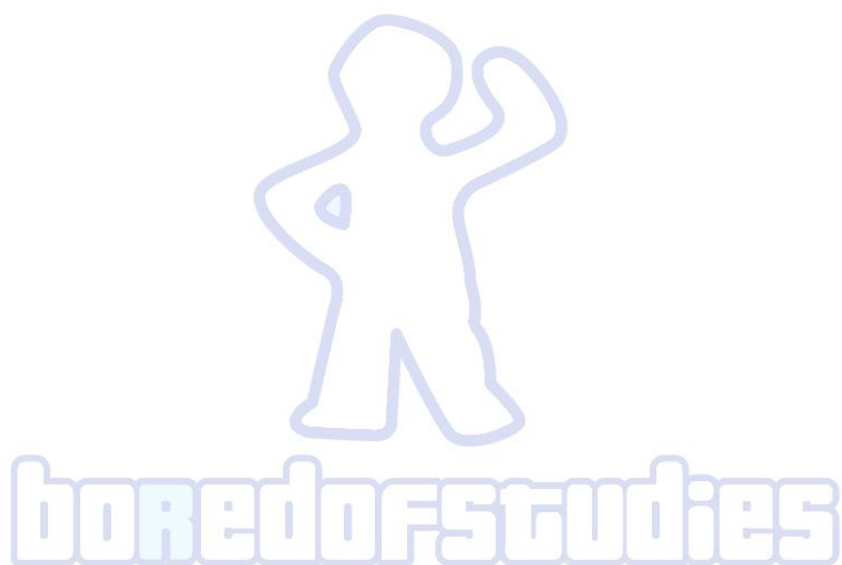




2015 Bored of Studies Trial Examinations

Mathematics Extension 2

Written by Carrotsticks & Trebla

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 1 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 7 – 22

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Total marks – 10

Attempt Questions 1 – 10

All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

1 The equation $x^2 - 2ix + k = 0$ has a root of $-1 + 2i$? What is the value of k ?

(A) $-1 + 2i$.

(B) $-1 - 2i$.

(C) 1.

(D) 5.

2 Let e be the eccentricity of a conic section with both foci on the x axis.

Which of the following is NOT always true?

(A) For the hyperbola, as $e \rightarrow \infty$, the asymptotes approach the y axis.

(B) If two ellipses have equal eccentricity, then they have the same equation.

(C) If two hyperbolae have equal eccentricity, then they share the same asymptotes.

(D) For the ellipse, as $e \rightarrow 0$, the directrices move further away from the origin whilst the foci approach the origin.

- 3 The hyperbolae $\mathcal{H}_1$ and $\mathcal{H}_2$ have equations $x^2 - y^2 = a^2$ and $xy = a^2$ respectively, where a is a positive constant.

Which of the following is a point of intersection of one of the directrices of $\mathcal{H}_2$ and one of the asymptotes of $\mathcal{H}_1$?

- (A) $\left(\frac{a}{2}, \frac{a}{2}\right)$.
- (B) $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$.
- (C) $(2a, 2a)$.
- (D) $(a\sqrt{2}, a\sqrt{2})$.

- 4 Sophia is given four options to transform any given curve.

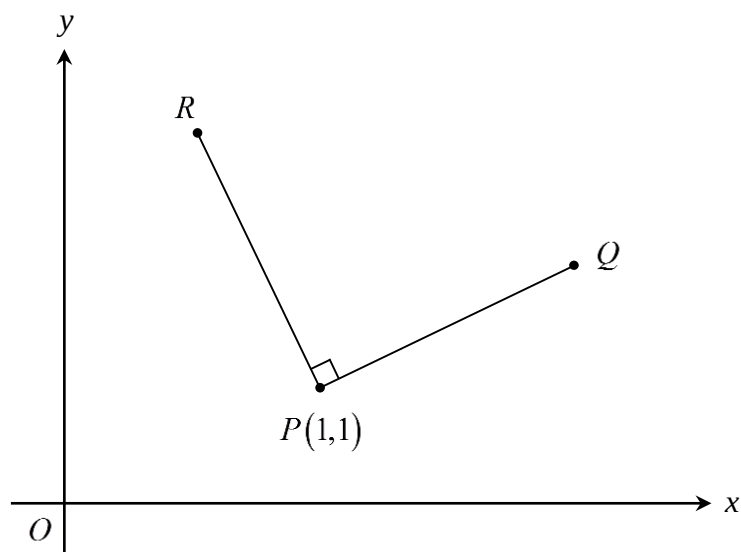
- (I) Translate 1 unit to the left, then double the x coordinates.
- (II) Double the x coordinates, then translate 1 unit to the left.
- (III) Reflect in the line $x = -1/2$, then multiply the x coordinates by -2 .
- (IV) Multiply the x coordinates by -2 , then reflect in the line $x = -1/2$.

She is then given a sketch of the curve $y = f(2x - 1)$.

Which of the above options should she select to obtain the graph of $y = f(x)$?

- (A) Options (I) or (III).
- (B) Options (I) or (IV).
- (C) Options (II) or (III).
- (D) Options (II) or (IV).

- 5 Let P be a point on the Argand diagram with coordinates $(1,1)$. Let Q be another point such that the complex number represented by vector PQ is given by $a + bi$.



The vector PQ is rotated about the point P anti-clockwise by an angle of $\frac{\pi}{2}$ to form the vector PR .

What are the coordinates of R ?

- (A) $(-b, a)$.
- (B) $(1-b, 1+a)$.
- (C) $(b, -a)$.
- (D) $(1+b, 1-a)$.

- 6 Let α be a non-zero complex number and $P(x)$ be a monic polynomial with real coefficients such that $P(\alpha) = P'(\alpha) = 0$.

Which of the following statements is always true?

- (A) The complex number α must be a real number.
 - (B) The polynomial $P(x)$ has a stationary point at $(\alpha, 0)$.
 - (C) If α is non-real, then the polynomial $P(x)$ must have even degree.
 - (D) If α is non-real, then the smallest possible degree of $P(x)$ is 4.
- 7 A solid of revolution is formed by rotating the region bounded by $y = 2x - x^2$, the line $x = 1$ and the x axis about the line $x = 1$.

By using the method of cylindrical shells, which of the following integrals is equal to the volume of the solid?

- (A) $\pi \int_0^1 (1-x)(2x-x^2) dx.$
- (B) $2\pi \int_0^1 x(2x-x^2) dx.$
- (C) $\pi \int_0^1 x dx.$
- (D) $2\pi \int_0^1 x dx.$

- 8 A particle moves in uniform circular motion about the origin with angular velocity ω .

Let x be the horizontal displacement of the particle.

Which of the following is the correct expression for the horizontal component of the particle's acceleration?

(A) $\ddot{x} = -\omega x$.

(B) $\ddot{x} = \omega x$.

(C) $\ddot{x} = -\omega^2 x$.

(D) $\ddot{x} = \omega^2 x$.

- 9 Let a , b and c be the lengths of the sides of $\triangle ABC$, where the longest side has length a .

If $\angle BAC$ is obtuse, which of the following inequalities is always true?

(A) $a^2 > 2bc$.

(B) $a^2 > b + c$.

(C) $a^2 < 2bc$.

(D) $a^2 < b + c$.

10 Let n and k be positive integers.

Which of the following is equivalent to $\int_0^\pi \sin^n(x) \cos(2kx) dx$?

(A) $\int_0^{\frac{\pi}{2}} \sin^n\left(\frac{x}{2}\right) \cos(kx) dx.$

(B) $\int_{\frac{\pi}{2}}^\pi \sin^n\left(\frac{x}{2}\right) \cos(kx) dx.$

(C) $\int_0^\pi \sin^n\left(\frac{x}{2}\right) \cos(kx) dx.$

(D) $\int_0^{2\pi} \sin^n\left(\frac{x}{2}\right) \cos(kx) dx.$

Total marks – 90

Attempt Questions 11 – 16

All questions are of equal value

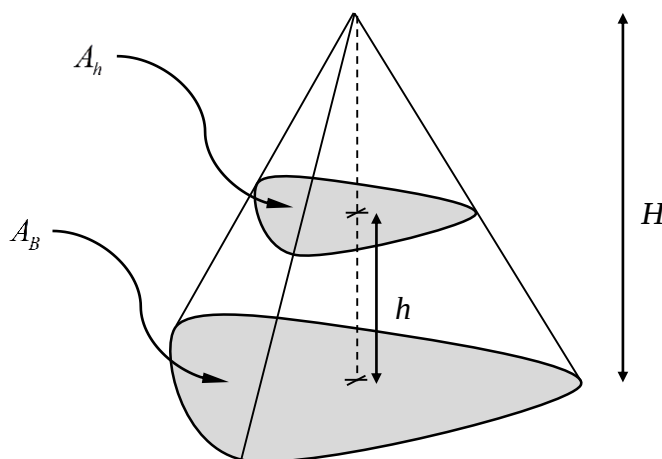
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{\sqrt{\sin \theta}}{\cos \theta \sin \theta} d\theta$. 3

- (b) The diagram below shows a right pyramid with height H and an irregularly shaped base with area A_B . 3

Cross sections taken parallel to the ground at height h have area A_h .



Show that the volume of the pyramid is $\frac{1}{3} A_B H$.

- (c) The polynomial $P(x) = x^3 - px^2 + p$ has roots α, β, γ . 4

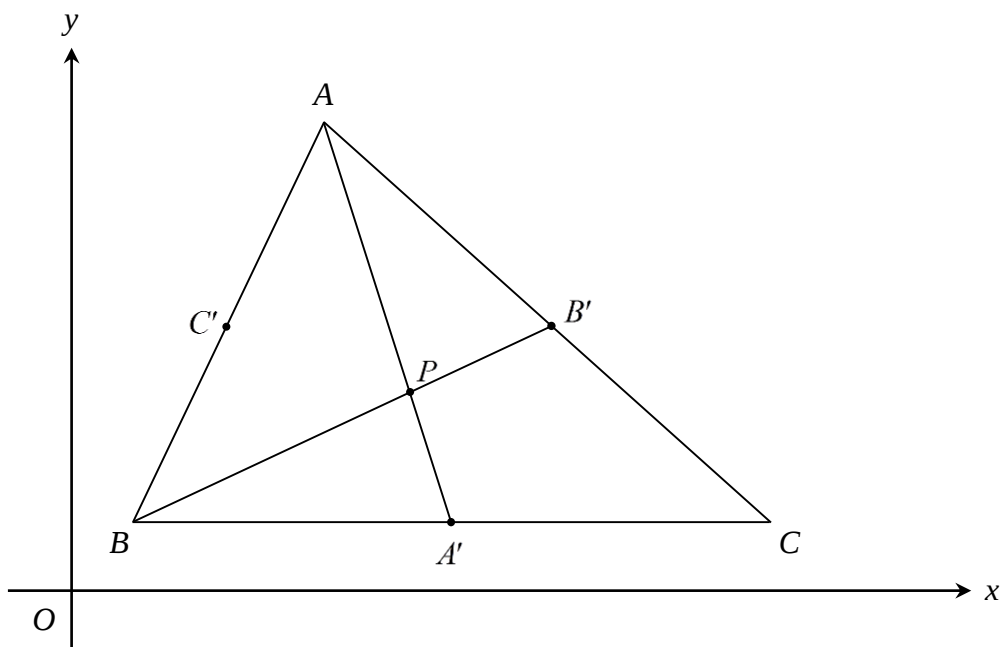
Find the values of p such that

$$\frac{1}{(\alpha + \beta)^2} + \frac{1}{(\beta + \gamma)^2} + \frac{1}{(\alpha + \gamma)^2} = 0.$$

Question 11 continues on page 8

Question 11 (continued)

- (d) Points A , B and C represent the complex numbers a , b and c respectively on the Argand diagram, as shown below.



Let A' , B' and C' be the midpoints of BC , AC and AB respectively.

Let the intersection of AA' and BB' be P , representing the complex number p .

You may assume, without proof, that $\frac{AP}{AA'} = \frac{BP}{BB'} = k$, where $0 < k < 1$.

- (i) Show that any point P on the interval AA' can be expressed as 1

$$p = (1-k)a + \frac{k}{2}(b+c).$$

- (ii) Use (i) to show that $p = \frac{a+b+c}{3}$. 3

- (iii) Deduce that P also lies on CC' . 1

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the curve $\mathcal{C}$ implicitly defined by the equation

$$x^3 - y^3 = 3\ln(xy).$$

(i) Show that $(1,1)$ is a critical point. 2

(ii) Let (x_0, y_0) be any point on $\mathcal{C}$. 1

Show that $(-y_0, -x_0)$ also lies on $\mathcal{C}$.

(iii) By using the fact that 1

$$\lim_{x \rightarrow \infty} \left(\frac{\ln x}{x^3} \right) = 0,$$

or otherwise, show that if $x \rightarrow \infty$, then $y \rightarrow \infty$.

(iv) By considering the expression 2

$$\left(\frac{y^3}{y^3 + 3\ln y} \right) \left(\frac{x^3 - 3\ln x}{x^3} \right),$$

or otherwise, show that $y = x$ is an asymptote.

(v) Hence, sketch the graph of the curve. 2

Question 12 continues on page 10

Question 12 (continued)

- (b) Consider the equation $z^n = 1$ with roots $\alpha_k = \cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$ for $k = 1, 2, 3, \dots, n$.

- (i) Show that 1

$$(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) \dots (1 - \alpha_{n-1}) = n.$$

- (ii) Hence, show that 3

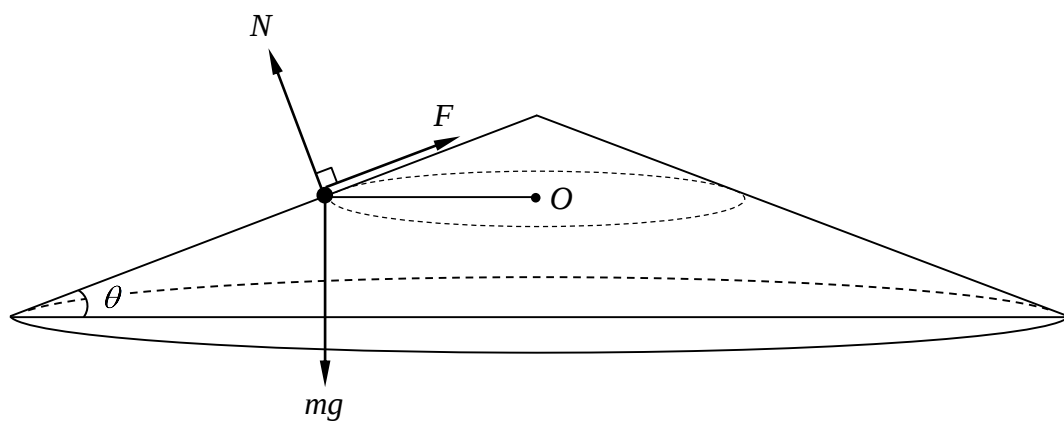
$$\sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \sin\left(\frac{3\pi}{n}\right) \dots \sin\left(\frac{n-1}{n}\pi\right) = \frac{n}{2^{n-1}}.$$

Question 12 continues on page 11

Question 12 (continued)

- (c) The diagram below shows a car of mass m on the cross section of a bend in a highway, which is part of a circle of radius r and centre O . The road surface is banked at an angle θ to the horizontal.

The car experiences a normal force N , a lateral force F directed up the bend and a gravitational force g .



- (i) Find the maximum possible velocity of the car before it leaves the surface of the cone. 2
- (ii) Hence, or otherwise, show that the maximum amount of lateral force the car will experience is 1

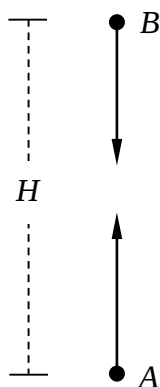
$$F = \frac{mg}{\sin \theta}.$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

- (a) A particle A of mass m is projected vertically with initial velocity u and reaches maximum height H . It experiences gravitational force mg and a resistive force mkv_A^2 , where v_A is the velocity of the particle and k is a constant.

At the same time, particle B of equal mass is dropped from height H above the ground. It experiences gravitational force mg and resistive force mkv_B^2 .



Let w be the terminal velocity of particle B .

- (i) Show that the displacement of particle A from the origin is 3

$$x_A = \frac{1}{2k} \ln \left(\frac{w^2 + u^2}{w^2 + v_A^2} \right),$$

where v_A is the velocity of particle A .

- (ii) The two particles collide when particle A reaches velocity w . 2

The displacement of particle B from the point of release is given by

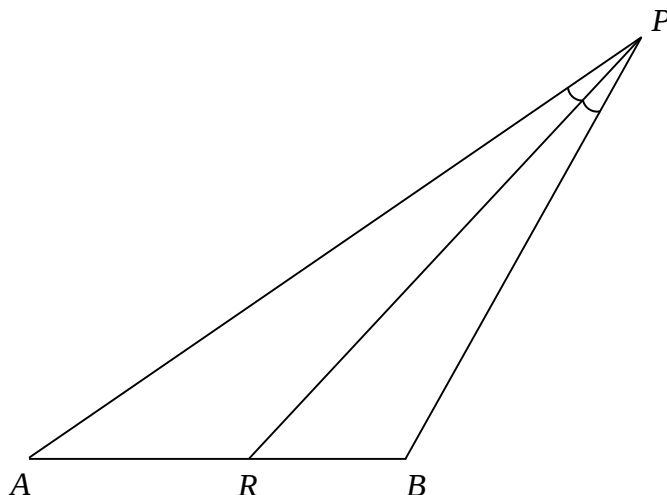
$$x_B = \frac{1}{2k} \ln \left(\frac{w^2}{w^2 - v_B^2} \right) \quad \text{(Do NOT prove this)}$$

Use this result to show that when the particles collide, particle B has velocity $v_B = \frac{w}{\sqrt{2}}$.

Question 13 continues on page 13

Question 13 (continued)

- (b) In $\triangle PAB$, the point R is chosen so that PR bisects $\angle APB$.

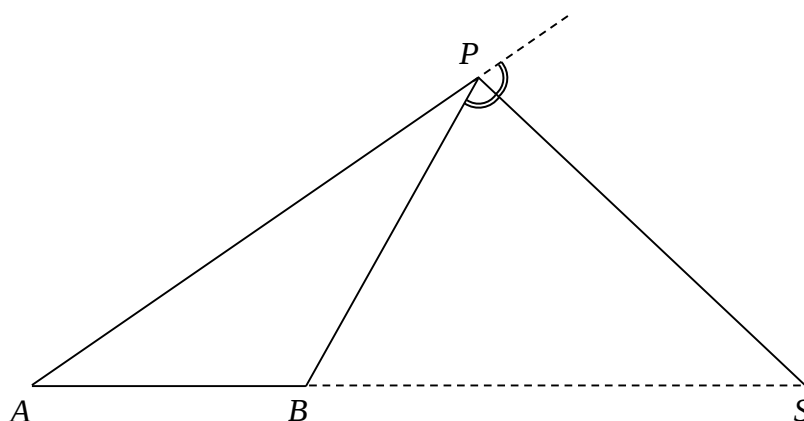


- (i) Prove that $\frac{AP}{BP} = \frac{AR}{BR}$.

2

It can similarly be shown that if PS bisects the exterior angle of $\angle APB$, then

$$\frac{AP}{BP} = \frac{AS}{BS}. \quad \text{(Do NOT prove this)}$$



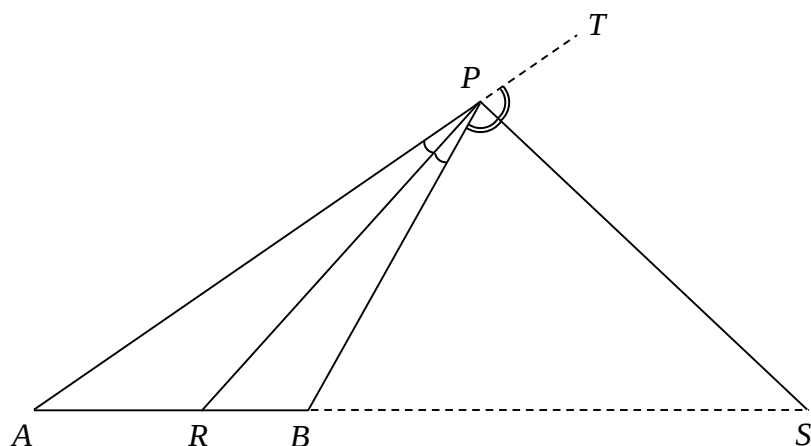
Question 13 continues on page 14

Question 13 (continued)

- (ii) The diagram below shows fixed points A and B , and a point P so that 2

$$\frac{PA}{PB} = k.$$

for some fixed $k > 1$.



The interval AP is produced to T and the points R and S are chosen such that PR and PS bisect $\angle APB$ and $\angle BPT$ respectively.

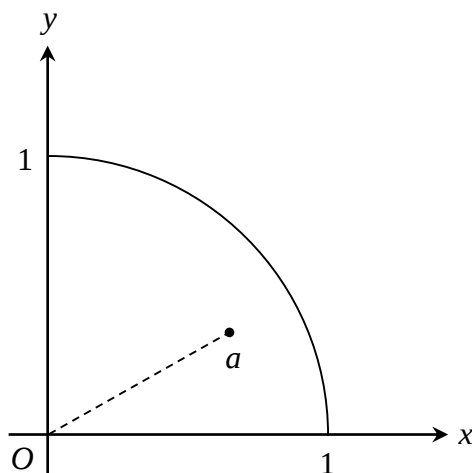
Use (i), or otherwise, to explain why as P varies, R and S remain fixed.

- (iii) Deduce that the locus of P is a circle with diameter RS . 2

Question 13 continues on page 15

Question 13 (continued)

- (c) The diagram below shows a fixed complex number a strictly inside the unit circle on the Argand diagram.



Copy the diagram into your writing booklet.

- (i) Use part (b), or otherwise, to explain why the locus of z satisfying 2

$$\left| \frac{z-a}{1-\bar{a}z} \right| = r,$$

where $r|a| > 1$, is a circle.

You do not need to find the centre and radius of the circle.

- (ii) Find the value of r such that the locus of z touches the unit circle. 2

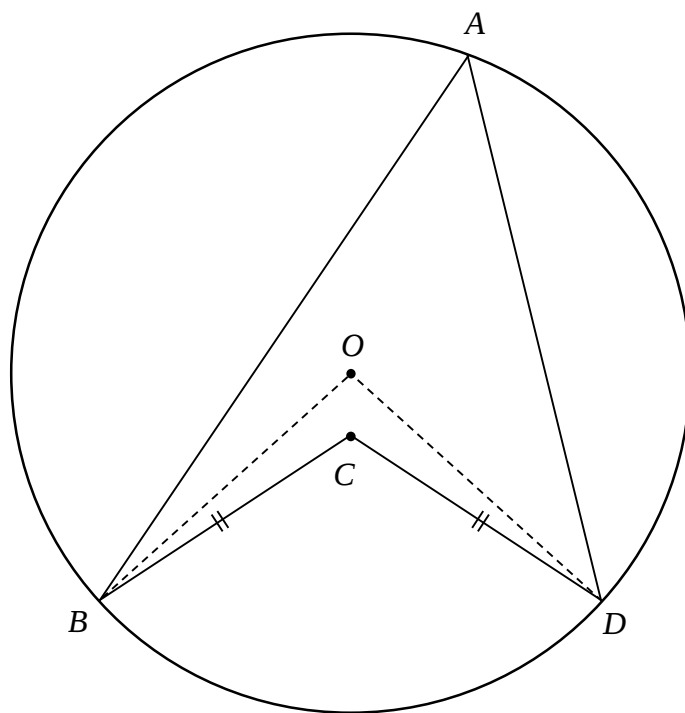
End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

- (a) The diagram below shows a quadrilateral $ABCD$, where $BC = CD$ and $\angle BCD = 2 \times \angle BAD$.

A circle is drawn to pass through points A , B and D .

Let the centre of this circle be O and assume that the point C is distinct from O .



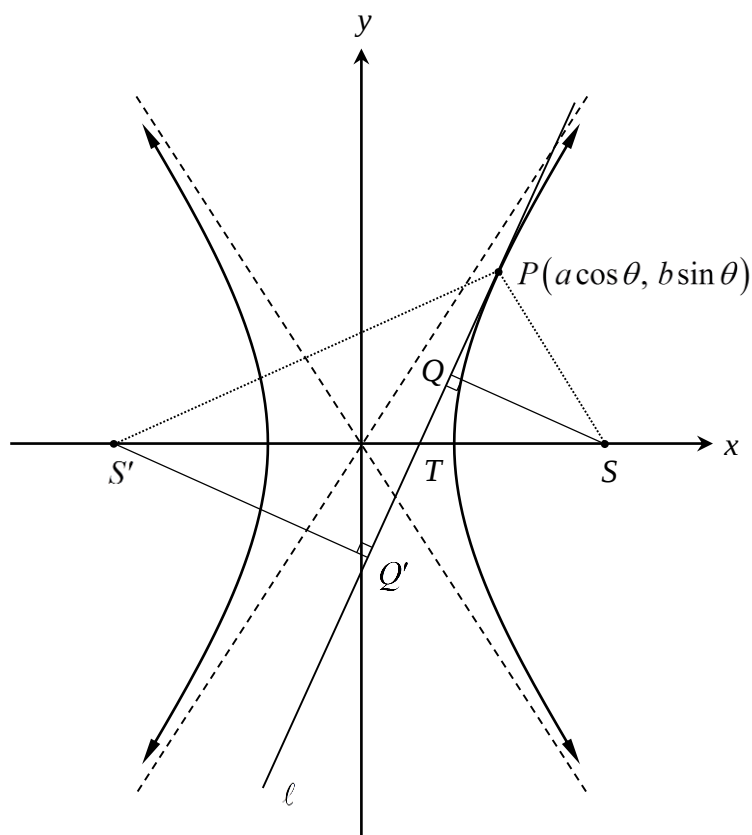
Copy the diagram into your writing booklet.

- | | |
|--|----------|
| (i) Show that $\triangle COB \equiv \triangle COD$. | 1 |
| (ii) Hence, prove that the point C cannot be distinct from O . | 2 |

Question 14 continues on page 17

Question 14 (continued)

- (b) The diagram below shows a point $P(a \cos \theta, b \sin \theta)$ on the positive branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci S and S' . From P , a tangent ℓ with equation $\frac{x \cos \theta}{a} - \frac{y \sin \theta}{b} = 1$ is drawn to intersect the x axis at T . Let Q and Q' be points on ℓ such that QS and $Q'S'$ are perpendicular to ℓ .



- (i) Show that $\frac{TS}{TS'} = \frac{PS}{PS'}$. 3
- (ii) Show that $\frac{QS}{Q'S'} = \frac{PS}{PS'}$. 2
- (iii) Deduce that the line ℓ bisects $\angle SPS'$. 1

Question 14 continues on page 18

Question 14 (continued)

(c) It can be shown that for positive integer values of n ,

$$1 + 2(\cos x + \cos 2x + \cos 3x + \dots + \cos nx) = \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \quad \text{(Do NOT prove this)}$$

(i) Show that 1

$$\int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)} dx = \frac{\pi}{2} + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)\right).$$

(ii) Show that 2

$$\left| \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx \right| < \frac{1}{2n+1} \left[2\sqrt{2} + \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} dx \right].$$

(iii) Hence, show that as $n \rightarrow \infty$, 2

$$\int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)} dx \rightarrow \pi.$$

(iv) Deduce that 1

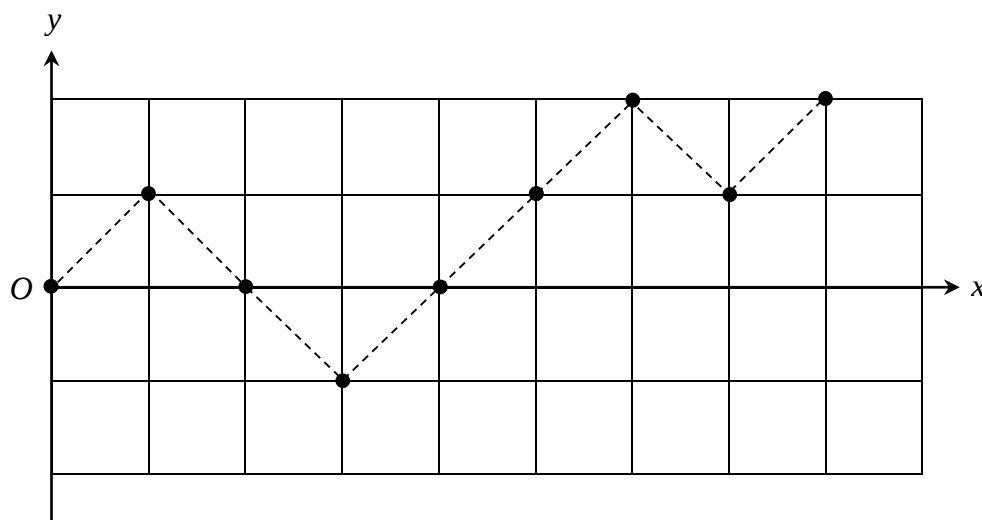
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

- (a) A sequence of letters is made using a copies of the letter U and b copies of the letter D in some order, where $a > b$. The sequence can be expressed graphically by matching the letters U and D to a series of movements. From the current position $P(x, y)$, the letter U moves P to $(x+1, y+1)$ whereas the letter D moves P to $(x+1, y-1)$.

The diagram below shows the corresponding graph of the sequence $UDDUUUDU$.



- (i) Explain why if the sequence starts from the origin, it will then terminate at the point $T(a+b, a-b)$. 1
- (ii) Write down the number of possible paths from the origin to T . 1
- (iii) Explain why the number of paths from $(1,1)$ to T that touches or crosses the x axis is equal to the number of paths from $(1,-1)$ to T . 1
- (iv) Hence, show that the number of paths from $(1,1)$ to T that do NOT touch or cross the x axis is 2

$$\frac{a-b}{a+b} \binom{a+b}{a}.$$

Question 15 continues on page 20

Question 15 (continued)

- (b) Two candidates A and B are polled against each other in a vote. Each vote is written on identical cards and placed in a bag. The tally is done by pulling a card randomly from the bag, recording it, then discarding the card. 2

Candidate A has p votes and Candidate B has q votes, where $p > q$.

Using part (a), or otherwise, find the probability that at all times during the vote count, Candidate A has a higher tally than Candidate B .

- (c) Sketch the graph of $y = \frac{x^2 - 1}{x^2 + 1}$, labelling any asymptotes, stationary points and intercepts. 2

- (d) Let $P(x) = kx^3 + kx + 2$ be a cubic polynomial with real and non-zero coefficients.

- (i) Express the polynomial $P(x)$ in the form 1

$$P(x) = (x^2 + a)(kx + b) - (x^2 - a),$$

for appropriate values of a and b .

- (ii) By considering the sketch from part (c), explain why $P(x)$ has exactly one real root. 1

- (iii) Describe the behaviour of the real root of $P(x)$ as $k \rightarrow \infty$. 1

- (iv) Hence describe the behaviour of the modulus and argument of the non-real roots of $P(x)$ as $k \rightarrow \infty$. 3

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let $f(x)$ be a non-linear continuous function in the interval $a \leq x \leq b$, where $a \neq b$.

Define

$$E = \frac{b-a}{2} [f(a) + f(b)] - \int_a^b f(x) dx.$$

(i) Show that 2

$$E = \int_a^b x f'(x) dx - \frac{a+b}{2} [f(b) - f(a)].$$

(ii) Show that 2

$$E = \frac{1}{2} \int_a^b (b-x)(x-a) f''(x) dx.$$

(iii) Suppose $f(x)$ is concave up for $a \leq x \leq b$. 2

Prove that $E > 0$ and explain the significance of this result.

Question 16 continues on page 22

Question 16 (continued)

- (b) A continuous function $f(x)$ has the property that for any $x_i > x_j$,

$$f'(x_i) \leq f'(x_j).$$

- (i) Explain, with the use of a graph, why for any $x_i > x_j$, 1

$$f'(x_i) \leq \frac{f(x_i) - f(x_j)}{x_i - x_j} \leq f'(x_j).$$

- (ii) Let α be a constant such that $0 < \alpha < 1$ and suppose that $x_1 > x_2$. 1

Show that $x_2 < \alpha x_1 + (1 - \alpha)x_2 < x_1$.

- (iii) Let α be a constant such that $0 < \alpha < 1$. 2

Use part (i), or otherwise, to show that for any x_1 and x_2 ,

$$f(\alpha x_1 + (1 - \alpha)x_2) \geq \alpha f(x_1) + (1 - \alpha)f(x_2).$$

- (iv) Let $a_1, a_2, a_3, \dots, a_n$ be any set of positive numbers such that 4

$$a_1 + a_2 + \dots + a_n = 1.$$

Use mathematical induction to prove that for $n \geq 2$

$$f(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) \geq a_1 f(x_1) + a_2 f(x_2) + \dots + a_n f(x_n).$$

- (v) By choosing an appropriate function for $f(x)$, show that 1

$$\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n},$$

for positive values of $x_1, x_2, \dots, x_n$.

End of Exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2015 Bored of Studies Trial Examinations

Mathematics Extension 2

SOLUTIONS

Multiple Choice

1. A
2. B
3. B
4. D
5. B
6. D
7. C
8. C
9. A
10. C

Brief Explanations

Question 1 The value of k can be found either by substituting it directly into the quadratic, as it is a root, or by considering the sum and product of roots.

Question 2 (B) is not always true because the eccentricity determines the ‘shape’ of the ellipse. So if one ellipse is a scaled version of the other, they will have the same eccentricity but different equations. For example, $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $\frac{x^2}{36} + \frac{y^2}{16} = 1$ both have eccentricity $\frac{5}{9}$.

This can also be observed by considering $e^2 = 1 - \frac{b^2}{a^2}$, where we can still generate the same value of e using differing values of b and a , as the eccentricity relies on the ratio of the two.

Question 3 The asymptotes of $\mathcal{H}_1$ are $y = \pm x$ and the directrices of $\mathcal{H}_2$ are $x + y = \pm a\sqrt{2}$. Observe that all of the options are in the first quadrant, so we solve $y = x$ simultaneously with $x + y = a\sqrt{2}$ to get (B).

Question 4 Option (I) yields $f(2(x+1)-1) = f(2x+1)$ and then

$f\left(2\left(\frac{x}{2}\right)+1\right) = f(x+1)$. Option (II) yields $f\left(2\left(\frac{x}{2}\right)-1\right) = f(x-1)$ and then

$f((x+1)-1) = f(x)$, so this is one possible option. Option (III) yields

$f(2(-x-1)-1) = f(-2x-3)$ and then $f\left(-2\left(-\frac{x}{2}\right)-3\right) = f(x-3)$. Option (IV) yields

$f\left(2\left(-\frac{x}{2}\right)-1\right) = f(-x-1)$ and then $f(-(-x-1)-1) = f(x)$, so this is also a possible

option. Hence options (II) or (IV) work.

Question 5 We have the vector equation $\overrightarrow{PR} = i\overrightarrow{PQ}$. Let r be the complex number represented by R , so $r - (1+i) = i(a+bi)$ noting that $a+bi$ represents the vector PQ . Hence $r = i(a+bi) + (1+i) = (1-b) + i(1+a)$, which is (B).

Question 6 A counter example for (A) is $P(x) = x^4 + 2x^2 + 1$, which has $P(i) = P'(i) = 0$. This also serves as a counter example for (B) since α is not real. The polynomial is not necessarily even because we can have any real linear factor multiplied to our counter example above, thus making it have odd degree. (D) is always true because if α is non-real, then both $(x-\alpha)^2$ and $(x-\bar{\alpha})^2$ must be factors for the conjugate root theorem to hold. So the polynomial must be in the form $P(x) = (x-\alpha)^2(x-\bar{\alpha})^2Q(x)$, which has degree $n \geq 4$.

Question 7 The method of cylindrical shells yields $V = 2\pi \int_0^1 (1-x)y dx$. Using the substitution $y = 2x - x^2$, we have $dy = 2(1-x)dx$ and so $V = \pi \int_0^1 y dy$, which is the same value as option (C) as y is just a dummy variable.

Question 8 As x moves along a circle about the origin, its x coordinate can be parameterised as $x = r \cos \theta$. Differentiate both sides with respect to t twice, noting that $\dot{\theta} = \omega$, we obtain (C). Alternatively, one could observe that in a one dimensional context, the circular motion is just simple harmonic motion so the answer is either (A) or (C). But only (C) will yield an appropriate period for the circular motion/simple harmonic motion.

Question 9 From the cosine rule, we have $\cos \angle BAC = \frac{b^2 + c^2 - a^2}{2bc} < 0$ since $\angle BAC$ is obtuse. So $a^2 > b^2 + c^2 = (b-c)^2 + 2bc \geq 2bc$. So the answer is (A).

Question 10 Using the substitution $u = 2x$, the given integral I is equivalent to

$\frac{1}{2} \int_0^{2\pi} \sin^n\left(\frac{x}{2}\right) \cos(kx) dx$, which eliminates (D). Observe that this integral is symmetric about $x = \pi$, so $\int_0^{2\pi} \sin^n\left(\frac{x}{2}\right) \cos(kx) dx = 2 \times \int_0^{\pi} \sin^n\left(\frac{x}{2}\right) \cos(kx) dx$ so for the original integral, we have $I = \frac{1}{2} \int_0^{2\pi} \sin^n\left(\frac{x}{2}\right) \cos(kx) dx = \int_0^{\pi} \sin^n\left(\frac{x}{2}\right) \cos(kx) dx$, which is (C).

Written Response

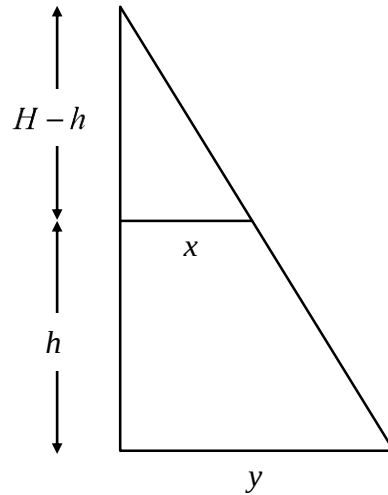
Question 11 (a)

Let $u^2 = \sin \theta \Rightarrow 2u \, du = \cos \theta \, d\theta$.

$$\begin{aligned}\int \frac{\sqrt{\sin \theta}}{\sin \theta \cos \theta} d\theta &= \int \frac{\cos \theta \sqrt{\sin \theta}}{\sin \theta \cos^2 \theta} d\theta \\&= \int \frac{\cos \theta \sqrt{\sin \theta}}{\sin \theta (1 - \sin^2 \theta)} d\theta \\&= 2 \int \frac{u^2}{u^2 (1 - u^4)} du \\&= 2 \int \frac{du}{1 - u^4} \\&= \int \frac{1}{1 - u^2} + \frac{1}{1 + u^2} du \\&= \int \frac{1/2}{1 + u} + \frac{1/2}{1 - u} + \frac{1}{1 + u^2} du \\&= \frac{1}{2} \ln \left(\frac{1 + u}{1 - u} \right) + \tan^{-1}(u) + C \\&= \frac{1}{2} \ln \left(\frac{1 + \sqrt{\sin \theta}}{1 - \sqrt{\sin \theta}} \right) + \tan^{-1}(\sqrt{\sin \theta}) + C\end{aligned}$$

Question 11 (b)

Consider an arbitrary ‘dimension’ x from the slice and the corresponding dimension y from the base.



By similar triangles, we have $\frac{H-h}{h} = \frac{x}{y}$.

But recall that the square of the ratio of corresponding sides is equal to the ratio of the areas,

in similar shapes. Hence $\left(\frac{x}{y}\right)^2 = \frac{A_h}{A_B}$ and so $\left(\frac{H-h}{H}\right)^2 = \frac{A_h}{A_B}$ or equivalently

$$A_h = A_B \left(\frac{H-h}{H}\right)^2.$$

The volume of an arbitrary slice at height h is $\delta V = A_h \delta h$.

$$\begin{aligned} V &= \int_0^H A_B \left(\frac{H-h}{H}\right)^2 dh \\ &= \frac{A_B}{H^2} \int_0^H (H-h)^2 dh \\ &= -\frac{A_B}{3H^2} (H-h)^3 \Big|_0^H \\ &= \frac{1}{3} A_B H \end{aligned}$$

Question 11 (c)

First obtain a polynomial with roots $A = \alpha + \beta$, $B = \beta + \gamma$ and $C = \alpha + \gamma$. We will then find the value of $\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2}$.

Let $u = \alpha + \beta$.

From the sum of roots, we have $\alpha + \beta + \gamma = p$ and so $u = p - \gamma \Rightarrow \gamma = p - u$.

Since γ is a root, $P(\gamma) = 0$.

$$\begin{aligned} P(p-u) &= (p-u)^3 - p(p-u)^2 + p \\ &= -u^3 + 2pu^2 - p^2u + p \end{aligned}$$

Hence, the polynomial $R(x) = x^3 - 2px^2 + p^2x - p$ has roots $A = \alpha + \beta$, $B = \beta + \gamma$ and $C = \alpha + \gamma$.

$$\begin{aligned} \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} &= \frac{A^2B^2 + A^2C^2 + B^2C^2}{A^2B^2C^2} \\ &= \frac{(AB + AC + BC)^2 - 2ABC(A + B + C)}{(ABC)^2} \\ &= \frac{p^4 - 4p^2}{p^2} \\ &= p^2 - 4 \\ &= 0 \quad \dots (\text{from the question}) \end{aligned}$$

Hence, the values of p such that $\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} = 0$ are $p = \pm 2$.

Question 11 (d) (i)

As vectors AP and AA' are collinear, we can say that $\overrightarrow{AP} = k\overrightarrow{AA'}$. If a' is the complex number representing A' , this becomes

$$p - a = k(a' - a).$$

But as A' is the midpoint of BC , we have $a' = \frac{b+c}{2}$.

$$\begin{aligned} p - a &= k\left(\frac{b+c}{2} - a\right) \\ p &= (1-k)a + \frac{k}{2}(b+c) \end{aligned}$$

Question 11 (d) (ii)

Similarly to (i), using B' instead, we have $p = (1-k)b + \frac{k}{2}(a+c)$, so equating p yields

$$\begin{aligned} (1-k)a + \frac{k}{2}(b+c) &= (1-k)b + \frac{k}{2}(a+c) \\ (1-k)(a-b) &= \frac{k}{2}(a+c-b-c) \\ (1-k)(a-b) &= \frac{k}{2}(a-b) \\ k &= \frac{2}{3} \end{aligned}$$

Substituting this back into $p = (1-k)a + \frac{k}{2}(b+c)$ yields the required result.

Question 11 (d) (iii)

Let Q be the point on the interval CC' , represented by the complex number q , such that $\overrightarrow{CQ} = r\overrightarrow{CC'}$ for some $0 < r < 1$.

Then by a similar argument to part (i), we have $q = (1-r)c + \frac{r}{2}(a+b)$

Substituting in $r = \frac{2}{3}$ yields $q = \frac{a+b+c}{3} = p$, and so P lies on CC' .

Question 12 (a) (i)

First split the log to obtain $x^3 - y^3 = 3(\ln x + \ln y)$.

Differentiate both sides with respect to x .

$$3x^2 - 3y^2 \frac{dy}{dx} = 3 \left(\frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} \right)$$

Rearrange to make $\frac{dy}{dx}$ the subject.

$$\frac{dy}{dx} = \frac{x^3 y - y}{xy^3 + x}$$

Substituting $(1,1)$ yields $\frac{dy}{dx} = 0$ and hence it is a critical point.

Question 12 (a) (ii)

Substitute in $(-y_0, -x_0)$.

$$\begin{aligned} \text{LHS} &= -y_0^3 + x_0^3 \\ &= x_0^3 - y_0^3 \\ &= 3\ln(x_0 y_0) \quad \dots (\text{as } (x_0, y_0) \text{ lies on the original curve}) \\ &= 3\ln((-x_0)(-y_0)) \\ &= \text{RHS} \end{aligned}$$

Geometrically, this means for every point (x_0, y_0) on the curve, there exists a corresponding point $(-y_0, -x_0)$ that also lies on the curve, and the effect of that is that the curve is symmetric in $y = -x$.

Question 12 (a) (iii)

Rearrange the equation as

$$x^3 - 3\ln(x) = y^3 + 3\ln(y)$$

$$x^3 \left[1 - 3 \left(\frac{\ln(x)}{x^3} \right) \right] = y^3 + 3\ln(y)$$

As $x \rightarrow \infty$, $\frac{\ln(x)}{x^3} \rightarrow 0$ and so LHS $\rightarrow \infty$. So it must be the case that $y \rightarrow \infty$.

Question 12 (a) (iv)

First, recall from (iii) that $x^3 - 3\ln(x) = y^3 + 3\ln(y)$ or equivalently $\frac{x^3 - 3\ln(x)}{y^3 + 3\ln(y)} = 1$.

So $\left(\frac{y^3}{y^3 + 3\ln y} \right) \left(\frac{x^3 - 3\ln x}{x^3} \right) = \frac{y^3}{x^3}$. By dividing the numerator and denominator of the left and right factors by y^3 and x^3 respectively, we have

$$\left[\frac{1}{1 + 3 \left(\frac{\ln y}{y^3} \right)} \right] \left[1 - 3 \left(\frac{\ln x}{x^3} \right) \right] = \frac{y^3}{x^3}$$

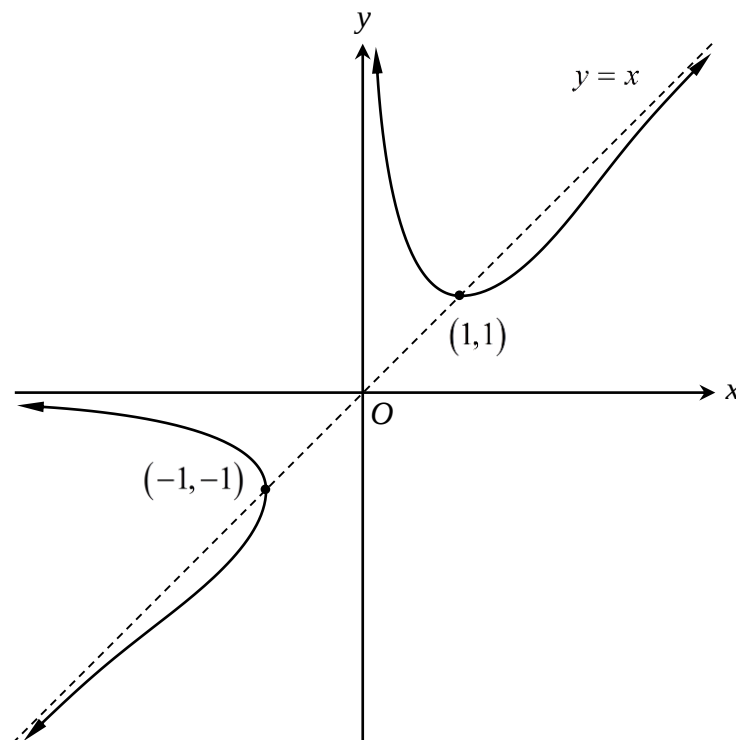
As $x \rightarrow \infty$, we know from part (iii) that $y \rightarrow \infty$. Using the given limit in part (iii), the left factor approaches 1 and so the right factor also approaches 1.

Hence $\left(\frac{y}{x} \right)^3 \rightarrow 1$ and therefore $y \rightarrow x$, meaning that $y = x$ is an asymptote.

Question 12 (a) (v)

We know that the curve has the following features:

- ✓ Critical point at $(1,1)$.
- ✓ Symmetric about $y = -x$
 - So $(-1,-1)$ is also a critical point.
- ✓ Has an oblique asymptote $y = x$.



Question 12 (b) (i)

We can factorise

$$\begin{aligned} P(z) &= z^n - 1 \\ &= (z - 1)(1 + z + z^2 + \dots + z^{n-1}) \end{aligned}$$

This polynomial has roots $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$.

But $\alpha_n = \cos(2\pi) + i \sin(2\pi) = 1$, so the roots of $S(z) = 1 + z + z^2 + \dots + z^{n-1} = 0$ must be $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-1}$.

So the linear decomposition of $S(z)$ is

$$\begin{aligned} S(z) &= 1 + z + z^2 + z^3 + \dots + z^{n-1} \\ &= (z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \dots (z - \alpha_{n-1}) \end{aligned}$$

and evaluating $S(1)$ yields

$$\begin{aligned} (1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) \dots (1 - \alpha_{n-1}) &= \underbrace{1 + 1 + 1 + \dots + 1}_{n \text{ times}} \\ &= n \end{aligned}$$

Question 12 (b) (ii)

Consider the general factor $1 - \alpha_k = 1 - \cos\left(\frac{2k\pi}{n}\right) - i \sin\left(\frac{2k\pi}{n}\right)$ for $k = 1, 2, 3, \dots, n-1$.

$$\begin{aligned}
 1 - \cos\left(\frac{2k\pi}{n}\right) - i \sin\left(\frac{2k\pi}{n}\right) &= 1 - \left(1 - 2\sin^2\left(\frac{k\pi}{n}\right)\right) - 2i \sin\left(\frac{k\pi}{n}\right) \cos\left(\frac{k\pi}{n}\right) \\
 &= 2\sin^2\left(\frac{k\pi}{n}\right) - 2i \sin\left(\frac{k\pi}{n}\right) \cos\left(\frac{k\pi}{n}\right) \\
 &= 2\sin\left(\frac{k\pi}{n}\right) \left[\sin\left(\frac{k\pi}{n}\right) - i \cos\left(\frac{k\pi}{n}\right) \right] \\
 &= 2\sin\left(\frac{k\pi}{n}\right) \left[\cos\left(\frac{\pi}{2} - \frac{k\pi}{n}\right) - i \sin\left(\frac{\pi}{2} - \frac{k\pi}{n}\right) \right] \\
 1 - \alpha_k &= 2\sin\left(\frac{k\pi}{n}\right) \left[\cos\left(\frac{\pi}{2n}(2k-n)\right) + i \sin\left(\frac{\pi}{2n}(2k-n)\right) \right]
 \end{aligned}$$

Compute the modulus of both sides to get $|1 - \alpha_k| = \left| 2\sin\left(\frac{k\pi}{n}\right) \right|$. Since $k = 1, 2, 3, \dots, n-1$, the expression $\sin\left(\frac{k\pi}{n}\right)$ is always positive and so $|1 - \alpha_k| = 2\sin\left(\frac{k\pi}{n}\right)$.

Multiplying these expressions for $k = 1, 2, 3, \dots, n-1$, we get

$$\begin{aligned}
 2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \dots \sin\left(\frac{n-1}{n}\pi\right) &= |1 - \alpha_1| |1 - \alpha_2| |1 - \alpha_3| \dots |1 - \alpha_{n-1}| \\
 2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \dots \sin\left(\frac{n-1}{n}\pi\right) &= |(1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) \dots (1 - \alpha_{n-1})| \\
 2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \dots \sin\left(\frac{n-1}{n}\pi\right) &= |n| \quad \dots (\text{from part (i)}) \\
 2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \dots \sin\left(\frac{n-1}{n}\pi\right) &= n \\
 \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \dots \sin\left(\frac{n-1}{n}\pi\right) &= \frac{n}{2^{n-1}}
 \end{aligned}$$

Question 12 (c) (i)

Resolving forces vertically and horizontally, we have

$$F \sin \theta + N \cos \theta = mg \quad \dots(1)$$

$$F \cos \theta - N \sin \theta = \frac{mv^2}{r} \quad \dots(2)$$

By considering $(1) \times \cos \theta - (2) \times \sin \theta$, we have

$$N = mg \cos \theta - \frac{mv^2}{r} \sin \theta .$$

From this expression, we can observe that there must exist some velocity $v = v_0$ where the car leaves the surface of the cone. To find this velocity let $N = 0$, which yields $v_0^2 = rg \cot \theta$.

Question 12 (c) (ii)

By considering $(1) \times \sin \theta + (2) \times \cos \theta$, we have $F = mg \sin \theta + \frac{mv^2}{r} \cos \theta$

To find the amount of lateral thrust F_0 that the car experiences at the maximum velocity v_0 , we can substitute the result from part (i) back into F .

$$\begin{aligned} F_0 &= mg \sin \theta + \frac{mv_0^2}{r} \cos \theta \\ &= mg \sin \theta + \frac{mgr \cot \theta}{r} \cos \theta \\ &= mg \sin \theta + mg \frac{\cos^2 \theta}{\sin \theta} \\ &= mg \left(\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta} \right) \\ &= \frac{mg}{\sin \theta} \end{aligned}$$

Alternatively

From our resolution of vertical forces, we had $F \sin \theta + N \cos \theta = mg$.

By letting $N = 0$, we obtain the result immediately.

Question 13 (a) (i)

First we'll find an expression for the terminal velocity w . Particle B has force equation

$$\begin{aligned} F_{net} &= ma \\ &= mg - mkv^2 \\ a &= g - kv^2 \end{aligned}$$

For terminal velocity, we have $a = 0$, $v = w$ and so $w^2 = \frac{g}{k}$.

The force equation for Particle A is $F_{net} = ma = -(mg + kv^2)$, so we have the acceleration-velocity equation $a = -(g + kv^2)$. Using the fact that $w^2 = \frac{g}{k}$, this is equivalent to

$$\begin{aligned} v \frac{dv}{dx} &= -(g + kv^2) \\ &= -k(w^2 + v^2) \end{aligned}$$

So by separating variables and integrating both sides with respect to appropriate limits, we have

$$\begin{aligned} -k \int_0^{x_A} dx &= \int_u^{v_A} \frac{v}{w^2 + v^2} dv \\ &= \frac{1}{2} \ln(w^2 + v^2) \Big|_u^{v_A} \\ -kx_A &= \frac{1}{2} \ln \left(\frac{w^2 + v_A^2}{w^2 + u^2} \right) \\ x_A &= -\frac{1}{2k} \ln \left(\frac{w^2 + v_A^2}{w^2 + u^2} \right) \\ &= \frac{1}{2k} \ln \left(\frac{w^2 + u^2}{w^2 + v_A^2} \right) \end{aligned}$$

Question 13 (a) (ii)

From part (i), we can find an expression for H by setting $v_A = 0$, $x = H$.

$$H = \frac{1}{2k} \ln \left(\frac{w^2 + u^2}{w^2} \right)$$

From the diagram, we can observe that when the particles collide, $x_A + x_B = H$ since x_A is the distance from the ground and x_B is the distance from maximum height H . Also, noting that they collide when $v_A = w$, we have

$$\begin{aligned} \frac{1}{2k} \ln \left(\frac{w^2 + u^2}{w^2 + w^2} \right) + \frac{1}{2k} \ln \left(\frac{w^2}{w^2 - v_B^2} \right) &= \frac{1}{2k} \ln \left(\frac{w^2 + u^2}{w^2} \right) \\ \ln \left(\frac{w^2 + u^2}{w^2 + w^2} \times \frac{w^2}{w^2 - v_B^2} \right) &= \ln \left(\frac{w^2 + u^2}{w^2} \right) \\ \frac{w^2 + u^2}{2w^2} \times \frac{w^2}{w^2 - v_B^2} &= \frac{w^2 + u^2}{w^2} \\ \frac{1}{2} \times \frac{w^2}{w^2 - v_B^2} &= 1 \\ w^2 &= 2w^2 - 2v_B^2 \\ w^2 &= 2v_B^2 \\ v_B &= \frac{w}{\sqrt{2}} \quad \dots (\text{as } v_B > 0) \end{aligned}$$

Question 13 (b) (i)

We first use the sine rule in triangles APR and BPR to respectively obtain

$$\frac{\sin \angle ARP}{AP} = \frac{\sin \angle APR}{AR} \quad \dots(1)$$

$$\frac{\sin \angle BRP}{BP} = \frac{\sin \angle BPR}{BR} \quad \dots(2)$$

Since, $\angle ARP$ and $\angle BRP$ are supplementary, $\sin \angle ARP = \sin(\pi - \angle BRP) = \sin \angle BRP$.

Also, it is given that $\angle APR = \angle BPR$, so we have $\sin \angle APR = \sin \angle BPR$.

Calculating $(2) \div (1)$:

$$\frac{\sin \angle BRP}{BP} \times \frac{AP}{\sin \angle ARP} = \frac{\sin \angle BPR}{BR} \times \frac{AR}{\sin \angle APR}$$

$$\frac{AP}{BP} = \frac{AR}{BR}$$

Question 13 (b) (ii)

From the given ratio, and from part (i), we have

$$\frac{AS}{BS} = \frac{AR}{BR} = k,$$

which is a fixed ratio.

As A and B are fixed points, and R and S are concurrent with AB , it must be the case that R and S are also fixed points regardless of the position of P .

Question 13 (b) (iii)

Since $\angle APT = \pi$ and $\angle RPS = \frac{1}{2} \angle APT$, we have $\angle RPS = \frac{\pi}{2}$.

But the points R and S are fixed, with P varying. Hence, as P varies, it must move along the circumference of a circle with diameter RS (angle in a semi-circle).

Question 13 (c) (i)

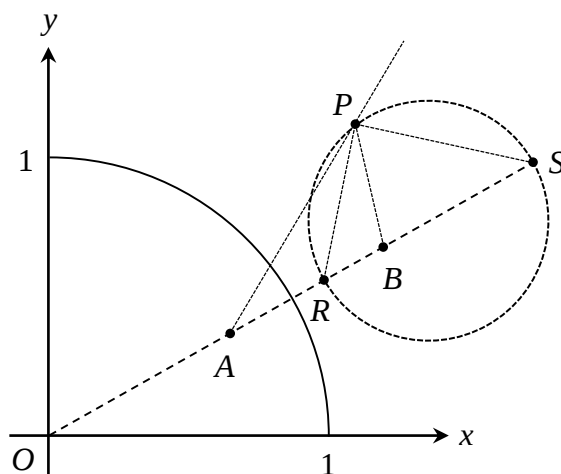
By first noting that $1 - \bar{a}z = \bar{a}\left(\frac{1}{\bar{a}} - z\right)$, and using the fact $|a| = |\bar{a}|$, we can deduce that the given locus condition is equivalent to

$$\frac{|z - a|}{\left|z - \frac{1}{\bar{a}}\right|} = r|a|.$$

As $r|a| > 1$, we can observe that this is similar to ratio $\frac{PA}{PB} = k$ from part (b), where $P \Rightarrow z$, $A \Rightarrow a$ and $B \Rightarrow \frac{1}{\bar{a}}$.

Hence, using the result of (b) (iii), we can deduce that the locus of z is a circle.

The diagram below shows this result, where A represents the complex number a and B represents the complex number $\frac{1}{\bar{a}}$. Note that the complex number $\frac{1}{\bar{a}}$ has the same argument as the complex number a , but with a reciprocal modulus. As a is inside the unit circle, the complex number $\frac{1}{\bar{a}}$ must be outside the unit circle.



Question 13 (c) (ii)

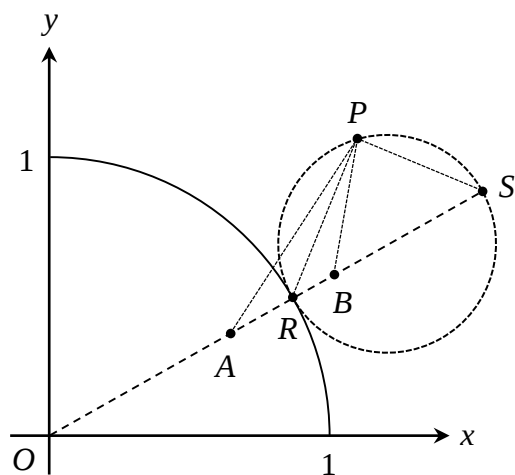


Diagram not to scale

From the ratio in part (b) (i), and from the locus condition, we have

$$\frac{PA}{PB} = \frac{RA}{RB} = r|a|,$$

where R lies on the unit circle in order for the locus of z to touch the unit circle.

So in the context of moduli, this translates to

$$\frac{RA}{RB} = \frac{1-|a|}{\left|\frac{1}{\bar{a}}\right|-1} = r|a|$$

So working with this, we have

$$\frac{1-|a|}{\frac{1}{|a|}-1} = r|a|$$

$$1-|a| = r|a|\left(\frac{1}{|a|}-1\right)$$

$$1-|a| = r(1-|a|)$$

$$r = 1$$

Question 14 (a) (i)

We have $CB = CD$ (given), and $OB = OD$ (radii), and OC is common.

Hence $\triangle COB \equiv \triangle COD$ (SSS).

Question 14 (a) (ii)

Since triangles are congruent, we have $\angle CBO = \angle CDO$. Adding all the angles in $\triangle COB$ and $\triangle COD$, we have

$$(\angle CBO + \angle BOC + \angle OCB) + (\angle CDO + \angle DOC + \angle OCD) = 2\pi$$

$$2\angle CBO + \angle BCD_{\text{reflex}} + \angle BOD = 2\pi$$

as $BODC$ is a quadrilateral.

But

$$\begin{aligned}\angle BCD &= 2 \times \angle BAD \quad \dots (\text{given}) \\ &= \angle BOD \quad \dots (\text{angle at the centre} = 2 \times \text{angle at the circumference})\end{aligned}$$

So then we have

$$\begin{aligned}2\angle CBO + (2\pi - \angle BCD) + \angle BOD &= 2\pi \\ 2\angle CBO + \angle BOD - \angle BCD &= 0 \\ 2\angle CBO &= 0 \quad \dots (\text{since } \angle BOD = \angle BCD) \\ \angle CBO &= 0\end{aligned}$$

And this forces O and C to coincide.

Question 14 (b) (i)

The x intercept of the tangent is $x = \frac{a}{\cos \theta}$ and so

$$\begin{aligned} TS &= ae - \frac{a}{\cos \theta} \\ &= \frac{a}{\cos \theta} (e \cos \theta - 1) \end{aligned}$$

Similarly, $TS' = \frac{a}{\cos \theta} (e \cos \theta + 1)$ so $\frac{TS}{TS'} = \frac{e \cos \theta - 1}{e \cos \theta + 1}$.

Recall that $PS = ePM$ and $PS' = ePM'$.

$$\begin{aligned} \frac{PS}{PS'} &= \frac{PM}{PM'} \\ &= \frac{a \cos \theta - \frac{a}{e}}{a \cos \theta + \frac{a}{e}} \\ &= \frac{e \cos \theta - 1}{e \cos \theta + 1} \end{aligned}$$

And so we have the required result.

Question 14 (b) (ii)

First, observe that

$\angle STQ = \angle S'TQ'$ (vertically opposite angles)

$\angle TQS = \angle TQ'S' = \frac{\pi}{2}$ (given).

Hence $\triangle TQS \parallel \triangle TQ'S'$ (equiangular).

Hence, by similar triangle ratios, we have

$$\begin{aligned} \frac{QS}{Q'S'} &= \frac{TS}{TS'} \\ &= \frac{PS}{PS'} \quad \dots (\text{from part (i)}) \end{aligned}$$

Question 14 (b) (iii)

In $\triangle PQS$ and $\triangle PQ'S'$, we have $\sin(\angle SPQ) = \frac{QS}{PS}$ and $\sin(\angle S'PQ') = \frac{Q'S'}{PS'}$ respectively.

If we divide these expressions, we have

$$\frac{\sin(\angle SPQ)}{\sin(\angle S'PQ')} = \frac{QS}{PS} \times \frac{PS'}{Q'S'}$$

$$= 1 \quad \dots (\text{from part (ii)})$$

So $\sin(\angle SPQ) = \sin(\angle S'PQ')$.

As the two angles must be acute (they both exist in right angled triangles), it must be the case that $\angle SPQ = \angle S'PQ'$ and hence the line ℓ bisects $\angle SPS'$.

Question 14 (c) (i)

Integrating both sides of the given identity, over the domain $0 \leq x \leq \frac{\pi}{2}$, we have

$$\int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)} dx = \int_0^{\frac{\pi}{2}} 1 + 2(\cos x + \cos 2x + \cos 3x + \dots + \cos nx) dx$$

$$= x + 2 \left(\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots + \frac{1}{n} \sin nx \right) \Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{1}{n} \sin \left(\frac{n\pi}{2} \right) \right) \quad \dots (\text{as } \sin k\pi = 0, k \in \mathbb{Z})$$

Question 14 (c) (ii)

We can use integration by parts on $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx$ by setting $u = \frac{1}{\sin\left(\frac{x}{2}\right)}$ and

$dv = \sin\left(n + \frac{1}{2}\right)x$. Noting that $\cos\left(n + \frac{1}{2}\right)\pi = 0$, we then have

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx &= -\frac{2}{2n+1} \frac{\cos\left(n + \frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} \Bigg|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2n+1} \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} \cos\left(n + \frac{1}{2}\right)x dx \\ &= -\frac{2}{2n+1} \left[0 - \sqrt{2} \cos\left(\frac{2n+1}{4}\pi\right) \right] - \frac{1}{2n+1} \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} \cos\left(n + \frac{1}{2}\right)x dx \\ &= \frac{1}{2n+1} \left[2\sqrt{2} \cos\left(\frac{2n+1}{4}\pi\right) - \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} \cos\left(n + \frac{1}{2}\right)x dx \right] \end{aligned}$$

Taking the absolute value of both sides and using the triangle inequality, we have

$$\begin{aligned} \left| \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx \right| &= \frac{1}{2n+1} \left| 2\sqrt{2} \cos\left(\frac{2n+1}{4}\pi\right) - \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} \cos\left(n + \frac{1}{2}\right)x dx \right| \\ &\leq \frac{1}{2n+1} \left[2\sqrt{2} \left| \cos\left(\frac{2n+1}{4}\pi\right) \right| + \left| \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} \cos\left(n + \frac{1}{2}\right)x dx \right| \right] \\ &\leq \frac{1}{2n+1} \left[2\sqrt{2} + \left| \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} dx \right| \right] \\ &= \frac{1}{2n+1} \left[2\sqrt{2} + \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(x/2)}{\sin^2(x/2)} dx \right] \quad \dots \left(\begin{array}{l} \text{since the integrand is} \\ \text{positive for all } \frac{\pi}{2} \leq x \leq \pi \end{array} \right) \end{aligned}$$

Question 14 (c) (iii)

First consider

$$\int_0^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx = \int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx$$

We can calculate what the left hand side becomes directly, using the given identity.

$$\begin{aligned} \int_0^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx &= \int_0^{\pi} 1 + 2(\cos x + \cos 2x + \cos 3x + \dots + \cos nx) dx \\ &= x + 2 \left(\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots + \frac{1}{n} \sin nx \right) \Big|_0^{\pi} \\ &= \pi \end{aligned}$$

So we now have

$$\pi = \int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx$$

From part (ii) we can deduce that as $n \rightarrow \infty$, $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin(n+1/2)x}{\sin(x/2)} dx \rightarrow 0$ and so we have the required result.

Question 14 (c) (iv)

From part (i), we had

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)} dx &= \frac{\pi}{2} + 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) \right) \\ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{1}{n} \sin\left(\frac{n\pi}{2}\right) &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin(n+1/2)x}{\sin(x/2)} dx - \frac{\pi}{4} \end{aligned}$$

Taking the limit of both sides as $n \rightarrow \infty$, we have $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{1}{2}(\pi) - \frac{\pi}{4} = \frac{\pi}{4}$

Question 15 (a) (i)

Regardless of whether it is a U or a D , the graph will progress to the right by 1 unit. As the sequence contains $a + b$ letters in total, the x coordinate will end at $x = a + b$.

As there are more U 's than D 's (given), the net vertical movement will be the number of U 's less by the number of D 's, which is $a - b$. Hence the y coordinate will end at $y = a - b$.

Question 15 (a) (ii)

The number of paths from the origin to T is the same as the number of ways of arranging a sequence of a copies of U and b copies of D , which is $\frac{(a+b)!}{a!b!}$.

Alternatively

Out of a sequence of $a + b$ letters, choose either a or b to become U 's, thus forcing the remainder of the letters to be D 's. So we have $\binom{a+b}{a}$ or $\binom{a+b}{b}$.

Question 15 (a) (iii)

Suppose the sequence from $(1,1)$ to T either touches or crosses the x axis. There must exist a 'first time' $(c,0)$ where it makes contact with the x axis. Reflect the path up until $(c,0)$ about the x axis. By doing so, we obtain a unique path from $(1,-1)$ to T .

Observe that by doing this, we obtain a one-to-one correspondence between the paths from $(1,1)$ to T , which touch or intersect the x axis, to the total number of paths from $(1,-1)$ to T , and visa versa. Hence the two sets of paths must be equal in size.

Question 15 (a) (iv)

The total number of paths from $(1,1)$ to T is $\binom{a+b-1}{a-1} = \frac{(a+b-1)!}{(a-1)!b!} \dots(1).$

The total number of paths from $(1,-1)$ to T is $\binom{a+b-1}{a} = \frac{(a+b-1)!}{a!(b-1)!} \dots(2).$

The number of paths from $(1,1)$ to T that do NOT touch or cross the x axis is the value in equation (1) less by the paths that DO touch or cross the x axis, which from part (iii) is given by equation (2).

Hence, we have

$$\begin{aligned}
 \frac{(a+b-1)!}{(a-1)!b!} - \frac{(a+b-1)!}{a!(b-1)!} &= \frac{(a+b-1)!}{(a-1)!(b-1)!} \left[\frac{1}{b} - \frac{1}{a} \right] \\
 &= \frac{(a+b-1)!}{(a-1)!(b-1)!} \left[\frac{a-b}{ab} \right] \\
 &= \frac{(a+b-1)!}{a!b!} (a-b) \\
 &= \frac{a-b}{a+b} \frac{(a+b)!}{a!b!} \\
 &= \frac{a-b}{a+b} \binom{a+b}{a}
 \end{aligned}$$

Question 15 (b)

Let an A card being drawn correspond to a U from part (a), and a B card being drawn correspond to a D . Observe that for Candidate A to always have a higher tally than Candidate B , the first card drawn is necessarily an A card. In other words, the sequence must begin with a U .

Candidate A always having a higher tally than Candidate B corresponds to a path from $(1,1)$ to $(p+q, p-q)$ that does not touch or cross the x axis, which is $\frac{p-q}{p+q} \binom{p+q}{p}$, from part (a) (iv).

The total number of possible vote sequences (sample space) corresponds to the total number of paths from the origin to $(p+q, p-q)$, which is $\binom{p+q}{p}$.

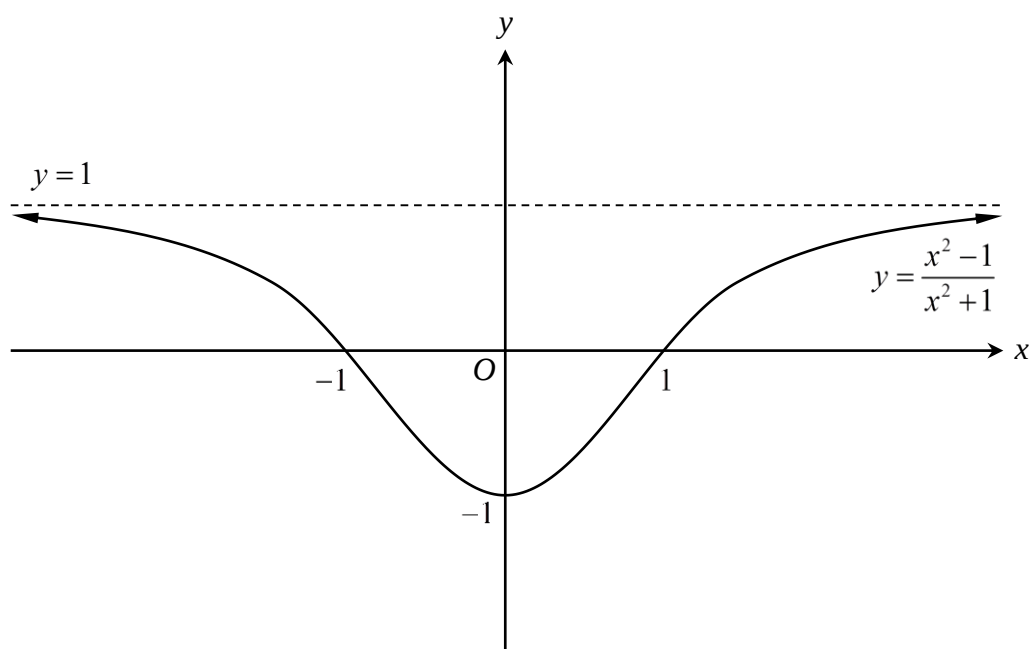
Hence, the required probability is $\frac{\frac{p-q}{p+q} \binom{p+q}{p}}{\binom{p+q}{p}} = \frac{p-q}{p+q}$.

Question 15 (c)

By observing the equation, we can immediately acquire the following features.

- ✓ y intercept at $(0, -1)$.
- ✓ x intercepts at $(\pm 1, 0)$.
- ✓ Horizontal asymptote $y = 1$.
- ✓ The asymptotes are approached from below, as the numerator is always less than the denominator.
- ✓ The function is even.

And this is sufficient to deduce the shape of the curve.



Question 15 (d) (i)

By equating the coefficient of x , we have $ak = k$, so $a = 1$.

By equating the constant term, we have $ab + a = 2$ and hence $b = 1$.

So $P(x) = (x^2 + 1)(kx + 1) - (x^2 - 1)$.

Question 15 (d) (ii)

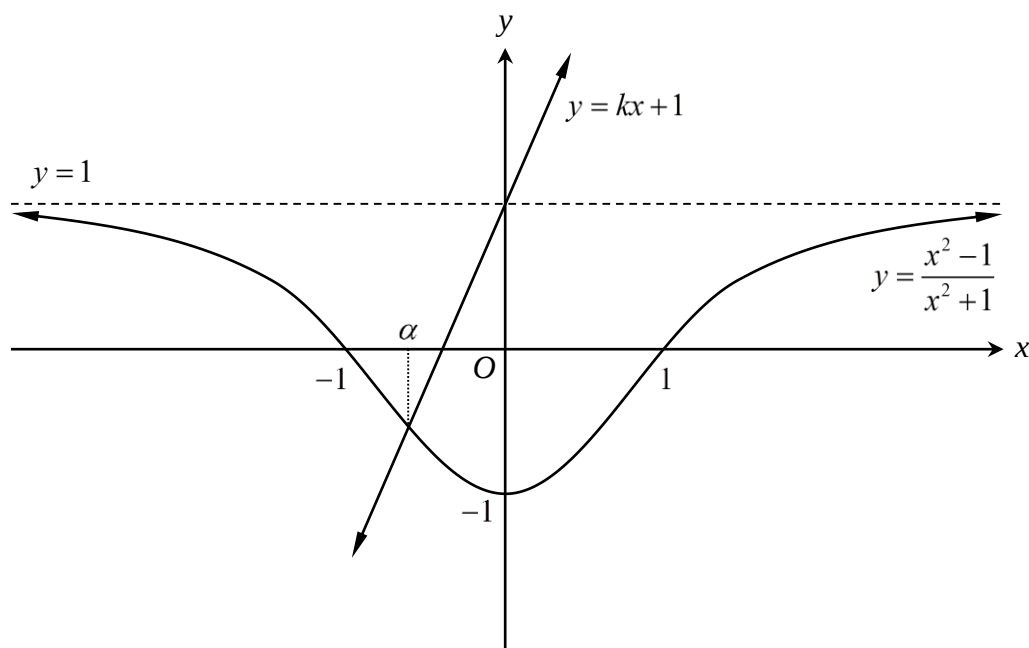
Let $P(x) = 0$.

$$(x^2 + 1)(kx + 1) = (x^2 - 1)$$

$$kx + 1 = \frac{x^2 - 1}{x^2 + 1}$$

So if $P(x)$ has exactly one real root α , it will correspond to the intersection point of the line

$y = kx + 1$ with the curve $y = \frac{x^2 - 1}{x^2 + 1}$. From the diagram below, we can observe that the line intersects the curve exactly once for any non-zero value of k . Hence, there is only one real root.

**Question 15 (d) (iii)**

From the diagram above, k approaching infinity corresponds to the line $y = kx + 1$ becoming steeper, which will result in $\alpha \rightarrow 0^-$.

Question 15 (d) (iv)

Let the roots of $P(x)$ be α, β and $\bar{\beta}$.

$$\begin{aligned}\sum_i \alpha_i &= \alpha + \beta + \bar{\beta} \\ &= \alpha + 2\operatorname{Re}(\beta) \\ &= 0 \quad \dots (\text{by observing the coefficients}) \\ \operatorname{Re}(\beta) &= -\frac{\alpha}{2}\end{aligned}$$

Since $\alpha \rightarrow 0^-$, from part (ii), we can deduce that as $k \rightarrow \infty$, $\operatorname{Re}(\beta) \rightarrow 0^+$.

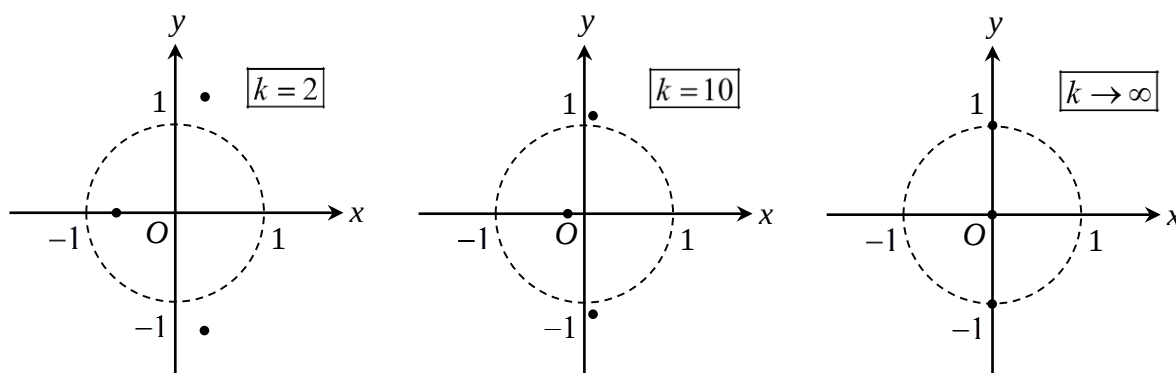
$$\begin{aligned}\sum_{i < j} \alpha_i \alpha_j &= \alpha\beta + \alpha\bar{\beta} + \beta\bar{\beta} \\ &= 2\alpha \operatorname{Re}(\beta) + |\beta|^2 \\ &= 1 \quad \dots (\text{by observing the coefficients}) \\ |\beta|^2 &= 1 - 2\alpha \operatorname{Re}(\beta)\end{aligned}$$

Since $\operatorname{Re}(\beta) \rightarrow 0^+$ and $\alpha \rightarrow 0^-$, we can deduce that $|\beta|^2 \rightarrow 1^+$.

So the non-real roots approach the unit circle from the outside, and the real component approaches zero. This means that the argument of the roots approach $\pm \frac{\pi}{2}$ as $k \rightarrow \infty$.

And so the non-real roots approach $\pm i$. This can be verified by expressing the polynomial as $x(x^2 + 1) = -\frac{2}{k}$ and taking the limit as $k \rightarrow \infty$.

The diagrams below show the behaviour of the roots for increasing values of k .



Question 16 (a) (i)

Using integration by parts on $\int_a^b f(x) dx$, we have

$$\begin{aligned}\int_a^b f(x) dx &= x f(x) \Big|_a^b - \int_a^b x f'(x) dx \\ &= b \cdot f(b) - a \cdot f(a) - \int_a^b x f'(x) dx\end{aligned}$$

Substituting this into E , we have

$$\begin{aligned}E &= \frac{b-a}{2} [f(a) + f(b)] - b \cdot f(b) + a \cdot f(a) + \int_a^b x f'(x) dx \\ &= \int_a^b x f'(x) dx - \left(\frac{a+b}{2} \right) [f(b) - f(a)]\end{aligned}$$

Question 16 (a) (ii)

Using integration by parts on $I = \frac{1}{2} \int_a^b (b-x)(x-a) f''(x) dx$, we have

$$\begin{aligned}
 I &= \frac{1}{2} \left[(b-x)(x-a) f'(x) \right]_a^b - \frac{1}{2} \int_a^b ((a+b)-2x) f'(x) dx \\
 &= -\frac{1}{2} \int_a^b ((a+b)-2x) f'(x) dx \\
 &= \int_a^b \left(x - \left(\frac{a+b}{2} \right) \right) f'(x) dx \\
 &= \int_a^b x f'(x) dx - \left(\frac{a+b}{2} \right) \int_a^b f'(x) dx \\
 &= \int_a^b x f'(x) dx - \left(\frac{a+b}{2} \right) [f(b) - f(a)] \\
 &= E
 \end{aligned}$$

Question 16 (b) (iii)

Since $f(x)$ is concave up in the domain $a \leq x \leq b$, we can assume that $f''(x) > 0$.

But observe also that $(b-x)(x-a) \geq 0$ for all $a \leq x \leq b$.

Hence $\int_a^b (b-x)(x-a) f''(x) dx > 0$ and so $E > 0$.

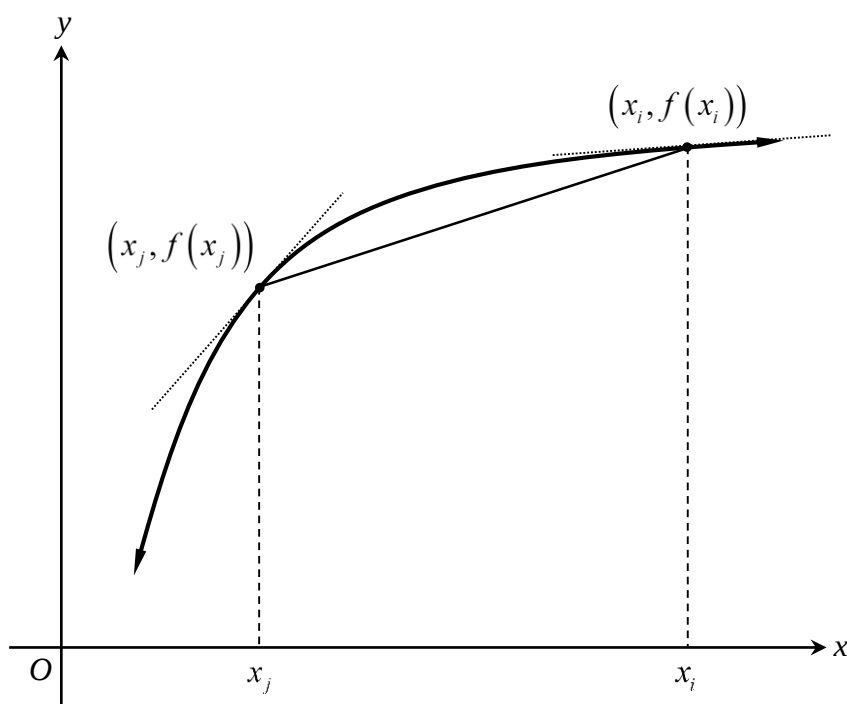
What this proves is that if a function is concave up in the domain $a \leq x \leq b$, applications of the trapezoidal rule within the same domain will always over-approximate the value of

$$\int_a^b f(x) dx.$$

Question 16 (b) (i)

The given condition implies that the function is concave down, where equality holds when the function is linear. The diagram below shows an example of such a curve.

Observe that from the diagram, the gradient of the secant lies between the gradient of the tangents at x_j and x_i . And so we have the required inequality.



Question 16 (b) (ii)

Let x_α be the value on the number line that divides the interval $x_2 \leq x \leq x_1$ internally in the ratio $\alpha : 1 - \alpha$.

Hence we have $x_\alpha = \frac{\alpha x_1 + (1 - \alpha)x_2}{\alpha + (1 - \alpha)}$. As x_α internally divides the interval, it must satisfy

$$x_2 < x_\alpha < x_1 \text{ and so } x_2 < \alpha x_1 + (1 - \alpha)x_2 < x_1.$$

Question 16 (b) (iii)

From the right hand inequality in part (i), we have

$$f(x_i) \leq f(x_j) + (x_i - x_j) f'(x_j)$$

Let $x_i = x_1$ and $x_j = \alpha x_1 + (1 - \alpha)x_2$.

$$f(x_1) \leq f(\alpha x_1 + (1 - \alpha)x_2) + (x_1 - \alpha x_1 - (1 - \alpha)x_2) f'(\alpha x_1 + (1 - \alpha)x_2)$$

$$f(x_1) \leq f(\alpha x_1 + (1 - \alpha)x_2) + (1 - \alpha)(x_1 - x_2) f'(\alpha x_1 + (1 - \alpha)x_2) \quad \dots(1)$$

From the left hand inequality from part (i), we have

$$f(x_j) \leq f(x_i) - (x_i - x_j) f'(x_i)$$

Let $x_i = \alpha x_1 + (1 - \alpha)x_2$ and $x_j = x_2$.

$$f(x_2) \leq f(\alpha x_1 + (1 - \alpha)x_2) - (\alpha x_1 + (1 - \alpha)x_2 - x_2) f'(\alpha x_1 + (1 - \alpha)x_2)$$

$$f(x_2) \leq f(\alpha x_1 + (1 - \alpha)x_2) - \alpha(x_1 - x_2) f'(\alpha x_1 + (1 - \alpha)x_2) \quad \dots(2)$$

Taking $(1) \times \alpha + (2) \times (1 - \alpha)$:

$$\alpha f(x_1) + (1 - \alpha) f(x_2) \leq f(\alpha x_1 + (1 - \alpha)x_2)$$

which is the required result.

Question 16 (b) (iv)

Base Case: $n = 2$

We have $a_1 + a_2 = 1$, so $a_2 = 1 - a_1$.

From part (ii), we have

$$\begin{aligned}
f(a_1 x_1 + a_2 x_2) &= f(a_1 x_1 + (1 - a_1) x_2) \\
&\geq a_1 f(x_1) + (1 - a_1) f(x_2) \\
&= a_1 f(x_1) + a_2 f(x_2)
\end{aligned}$$

as required.

Inductive Hypothesis: $n = k$

$$f(a_1 x_1 + a_2 x_2 + \dots + a_k x_k) \geq a_1 f(x_1) + a_2 f(x_2) + \dots + a_k f(x_k),$$

where $a_1, a_2, a_3, \dots, a_k$ is any set of positive numbers such that $a_1 + a_2 + \dots + a_k = 1$

Inductive Step: $n = k \Rightarrow n = k + 1$

Required to prove

If $a_1, a_2, a_3, \dots, a_k, a_{k+1}$ is any set of positive numbers such that $a_1 + a_2 + \dots + a_k + a_{k+1} = 1$,

$$f(a_1 x_1 + a_2 x_2 + \dots + a_k x_k + a_{k+1} x_{k+1}) \geq a_1 f(x_1) + a_2 f(x_2) + \dots + a_k f(x_k) + a_{k+1} f(x_{k+1}).$$

$$\begin{aligned}
\text{LHS} &= f(a_1x_1 + a_2x_2 + \dots + a_kx_k + a_{k+1}x_{k+1}) \\
&= f((a_1x_1 + a_2x_2 + \dots + a_kx_k) + (a_{k+1}x_{k+1})) \\
&= f\left((a_1 + a_2 + \dots + a_k)\left(\frac{a_1x_1 + a_2x_2 + \dots + a_kx_k}{a_1 + a_2 + \dots + a_k}\right) + (1 - (a_1 + a_2 + \dots + a_k))(x_{k+1})\right) \\
&\geq (a_1 + a_2 + \dots + a_k) f\left(\frac{a_1x_1 + a_2x_2 + \dots + a_kx_k}{1 - a_{k+1}}\right) + (1 - (a_1 + a_2 + \dots + a_k)) f(x_{k+1}) \quad \dots(\text{part (ii)}) \\
&= (1 - a_{k+1}) f\left(\frac{a_1}{1 - a_{k+1}}x_1 + \frac{a_2}{1 - a_{k+1}}x_2 + \frac{a_3}{1 - a_{k+1}}x_3 + \dots + \frac{a_k}{1 - a_{k+1}}x_k\right) + a_{k+1} f(x_{k+1}) \\
&= (1 - a_{k+1}) f(A_1x_1 + A_2x_2 + A_3x_3 + \dots + A_kx_k) + a_{k+1} f(x_{k+1}) \quad \dots\left(\text{where } A_k = \frac{a_k}{1 - a_{k+1}}\right)
\end{aligned}$$

However, observe that $A_1 + A_2 + A_3 + \dots + A_k = \frac{a_1 + a_2 + a_3 + \dots + a_k}{1 - a_{k+1}} = 1$.

Hence, we can apply the inductive hypothesis. So continuing on, we then have

$$\begin{aligned}
\text{LHS} &\geq (1 - a_{k+1}) f(A_1x_1 + A_2x_2 + A_3x_3 + \dots + A_kx_k) + a_{k+1} f(x_{k+1}) \\
&\geq (1 - a_{k+1}) [A_1 f(x_1) + A_2 f(x_2) + \dots + A_k f(x_k)] + a_{k+1} f(x_{k+1}) \quad \dots(\text{inductive hypothesis}) \\
&= (1 - a_{k+1}) A_1 f(x_1) + (1 - a_{k+1}) A_2 f(x_2) + \dots + (1 - a_{k+1}) A_k f(x_k) + a_{k+1} f(x_{k+1}) \\
&= a_1 f(x_1) + a_2 f(x_2) + \dots + a_k f(x_k) + a_{k+1} f(x_{k+1}) \\
&= \text{RHS, as required.}
\end{aligned}$$

Question 16 (b) (v)

The function $f(x) = \ln(x)$ is a suitable candidate because it is concave down.

Also, set $a_1 = a_2 = a_3 = \dots = a_n = \frac{1}{n}$.

From the result in part (iii), and using log laws, we have

$$\ln\left(\frac{x_1 + x_2 + x_3 + \dots + x_n}{n}\right) \geq \frac{1}{n}(\ln(x_1) + \ln(x_2) + \dots + \ln(x_n))$$

$$\ln\left(\frac{x_1 + x_2 + x_3 + \dots + x_n}{n}\right) \geq \ln\left[\left(x_1 x_2 x_3 \dots x_n\right)^{\frac{1}{n}}\right]$$

$$\frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 x_3 \dots x_n}$$