h-h^2}} \\ &= \frac{20(2h-h^2)}{\sqrt{2h-h^2}} \quad \text{note that } 0<h<1 \\ &= 20\sqrt{2h-h^2}\end{aligned}$$

It is given that S is the area of the tank's cylindrical surface that is in contact with the oil after t weeks. This is equal to the rectangular area at the bottom of the cylindrical tank plus twice the area of the segment at each end of the tank.

The area of the rectangle is the arc length of the segment multiplied by 10 metres and the area of the segment is A, as defined earlier above.

$$\begin{aligned}
S &= 1\times\theta\times 10 + 2A \quad \text{but } \frac{\theta}{2} = \cos^{-1}(1-h) \text{ and } V = 10A \\
&= 20\cos^{-1}(1-h) + \frac{V}{5} \\
\frac{dS}{dh} &= \frac{-20(-1)}{\sqrt{1-(1-h)^2}} + \frac{1}{5}\cdot\frac{dV}{dh} \\
&= \frac{20}{\sqrt{2h-h^2}} + 4\sqrt{2h-h^2}
\end{aligned}$$

Using the chain rule and given $\dfrac{dV}{dt} = -5$

$$\begin{aligned}
\frac{dS}{dh} &= \frac{dS}{dh}\times\frac{dh}{dV}\times\frac{dV}{dt} \\
&= \left(\frac{20}{\sqrt{2h-h^2}} + 4\sqrt{2h-h^2}\right)\times\frac{1}{20\sqrt{2h-h^2}}\times(-5) \\
&= -\frac{5}{2h-h^2} - 1 \\
&= \frac{5}{h(h-2)} - 1
\end{aligned}$$

(d) A new streak is defined everytime the current outcome differs from the previous outcome. For example, in the sequence $\{HTTTTHHHTH\}$ there are 5 streaks but there are 4 occasions where the outcome changes, which can be visualised as $\{H|TTTT|HHH|T|H\}$. This means that for $k+1$ streaks to occur, there must be k instances where the outcome changes in the sequence.

For any pair of coin tosses, let X_j be a Bernoulli random variable for $j = 1, 2, 3, \ldots, n-1$ which takes the value 0 if the outcomes between two consecutive tosses is the same and 1 if the outcomes between two consecutive tosses are different. Note that these are equally likely so $X_j \sim \text{Bin}\left(1, \frac{1}{2}\right)$.

Repeating this across $n-1$ pairs (which is equivalent to n coin tosses) means the sum of these Bernoulli random variables $Y = X_1 + X_2 + X_3 + \cdots + X_{n-1}$, represents number of occasions that two consecutive tosses have a different outcome, so $Y \sim \text{Bin}\left(n-1, \frac{1}{2}\right)$. However, S is the random variable represent the number of streaks, so $S = Y + 1$. Hence

$$\begin{aligned}
E(S) &= E(Y+1) \\
&= E(Y) + 1 \\
&= (n-1)\times\frac{1}{2} + 1 \\
&= \frac{n+1}{2}
\end{aligned}$$