2012 Bored of Studies Trial Examinations

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 1 – 7

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 8 – 23

90 marks

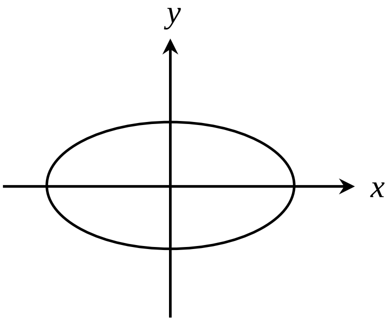
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

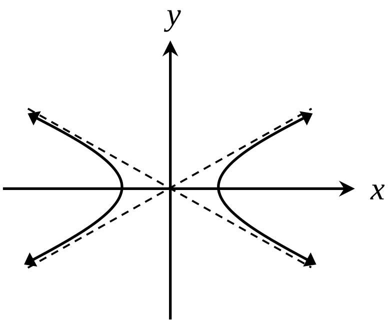
Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

1 Which of the following is the correct sketch of a conic section with eccentricity $0 < e < 1$?

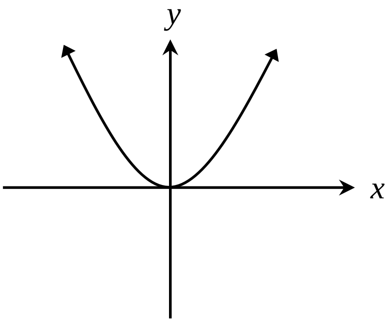
(A)



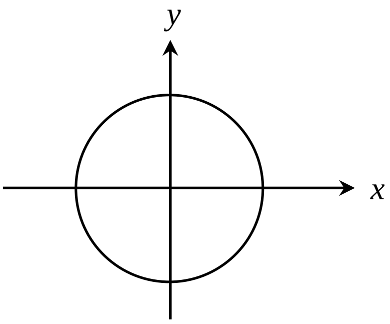
(B)



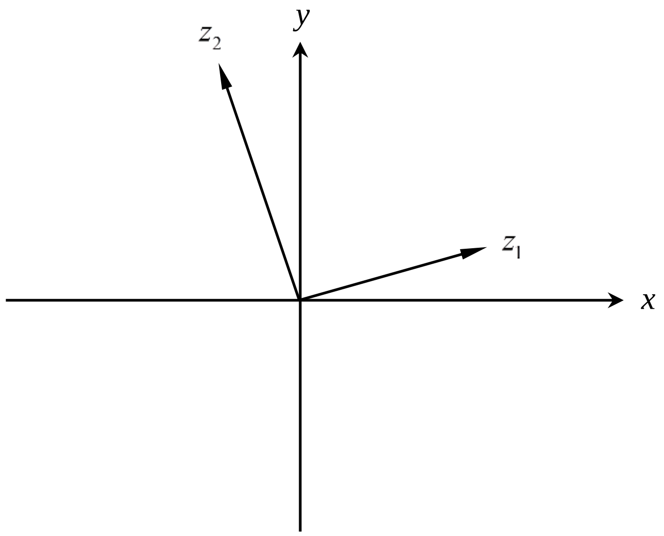
(C)



(D)

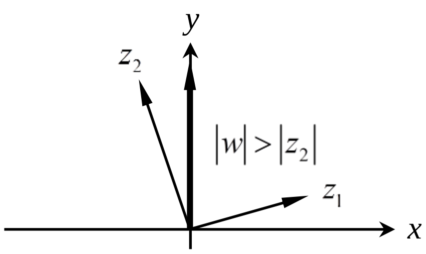


2 The following diagram shows vectors z_1 and z_2, where $|z_1| < 1$ and $|z_2| = 1$.

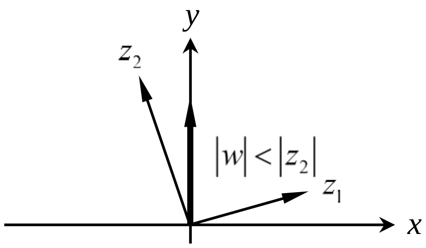


Which of the following diagrams most accurately shows the vector $w = \dfrac{z_2}{z_1}$?

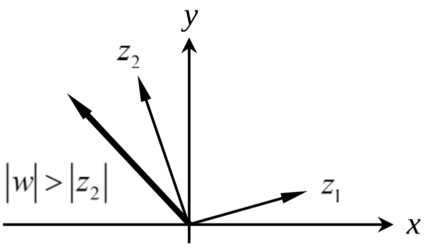
(A)



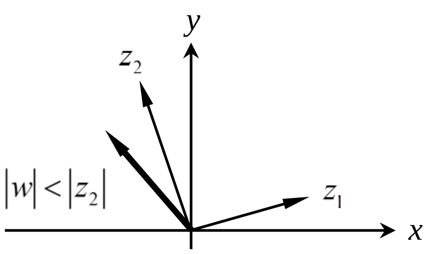
(B)



(C)



(D)



3 By letting $t=\tan\left(\dfrac{x}{2}\right)$, evaluate $\displaystyle\int \frac{dx}{1-\sin x+\cos x}$.

(A) $\ln\left|\tan\left(\dfrac{x}{2}\right)-1\right|+C$.

(B) $\ln\left|\tan\left(\dfrac{x}{2}\right)+1\right|+C$.

(C) $-\ln\left|\tan\left(\dfrac{x}{2}\right)+1\right|+C$.

(D) $-\ln\left|\tan\left(\dfrac{x}{2}\right)-1\right|+C$.

4 Let ω be a complex root such that $\omega^n=1$, $\omega\neq 1$.

Find the value of $\displaystyle\sum_{k=0}^{n}\left(\omega^k+\frac{1}{\omega^k}\right)$.

(A) 0.

(B) 1.

(C) 2.

(D) 3.

5 Which of the following statements is not necessarily true?

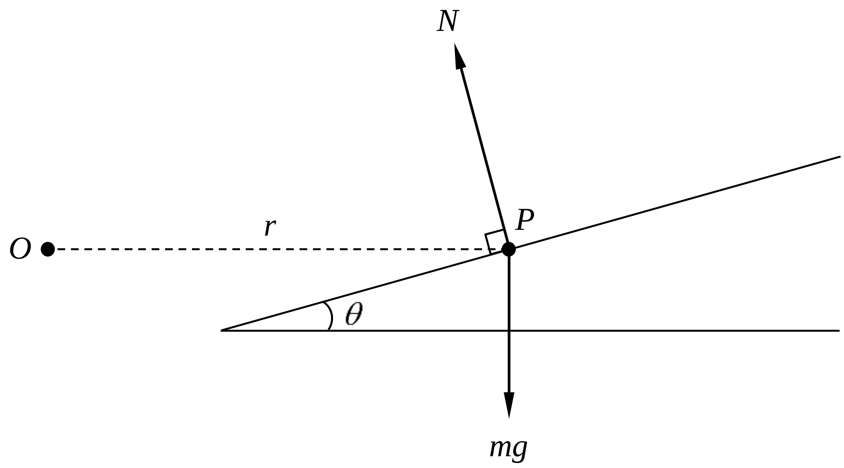
(A) $\displaystyle\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$.

(B) If $f(x)<g(x)$ for $0\le x\le a$, then $\displaystyle\int_0^a f(x)\,dx<\int_0^a g(x)\,dx$.

(C) If a polynomial has a root of multiplicity *n*, then the polynomial has degree n.

(D) The expression $z^n=1$ has exactly $n-1$ non-real roots, if *n* is odd.

6 A body moves around a track with radius r, that is banked at some angle θ to the horizontal. It moves with a constant velocity $v\,\mathrm{ms}^{-1}$. The body has some mass m and experiences a gravitational force mg, a normal reaction N to the road, and some lateral force F, which points either up or down the track, depending on the velocity.



Given that the velocity satisfies

$$0 < v < \sqrt{gr\tan\theta}\,,$$

which of the following expressions correctly resolves the force?

(A) Horizontally: $N\sin\theta + F\cos\theta = mv^2/r$.

Vertically: $N\cos\theta - F\sin\theta = mg$.

(B) Horizontally: $N\sin\theta - F\cos\theta = mv^2/r$.

Vertically: $N\cos\theta + F\sin\theta = mg$.

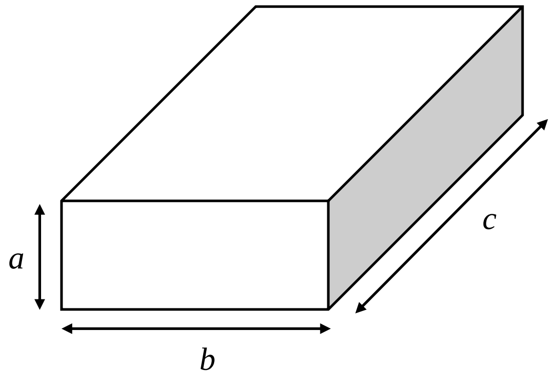
(C) Horizontally: $F\cos\theta - N\sin\theta = mv^2/r$.

Vertically: $F\sin\theta + N\cos\theta = mg$.

(D) Horizontally: $F\cos\theta + N\sin\theta = mv^2/r$.

Vertically: $F\sin\theta - N\cos\theta = mg$.

7 Consider the following rectangular prism with side lengths *a*, *b* and *c* such that the sum of the lengths of the edges, is equal to 4.



Find the maximum possible surface area.

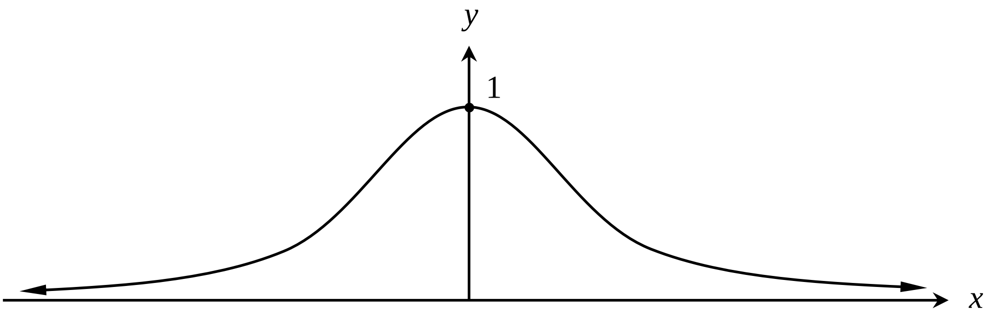
(A) $1/3$.

(B) $2/3$.

(C) 1.

(D) $4/3$.

8 Which of the following equations best describes the following curve?



(A) $y = e^{x^2}$.

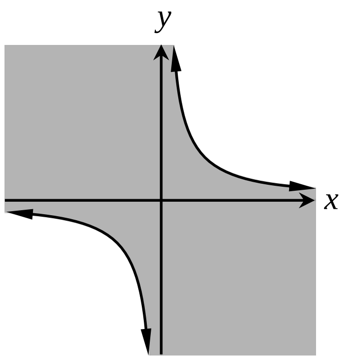
(B) $y = e^{-x^2-x}$.

(C) $y = \dfrac{1}{(x+1)^2}$.

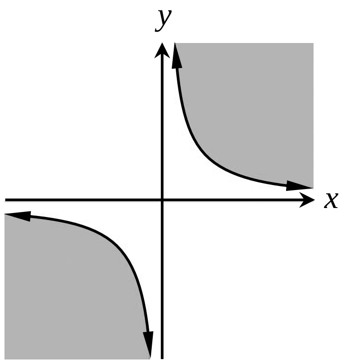
(D) $y = \dfrac{1}{x^2+1}$.

9 Which of the following is the locus of the condition $\mathrm{Im}\left(z^2\right) \leq 2$?

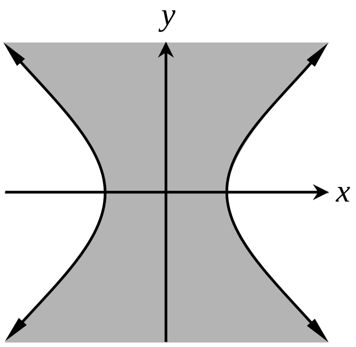
(A)



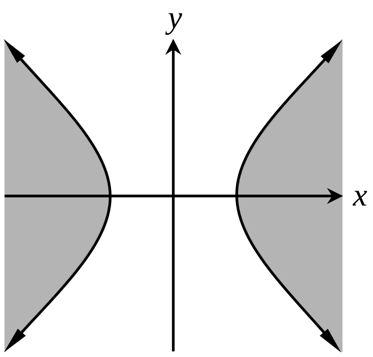
(B)



(C)



(D)



10 The polynomial $P(x) = x^6 - 4x^5 - x^4 + 24x^3 - 5x^2 - 100x + 125$ has a double root $2 + i$.

How many of the roots of $P(x)$ are real?

(A) 0.

(B) 1.

(C) 2.

(D) 4.

Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int_0^1 \frac{e^{2x}-1}{e^{2x}+1}\,dx$. **2**

(b) Consider the integral $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x - 4\sin^2 x}$.

(i) Find functions $A(x)$ and $B(x)$ such that **3**

$$\frac{1}{9\cos^2 x - 4\sin^2 x} = \frac{A(x)}{3\cos x - 2\sin x} + \frac{B(x)}{3\cos x + 2\sin x}.$$

Hint: $\sin^2 x + \cos^2 x = 1$.

(ii) Hence evaluate the integral. **2**

(c) The region bounded by the curve $y = x$ and $y = x^n$, where $n > 1$, is rotated about the line $x = 1$.

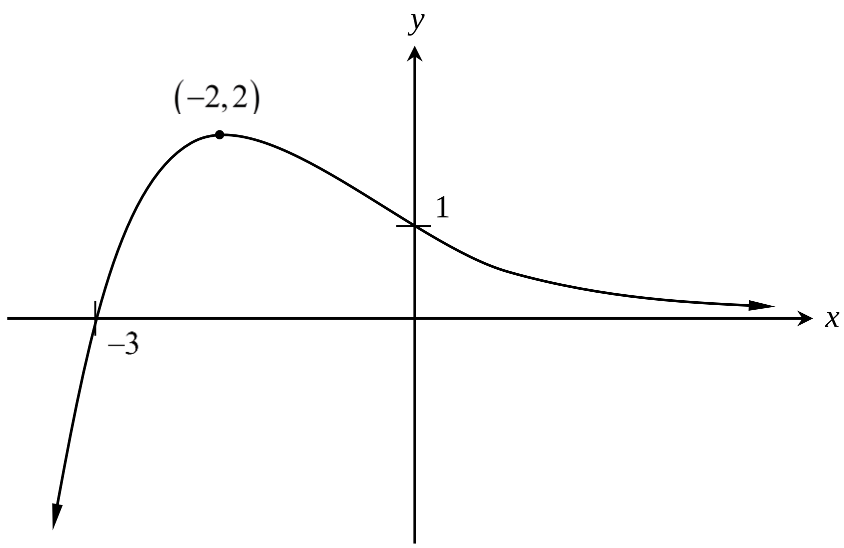
(i) Use the method of cylindrical shells to find the volume of the solid generated. **3**

(ii) Find the limiting volume of the solid as $n \to \infty$. **1**

(iii) Interpret your result geometrically. **1**

Question 11 continues on page 9

(d) The following is the graph of the function $y = f\left(x\right)$. **3**



Copy the diagram into your writing booklet.

Sketch the function $y = f\left(\dfrac{1}{x}\right)$, labeling all features.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

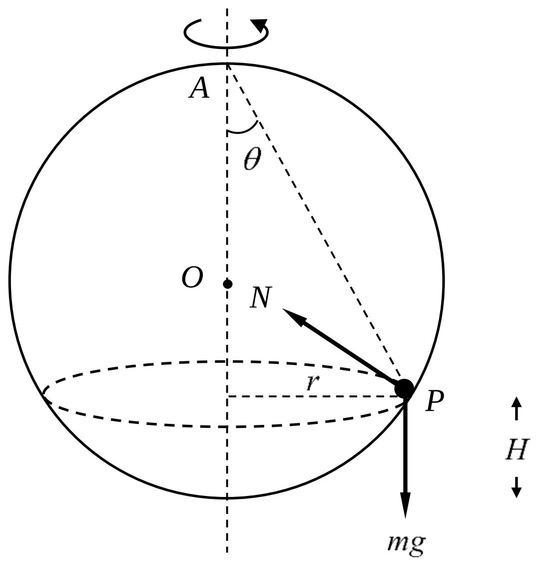
(a) Consider the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$, $c \neq 0$. A tangent is constructed in the interval $0 < x < c$, and this tangent meets the x and y axes at $A(a,0)$ and $B(0,b)$ respectively.

(i) Show that $\dfrac{dy}{dx}=-\sqrt{\dfrac{y}{x}}$. **1**

(ii) Hence prove that for any tangent in the interval $0 < x < c$, **3**

$$a+b=c\,.$$

(b) A frictionless sphere with radius R and centre O rotates about its vertical diameter with uniform angular velocity ω radians per second. A ball of mass m rolls freely inside the sphere, and experiences a gravitational force mg and a normal force N. The path of the marble describes a circle of radius $r \leq R$. Let $\angle OAP=\theta$ and let the height of the ball from the floor be H. **4**

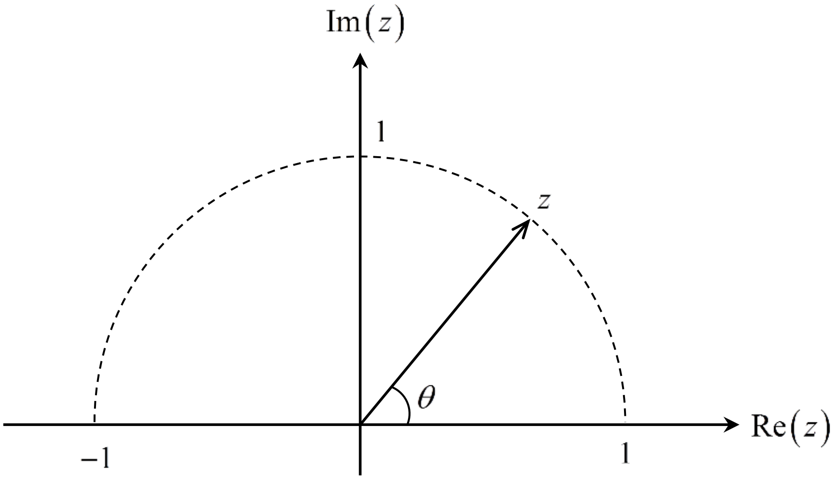


The marble rolls with some angular velocity ω such that $0 \leq h_1 \leq H \leq h_2 \leq R$. Show that

$$\sqrt{\frac{2gh_1}{r^2-h_1^{\,2}}} \leq \omega \leq \sqrt{\frac{2gh_2}{r^2-h_2^{\,2}}}\,.$$

Question 12 continues on page 11

(c) A complex number z lies on the upper half of a unit semi-circle.



Copy the diagram into your writing booklet.

(i) Sketch the vectors $z+1$ and $z-1$ on the same diagram. **2**

(ii) Use your diagram, or otherwise, to prove that **2**

$$\operatorname{Arg}\left(\frac{z-1}{z+1}\right)=\frac{\pi}{2}.$$

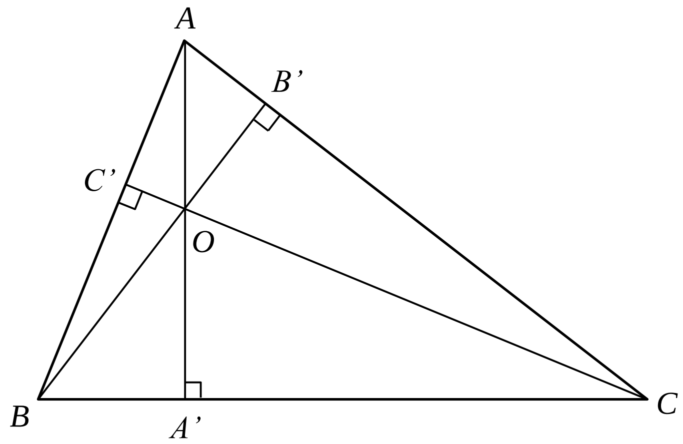
(iii) Hence prove that **1**

$$\operatorname{Arg}(z)=2\tan^{-1}\left|\frac{z-1}{z+1}\right|.$$

Question 12 continues on page 12

Question 12 (continued)

(d) Consider an arbitrary triangle ABC. Three altitudes AA', BB' and CC' are drawn , and they are concurrent at the point O **(Do NOT prove this)**.



(i) Show that **1**

$$\frac{OA'}{AA'} = \frac{|BOC|}{|ABC|},$$

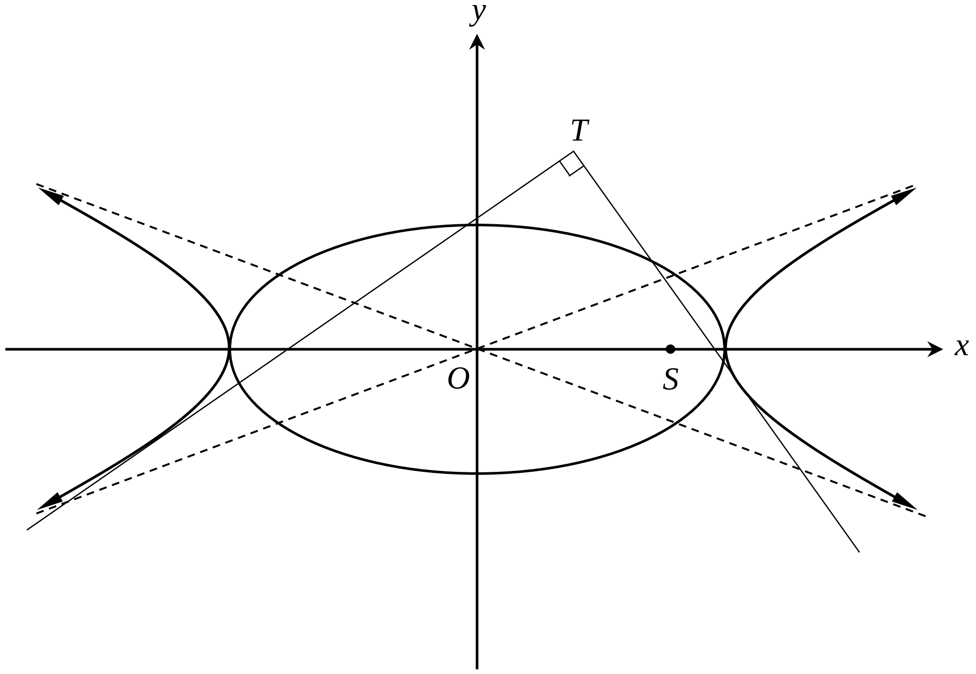
where $|XYZ|$ denotes the area of ΔXYZ .

(ii) Hence prove that $\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>b$, on the same set of axes. Let the positive focus of the ellipse be S. From two points on the hyperbola, mutually perpendicular tangents are drawn and intersect each other at T.



(i) Show that all lines in the form **3**

$$y = mx \pm \sqrt{a^2m^2 - b^2},$$

where m is some real constant, are tangential to the hyperbola.

(ii) Show that the locus of T is the circle $x^2+y^2=a^2-b^2$. **2**

(iii) Deduce that the triangle described by OTS is isosceles. **1**

Question 13 continues on page 14

(b) A body of mass m is projected vertically from the ground with initial velocity U. It experiences a resistance force with magnitude mkV^4, where k is a positive constant, and a gravitational acceleration g. Let its velocity at some time t be V, and also let its vertical displacement from the ground be x.

(i) Prove that **3**

$$x = \frac{1}{2\sqrt{kg}} \tan^{-1}\left(\frac{\sqrt{kg}\left(U^2 - V^2\right)}{g + kU^2V^2} \right).$$

You may assume, without proof, that $\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\frac{A-B}{1+AB} \right)$.

(ii) Prove that the maximum height is **1**

$$H = \frac{1}{2\sqrt{kg}} \tan^{-1}\left(U^2 \sqrt{\frac{k}{g}} \right).$$

(iii) Hence find the limiting height H as $U \to \infty$. **1**

(c) A group of n people are to be arranged in a straight line. Of the n people, m particular people, where $n \geq 2m$, wish be seated together in a group. Let the number of such permutations be P_T.

All m people are now to be seated such that they are all separated by at least one person from the remaining $n - m$ people. Let the number of such permutations be P_S.

(i) Show that **3**

$$\frac{P_S}{P_T} = \frac{1}{m} \times \binom{n-m}{m-1}.$$

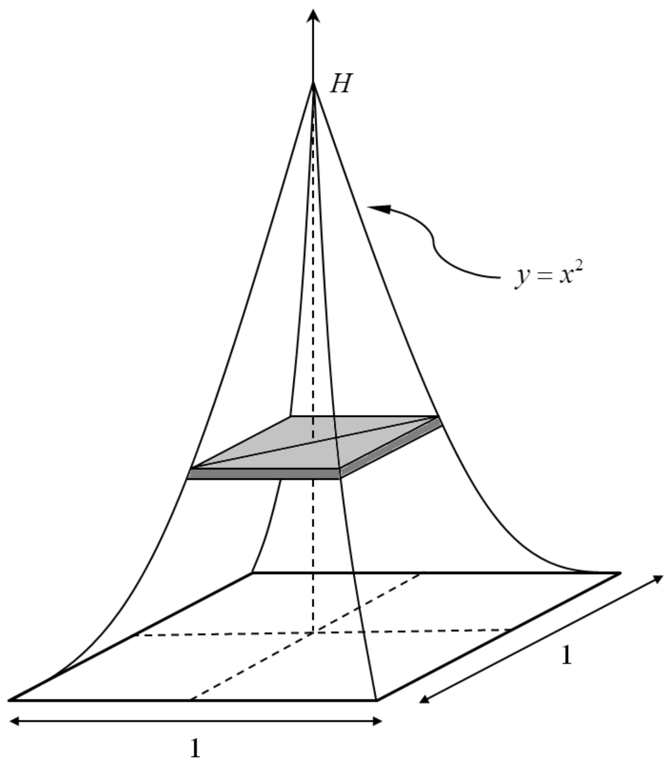
(ii) Deduce that if n is even, and m is exactly half of n, then $P_S = P_T$. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A pyramid-like structure with curved edges has a square base of unit length. Cross sections taken parallel to the base are squares, and the 'pyramid' eventually ends at the tip with some height H. All the curved edges follow the shape of the curve $y = x^2$, with the corners of the base being the vertex of the parabola.

Let the height, from the base, of an arbitrary slice be h.



(i) Show that the length of the diagonal of the slice is **3**

$$d = \sqrt{2}\left(1 - \sqrt{2h}\right).$$

(ii) Show that $H = \dfrac{1}{2}$. **1**

(iii) Hence, find the volume of the solid. **2**

Question 14 continues on page 16

(b) Consider the expression $z^n = 1$, which has n complex roots z_k, where $k = 1, 2, 3, \ldots, n$. You may assume, without loss of generality, that

$$\operatorname{Arg}(z_k) = \frac{2k\pi}{n}.$$

(i) Prove that for any two roots z_k and z_{k-1}, we have **2**

$$|z_k - z_{k-1}| = \sqrt{2 - 2\cos\frac{2\pi}{n}}.$$

(ii) The n roots of $z^n = 1$ form a regular polygon with some perimeter P_n. **1**

Explain why $P_n = n|z_k - z_{k-1}|$.

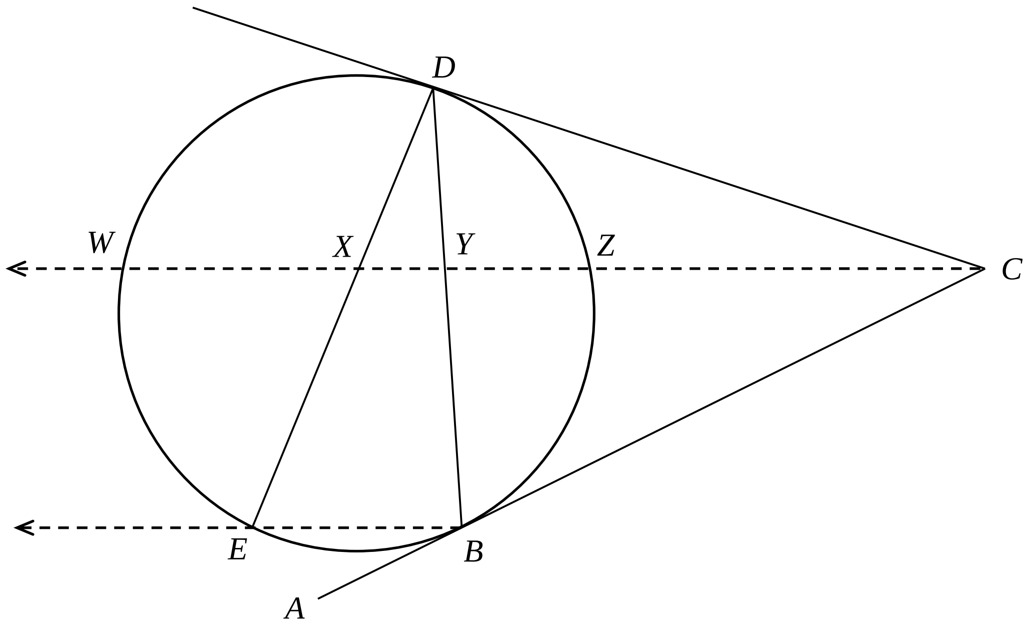
(iii) By letting $\theta = \dfrac{2\pi}{n}$, or otherwise, prove that **2**

$$\lim_{n\to\infty} P_n = 2\pi.$$

(iv) Explain the result in (iii) geometrically. **1**

Question 14 continues on page 17

(c) An interval AC is drawn such that it is tangential to a circle at B. From C, another tangent is constructed to meet the circle again at D. Also, a horizontal line is drawn such that it meets the circle at points Z and W shown below. A chord DB is drawn and intersects CW at Y. From point D, another chord is drawn to meet a horizontal from B at E, which lies on the circle, such that it intersects CW at X.



(i) A circle is drawn through points DXY. Explain why CD is a mutual tangent to both circles at D. **1**

(ii) Deduce that $\dfrac{XW}{XC} = \dfrac{ZY}{ZC}$. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Define the following integral, where n is a positive integer.

$$I_n = \lim_{a \to \infty} \int_0^a x^{n-1} e^{-x}\, dx.$$

You may assume that for all real $n > 0$, $\lim_{x \to \infty} x^n e^{-x} = 0$.

(i) Show that $I_1 = 1$. **1**

(ii) Show that $I_{n+1} = nI_n$. **2**

(iii) Hence prove that $I_{n+1} = n!$. **1**

(iv) Define another integral **2**

$$J_n = \lim_{b \to 0} \int_b^1 (\ln x)^n\, dx.$$

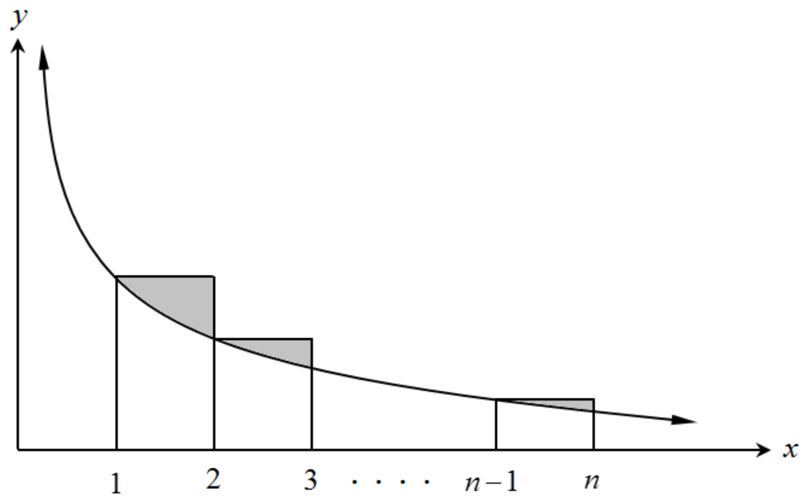
Show that

$$\frac{I_n}{J_n} = \frac{(-1)^n}{n}.$$

Question 15 continues on page 19

(b) Consider the function $f(x) = \frac{1}{x}$,where n is an integer. Upper bound rectangles are constructed in the domain $1 \leq x \leq n$, each with unit width.

Let E_n be the total excess area between the upper bound rectangles and the area under the curve, indicated by the grey regions in the diagram below.



(i) Let $H_n = \sum_{r=1}^{n} \frac{1}{r}$. Show that **2**

$$H_n - \ln n = E_n + \frac{1}{n}.$$

(ii) Given that as $n \to \infty$, E_n approaches some finite limit γ, show that **1**

$$\lim_{n \to \infty} \left(H_n - \ln n \right) = \gamma.$$

Question 15 continues on page 20

Question 15 (continued)

(iii) Define a series $S_n = \sum_{r=1}^{n} \frac{(-1)^r}{r}$. **3**

Use Mathematical Induction to prove that

$$S_{2n} = H_n - H_{2n},$$

for all positive integers *n*.

(iv) Consider I_n and J_n from part (a). Use (ii) and (iii) from part (b) to prove that **3**

$$\lim_{n\to\infty} \sum_{r=0}^{n} \frac{r!}{J_{r+1}} = \ln\left(\frac{1}{2}\right).$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Using the extended Triangle Inequality **(Do NOT prove this result)**, **2**

$$|z_0 + z_1 + z_2 + \ldots + z_n| \le |z_0| + |z_1| + |z_2| + \ldots + |z_n|$$

prove that

$$|z_0 + z_1 + z_2 + \ldots + z_n| \ge |z_0| - |z_1| - |z_2| - \ldots - |z_n|$$

(b) Let x be defined for all complex values.

(i) Let $R(x) = \sum_{k=0}^{n} b_k x^k$, where $n \ge 2$ and $0 < b_n \le b_{n-1} \le \ldots \le b_1 \le b_0$,
which has n distinct roots.

(1) Use the result in part (a) to show that **2**

$$|(1-x)R(x)| \ge b_0 - \sum_{k=1}^{n} (b_{k-1} - b_k)|x|^k - b_n |x|^{n+1}.$$

(2) Hence deduce that if $|x| < 1$, then **1**

$$|(1-x)R(x)| > 0.$$

(3) Suppose that α is a root of $R(x)$. Explain why $|\alpha| \ge 1$. **1**

Question 16 continues on page 22

Question 16 (continued)

(ii) Let $S(x)=\sum_{k=0}^{n} c_k x^k$, where $n \geq 2$ and $0 < c_0 \leq c_1 \leq \ldots \leq c_{n-1} \leq c_n$.

(1) Define $T(x)=\sum_{k=0}^{n} c_{n-k} x^k$. Show that if β is a root of $S(x)$, then $\dfrac{1}{\beta}$ is a root of $T(x)$. **1**

(2) Hence by considering $\left|(1-x)T(x)\right|$, or otherwise, show that **2**

$$|\beta| \leq 1.$$

Question 16 continues on page 23

Question 16 (continued)

(c) For all complex values of x, define $P(x)=\sum_{k=0}^{n} a_k x^k$, where $a_k > 0$ for $k = 0,1,2,...,n$.

For all $1 \le k \le n$, let

- A be the minimum value of $\left\{\frac{a_0}{a_1},\frac{a_1}{a_2},\frac{a_2}{a_3},...,\frac{a_{n-1}}{a_n}\right\}$.
- B be the maximum value of $\left\{\frac{a_0}{a_1},\frac{a_1}{a_2},\frac{a_2}{a_3},...,\frac{a_{n-1}}{a_n}\right\}$.

(i) Show that **1**

$$0 < a_n A^n \le a_{n-1}A^{n-1} \le ... \le a_2 A^2 \le a_1 A \le a_0 .$$

(ii) Let $b_k = a_k A^k$, for all $k = 0,1,2,...,n$. **2**

Prove that if ω is a root of $P(x)$, then $\frac{\omega}{A}$ is a root of $R(x)$, as defined in part (b) (i).

(iii) By letting $c_k = a_k B^k$, for all $k = 0,1,2,...,n$, show that $\frac{\omega}{B}$ is a root of $S(x)$, as defined in part (b) (ii). **2**

(iv) Hence, using (b), prove that for any root ω of $P(x)$, **1**

$$A \le |\omega| \le B .$$

End of Exam

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2012 Bored of Studies Trial Examinations

Mathematics Extension 2

SOLUTIONS

Disclaimer: These solutions may contain small errors. If any are found, please feel free to contact either Carrotsticks or Trebla on www.boredofstudies.org, regarding them.

Thanks: To Trebla, for his many hours spent verifying solutions and suggesting alternate methods.

Multiple Choice

1. A
2. A
3. D
4. C
5. C
6. B
7. B
8. D
9. A
10. A

Brief Explanations

Question 1 Definition of ellipse.

Question 2 Vector must be longer, since we have the reciprocal of a vector that has modulus less than 1. The arguments must also be subtracted, since we are dividing two vectors.

Question 3 Standard *t* substitution integral.

Question 4 Recall that $\omega^n = 1 \Rightarrow \omega^n - 1 = 0 \Rightarrow (\omega - 1)\left(\omega^{n-1} + \omega^{n-2} + ... + \omega + 1\right) = 0$. So we therefore have $\sum_{k=0}^{n-1} \omega^k = 0$. Also, a further manipulation $\omega^k + \frac{1}{\omega^k} = \frac{\omega^{2k} + 1}{\omega^k}$ along with $\omega^{n+1} = \omega^n \times \omega^1 = \omega$ yields the final result.

Question 5 Consider $P(x) = (x-k)^n Q(x)$, where $k > 0$ and $n \in \mathbb{N}$.

Question 6 Standard resolution of forces. Note that *F* points upwards since *V* is less than the optimal speed.

Question 7 Use the inequality $a^2 + b^2 + c^2 \geq ab + bc + ac$ combined with $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac)$ to acquire the answer. Note that $a + b + c = 1$.

Question 8 (B) and (C) are not even. (A) diverges.

Question 9 The locus is just $xy \leq 1$.

Question 10 Use the Conjugate Root Theorem to factorise, then the discriminant.

Written Response

Question 11 (a)

First divide numerator and denominator by e^x, allowed since $e^x > 0$ for all $x \in \mathbb{R}$.

$$\begin{aligned}\int_0^1 \frac{e^{2x}-1}{e^{2x}+1}\,dx &= \int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx \\ &= \ln\left(e^x + e^{-x}\right)\Big|_0^1 \\ &= \ln\left(\frac{e+e^{-1}}{1+1}\right) \\ &= \ln\left(\frac{e^2+1}{2e}\right)\end{aligned}$$

Question 11 (b) (i)

Cross multiply the expressions.

$$\frac{1}{9\cos^2 x - 4\sin^2 x} = \frac{A(x)(3\cos x + 2\sin x) + B(x)(3\cos x - 2\sin x)}{9\cos^2 x - 4\sin^2 x}$$

But recall that $\sin^2 x + \cos^2 x = 1$, so we therefore have, by equating the numerator:

$$\sin^2 x + \cos^2 x = A(x)(3\cos x + 2\sin x) + B(x)(3\cos x - 2\sin x)$$

Group the sines and cosines:

$$\sin^2 x + \cos^2 x = 2\sin x\big(A(x) - B(x)\big) + 3\cos x\big(A(x) + B(x)\big)$$

So therefore, we have:

$$A(x) + B(x) = \frac{1}{3}\cos x \qquad\qquad A(x) - B(x) = \frac{1}{2}\sin x$$

Solving simultaneously, we immediately acquire:

$$A(x) = \frac{1}{2}\left(\frac{1}{3}\cos x + \frac{1}{2}\sin x\right) = \frac{1}{12}(2\cos x + 3\sin x)$$

Similarly:

$$B(x) = \frac{1}{2}\left(\frac{1}{3}\cos x - \frac{1}{2}\sin x\right) = \frac{1}{12}(2\cos x - 3\sin x)$$

Question 11 (b) (ii)

Our integral is now:

$$\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x - 4\sin^2 x} = \frac{1}{12}\int_0^{\frac{\pi}{4}} \left(\frac{2\cos x + 3\sin x}{3\cos x - 2\sin x} + \frac{2\cos x - 3\sin x}{3\cos x + 2\sin x} \right) dx$$

$$= \frac{1}{12}\int_0^{\frac{\pi}{4}} \left(-\frac{-3\sin x - 2\cos x}{3\cos x - 2\sin x} + \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x} \right) dx$$

$$= \frac{1}{12}\int_0^{\frac{\pi}{4}} \left(\frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x} - \frac{-3\sin x - 2\cos x}{3\cos x - 2\sin x} \right) dx$$

$$= \frac{1}{12}\Big[\ln\left(3\cos x + 2\sin x\right) - \ln\left(3\cos x - 2\sin x\right) \Big]_0^{\frac{\pi}{4}}$$

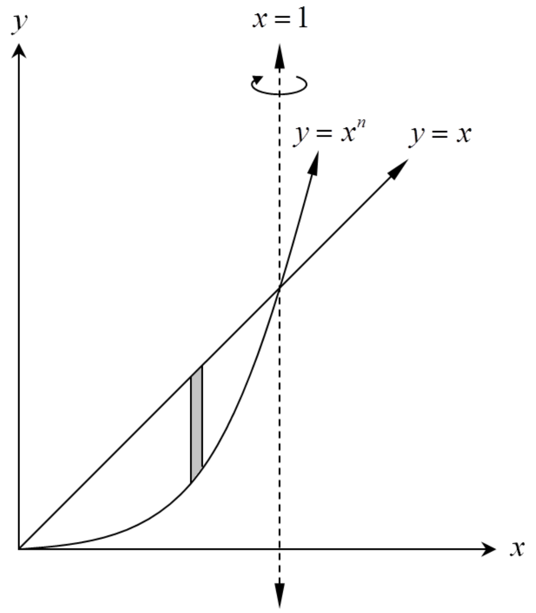
$$= \frac{1}{12}\left[\ln\left(\frac{3\cos x + 2\sin x}{3\cos x - 2\sin x} \right) \right]_0^{\frac{\pi}{4}}$$

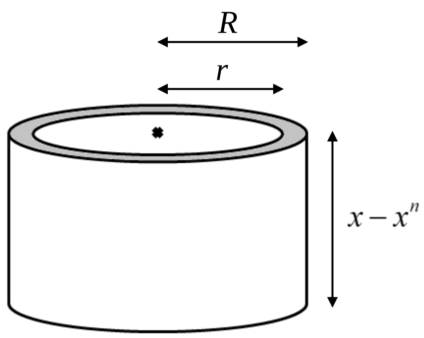
$$= \frac{1}{12}\left[\ln\left(\frac{\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}}}{\frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}}} \right) - \ln\left(1\right) \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{12}\left[\ln\left(5\right) - \ln\left(1\right) \right]$$

$$= \frac{1}{12}\ln 5$$

Question 11 (c) (i)





$R=(1-x)+\delta x$
$r=1-x$

Computing the volume of the shell, we have:

$$\begin{aligned}
\delta V_n &= \pi\left(R^2-r^2\right)\left(x-x^n\right)\\
&= \pi\left(\left((1-x)+\delta x\right)^2-(1-x)^2\right)\left(x-x^n\right)\\
&= \pi\left((1-x)^2+2(1-x)\delta x+(\delta x)^2-(1-x)^2\right)\left(x-x^n\right)\\
&= \pi\left(2(1-x)\delta x+(\delta x)^2\right)\left(x-x^n\right)\\
&\approx 2\pi(1-x)\left(x-x^n\right)\delta x\\
&= 2\pi\left(x-x^n-x^2+x^{n+1}\right)\delta x\\
&= 2\pi\left(x-x^2-x^n+x^{n+1}\right)\delta x
\end{aligned}$$

Summing the shells, we have:

$$\begin{aligned}
V_n &= \lim_{\delta x\to 0}\sum_{0\le x\le 1} 2\pi\left(x-x^2-x^n+x^{n+1}\right)\delta x\\
&= \int_0^1 2\pi\left(x-x^2-x^n+x^{n+1}\right)\delta x\\
&= 2\pi\int_0^1\left(x-x^2-x^n+x^{n+1}\right)\delta x
\end{aligned}$$

$$\begin{aligned}
&= 2\pi\left[\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{n+1}x^{n+1}+\frac{1}{n+2}x^{n+2}\right]_0^1\\
&= 2\pi\left[\frac{1}{2}-\frac{1}{3}-\frac{1}{n+1}+\frac{1}{n+2}\right]\\
&= 2\pi\left[\frac{1}{6}-\frac{1}{n+1}+\frac{1}{n+2}\right]
\end{aligned}$$

Question 11 (c) (ii)

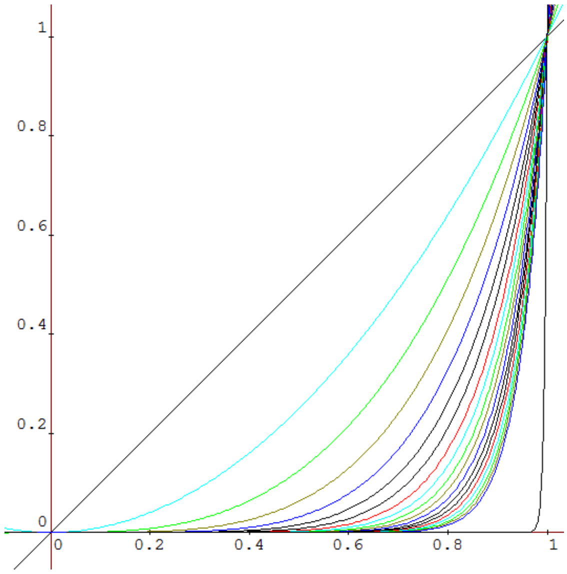
Take the limit as $n \to \infty$.

$$\lim_{n\to\infty} V_n = 2\pi \times \frac{1}{6}$$
$$= \frac{\pi}{3}$$

Question 11 (c) (iii)

We observe what happens to $y = x^n$ as $n \to \infty$.

Since $0 \le x \le 1$, we have $0 \le y \le 1$. We know that as *n* gets large, the curve $y = x^n$ gets lower, in the domain $0 \le x \le 1$.

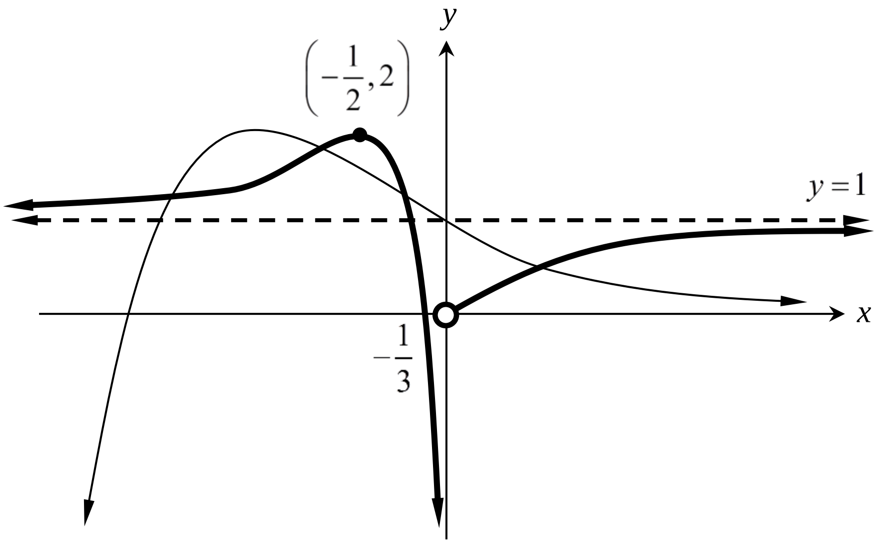


Hence we see that if *n* gets large, then the area between the two curves will approach a right-angled triangle (the lone curve on the right, in the above diagram, is $y = x^{200}$).

So if we rotate this 'triangle' about the line $x = 1$, we will acquire a cone with unit base radius, and unit height.

Using the formula $V_{\text{cone}} = \frac{1}{3}\pi r^2 h$, where $r = 1$ and $h = 1$, we indeed acquire $V_{\text{cone}} = \frac{\pi}{3}$.

Question 11 (d)



Question 12 (a) (i)

Differentiate the expression implicitly.

$$\sqrt{x}+\sqrt{y}=\sqrt{c}$$

$$\begin{aligned}
\frac{1}{2\sqrt{x}}+\frac{y'}{2\sqrt{y}}&=0\\
\frac{1}{\sqrt{x}}+\frac{y'}{\sqrt{y}}&=0\\
\frac{y'}{\sqrt{y}}&=-\frac{1}{\sqrt{x}}\\
y'&=-\frac{\sqrt{y}}{\sqrt{x}}\\
&=-\sqrt{\frac{y}{x}} \qquad \square
\end{aligned}$$

Question 12 (a) (ii)

Using the point-gradient formula:

$$y-y_1=-\sqrt{\frac{y_1}{x_1}}\left(x-x_1\right)$$

The line passes through the point $\left(a,0\right)$:

$$\begin{aligned}-y_1&=-\sqrt{\frac{y_1}{x_1}}\left(a-x_1\right)\\ \sqrt{y_1}&=\frac{1}{\sqrt{x_1}}\left(a-x_1\right)\\ a-x_1&=\sqrt{x_1y_1}\\ a&=\sqrt{x_1y_1}+x_1\\ &=\sqrt{x_1}\left(\sqrt{x_1}+\sqrt{y_1}\right)\\ &=\sqrt{x_1}\times\sqrt{c}\end{aligned}$$

Similarly for the point $\left(0,b\right)$:

$$\begin{aligned}b-y_1&=-\sqrt{\frac{y_1}{x_1}}\left(-x_1\right)\\ &=\sqrt{x_1y_1}\\ b&=\sqrt{x_1y_1}+y_1\\ &=\sqrt{y_1}\left(\sqrt{x_1}+\sqrt{y_1}\right)\\ &=\sqrt{y_1}\times\sqrt{c}\end{aligned}$$

So adding them, we have:

$$\begin{aligned}a+b&=\sqrt{x_1}\times\sqrt{c}+\sqrt{y_1}\times\sqrt{c}\\ &=\sqrt{c}\left(\sqrt{x_1}+\sqrt{y_1}\right)\\ &=\sqrt{c}\times\sqrt{c}\\ &=c \qquad \square\end{aligned}$$

Alternatively

$$y - y_1 = -\sqrt{\frac{y_1}{x_1}}\left(x - x_1\right)$$

We know this passes through $\left(a, 0\right)$:

$$\begin{aligned}
-y_1 &= -\sqrt{\frac{y_1}{x_1}}\left(a - x_1\right) \\
\sqrt{y_1} &= \frac{a - x_1}{\sqrt{x_1}} \\
\sqrt{x_1}\sqrt{y_1} &= a - x_1 \qquad (1)
\end{aligned}$$

Similarly, it passes through $\left(0, b\right)$:

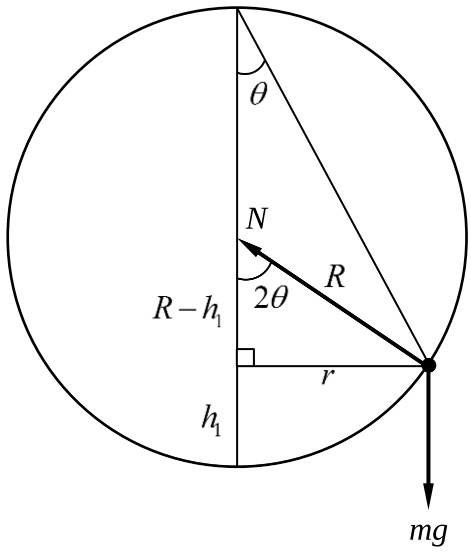
$$\begin{aligned}
b - y_1 &= -\sqrt{\frac{y_1}{x_1}}\left(-x_1\right) \\
b - y_1 &= \sqrt{x_1}\sqrt{y_1} \qquad (2)
\end{aligned}$$

Add the two equations:

$$\begin{aligned}
2\sqrt{x_1}\sqrt{y_1} &= a + b - x_1 - y_1 \\
&= a + b - \left(x_1 + y_1\right) \\
a + b &= x_1 + 2\sqrt{x_1}\sqrt{y_1} + y_1 \\
&= \left(\sqrt{x_1} + \sqrt{y_1}\right)^2 \\
&= \left(\sqrt{c}\right)^2 \\
&= c \qquad \square
\end{aligned}$$

Question 12 (b)

Consider the following diagram.



Resolving forces vertically: $N\cos 2\theta = mg$

Resolving forces horizontally: $N\sin 2\theta = mr\omega^2$

Divide the two expressions to acquire $\tan 2\theta = \dfrac{r\omega^2}{g}$. But we also know that $\tan 2\theta = \dfrac{r}{R-h_1}$, using the right-angled triangle. Equating the two exprssions yields:

$$\begin{aligned}\frac{r\omega^2}{g} &= \frac{r}{R-h_1}\\ \frac{\omega^2}{g} &= \frac{1}{R-h_1}\\ \omega^2 &= \frac{g}{R-h_1}\end{aligned}$$

Our expression is in terms of R. The required expression is in terms of r, so we will use Pythagoras' Theorem to form a relatonship between the two variables.

$$\begin{aligned}r^2 &= R^2 - \left(R-h_1\right)^2\\ &= R^2 - R^2 + 2Rh_1 - h_1^2\\ &= 2Rh_1 - h_1^2\\ R &= \frac{r^2 + h_1^2}{2h_1}\end{aligned}$$

Substitute this into our expression $\omega^2 = \dfrac{g}{R - h_1}$:

$$\begin{aligned}
\omega^2 &= \frac{g}{\dfrac{r^2 + h_1^{\,2}}{2h_1} - h_1} \\
&= \frac{g}{\dfrac{r^2 + h_1^{\,2} - 2h_1^{\,2}}{2h_1}} \\
&= \frac{2gh_1}{r^2 - h_1^{\,2}}
\end{aligned}$$

So therefore to maintain a height greater than h_1, the particle must have angular velocity

$$\omega \geq \sqrt{\frac{2gh_1}{r^2 - h_1^{\,2}}}$$

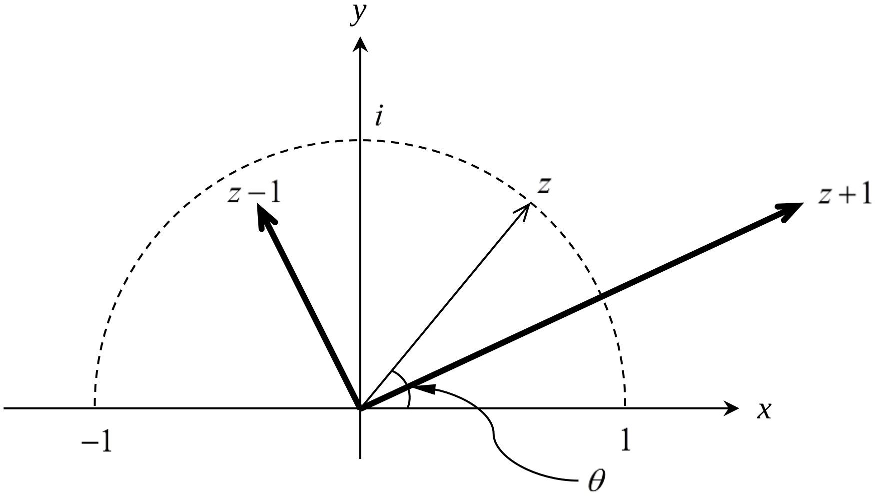
Similarly for h_2, the particle must have angular velocity $\omega \leq \sqrt{\dfrac{2gh_2}{r^2 - h_2^{\,2}}}$.

Therefore we have the required expression:

$$\sqrt{\frac{2gh_1}{r^2 - h_1^{\,2}}} \leq \omega \leq \sqrt{\frac{2gh_2}{r^2 - h_2^{\,2}}} \qquad \square$$

Question 12 (c) (i)

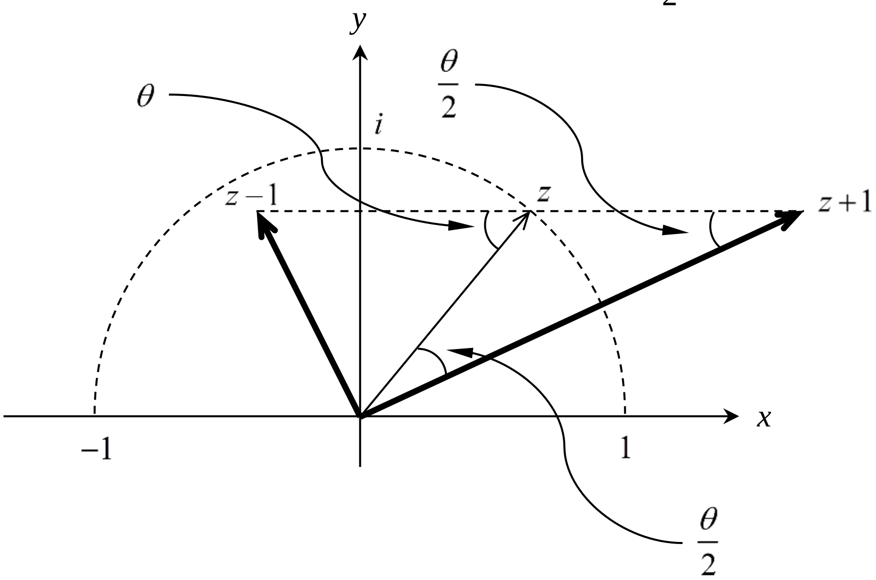
We simply extend the real component one unit forwards and backwards.



Note that as a consequence z, $z-1$ and $z+1$ are now all collinear.

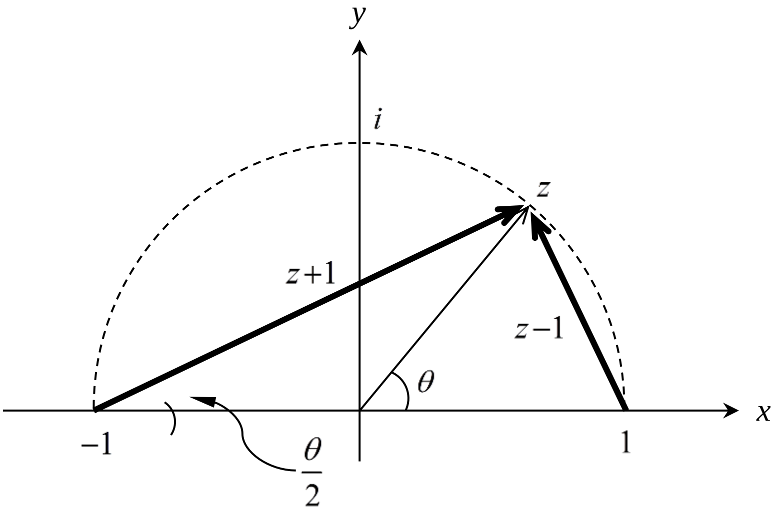
Question 12 (c) (ii)

We will find $\operatorname{Arg}(z-1)$ and $\operatorname{Arg}(z+1)$, and show that they differ by $\frac{\pi}{2}$.



All we need to really spot, is that $\operatorname{Arg}(z)-\operatorname{Arg}(z+1)=\frac{\theta}{2}$, since the triangle enclosed by the origin, z and $z+1$, is an isosceles triangle. A basic angle chase using the exterior angle theorem and the angle sum of triangle quickly yields $\operatorname{Arg}(z-1)-\operatorname{Arg}(z)=\frac{\pi-\theta}{2}$, where it follows immediately that $\operatorname{Arg}(z-1)-\operatorname{Arg}(z+1)=\frac{\pi}{2}$ and thus the result. □

Alternatively



We see immediately that since we have a semi-circle, the angle subtended by the points 1, -1 and z form a right-angled triangle (Thales' Theorem), so therefore we have

$\operatorname{Arg}(z-1)-\operatorname{Arg}(z+1)=\frac{\pi}{2}$, and thus the result. $\square$

Question 12 (c) (iii)

The required statement is equivalent to proving that $\left|\frac{z-1}{z+1}\right|=\tan\left(\frac{\theta}{2}\right)$, noting that $\operatorname{Arg}(z)=\theta$.

From diagram, we can easily see that:

$$\begin{aligned}\tan\left(\frac{\theta}{2}\right)&=\frac{|z-1|}{|z+1|}\\&=\left|\frac{z-1}{z+1}\right| \qquad \square\end{aligned}$$

Question 12 (d) (i)

We know that $\left|BOC\right| = \frac{1}{2} \times BC \times OA'$. We also know that $\left|ABC\right| = \frac{1}{2} \times BC \times AA'$.

Hence putting the expressions in a fraction, we immediately acquire:

$$\begin{aligned}\frac{\left|BOC\right|}{\left|ABC\right|} &= \frac{\frac{1}{2} \times BC \times OA'}{\frac{1}{2} \times BC \times AA'} \\ &= \frac{OA'}{AA'} \qquad \square\end{aligned}$$

Question 12 (d) (ii)

From (i), we have $\frac{OA'}{AA'} = \frac{\left|BOC\right|}{\left|ABC\right|}$.

Repeat for other sides to obtain $\frac{OB'}{BB'} = \frac{\left|AOC\right|}{\left|ABC\right|}$ and $\frac{OC'}{CC'} = \frac{\left|AOB\right|}{\left|ABC\right|}$.

Add all the expressions to acquire:

$$\begin{aligned}\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} &= \frac{\left|BOC\right|}{\left|ABC\right|} + \frac{\left|AOC\right|}{\left|ABC\right|} + \frac{\left|AOB\right|}{\left|ABC\right|} \\ &= \frac{\left|BOC\right| + \left|AOC\right| + \left|AOB\right|}{\left|ABC\right|} \\ &= \frac{\left|ABC\right|}{\left|ABC\right|} \\ &= 1 \qquad \square\end{aligned}$$

Question 13 (a) (i)

Consider the line $y = mx + c$. Let this line be tangential to the hyperbola.

To do so, we will substitute the line into the equation of the hyperbola, then let the discriminant be zero

The equation of the hyperbola is $b^2x^2 - a^2y^2 = a^2b^2$. Substitute $y = mx + c$:

$$b^2x^2 - a^2\left(mx + c\right)^2 = a^2b^2$$

Expand and re-arrange as a quadratic in terms of *x*:

$$\begin{aligned} b^2x^2 - a^2\left(m^2x^2 + 2mcx + c^2\right) &= a^2b^2 \\ b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 &= a^2b^2 \\ x^2\left(b^2 - a^2m^2\right) - 2a^2mcx - a^2\left(b^2 + c^2\right) &= 0 \end{aligned}$$

Let the discriminant be equal to zero:

$$\begin{aligned} 4a^4m^2c^2 + 4a^2\left(b^2 - a^2m^2\right)\left(b^2 + c^2\right) &= 0 \\ a^2m^2c^2 + \left(b^2 - a^2m^2\right)\left(b^2 + c^2\right) &= 0 \\ a^2m^2c^2 + b^4 + b^2c^2 - a^2m^2b^2 - a^2m^2c^2 &= 0 \\ b^4 + b^2c^2 - a^2m^2b^2 &= 0 \\ b^2 + c^2 - a^2m^2 &= 0 \\ c^2 &= a^2m^2 - b^2 \end{aligned}$$

Substitute back into $y = mx + c$:

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad \square$$

Question 13 (a) (ii)

We will re-arrange $y = mx \pm \sqrt{a^2m^2 - b^2}$ be a quadratic in terms of *m*.

$$\begin{aligned} \pm\sqrt{a^2m^2 - b^2} &= mx - y \\ a^2m^2 - b^2 &= (mx - y)^2 \\ a^2m^2 - b^2 &= m^2x^2 - 2mxy + y^2 \end{aligned}$$

Hence our quadratic is $m^2\left(a^2 - x^2\right) + 2mxy - \left(b^2 + y^2\right) = 0$.

Since the two tangents are to be perpendicular to each other, we will let the product of roots be equal to -1, such that $m_1m_2 = -1$.

So immediately, we acquire:

$$\begin{aligned} \frac{-\left(b^2 + y^2\right)}{a^2 - x^2} &= -1 \\ \frac{b^2 + y^2}{a^2 - x^2} &= 1 \\ b^2 + y^2 &= a^2 - x^2 \\ x^2 + y^2 &= a^2 - b^2 \qquad \square \end{aligned}$$

Question 13 (a) (iii)

We know that OT is the radius of the circle, so $OT^2 = a^2 - b^2$.

We also know that $OS = ae$, since S is the focus of the ellipse.

Squaring both sides, we have $OS^2 = a^2e^2$. However, recall the formula $b^2 = a^2\left(1-e^2\right)$:

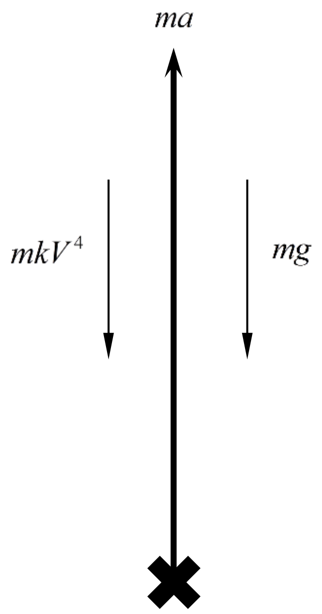
$$\begin{aligned} 1-e^2 &= \frac{b^2}{a^2} \\ e^2 &= 1-\frac{b^2}{a^2} \\ &= \frac{a^2-b^2}{a^2} \end{aligned}$$

So therefore:

$$\begin{aligned} OS^2 &= a^2e^2 \\ &= a^2\left(\frac{a^2-b^2}{a^2}\right) \\ &= a^2-b^2 \\ &= OT^2 \end{aligned}$$

Thus $OS = OT$, so therefore triangle OTS is isosceles.

Question 13 (b) (i)



Balancing the forces, we have:

$$
\begin{aligned}
ma &= -mg - mkV^4 \\
a &= -\left(g + kV^4\right) \\
V\frac{dV}{dx} &= -\left(g + kV^4\right)
\end{aligned}
$$

Re-arrange and integrate both sides, using appropriate limits:

$$
\begin{aligned}
\int_0^{x_t} dx \quad &= -\int_U^{V_t} \frac{V}{g + kV^4}\, dV \\
x_t &= \int_{V_t}^{U} \frac{V}{g + kV^4}\, dV
\end{aligned}
$$

To integrate this, we let $y = V^2$:

$$dy = 2V\, dV \Rightarrow \frac{dy}{2} = V\, dV$$

Finding limits:

$$
\begin{aligned}
V &= U \Rightarrow y = U^2 \\
V &= V_t \Rightarrow y = V_t^2
\end{aligned}
$$

So our integral becomes:

$$x=\frac{1}{2}\int_{V_t^2}^{U^2}\frac{dy}{g+ky^2}$$

$$=\frac{1}{2\sqrt{kg}}\tan^{-1}\left(\frac{y\sqrt{k}}{\sqrt{g}}\right)\Bigg|_{V_t^2}^{U^2}$$

$$=\frac{1}{2\sqrt{kg}}\left[\tan^{-1}\left(\frac{U^2\sqrt{k}}{\sqrt{g}}\right)-\tan^{-1}\left(\frac{V_t^2\sqrt{k}}{\sqrt{g}}\right)\right]$$

$$=\frac{1}{2\sqrt{kg}}\tan^{-1}\left(\frac{\frac{U^2\sqrt{k}}{\sqrt{g}}-\frac{V_t^2\sqrt{k}}{\sqrt{g}}}{1+\frac{U^2\sqrt{k}}{\sqrt{g}}\times\frac{V_t^2\sqrt{k}}{\sqrt{g}}}\right)$$

$$=\frac{1}{2\sqrt{kg}}\tan^{-1}\left(\frac{\frac{\sqrt{k}}{\sqrt{g}}\left(U^2-V_t^2\right)}{1+\frac{kU^2V_t^2}{g}}\right)$$

$$=\frac{1}{2\sqrt{kg}}\tan^{-1}\left(\frac{\frac{\sqrt{k}}{\sqrt{g}}\left(U^2-V_t^2\right)}{\frac{g+kU^2V_t^2}{g}}\right)$$

$$=\frac{1}{2\sqrt{kg}}\tan^{-1}\left(\frac{\sqrt{k}\left(U^2-V_t^2\right)}{\frac{g+kU^2V_t^2}{\sqrt{g}}}\right)$$

$$=\frac{1}{2\sqrt{kg}}\tan^{-1}\left(\frac{\sqrt{kg}\left(U^2-V_t^2\right)}{g+kU^2V_t^2}\right) \qquad \square$$

Question 13 (b) (iii)

For maximum height, let $V = 0$:

$$\begin{aligned}
x &= \frac{1}{2\sqrt{kg}} \tan^{-1}\left(\frac{U^2\sqrt{kg}}{g}\right) \\
&= \frac{1}{2\sqrt{kg}} \tan^{-1}\left(\frac{U^2\sqrt{k}}{\sqrt{g}}\right) \\
&= \frac{1}{2\sqrt{kg}} \tan^{-1}\left(U^2\sqrt{\frac{k}{g}}\right) \qquad \square
\end{aligned}$$

Question 13 (b) (iii)

Finding the limit as $U \to \infty$.

$$\begin{aligned}
\lim_{U\to\infty}\left(\frac{1}{2\sqrt{kg}} \tan^{-1}\left(U^2\sqrt{\frac{k}{g}}\right)\right) &= \frac{1}{2\sqrt{kg}} \lim_{U\to\infty}\left(\tan^{-1}\left(U^2\sqrt{\frac{k}{g}}\right)\right) \\
&= \frac{1}{2\sqrt{kg}} \times \frac{\pi}{2} \\
&= \frac{\pi}{4\sqrt{kg}}
\end{aligned}$$

Question 13 (c) (i)

We will find P_T, then P_S, then arrange them as a ratio.

To find P_T we group the *m* people and we will have $n-m$ people remaining. However, we must also include the group as a single element, so we have $(n-m+1)!$ permutations for the group + everybody else. We then permute the *m* people in the group, which yields $m!$ and therefore $P_T = (n-m+1)!m!$.

To find P_S, we will consider the remaining $n-m$ people. If we put these people next to each other in a line, there will be $n-m+1$ 'gaps', in which we can place the *m* people, who are to be separated. This way, we are guaranteed that they are all separated. Hence to choose positions for the *m* people, we have $\binom{n-m+1}{m}$. We then permute the *m* people, which gives us $m!$ and then the remaining $n-m$ people, giving us $(n-m)!$ Therefore

$$P_S = \binom{n-m+1}{m}(n-m)!m!$$

So putting it as a ratio, we have:

$$\begin{aligned}
\frac{P_S}{P_T} &= \frac{\binom{n-m+1}{m}(n-m)!m!}{(n-m+1)!m!} \\
&= \frac{\binom{n-m+1}{m}}{(n-m+1)} \\
&= \frac{\frac{(n-m+1)!}{m!(n-2m+1)!}}{(n-m+1)} \\
&= \frac{(n-m+1)!}{(n-m+1)m!(n-2m+1)!} \\
&= \frac{(n-m)!}{m!(n-2m+1)!} \\
&= \frac{1}{m}\times\frac{(n-m)!}{(m-1)!(n-2m+1)!} \\
&= \frac{1}{m}\times\binom{n-m}{m-1} \qquad \square
\end{aligned}$$

Question 13 (c) (ii)

Suppose that n is even, we can simply substitute $m = \dfrac{n}{2}$.

$$\begin{aligned}
\frac{P_S}{P_T} &= \frac{1}{n/2} \times \binom{n - n/2}{n/2 - 1} \\
&= \frac{2}{n} \times \binom{n/2}{n/2 - 1} \\
&= \frac{2}{n} \times \frac{(n/2)!}{(n/2 - 1)!(n/2 - n/2 + 1)!} \\
&= \frac{2}{n} \times \frac{(n/2)!}{(n/2 - 1)!} \\
&= \frac{2}{n} \times \frac{n}{2} \\
&= 1
\end{aligned}$$

Therefore $P_S = P_T$.

Alternatively

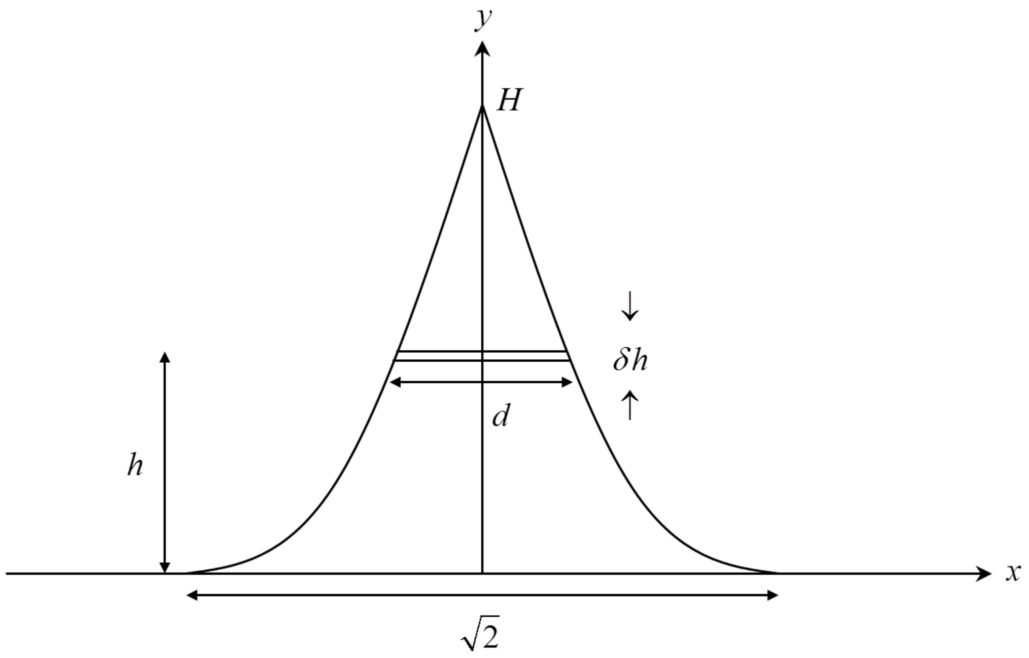
You could substitute $n = 2m$:

$$\begin{aligned}
\frac{P_S}{P_T} &= \frac{1}{m} \times \binom{2m - m}{m - 1} \\
&= \frac{1}{m} \times \binom{m}{m - 1} \\
&= \frac{1}{m} \times m \\
&= 1
\end{aligned}$$

And hence the result. $\square$

Question 14 (a) (i)

If we take the *xy* plane to be along the diagonal of the base, we have the following situation.



However, recall that the base of the vertex lies on the corner of the base, meaning that the vertex is exactly $\frac{\sqrt{2}}{2}$ units away from the base. We therefore have the equation of the parabolas being:

$$y=\left(x-\frac{\sqrt{2}}{2}\right)^2,\ 0\le x\le\frac{\sqrt{2}}{2} \qquad\qquad y=\left(x+\frac{\sqrt{2}}{2}\right)^2,\ -\frac{\sqrt{2}}{2}\le x\le 0$$

To find the value of *h* in terms of *d*, we can just use the function. From the diagram, we see that when $x=\frac{d}{2}$, we have $y=h$.

Using the equation of the parabola on the right, we have:

$$\begin{aligned}
h &= \left(\frac{d}{2}-\frac{\sqrt{2}}{2}\right)^2 \\
\pm\sqrt{h} &= \frac{1}{2}\left(d-\sqrt{2}\right) \qquad \ldots\left(\text{Since } d\le\sqrt{2}\text{, we take the negative root}\right) \\
d-\sqrt{2} &= -2\sqrt{h} \\
d &= \sqrt{2}-2\sqrt{h} \\
&= \sqrt{2}\left(1-\sqrt{2h}\right) \qquad \square
\end{aligned}$$

Question 14 (a) (ii)

Observe that H is simply the y intercept, so it immediately follows, by letting $x = 0$, that $H = \frac{1}{2}$.

Question 14 (a) (iii)

From the diagonal of the slice, we can deduce the area of the slice.

Since the slice is a square, we can use Pythagoras' Theorem. Let the side length of the square be s.

$$
\begin{aligned}
s^2 + s^2 &= d^2 \\
2s^2 &= d^2 \\
&= 2\left(\sqrt{2h} + 1\right)^2 \\
s^2 &= \left(\sqrt{2h} + 1\right)^2
\end{aligned}
$$

So therefore, the volume of the slice is:

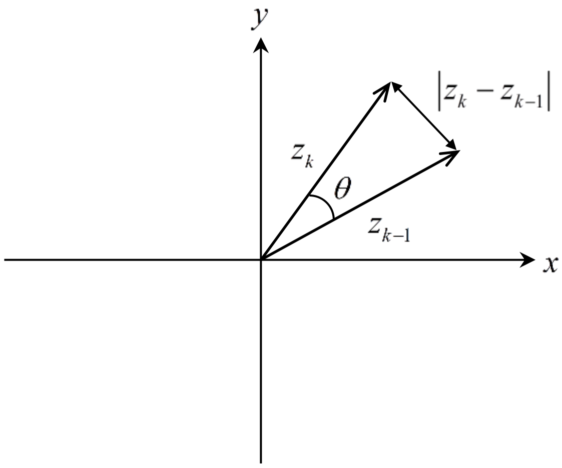
$$\delta V = \left(\sqrt{2h} + 1\right)^2 \delta h$$

Integrate from 0 to $\frac{1}{2}$:

$$
\begin{aligned}
V &= \lim_{\delta h \to 0} \sum_{0 \le h \le 1/2} \left(\sqrt{2h} + 1\right)^2 \delta h \\
&= \int_0^{\frac{1}{2}} \left(\sqrt{2h} + 1\right)^2 dh \\
&= \int_0^{\frac{1}{2}} \left(2h + 2\sqrt{2}h^{\frac{1}{2}} + 1\right) dh \\
&= h^2 - 2\sqrt{2} \times \frac{2}{3} h\sqrt{h} + h \Big|_0^{\frac{1}{2}} \\
&= h^2 - \frac{4\sqrt{2}}{3} h\sqrt{h} + h \Big|_0^{\frac{1}{2}} \\
&= \frac{1}{4} - \frac{4\sqrt{2}}{3} \times \frac{1}{2\sqrt{2}} + \frac{1}{2} \\
&= \frac{1}{12}
\end{aligned}
$$

Question 14 (b) (i)

We will use the Cosine Rule.



Using the Cosine Rule, we have:

$$\begin{aligned}
\left|z_k - z_{k-1}\right|^2 &= \left|z_k\right|^2 + \left|z_k\right|^2 - 2\left|z_k\right|\left|z_k\right|\cos\left(\frac{2\pi}{n}\right) \\
&= 1 + 1 - 2\cos\left(\frac{2\pi}{n}\right) \qquad \ldots\left(\text{since } \left|z_k\right| = 1\right) \\
&= 2 - 2\cos\left(\frac{2\pi}{n}\right) \\
\left|z_k - z_{k-1}\right| &= \sqrt{2 - 2\cos\left(\frac{2\pi}{n}\right)} \quad \square
\end{aligned}$$

Alternatively

For simplicity, we let $x = \dfrac{2k\pi}{n}$ and $y = \dfrac{2(k-1)\pi}{n}$.

$$\begin{aligned}
|z_k - z_{k-1}| &= |\cos x + i\sin x - \cos y - i\sin y| \\
&= |\cos x - \cos y + i[\sin x - \sin y]| \\
&= \sqrt{(\cos x - \cos y)^2 + (\sin x - \sin y)^2} \\
&= \sqrt{\cos^2 x - 2\cos x\cos y + \cos^2 y + \sin^2 x - 2\sin x\sin y + \sin^2 y} \\
&= \sqrt{\cos^2 x + \sin^2 x + \cos^2 y + \sin^2 y - 2(\cos x\cos y + \sin x\sin y)} \\
&= \sqrt{2 - 2\cos\left(\frac{2k\pi}{n} - \frac{2(k-1)\pi}{n}\right)} \\
&= \sqrt{2 - 2\cos\left(\frac{2k\pi - 2(k-1)\pi}{n}\right)} \\
&= \sqrt{2 - 2\cos\left(\frac{2k\pi - 2k\pi + 2\pi}{n}\right)} \\
&= \sqrt{2 - 2\cos\left(\frac{2\pi}{n}\right)} \qquad \square
\end{aligned}$$

Question 14 (b) (ii)

Geometrically, $|z_k - z_{k-1}|$ is the distance between the endpoints of the two vectors.

So it is the equivalent of the length of one side of the polygon. But we have an *n*-sided regular polygon, in which case the perimeter would be $P_n = n|z_k - z_{k-1}|$.

Question 14 (b) (iii)

Let $\theta = \dfrac{2\pi}{n}$, so as $n \to \infty$, $\theta \to 0$.

$$\begin{aligned}
\lim_{n\to\infty} P_n &= \lim_{n\to\infty} n\sqrt{2-2\cos\left(\frac{2\pi}{n}\right)} \\
&= \lim_{\theta\to 0} \frac{2\pi}{\theta}\sqrt{2-2\cos\theta} \\
&= \lim_{\theta\to 0} \frac{2\pi}{\theta}\sqrt{2-2\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)} \\
&= \lim_{\theta\to 0} \frac{2\pi}{\theta}\sqrt{2-2+4\sin^2\left(\frac{\theta}{2}\right)} \\
&= \lim_{\theta\to 0} \frac{2\pi}{\theta}\times 2\sin\left(\frac{\theta}{2}\right) \\
&= \lim_{\theta\to 0} \frac{4\pi}{\theta}\times \sin\left(\frac{\theta}{2}\right) \\
&= \lim_{\theta\to 0} 4\pi\times\frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)}\times\frac{1}{2} \\
&= 4\pi\times\frac{1}{2} \\
&= 2\pi \quad \square
\end{aligned}$$

Alternatively

Let $\theta = \dfrac{\pi}{n}$, so as $n \to \infty$, $\theta \to 0$.

$$\begin{aligned}
\lim_{n\to\infty} P_n &= \lim_{n\to\infty} n\sqrt{2-2\cos\left(\frac{2\pi}{n}\right)} \\
&= \lim_{\theta\to 0} \frac{\pi}{\theta}\sqrt{2-2\cos 2\theta} \\
&= \lim_{\theta\to 0} \frac{\pi}{\theta}\sqrt{2(1-\cos 2\theta)} \\
&= \lim_{\theta\to 0} \frac{\pi}{\theta}\sqrt{4\sin^2\theta} \\
&= \lim_{\theta\to 0} \frac{2\pi\sin\theta}{\theta} \\
&= 2\pi \quad \square
\end{aligned}$$

Question 14 (b) (iv)

The above result is the perimeter of a circle, since the circle is a limiting case of an n-gon when n gets infinitely large.

Question 14 (c) (i)

The point D is a common point.

We know that a circle is unique to a set of three distinct points, in this case D, X and Y. So a circle can indeed be constructed through those points. Since D is a mutual point, CD is also a mutual tangent.

Alternatively

We could do a relatively simple angle chase:

$\angle CDY = \angle DEB$ (Alternate Segment Theorem)

$\angle DEB = \angle DXY$ (Corresponding Angles, since $CW \parallel BE$)

Hence $\angle CDY = \angle DXY$, satisfying the converse of the Alternate Segment Theorem.

Question 14 (c) (ii)

Using the Secant-Chord Theorem with circle DXY, we have:

$$DC^2 = CY \times CX$$

Similarly for circle DBE, we have:

$$DC^2 = CZ \times CW$$

Equate the two ratios:

$$CY \times CX = CZ \times CW$$

Re-arrange the letters to fit the terms of the question:

$$CY \times XC = ZC \times CW$$

Re-arrange the expressions:

$$\frac{CW}{XC} = \frac{CY}{ZC}$$

But note that $CW = CX + XW$ and $CY = CZ + ZY$. Substitute this in:

$$\begin{aligned}
\frac{CX + XW}{XC} &= \frac{CZ + ZY}{ZC} \\
1 + \frac{XW}{XC} &= 1 + \frac{ZY}{ZC} \\
\frac{XW}{XC} &= \frac{ZY}{ZC} \qquad \square
\end{aligned}$$

Question 15 (a) (i)

We evaluate $I_{n+1} = \lim_{a\to\infty} \int_0^a e^{-x}\, dx$.

$$\begin{aligned} I_{n+1} &= \lim_{a\to\infty} \int_0^a e^{-x}\, dx. \\ &= \lim_{a\to\infty} \left[-e^{-x}\right]_0^a \\ &= -\lim_{a\to\infty} \left[e^{-x}\right]_0^a \\ &= -\lim_{a\to\infty} \left[e^{-a} - e^0\right] \\ &= -(-1) \\ &= 1 \quad \square \end{aligned}$$

Question 15 (a) (ii)

Shift the subscript up by 1 to get $I_{n+1} = \lim_{a\to\infty} \int_0^a x^n e^{-x}\, dx$.

Using Integration by Parts, we have:

$$\begin{aligned} u &= x^n & dv &= e^{-x} dx \\ du &= nx^{n-1} dx & v &= -e^{-x} \end{aligned}$$

$$\begin{aligned} I_{n+1} &= \lim_{a\to\infty} -x^n e^{-x}\Big|_0^a + \lim_{a\to\infty} n \int_0^a x^{n-1} e^{-x}\, dx \\ &= \lim_{a\to\infty} -a^n e^{-a} + nI_n \\ &= nI_n \quad \square \end{aligned}$$

Question 15 (a) (iii)

Substitute into itself recursively:

$$\begin{aligned} I_{n+1} &= nI_n \\ &= n(n-1)I_{n-1} \\ &= n(n-1)(n-2)I_{n-2} \\ &= \ldots \\ &= n\times(n-1)\times(n-2)\times\ldots 2\times 1\times I_1 \end{aligned}$$

But $I_1 = 1$, and hence $I_{n+1} = n!$ $\square$

Question 15 (a) (iv)

Using the previous integral $I_{n+1} = \lim_{a\to\infty} \int_0^a x^n e^{-x}\, dx,$ we make a substitution.

Let $x = -\ln t$, such that $dx = -\dfrac{dt}{t}$.

$x = a \Rightarrow t = e^{-a}$
$x = 0 \Rightarrow t = 1$

But since in the previous integral, we had $a \to \infty$, we will similarly have $t \to 0$ and thus the new limit of integration will be some value *b* getting infinitely small.

$$\begin{aligned}
I_{n+1} &= \lim_{b\to 0} \int_1^b (-\ln t)^n e^{\ln t} \times \frac{-dt}{t} \\
&= \lim_{b\to 0} \int_1^b (-1)^{n+1} (\ln t)^n\, t \times \frac{dt}{t} \\
&= \lim_{b\to 0} \int_1^b (-1)^{n+1} (\ln t)^n\, dt \\
&= \lim_{b\to 0} \int_b^1 (-1)^n (\ln t)^n\, dt \qquad \ldots(\text{swapping limits uses up a } -1) \\
&= (-1)^n \lim_{b\to 0} \int_b^1 (\ln t)^n\, dt \\
&= (-1)^n J_n
\end{aligned}$$

So thus we have $(-1)^n J_n = I_{n+1}$. But from (i), we have $I_{n+1} = nI_n$, which implies $(-1)^n J_n = nI_n$.

Hence $\dfrac{I_n}{J_n} = \dfrac{(-1)^n}{n}$ □

Alternatively

You could work out the actual reduction formula for J_n, then find the ratio $\frac{I_n}{J_n}$.

$$u = (\ln t)^n \qquad dv = dt$$

$$du = n \times \frac{(\ln t)^{n-1}}{t} dt \qquad v = t$$

$$\begin{aligned}
J_n &= \lim_{b \to 0} \int_b^1 (\ln t)^n \, dt \\
&= \lim_{b \to 0} \Big[t(\ln t)^n \Big]_b^1 - n \times \lim_{b \to 0} \int_b^1 (\ln t)^{n-1} \, dt \\
&= \lim_{b \to 0} \Big[0 - b(\ln b)^n \Big] - n \times \lim_{b \to 0} \int_b^1 (\ln t)^{n-1} \, dt \\
&= -n \times \lim_{b \to 0} \int_b^1 (\ln t)^{n-1} \, dt \qquad \ldots(\text{see Note}) \\
&= -nJ_{n-1}
\end{aligned}$$

> **NOTE**
>
> All people, who attempted this question and acquired the result, assumed the limit $\lim_{b \to 0} b(\ln b)^n = 0$ without proof.
>
> However, they were not penalised because the average would have otherwise been too low.
>
> The proof for this, which has been omitted from this document, uses the *Squeeze Law* and *L'Hopital's Principle*.

Computing J_0:

$$\begin{aligned}
J_0 &= \lim_{b \to 0} \int_b^1 (\ln t)^0 \, dt \\
&= \lim_{b \to 0} \int_b^1 dt \\
&= 1
\end{aligned}$$

So recursively, we acquire $J_n = (-1)^n n!$.

Placing this as a ratio with I_n:

$$\begin{aligned}
\frac{I_n}{J_n} &= \frac{(n-1)!}{(-1)^n n!} \\
&= \frac{(-1)^n}{n} \qquad \square \qquad \ldots\left(\text{since } \frac{1}{(-1)^n} = (-1)^n\right)
\end{aligned}$$

Question 15 (b) (i)

Summing the areas of the upper rectangles, we have

$$\begin{aligned}U &= f(1)\times 1 + f(2)\times 1 + f(3)\times 1 + \ldots + f(n-1)\times 1\\ &= f(1) + f(2) + f(3) + \ldots + f(n-1)\\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\\ &= H_{n-1}\end{aligned}$$

Observe that:

$$\begin{aligned}E_n &= H_{n-1} - \int_1^n \frac{1}{x}\,dx\\ &= H_{n-1} - \ln n\end{aligned}$$

Adding $\frac{1}{n}$ to both sides, we acquire:

$$\begin{aligned}E_n + \frac{1}{n} &= H_{n-1} - \ln n + \frac{1}{n}\\ &= H_n - \ln n \qquad \square\end{aligned}$$

Question 15 (b) (ii)

Taking the limit as $n \to \infty$, we have:

$$\begin{aligned}\lim_{n\to\infty}\left(H_n - \ln n\right) &= \lim_{n\to\infty} E_n + \frac{1}{n}\\ &= \gamma\end{aligned}$$

Since for large values of *n*, the term $\frac{1}{n}$ approaches 0.

Question 15 (b) (iii)

We will use Mathematical Induction on *n*.

Base Case: $n=1$**.**

$$
\begin{aligned}
LHS &= S_2 \\
&= \sum_{r=1}^{2} \frac{(-1)^r}{r} \\
&= -\frac{1}{1}+\frac{1}{2} \\
&= -\frac{1}{2}
\end{aligned}
\qquad\qquad
\begin{aligned}
RHS &= H_1 - H_2 \\
&= \sum_{r=1}^{1} \frac{1}{r} - \sum_{r=1}^{2} \frac{1}{r} \\
&= \frac{1}{1}-\frac{1}{2}-\frac{1}{1} \\
&= -\frac{1}{2}
\end{aligned}
$$

Thus the base case has been proven.

Inductive Hypothesis: $n=k$**.**

$S_{2k} = H_k - H_{2k}$, for all $k \in \mathbb{N}$.

Inductive Step: $k \Rightarrow k+1$**.**

Required to prove: $S_{2k+2} = H_{k+1} - H_{2k+2}$

$$
\begin{aligned}
S_{2k+2} &= \sum_{r=1}^{2k+2} \frac{(-1)^r}{r} \\
&= \sum_{r=1}^{2k} \frac{(-1)^r}{r} + \sum_{r=2k+1}^{2k+2} \frac{(-1)^r}{r} \\
&= S_{2k} - \frac{1}{2k+1} + \frac{1}{2k+2} \\
&= H_k - H_{2k} - \frac{1}{2k+1} + \frac{1}{2(k+1)} \\
&= H_k - H_{2k} - \frac{1}{2k+1} + \frac{1}{k+1} - \frac{1}{2(k+1)} \\
&= \left(H_k + \frac{1}{k+1}\right) - \left(H_{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}\right) \\
&= H_{k+1} - H_{2k+2} \\
&= RHS
\end{aligned}
$$

Thus, via proof of induction, true for all $n \in \mathbb{N}$. □

Question 15 (b) (iv)

Recall that $I_{r+1} = r!$

We know that $S_{2n} = H_n - H_{2n}$.

Using previous parts, we have:

$$\lim_{n\to\infty}\left(H_n - \ln n\right) = \gamma \qquad \lim_{n\to\infty}\left(H_{2n} - \ln 2n\right) = \gamma$$

So using the expression $S_{2n} = H_n - H_{2n}$, we have:

$$\begin{aligned}
S_{2n} &= \left(H_n - \ln n\right) - \left(H_{2n} - \ln 2n\right) + \left(\ln n - \ln 2n\right) \\
&= \left(H_n - \ln n\right) - \left(H_{2n} - \ln 2n\right) + \ln\left(\frac{n}{2n}\right) \\
&= \left(H_n - \ln n\right) - \left(H_{2n} - \ln 2n\right) + \ln\left(\frac{1}{2}\right)
\end{aligned}$$

Taking the limit as $n \to \infty$ of both sides:

$$\begin{aligned}
\lim_{n\to\infty} S_{2n} &= \lim_{n\to\infty}\left[\left(H_n - \ln n\right) - \left(H_{2n} - \ln 2n\right) + \ln\left(\frac{1}{2}\right)\right] \\
&= \gamma - \gamma + \ln\left(\frac{1}{2}\right) \\
&= \ln\left(\frac{1}{2}\right)
\end{aligned}$$

But we observe that the summands are $\dfrac{I_r}{J_r}$, so:

$$\begin{aligned}
\lim_{n\to\infty} S_{2n} &= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(-1)^r}{r} \\
&= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{I_r}{J_r} \\
&= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(r-1)!}{J_r} \\
&= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{r!}{J_{r+1}}
\end{aligned}$$

Hence $\displaystyle\lim_{n\to\infty}\sum_{r=0}^{n}\frac{r!}{J_{r+1}} = \ln\left(\frac{1}{2}\right)$ □

Question 16 (a)

We first prove a standard result: $|x-y| \le |x|+|y|$.

$$
\begin{aligned}
|x-y| &= |x+(-y)| \\
&\le |x|+|-y| \\
&= |x|+|y| \qquad \square
\end{aligned}
$$

Using the above inequality, we have

$$
\begin{aligned}
\left|(z_0+z_1+z_2+\ldots+z_n)-(z_1+z_2+\ldots+z_n)\right| &\le |z_0+z_1+z_2+\ldots+z_n|+|z_1+z_2+\ldots+z_n| \\
|z_0| &\le |z_0+z_1+z_2+\ldots+z_n|+|z_1+z_2+\ldots+z_n| \\
&\le |z_0+z_1+z_2+\ldots+z_n|+|z_1|+|z_2|+\ldots+|z_n| \\
|z_0|-|z_1|-|z_2|-\ldots-|z_n| &\le |z_0+z_1+z_2+\ldots+z_n| \qquad \square
\end{aligned}
$$

Question 16 (b) (i) (1)

$$
\begin{aligned}
|(1-x)R(x)| &= \left|(1-x)\sum_{k=0}^{n} b_k x^k\right| \\
&= \left|\sum_{k=0}^{n} b_k x^k - x\sum_{k=0}^{n} b_k x^k\right| \\
&= \left|\sum_{k=0}^{n} b_k x^k - \sum_{k=0}^{n} b_k x^{k+1}\right| \\
&= \left|\left(b_0+b_1x+b_2x^2+\ldots+b_nx^n\right)-\left(b_0x+b_1x^2+b_2x^3+\ldots+b_nx^{n+1}\right)\right| \\
&= \left|b_0+(b_1-b_0)x+(b_2-b_1)x^2+\ldots+(b_n-b_{n-1})x^n-b_nx^{n+1}\right| \\
&\ge |b_0|-|(b_1-b_0)x|-|(b_2-b_1)x^2|-\ldots-|(b_n-b_{n-1})x^n|-|b_nx^{n+1}| \\
&= |b_0|-|b_1-b_0||x|-|b_2-b_1||x|^2-\ldots-|b_n-b_{n-1}||x|^n-|b_n||x|^{n+1} \\
&= |b_0|-(b_0-b_1)|x|-(b_1-b_2)|x|^2-\ldots-(b_{n-1}-b_n)|x|^n-|b_n||x|^{n+1} \qquad \ldots(\text{since } \mathrm{b}_n \le b_{n-1}) \\
&= b_0-\sum_{k=1}^{n}(b_{k-1}-b_k)|x|^k-b_n|x|^{n+1} \qquad \square
\end{aligned}
$$

Question 16 (b) (i) (2)

When $|x|<1$, then the inequality becomes

$$\begin{aligned}\left|(1-x)R(x)\right| &\ge b_0-\sum_{k=1}^{n}\left(b_{k-1}-b_k\right)|x|^k-b_n|x|^{n+1}\\ &> b_0-\sum_{k=1}^{n}\left(b_{k-1}-b_k\right)-b_n\\ &=0 \quad \square \qquad \text{...(since we have a Telescoping Sum)}\end{aligned}$$

Question 16 (b) (i) (3)

Since $\left|(1-x)R(x)\right|>0$ when $|x|<1$, we have $\left|R(x)\right|>0$. This means that since $R(x)\neq 0$ when $|x|<1$, any root α of $R(x)$ must satisfy $|\alpha|\ge 1$.

Question 16 (b) (ii) (1)

Since β is a root of $S(x)$, we have $S(\beta)=c_0+c_1\beta+c_2\beta^2+...+c_n\beta^n=0$.

We now evaluate $T\left(\frac{1}{\beta}\right)$.

$$\begin{aligned}T\left(\frac{1}{\beta}\right) &= \sum_{k=0}^{n}c_{n-k}\frac{1}{\beta^k}\\ &=\sum_{k=0}^{n}c_{n-k}\beta^{-k}\\ &=\frac{1}{\beta^n}\times\sum_{k=0}^{n}c_{n-k}\beta^{n-k}\\ &=\frac{1}{\beta^n}\times\left(c_n\beta^n+c_{n-1}\beta^{n-1}+...+c_1\beta+c_0\right)\\ &=0 \quad \square\end{aligned}$$

Question 16 (b) (ii) (2)

Likewise for $R(x)$, we acquire:

$$\left|(1-x)T(x)\right| \geq |c_n| - \sum_{k=1}^{n} |c_k - c_{k-1}||x|^{n-k} - |c_0||x|^{n+1}$$

Note that the exponent is $n-k$ and that we have $c_k - c_{k-1}$ as opposed to $c_{k-1} - c_k$ like before. This is due to the nature of the coefficients (in a sense 'reversed') of $T(x)$.

Again, similarly to (i), having $|x|<1$ yields us another Telescoping Sum, which results in $\left|(1-x)T(x)\right| > 0$. Suppose γ is some root of $T(x)$. We then have $|\gamma| \geq 1$, similarly to (i).

However, recall that $\gamma = \dfrac{1}{\beta}$ due to (b) (ii) (1), hence $\left|\dfrac{1}{\beta}\right| \geq 1$ and therefore $|\beta| \leq 1$. □

Question 16 (c) (i)

We know that $A \leq \dfrac{a_{k-1}}{a_k}$, for all $1 \leq k \leq n$, since A is defined to be the minimum.

Re-arranging, we have:

$$\frac{A^k}{A^{k-1}} \leq \frac{a_{k-1}}{a_k}$$
$$a_k A^k \leq a_{k-1} A^{k-1}$$

And hence we have (noting that all the values are positive since $a_k > 0 \Rightarrow A > 0$) :

$$0 < a_n A^n \leq a_{n-1} A^{n-1} \leq \ldots \leq a_2 A^2 \leq a_1 A \leq a_0 \quad \square$$

Question 16 (c) (ii)

If we substitute the expression $b_k = a_k A^k$ into the inequality, we have

$0 < b_n \leq b_{n-1} \leq \ldots \leq b_1 \leq b_0$

Since $P(\omega) = 0$, then

$$\begin{aligned}
\sum_{k=0}^{n} a_k \omega^k &= 0 \\
\sum_{k=0}^{n} b_k \frac{\omega^k}{A^k} &= 0 \\
\sum_{k=0}^{n} b_k \left(\frac{\omega}{A}\right)^k &= 0 \\
R\left(\frac{\omega}{A}\right) &= 0
\end{aligned}$$

The claim, that the polynomial *R* appears here, is valid because of the inequality with the coefficients.

Hence $\dfrac{\omega}{A}$ is a root of $R(x)$. □

Question 16 (c) (iii)

Similarly to (c) (ii), we have $a_{k-1}B^{k-1} \geq a_k B^k$ and hence $0 < a_0 \leq a_1 B \leq \ldots \leq a_{n-1}B^{n-1} \leq a_n B^n$.

Using the substitution $c_k = a_k B^k$, we have $0 < c_0 \leq c_1 \leq \ldots \leq c_{n-1} \leq c_n$. Hence

$$\begin{aligned}
\sum_{k=0}^{n} a_k \omega^k &= 0 \\
\sum_{k=0}^{n} c_k \frac{\omega^k}{B^k} &= 0 \\
\sum_{k=0}^{n} c_k \left(\frac{\omega}{B}\right)^k &= 0 \\
S\left(\frac{\omega}{B}\right) &= 0
\end{aligned}$$

The claim, that we have the polynomial *S*, is valid here because of the inequality with the coefficients.

And hence $\dfrac{\omega}{B}$ is a root of $S(x)$.

Question 16 (d)

Since $\beta = \dfrac{\omega}{B}$ and $|\beta| \le 1$, we have $\left|\dfrac{\omega}{B}\right| \le 1$ and hence $|\omega| \le B$.

Similarly, since $\alpha = \dfrac{\omega}{A}$ and $|\alpha| \ge 1$, we have $\left|\dfrac{\omega}{A}\right| \ge 1$ and hence $|\omega| \ge A$.

Therefore, we obtain the inequality $A \le |\omega| \le B$. $\square$