2016 Bored of Studies Trial Examinations

Mathematics Extension 2

3rd October 2016

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 1 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 7 – 25

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

1 A particle of mass m is attached to a fixed point on a smooth table by an inextensible string of length r. It is then spun around in uniform circular motion with angular velocity ω. Let the tension that the string exerts be T.

The radius, angular velocity and mass of the particle is allowed to vary. Which of the following changes will result in a DIFFERENT value of T from the other options?

(A) The mass is quadrupled, but the angular velocity is halved.

(B) The mass halved, but the angular velocity quadrupled.

(C) The radius is halved, but the mass is doubled.

(D) The mass is halved, and the radius is doubled.

2 Which of the following statements is always TRUE?

(A) If $P(x)$ has real coefficients, and $x = a + bi$ is a root, then $x = a - bi$ is also a distinct root, for *all* real values of a and b.

(B) A cubic polynomial can *never* have exactly one non-real root.

(C) If a polynomial with real coefficients has n distinct non-real roots, then it *must* have degree $2n$.

(D) If a monic polynomial with integer coefficients has no integer roots, then its roots *must* be either irrational or non-real.

3 Consider the relation $e^y \sin x + e^x \sin y = 0$.

Which of the following is a correct expression for $\dfrac{dy}{dx}$?

(A) $e^{y-x}\left(\dfrac{\sin x - \cos x}{\cos y + \sin y}\right)$

(B) $e^{y-x}\left(\dfrac{\sin x + \cos x}{\cos y - \sin y}\right)$

(C) $e^{y-x}\left(\dfrac{\sin x - \cos x}{\cos y - \sin y}\right)$

(D) $e^{y-x}\left(\dfrac{\sin x + \cos x}{\cos y + \sin y}\right)$

4 Let $y = f(x)$ be any function that has an x intercept at $x = \alpha$.

Which of the following is ALWAYS true?

(A) The graph of $y = \left(f(x)\right)^2$ will have a stationary point at $x = \alpha$.

(B) The graph of $y = \left(f(x)\right)^3$ will have a horizontal point of inflexion at $x = \alpha$.

(C) The graph of $y = \sqrt{f(x)}$ will have a vertical tangent at $x = \alpha$.

(D) The graph of $y = \tan^{-1}\left(f(x)\right)$ will be parallel to $y = f(x)$ at $x = \alpha$.

5 Let $a > 0$ be a fixed positive constant. Consider a continuous odd function that has been shifted upwards by a units. Let this shifted function be $y = f(x)$. The function must satisfy the following conditions

- $f(-a) = 0$
- $f(a) = 2a$
- $f(x) \geq 0$

Which of the following does not have a fixed value, for ANY $f(x)$ satisfying the above properties?

(A) $\int_{-a}^{a} f(x)\,dx.$

(B) $\int_{-a}^{a} f(x) + f(-x)\,dx.$

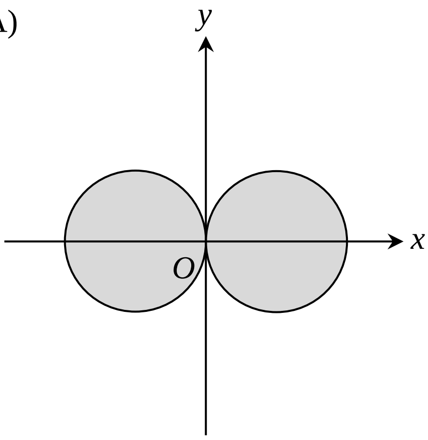
(C) $\int_{0}^{a} f(x) - f(a-x)\,dx.$

(D) $\int_{-a}^{a} f(x) + f(a-x)\,dx.$

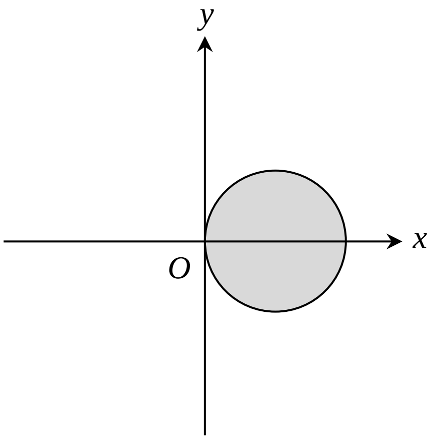
6 Which of the following diagrams best fits the locus equation described by

$$\left|\frac{1}{z}+\frac{1}{\overline{z}}\right|\geq 1$$

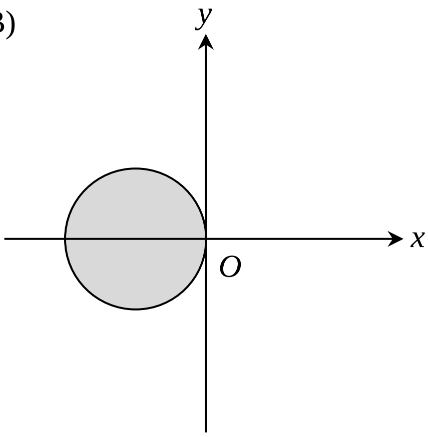
(A)



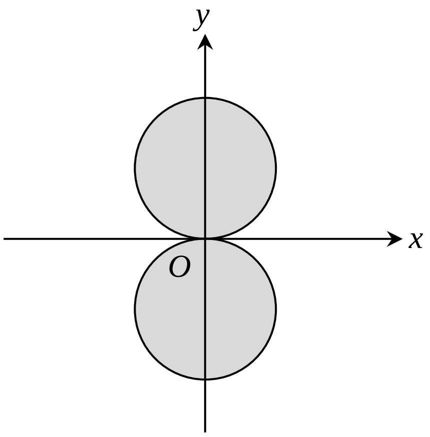
(C)



(B)



(D)



7 The integral

$$\int_{-2}^{-1} \frac{\sqrt{x^2-1}-2x^2\sqrt{3}}{x}\,dx,$$

is evaluated using the trigonometric substitution $x = \sec\theta$.

Which of the following is the correct value of the integral?

(A) $4\sqrt{3}-\dfrac{\pi}{3}$

(B) $4\sqrt{3}+\dfrac{\pi}{3}$

(C) $2\sqrt{3}-\dfrac{\pi}{3}$

(D) $2\sqrt{3}+\dfrac{\pi}{3}$

8 An ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, with $a > b > 0$, has one of its directrices being $x = 2$ and the minor axis length $2b$ is equal to k units.

For what values of k will such an ellipse exist?

(A) $0 < k \le 2$.

(B) $-2 \le k \le 2$.

(C) $-1 \le k \le 1$.

(D) $0 < k \le 1$.

9 A car of mass m moves along a banked track with speed v in a circle of radius r. The banking angle of the track is θ. Let g be the acceleration due to gravity.

The equation below is the expression for *friction* after forces have been resolved.

$$F = mg\sin\theta - \frac{mv^2}{r}\cos\theta \text{, where } F > 0 \text{.}$$

As v decreases, the value of F increases. When $v = v_F$, the magnitude of F attains a maximum $F_{\max}$, but the car will not slip. For all $0 < v < v_F$, the friction F maintains the same value $F_{\max}$ but the car will begin to slip sideways.

It is observed that at a particular moment, the car has some speed $0 < v < v_F$. Which of the following actions will best prevent the car from slipping further?

(A) Decrease radius AND increase mass.

(B) Decrease radius AND decrease mass.

(C) Increase radius AND increase mass.

(D) Increase radius AND decrease mass.

10 Let a, b and c be positive numbers. Which of the following expressions has the smallest minimum value?

(A) $\dfrac{(a+b)(b+c)(a+c)}{abc}$.

(B) $\dfrac{a+b}{c} + \dfrac{b+c}{a} + \dfrac{a+c}{b}$.

(C) $\left(a+\dfrac{1}{a}\right)\left(b+\dfrac{1}{b}\right)\left(c+\dfrac{1}{c}\right)$.

(D) $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose $f(x)$ is a non-linear polynomial where $f'(\alpha) = f(\alpha) = 0$ for a particular point $x = \alpha$.

(i) Show that $f(x)$ can be expressed in the form **2**

$$f(x) = (x - \alpha)^2 g(x),$$

where $g(x)$ is some function that is continuous at $x = \alpha$.

(ii) Suppose $f(x)$ is not a polynomial. **1**

By finding a counter-example, show that the result in (i) may no longer be true.

(b) (i) Sketch the graph of $y = \ln x - x$. **2**

(ii) Hence, or otherwise, sketch the graph of $y = e^{x+y}$. **2**

Question 11 continues on page 8

(c) Every minute, the probability that a certain bacterium will divide into two identical bacteria is p. The probability that it will die is $1-p$. A bacterium will either die or divide independently of the other bacteria in the population. Initially, the population consists of exactly one bacterium.

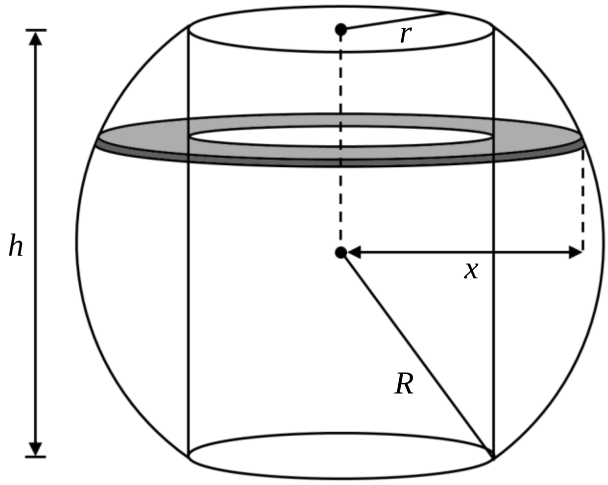
Let Q be the probability that the population eventually dies out.

(i) Explain why $Q=(1-p)+pQ^2$. **2**

(ii) Hence show that if $0 \le p \le \frac{1}{2}$, then the population is guaranteed to eventually become zero. **2**

Question 11 continues on page 9

(d) The diagram below shows a sphere of radius R. A hole of height h and constant radius r is drilled through the centre of the sphere, leaving behind a solid.



A typical horizontal cross section with outer radius x is shaded.

(i) Show that the volume of the solid is $\dfrac{\pi}{6}h^3$ units^3. **3**

(ii) Consider two spheres with unequal radii. A hole of fixed height h is drilled through the centre of both spheres, forming solids $\mathcal{S}_1$ and $\mathcal{S}_2$. **1**

What can be said about the volume of the two solids?

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m is projected vertically upwards through a medium with initial velocity u. It experiences gravitational acceleration g and a resistive force mkv^2, where k is a positive constant and v is the velocity of the particle at time t. The particle's displacement from the projection point, after t seconds, is given by x.

Let the terminal speed of the particle, when it falls back down through the same medium, be w.

(i) Show that the terminal velocity is given by $w^2 = \dfrac{g}{k}$. **1**

(ii) Show that particle has velocity **3**

$$v = w\left(\frac{u - w\tan\left(gt/w\right)}{w + u\tan\left(gt/w\right)}\right).$$

You may use the fact that if $0 < b < a$, then

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right).$$

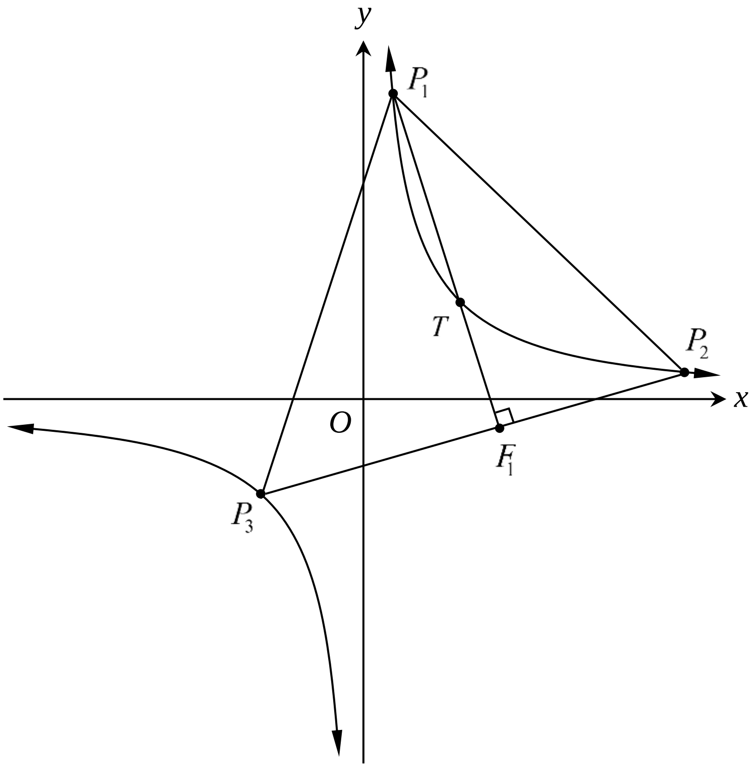
without proof.

(iii) Show that particle has displacement **2**

$$x = \frac{w^2}{g}\ln\left(\frac{w\cos\left(gt/w\right) + u\sin\left(gt/w\right)}{w}\right).$$

Question 12 continues on page 11

(b) Let $\left(ct, \frac{c}{t}\right)$, where $c > 0$, be any point on the hyperbola $xy = c^2$. The diagram below shows three points P_1, P_2 and P_3 with parameters t_1, t_2, t_3 respectively. The line F_1P_1 is perpendicular to P_2P_3 and intersects the hyperbola at T.



(i) Show that the equation of F_1P_1 is **2**

$$t_1t_2t_3x - t_1y = c\left(t_1^2t_2t_3 - 1\right).$$

(ii) Show that T has coordinates **2**

$$\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right).$$

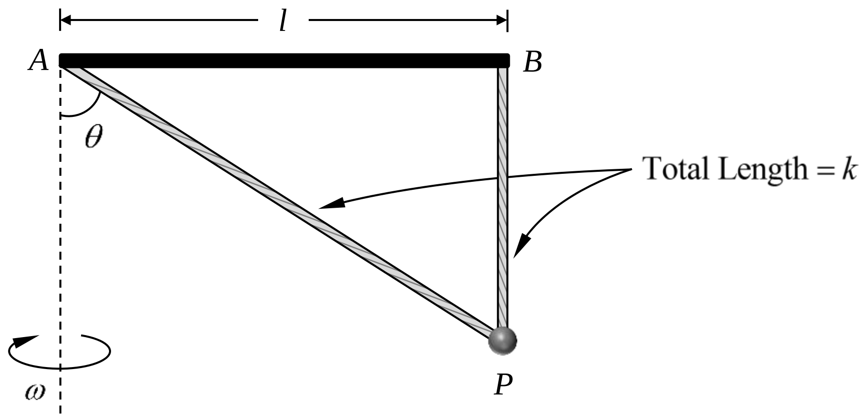
Question 12 continues on page 12

(iii) Let F_2 and F_3 be the feet of the perpendiculars from P_2 and P_3 to P_1P_3 and P_1P_2 respectively. **2**

Deduce that F_1P_1, F_2P_2 and F_3P_3 are concurrent at T.

Question 12 continues on page 13

(c) The diagram shows a straight rod AB of length l rotating in a horizontal plane about A. A light inextensible string of length k is attached to A and B. It is threaded through the centre of a bead P with mass m, which is free to slide smoothly along the string. When the angular speed of AB is ω, the bead remains vertically below B, and AP makes an acute angle θ with the vertical axis. Let g be the acceleration due to gravity.



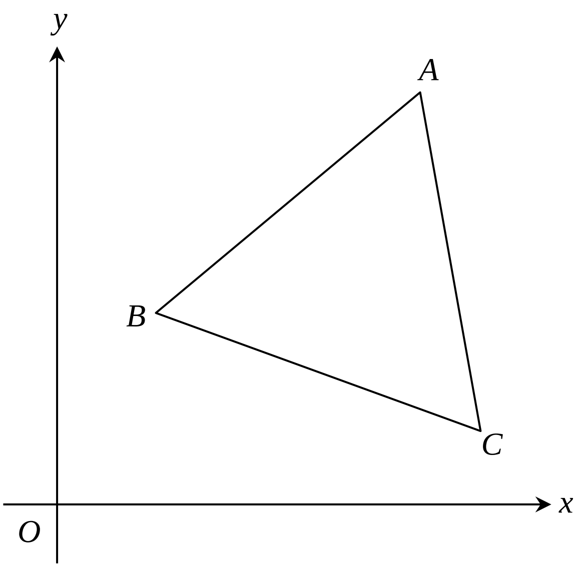
(i) Show that $\tan\left(\frac{\theta}{2}\right) = \frac{l\omega^2}{g}$. **2**

(ii) Hence prove that $\omega^2 = \frac{g}{k}$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Let a, b and c be complex numbers representing the vertices of $\triangle ABC$, labelled anti-clockwise in the Argand diagram. Let $\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$.



(i) Suppose $\triangle ABC$ is equilateral. **2**

Show that $a + b\omega + c\omega^2 = 0$.

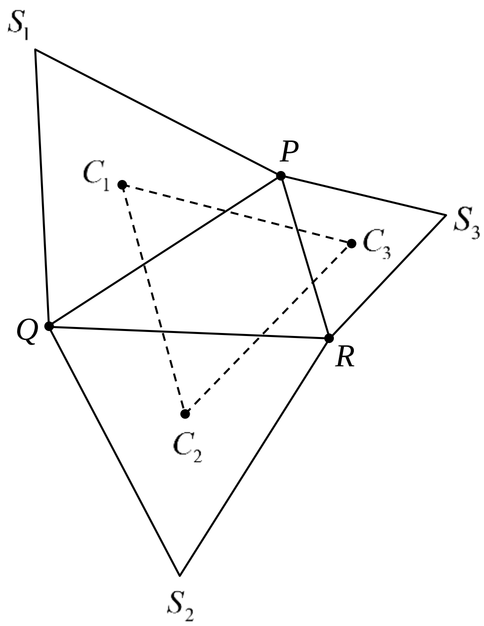
(ii) Show that conversely, if $a + b\omega + c\omega^2 = 0$, then $\triangle ABC$ is an equilateral triangle. **2**

Question 13 continues on page 15

Question 13 (continued)

(b) Let p, q and r be complex numbers representing the vertices of $\triangle PQR$. **2**

Each side of $\triangle PQR$ is one of the sides of an equilateral triangle with the third vertex being S_1, S_2 and S_3, as shown in the diagram below, represented by complex numbers s_1, s_2 and s_3.



It can be shown that the centre C_1 of $\triangle PQS_1$ is represented by the complex number

$$c_1 = \frac{p+q+s_1}{3},$$

and similar results hold for c_2 and c_3 corresponding to C_2 and C_3 respectively.

Use part (a) to prove that $\triangle C_1C_2C_3$ is equilateral.

Question 13 continues on page 16

(c) Let $P(x)$ be a monic polynomial of degree n with real distinct roots α_k, where $k = 1, 2, 3, \ldots, n$. Using partial fractions, $\dfrac{1}{P(x)}$ can be expressed in the form

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{c_k}{x-\alpha_k},$$

for some real c_k.

(i) Show that **1**

$$c_1 = \frac{x-\alpha_1}{P(x)} - \sum_{k=2}^{n} c_k \left(\frac{x-\alpha_1}{x-\alpha_k} \right)$$

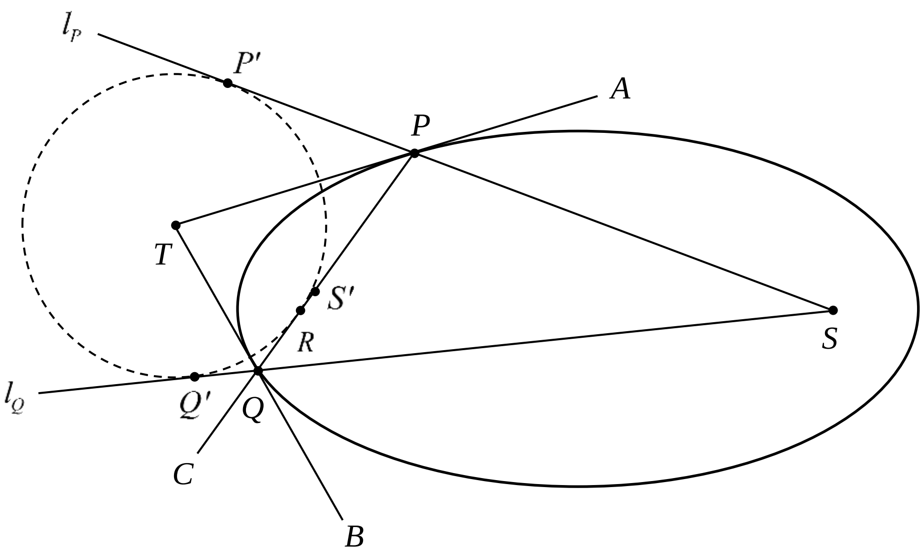
for some real c_k.

(ii) Hence show that **2**

$$\frac{1}{P(x)} = \sum_{k=1}^{n} \frac{1}{P'(\alpha_k)(x-\alpha_k)}.$$

Question 13 continues on page 17

(d) The diagram below shows a focal chord PQ on an arbitrary ellipse with foci S and S'. The chords SP and SQ are produced to form lines l_P and l_Q respectively. The tangents drawn from P and Q intersect at T.



(i) Show that T is the centre of a circle $\mathcal{C}$ that is tangential to l_P, l_Q and PQ at P', Q' and R respectively. **3**

(ii) Let R be the point of contact between the chord PQ and $\mathcal{C}$. Show that $PR + PS = QR + QS$. **1**

(iii) Deduce that R and S' coincide. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let x be a positive real number and $n \geq 2$ be an integer.

(i) Use mathematical induction to prove that **3**

$$(1+x)^n \geq 1+nx\,.$$

(ii) Hence show that if $x > \dfrac{n}{n-1}$, then **2**

$$x^n > x+1\,.$$

(b) Let $P(z) = z^n + z + 1$ be a non-linear polynomial. **2**

Use part (a) to prove that the complex roots of $P(z)$ lie in the region

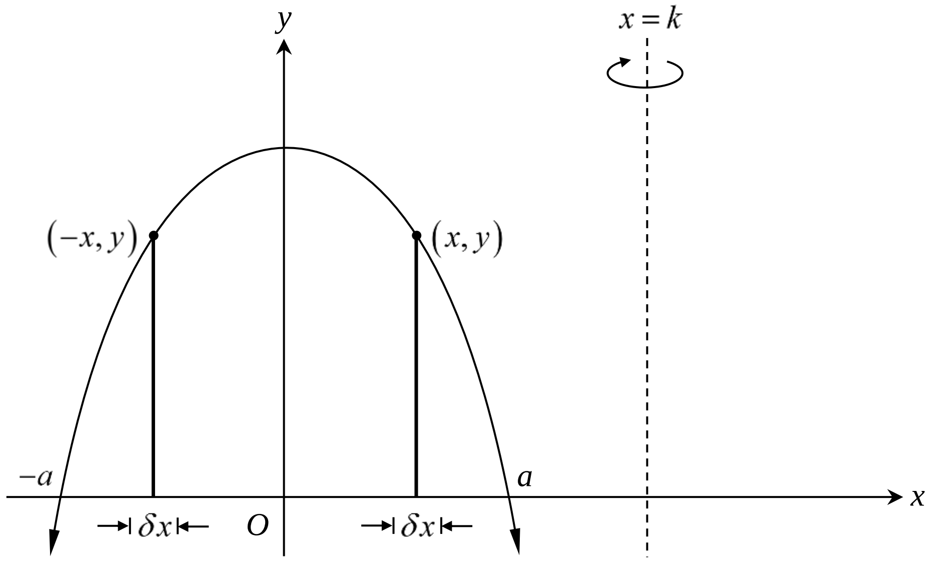
$$|z| \leq \frac{n}{n-1}\,.$$

Question 14 continues on page 19

(c) Let $y = f(x)$ be an even positive function in the domain $-a \le x \le a$ with $f(a) = f(-a) = 0$.

The region bounded by the curve and the x axis is rotated about the line $x = k$, forming a solid with volume V with cross sectional area A.

The diagram below shows two typical vertical strips of width δx at (x, y) and $(-x, y)$, which form shells with volume δV_1 and δV_2 respectively when rotated about $x = k$.



(i) Show that $\delta V_2 \approx 2\pi f(x)(k + x)\delta x$ and state a similar result for δV_1. **2**

(ii) Deduce that the ratio between V and A is independent of the function. **1**

Question 14 continues on page 20

(d) Consider the integral

$$I_n = \int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2}\,dx\,.$$

where n is a positive integer.

(i) Show that **2**

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = P(x) + \frac{2^{2n}(-1)^n}{1+x^2}\,,$$

where $P(x)$ is a polynomial of degree $8n-2$.

(ii) Show that $I_n \to 0$ as $n \to \infty$. **1**

(iii) Show that **1**

$$I_n = 2^{2n-2}(-1)^n \pi + R_n\,,$$

where R_n is some rational number.

(iv) Hence, explain the significance of the number $\left|\frac{R_n}{2^{2n}}\right|$ for large values of n. **1**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Prove that for $0 < x < \dfrac{\pi}{2}$, **3**

$$\cot x < \frac{1}{x} < \operatorname{cosec} x .$$

(b) Consider the equation

$$\left(\frac{z+i}{z-i}\right)^{2n} = -1,$$

where n is a positive integer.

(i) Show that the roots are $z = \cot\left(\dfrac{(2k-1)\pi}{4n}\right)$ where $k = 1, 2, \ldots, 2n$. **2**

(ii) Hence, or otherwise, find the roots of the polynomial **2**

$$P(x) = \binom{2n}{0}x^n - \binom{2n}{2}x^{n-1} + \binom{2n}{4}x^{n-2} + \ldots + (-1)^n\binom{2n}{2n}.$$

(iii) Deduce that **2**

$$\operatorname{cosec}^2\left(\frac{\pi}{4n}\right) + \operatorname{cosec}^2\left(\frac{3\pi}{4n}\right) + \operatorname{cosec}^2\left(\frac{5\pi}{4n}\right) + \ldots + \operatorname{cosec}^2\left(\frac{(2n-1)\pi}{4n}\right) = 2n^2 .$$

Question 15 continues on page 22

(iv) Use the results in part (a) to deduce that **2**

$$\frac{\pi^2}{16}\left(\frac{2n-1}{n}\right) < \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{(2n-1)^2} < \frac{\pi^2}{8}.$$

(v) Hence, find the limiting value of **1**

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots$$

as n gets large.

(c) (i) By considering a suitable binomial expansion, show that **2**

$$2^n\cos^n\theta\cos n\theta = \sum_{k=0}^{n}\binom{n}{k}\cos 2k\theta .$$

(ii) Hence, find **1**

$$\int_{-\pi}^{\pi}\cos^n\theta\cos n\theta\,d\theta .$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A bag has m white balls and $n - m$ black balls. An integer k from 0 to n inclusive is chosen randomly, and then k balls are drawn from the bag without replacement. **2**

Find the probability that all drawn balls are white.

(b) Suppose there are m white balls, $n - m$ black balls and one red ball arranged randomly in a row from left to right.

(i) Explain why the probability that the red ball is to the left of all the black balls is $\dfrac{1}{n-m+1}$. **1**

(ii) Use part (a), or otherwise, to simplify **2**

$$\frac{\binom{m}{0}}{\binom{n}{0}}+\frac{\binom{m}{1}}{\binom{n}{1}}+\frac{\binom{m}{2}}{\binom{n}{2}}+\ldots+\frac{\binom{m}{m}}{\binom{n}{m}}.$$

Question 16 continues on page 24

(c) Let r be a rational number $\frac{a}{b}$ where a and b are non-zero integers. Suppose e^r can be expressed in the form $\frac{p}{q}$, where p and q are integers.

Define the family of integrals

$$I_n = \frac{a^{2n+1}pq}{n!}\int_{-1}^{1}\left(1-x^2\right)^n e^{rx}\,dx,$$

for $n = 0, 1, 2, 3, \ldots$

You are given that $I_1 = 2ab^2\left(p^2+q^2\right) - 2b^3\left(p^2-q^2\right)$. (Do NOT prove this.)

(i) Show that **4**

$$I_n = 4a^2b^2I_{n-2} - 2(2n-1)b^2I_{n-1},$$

for all integers $n \geq 2$.

(ii) Prove that **2**

$$0 < \left|I_n\right| \leq 2pq\frac{|a|^{2n+1}}{n!}e^{|r|}.$$

Question 16 continues on page 25

Question 16 (continued)

(iii) It can be shown that if R is any real number, then as $n \to \infty$, **3**

$$\frac{R^n}{n!} \to 0 .$$

Use this to prove that if r is any non-zero rational number, then e^r is irrational.

(iv) Deduce that if r is a positive rational number other than 1, then $\log_e(r)$ is also irrational. **1**

End of Exam

2016 Bored of Studies Trial Examinations

Mathematics Extension 2

Solutions

Multiple Choice

1. B
2. D
3. C
4. D
5. D
6. A
7. B
8. A
9. A
10. B

Brief Explanations

Question 1 Resolving the forces yields $T = mr\omega^2$. Halving the mass and quadrupling the result yields $T_{\text{new}} = (m/2)r(4\omega)^2 = 8mr\omega^2 = 8T$, which is not the same as the original tension. But all the other options yield the same original tension, hence the answer is (B).

Question 2 According to (A), if $2+0i$ is a root, then $2-0i$ must also be a root, which is obviously not true as this implies that all roots occur in conjugate pairs. (B) would be correct if all the coefficients are real, which was not given. (C) is incorrect because all we can deduce from it is that it has degree *at least* $2n$, but this does not mean that it must have degree $2n$. (D) is correct because if the polynomial is monic and has integer coefficients, then the roots are either integers or irrational (this is actually a direct result of the Rational Root Theorem, but can be deduced from just the Integer Root Theorem). Since the roots are not integers, they must be irrational.

Question 3 The derivative $\frac{dy}{dx}$ can be found using implicit differentiation. The initial result is $\frac{dy}{dx} = -\left(\frac{e^y \cos x + e^x \sin y}{e^y \sin x + e^x \cos y}\right)$. Then substitute the original relation back in to obtain a simplified expression for $\frac{dy}{dx}$.

Question 4 (A) does not work when $f(x)=\sqrt{x}$. (B) does not work when $f(x)=\sqrt[3]{x}$. (C) does not work when $f(x)=x^2$. (D) works because $y'=\dfrac{f'(x)}{1+(f(x))^2}$ and at $x=\alpha$, this is equal to $f'(\alpha)$, which is the gradient of the tangent of $y=f(x)$ at $x=\alpha$.

Question 5 Note that $\int_{-a}^{a} f(x)-a\,dx=0.$ By splitting the integral and re-arranging, the integral in (A) can be found to be $2a^2$. Since the curve is point-wise symmetric about $(0,a)$, we have $f(x)+f(-x)=2a$ and the integral is also $4a^2$. If we split up the integral in (C), we realise quickly that it becomes 0 since $\int_0^a f(x)\,dx$ is equal to $\int_0^a f(a-x)\,dx.$ The mapping $x\mapsto a-x$ takes the domain $-a\le x\le a$ to $0\le x\le 2a$. Hence we cannot conclude anything about the value of $\int_{-a}^{a} f(a-x)\,dx,$ since we only have information about the integrand in the domain $0\le x\le 2a$ (the same properties provided in the question).

Question 6 Re-arranging, we have $|z+\bar{z}|\ge|z\bar{z}|$, which is the same thing as $|2x|\ge x^2+y^2$. Splitting into cases, this yields two circles that have been horizontally shifted, hence (A).

Question 7 To find the new limits, we set $\sec\theta=-1$ and $\sec\theta=-2$. When we solve our limits, we want to find them in the θ domain such that $\sec\theta$ is one-to-one. We pick $2\pi/3\le\theta\le\pi$. In this domain, we actually have $\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=-\tan\theta$. Once these facts have been established, we compute the integral as usual and find the answer to be (B).

Question 8 We know that $a/e=2\Rightarrow a^2=4e^2$ and $b=k/2\Rightarrow b^2=k^2/4$. Using the fact that $b^2=a^2(1-e^2)$, we have $a^2(1-e^2)=k^2/4$ and hence $16e^2(1-e^2)=k^2$. This is a quadratic in terms of *e*, so $16e^4-16e^2+k^2=0$. We want the discriminant to be positive so that solutions exist, so $256-64k^2\ge0$ and hence $-2\le k\le2$. But since *k* is a length, we have $0<k\le2$. This agrees with the condition for the ellipse $0<e<1$.

Question 9 As v gets smaller, the value of F increases until we reach $v = v_F$, in which case the value of F attains a maximum $F_{\max}$. If we decrease v further, the value of F can no longer increase accordingly, so the other constants in the *RHS* have to change so that the value of F remains at $F_{\max}$, hence preventing the car from slipping further. If we focus on the term $\frac{mv^2}{r} = mv^2 \times \frac{1}{r}$, we see that there are two ways to 'counteract' the effect of v getting smaller. We may choose to increase m or increase $\frac{1}{r}$, which corresponds to decreasing r.

Question 10 Using repeated applications of the standard result $\frac{a^2 + b^2}{2} \geq ab$ and modified versions of it, we deduce that $(\mathrm{A}) \geq 8$, $(\mathrm{B}) \geq 6$, $(\mathrm{C}) \geq 8$ and $(\mathrm{D}) \geq 9$. Hence (B) has the smallest minimum value.

Written Response

Question 11 (a) (i)

Since $f(\alpha)=0$, then by the factor theorem we have

$$f(x)=(x-\alpha)g_1(x),$$

where $g_1(x)$ is some polynomial. Using the product rule, we have

$$f'(x)=g_1(x)+(x-\alpha)g_1{}'(x).$$

Recall that $f'(\alpha)=0$, so $g_1(\alpha)=0$. Again, by the factor theorem, this means that

$$g_1(x)=(x-\alpha)g_2(x),$$

where $g_2(x)$ is some polynomial.

Substituting this back into our expression for $f(x)$, we have

$$\begin{aligned} f(x) &= (x-\alpha)g_1(x) \\ &= (x-\alpha)(x-\alpha)g_2(x) \\ &= (x-\alpha)^2 g_2(x) \end{aligned}$$

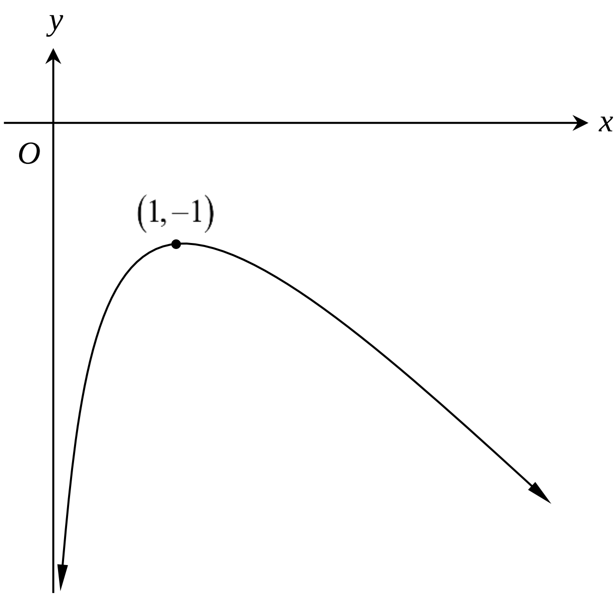
Question 11 (a) (ii)

Recall from curve sketching that if $y = f(x)$ has a 'single root' $x = \alpha$ with continuous derivative at $x = \alpha$, then $y = (f(x))^2$ has a 'double root' at $x = \alpha$. This idea can be used to generate many counter-examples. For example, observe that if $f(x) = (\ln x)^2$, then we have $f'(1) = f(1) = 0$. But this contradicts the statement in the question since

$$\begin{aligned} f(x) &= (\ln x)^2 \\ &= (x-1)^2 \underbrace{\frac{(\ln x)^2}{(x-1)^2}}_{g(x)} \end{aligned}$$

And we see that $g(x)$ is not continuous at $x = 1$.

Question 11 (b) (i)

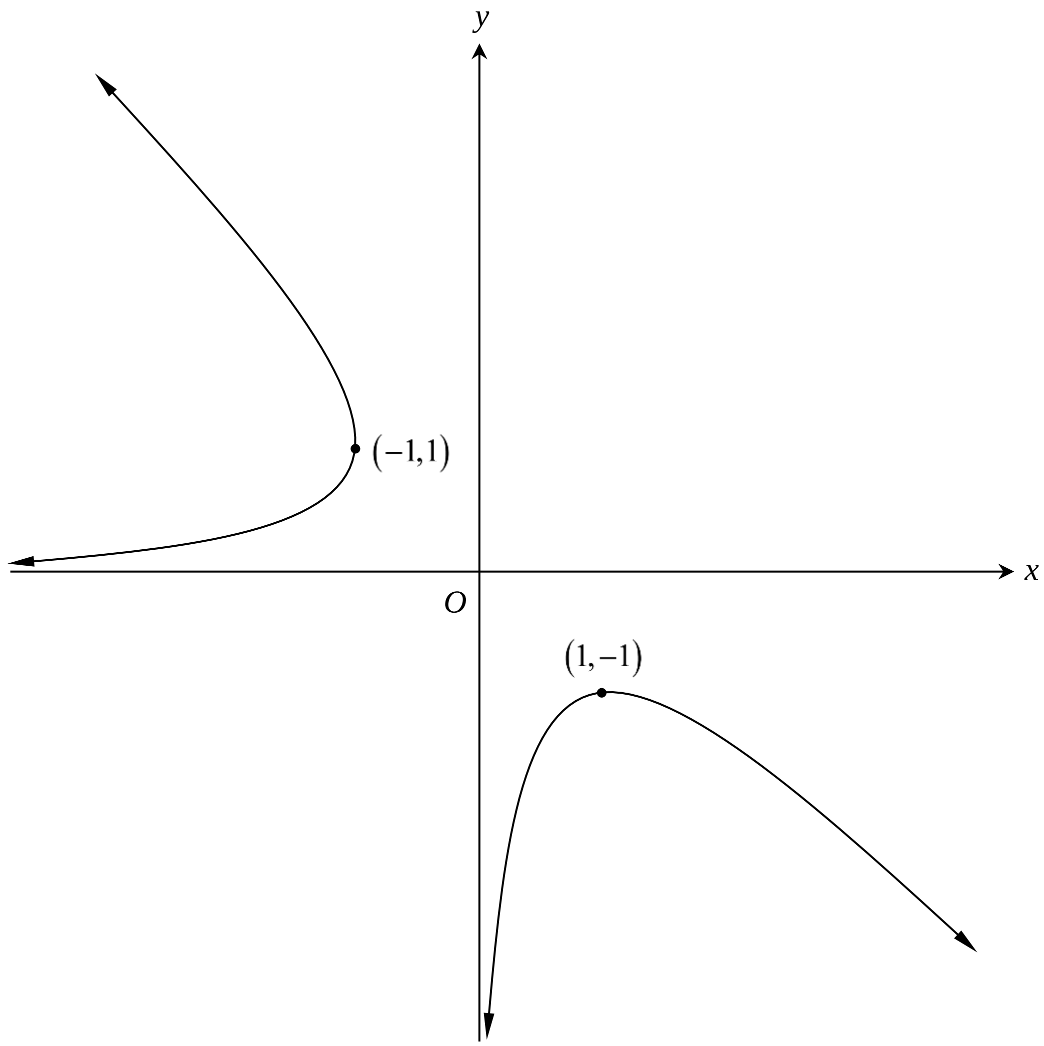


Question 11 (b) (ii)

Swap x and y in the equation from part (i).

$$\begin{aligned} \ln y - y &= x \\ \ln y &= x + y \\ y &= e^{x+y} \end{aligned}$$

Hence, the graph of $y = e^{x+y}$ will just be a reflection of the graph in (i) in the line $y = x$.



Question 11 (c) (i)

We partition the event of Q. There are two possible scenarios that can occur from the first bacterium. The first case is that it dies, which occurs with probability $1-p$. Of course, this implies that the population has died out. The second case is that it divides, which has probability p. The potential populations, from each of the two new bacteria formed, have probability Q of dying out. As these are independent events, the probability of *both* dying out is Q^2. Hence, the overall probability is $Q=(1-p)+pQ^2$.

Question 11 (c) (i)

Use the quadratic formula to solve for Q. Re-arranging yields $pQ^2-Q+(1-p)=0$.

$$
\begin{aligned}
Q &= \frac{1\pm\sqrt{1-4p(1-p)}}{2p} \\
&= \frac{1\pm\sqrt{1-4p+4p^2}}{2p} \\
&= \frac{1\pm(1-2p)}{2p}
\end{aligned}
$$

Hence $Q=1$ or $Q=\frac{1-p}{p}$. If $0<p<\frac{1}{2}$, then $Q=\frac{1-p}{p}>1$, which cannot be the case as Q is a probability. So it must be the case that $Q=1$. We now treat the extreme cases $p=0$ and $p=\frac{1}{2}$ individually, but in actual fact, both yield $Q=1$. Hence we take the case where $Q=1$, which means that the population is guaranteed to eventually become extinct.

Question 11 (d) (i)

Let the thickness of the arbitrary slice be δy, where *y* is the vertical distance measured from the centre of the circle. The volume of an arbitrary slice with outer radius *x* is then $\delta V = \pi\left(x^2 - r^2\right)\delta y$. But $x^2 = R^2 - y^2$ and so the volume becomes $\delta V = \pi\left(R^2 - y^2 - r^2\right)\delta y$.

But also observe that $R^2 - r^2 = \left(\frac{h}{2}\right)^2$, and so

We can now compute the volume.

$$\begin{aligned}
V &= 2\pi\int_0^{\frac{h}{2}}\left(\left(\frac{h}{2}\right)^2 - y^2\right)dy \\
&= 2\pi\left[\frac{h^2}{4}y - \frac{1}{3}y^3\right]_0^{\frac{h}{2}} \\
&= 2\pi\left[\frac{h^3}{8} - \frac{h^3}{24}\right] \\
&= \frac{\pi}{6}h^3
\end{aligned}$$

Question 11 (d) (ii)

From the result in (i), the volume is entirely dependent upon the height of the hole drilled through the centre and not dependent on *R*. So as long as they are the same, the volume of the remaining solids are the same.

Question 12 (a) (i)

Resolving forces, we have $a = g - kv^2$. When $v = w$, we have $a = 0$ and hence the result.

Question 12 (a) (ii)

For the upwards journey, we have

$$\begin{aligned} a &= -\left(g + kv^2\right) \\ &= -k\left(\frac{g}{k} + v^2\right) \\ \frac{dv}{dt} &= -k\left(w^2 + v^2\right) \end{aligned}$$

Solving this differential equation explicitly, using given initial conditions, we have

$$\begin{aligned} \int_0^t -k\,dt &= \int_u^v \frac{dv}{w^2 + v^2} \\ -kt &= \frac{1}{w}\tan^{-1}\left(\frac{v}{w}\right)\Bigg|_u^v \\ wkt &= \tan^{-1}\left(\frac{u}{w}\right) - \tan^{-1}\left(\frac{v}{w}\right) \\ &= \tan^{-1}\left(\frac{\frac{u}{w} - \frac{v}{w}}{1 + \left(\frac{u}{w}\right)\left(\frac{v}{w}\right)}\right) \\ &= \tan^{-1}\left(\frac{w(u - v)}{w^2 + uv}\right) \end{aligned}$$

Now make v the subject.

$$\begin{aligned}
\tan\left(wkt\right) &= \frac{w\left(u-v\right)}{w^2+uv} \\
\tan\left(gt/w\right) &= \frac{w\left(u-v\right)}{w^2+uv} \\
\left(w^2+uv\right)\tan\left(gt/w\right) &= w\left(u-v\right) \\
v\left(w+u\tan\left(gt/w\right)\right) &= w\left(u-w\tan\left(gt/w\right)\right) \\
v &= w\left(\frac{u-w\tan\left(gt/w\right)}{w+u\tan\left(gt/w\right)}\right)
\end{aligned}$$

Question 12 (a) (iii)

Multiplying both sides of the result in (ii) by $\cos\left(gt/w\right)$, we have

$$\begin{aligned}
\frac{dx}{dt} &= w\left(\frac{u-w\tan\left(gt/w\right)}{w+u\tan\left(gt/w\right)}\right) \\
&= w\left(\frac{u\cos\left(gt/w\right)-w\sin\left(gt/w\right)}{w\cos\left(gt/w\right)+u\sin\left(gt/w\right)}\right)
\end{aligned}$$

And we recognise that this is a log integral.

$$\begin{aligned}
x &= w\int\frac{u\cos\left(gt/w\right)-w\sin\left(gt/w\right)}{w\cos\left(gt/w\right)+u\sin\left(gt/w\right)}dt \\
&= w\left(\frac{w}{g}\right)\int\frac{\frac{g}{w}\left(u\cos\left(gt/w\right)-w\sin\left(gt/w\right)\right)}{w\cos\left(gt/w\right)+u\sin\left(gt/w\right)}dt \\
&= \frac{w^2}{g}\ln\left(w\cos\left(gt/w\right)+u\sin\left(gt/w\right)\right)+C
\end{aligned}$$

But $t=0,\ x=0 \Rightarrow C=-\frac{w^2}{g}\ln\left(w\right)$

$$\begin{aligned}
x &= \frac{w^2}{g}\ln\left(w\cos\left(gt/w\right)+u\sin\left(gt/w\right)\right)-\frac{w^2}{g}\ln\left(w\right) \\
&= \frac{w^2}{g}\ln\left(\frac{w\cos\left(gt/w\right)+u\sin\left(gt/w\right)}{w}\right)
\end{aligned}$$

Question 12 (b) (i)

We can find the gradient of F_1P_1 by first finding the gradient of P_2P_3, since they are perpendicular.

$$\begin{aligned} m_{P_2P_3} &= \frac{c/t_3 - c/t_2}{ct_3 - ct_2} \\ &= -\frac{1}{t_2t_3} \end{aligned}$$

Hence $m_{F_1P_1} = t_2t_3$. We can use the point-gradient formula using the point P_1.

$$\begin{aligned} y - \frac{c}{t_1} &= t_2t_3\left(x - ct_1\right) \\ t_1y - c &= t_1t_2t_3\left(x - ct_1\right) \\ t_1y - c &= t_1t_2t_3\left(x - ct_1\right) \\ t_1t_2t_3x - t_1y &= c\left(t_1^2t_2t_3 - 1\right) \end{aligned}$$

Question 12 (b) (ii)

Solve F_1P_1 simultaneously with the equation of the hyperbola $xy = c^2$.

$$\begin{aligned} t_1t_2t_3x - t_1\left(\frac{c^2}{x}\right) &= c\left(t_1^2t_2t_3 - 1\right) \\ t_1t_2t_3x^2 - c^2t_1 &= c\left(t_1^2t_2t_3 - 1\right)x \\ t_1t_2t_3x^2 - c\left(t_1^2t_2t_3 - 1\right)x - c^2t_1 &= 0 \end{aligned}$$

This is a quadratic in terms of *x*, and the solutions are the *x* coordinates of P_1 and F_1, which we will call x_P and x_F respectively.

Using the product of roots, we have $x_P\, x_F = -\frac{c}{t_2t_3}$. But we know that $x_P = ct_1$.

Hence $\left(ct_1\right)x_F = -\frac{c^2t_1}{t_1t_2t_3}$ and therefore $x_F = -\frac{c}{t_1t_2t_3}$. Substituting this into $xy = c^2$ yields the *y* coordinate.

Question 12 (b) (iii)

We notice that the equations of F_1P_1, F_2P_2 and F_3P_3 will be similar, except with the parameters cycled around. So we have

$$t_1t_2t_3x - t_2y = c\left(t_1t_2^2t_3 - 1\right)$$
$$t_1t_2t_3x - t_3y = c\left(t_1t_2t_3^2 - 1\right)$$

We will show that T lies on one of the lines.

$$\begin{aligned}
LHS &= t_1t_2t_3x - t_2y \\
&= t_1t_2t_3\left(-\frac{c}{t_1t_2t_3}\right) - t_2\left(-ct_1t_2t_3\right) \\
&= -c + ct_1t_2^2t_3 \\
&= c\left(t_1t_2^2t_3 - 1\right) \\
&= RHS
\end{aligned}$$

Similarly, for the other line, we have

$$\begin{aligned}
LHS &= t_1t_2t_3x - t_2y \\
&= t_1t_2t_3\left(-\frac{c}{t_1t_2t_3}\right) - t_3\left(-ct_1t_2t_3\right) \\
&= -c + ct_1t_2t_3^2 \\
&= c\left(t_1t_2t_3^2 - 1\right) \\
&= RHS
\end{aligned}$$

Hence, the three lines are concurrent at *T*.

Question 12 (c) (i)

Resolving forces, we have

$$
\begin{aligned}
T\cos\theta + T &= mg \\
T\sin\theta &= ml\omega^2
\end{aligned}
$$

Eliminating T

$$\frac{\sin\theta}{1+\cos\theta} = \frac{l\omega^2}{g}$$

Using double-angle formulae

$$
\begin{aligned}
\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} &= \frac{l\omega^2}{g} \\
\tan\left(\frac{\theta}{2}\right) &= \frac{l\omega^2}{g}
\end{aligned}
$$

Question 12 (c) (ii)

In $\triangle ABP$, we have $\sin\theta = \dfrac{l}{AP}$ and $\cos\theta = \dfrac{BP}{AP}$. But recall from earlier that $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{l\omega^2}{g}$

.

Substituting this in, we have

$$
\begin{aligned}
\frac{L/AP}{1+BP/AP} &= \frac{l\omega^2}{g} \\
\frac{l}{AP+BP} &= \frac{l\omega^2}{g}
\end{aligned}
$$

But from the diagram, we have $AP + BP = k$.

$$\frac{l}{k} = \frac{l\omega^2}{g}$$

$$\omega^2 = \frac{g}{k}$$

Question 13 (a) (i)

First, note that we have $\omega^3 = 1$ and hence $1 + \omega + \omega^2 = 0$. Hence, we have

$$\begin{aligned} \text{cis}\left(\frac{\pi}{3}\right) &= -\text{cis}\left(-\frac{2\pi}{3}\right) \\ &= -\omega^2 \\ &= 1 + \omega \end{aligned}$$

So from the diagram, we have

$$\begin{aligned} \overrightarrow{BC}\ \text{cis}\left(\frac{\pi}{3}\right) &= \overrightarrow{BA} \\ (c-b)(1+\omega) &= a - b \\ a + b\omega - c - c\omega &= 0 \\ a + b\omega - c(1+\omega) &= 0 \\ a + b\omega + c\omega^2 &= 0 \end{aligned}$$

where the last line was acquired using the fact that $1 + \omega + \omega^2 = 0$.

Question 13 (a) (ii)

Working backwards from the final steps in the previous question, we may obtain the equation $(c-b)\omega = a - c$ from the given fact that $a + b\omega + c\omega^2 = 0$.

Taking the modulus of both sides, we have $\left|\overrightarrow{BC}\right| = \left|\overrightarrow{CA}\right|$. In other words, the lengths *AC* and *BC* are equal in length.

Similarly, by using the fact that $\omega = -1 - \omega^2$, we obtain

$$
\begin{aligned}
a + b\omega + c\omega^2 &= 0 \\
a + b\left(-1 - \omega^2\right) + c\omega^2 &= 0 \\
a - b &= \left(b - c\right)\omega^2
\end{aligned}
$$

Again, taking the modulus of both sides, we establish that the lengths *AB* and *BC* are equal. Hence, the triangle is equilateral.

Question 13 (b)

Use part (a) (i) on each of the three equilateral triangles to obtain the system of equations

$$q + p\omega + s_1\omega^2 = 0 \qquad (1)$$
$$r + q\omega + s_2\omega^2 = 0 \qquad (2)$$
$$p + r\omega + s_3\omega^2 = 0 \qquad (3)$$

Now consider $(1)\times\omega + (2)\times\omega^2 + (3)$ and re-arrange, using the fact that $\omega^3 = 1$.

$$q\omega + p\omega^2 + s_1 + r\omega^2 + q + s_2\omega + p + r\omega + s_3\omega^2 = 0$$
$$(p + q + s_1) + \omega(q + r + s_2) + \omega^2(p + r + s_3) = 0$$

Divide both sides of the equation by 3.

$$\left(\frac{p+q+s_1}{3}\right) + \omega\left(\frac{q+r+s_2}{3}\right) + \omega^2\left(\frac{p+r+s_3}{3}\right) = 0$$

But from the definition of c_1, c_2, c_3, this then becomes

$$c_1 + c_2\omega + c_3\omega^2 = 0$$

And by (a) (ii), this is sufficient to deduce that $\Delta C_1C_2C_3$ is equilateral.

Question 13 (c) (i)

Multiply both sides of the given equation by $\left(x-\alpha_1\right)$ and re-arrange to make c_1 the subject.

$$\begin{aligned}
\frac{1}{P(x)} &= \sum_{k=1}^{n} \frac{c_k}{x-\alpha_k} \\
\frac{x-\alpha_1}{P(x)} &= c_1 + \sum_{k=2}^{n} \frac{c_k\left(x-\alpha_1\right)}{x-\alpha_k} \\
c_1 &= \frac{x-\alpha_1}{P(x)} - \sum_{k=2}^{n} c_k\left(\frac{x-\alpha_1}{x-\alpha_k}\right)
\end{aligned}$$

Question 13 (c) (ii)

Generalising the result in (i), we have for a general c_i

$$\begin{aligned}
c_i &= \frac{x-\alpha_i}{P(x)} - \sum_{\substack{k=1 \\ k\neq i}}^{n} c_i\left(\frac{x-\alpha_i}{x-\alpha_k}\right) \\
&= \frac{x-\alpha_i}{P(x)-P(\alpha_i)} - \sum_{\substack{k=1 \\ k\neq i}}^{n} c_i\left(\frac{x-\alpha_i}{x-\alpha_k}\right)
\end{aligned}$$

In the last line, the term $P\left(\alpha_i\right)$ was added to the denominator because it is equal to zero anyway, since α_i is a root.

Taking the limit as $x \to \alpha_i$, each of the terms in the sum vanish. Also, recall that

$$\lim_{x\to\alpha_i}\left(\frac{P(x)-P(\alpha_i)}{x-\alpha_i}\right) = P'\left(\alpha_i\right)$$

by the definition of the derivative. Hence, we have

$$\lim_{x\to\alpha_i}\left(c_i\right) = \lim_{x\to\alpha_i}\left(\frac{x-\alpha_i}{P(x)-P(\alpha_i)} - \underbrace{\sum_{\substack{k=1 \\ k\neq i}}^{n} c_i\left(\frac{x-\alpha_i}{x-\alpha_k}\right)}_{\text{each term is zero}}\right) = \frac{1}{P'\left(\alpha_i\right)},$$

and thus the result for each c_k.

Question 13 (d) (i)

We first aim to prove that $\Delta PP'T \equiv \Delta PRT$.

$$\begin{aligned} \angle APS &= \angle TPR && (\text{reflection property}) \\ \angle APS &= \angle P'PT && (\text{vertically opposite angles}) \end{aligned}$$

Hence $\angle TPR = \angle P'PT$. Also, PT is common and $PP' = PR$, as they are tangents from an external point *T*. Hence $\Delta PP'T \equiv \Delta PRT$ (SAS), from which we deduce that $P'T = RT$ as they are corresponding sides of congruent triangles.

Similarly, we can prove that $\Delta QQ'T \equiv \Delta QRT$, from which we deduce that $Q'T = RT$.

Since $P'T = RT = Q'T$, we conclude that *T* is uniquely the centre of a circle passing through P', *R* and Q'.

Question 13 (d) (ii)

Using the "tangents from an external point" theorem, we have $SP' = SQ'$. But $SP' = SP + PP'$ and $SQ' = SQ + QQ'$. So therefore $SP + PP' = SQ + QQ'$. But using the same theorem, we also have $PP' = PR$ and $QQ' = QR$. Hence the required result $SP + PR = SQ + QR$.

Question 13 (d) (iii)

Note that $PR = PS' + S'R$ and $QS' = QR + S'R$. Substitute this into the result in (ii).

$$\begin{aligned} PS' + S'R + PS &= QS' - S'R + QS \\ PS + PS' + 2(S'R) &= QS + QS' \end{aligned}$$

But recall the fact that $PS + PS' = QS + QS' = 2a$.

$$\begin{aligned} 2a + 2(S'R) &= 2a \\ S'R &= 0 \end{aligned}$$

And hence R and S' coincide.

Question 14 (a) (i)

We start our induction from $n=1$. This is because it is easier to consider $n=1$, and by proving it is true for all $n \geq 1$, we of course also prove that it is true for $n \geq 2$.

<u>*Base Case:*</u> $\text{LHS} = (1+x)^1 = 1+x = \text{RHS}$.

<u>*Inductive Hypothesis:*</u> $(1+x)^k \geq 1+kx$, where $k \geq 1$ and $x \geq 0$.

<u>*Inductive Step:*</u> Let $f(x) = (1+x)^{k+1} - (1+(k+1)x)$. It suffices to prove that $f(x) \geq 0$.

$$\begin{aligned} f(x) &= (1+x)^{k+1} - 1 - kx - x \\ &\geq (1+x)(1+kx) - 1 - kx - x \\ &= kx^2 \\ &\geq 0 \end{aligned}$$

Hence true by induction.

Question 14 (a) (ii)

Since $x > \dfrac{n}{n-1} > 1$, we have $x-1>0$. Hence, the inequality in (i) is equivalent to

$$\begin{aligned} (1+(x-1))^n &\geq 1+n(x-1) \\ x^n &\geq 1+n(x-1) \end{aligned}$$

which is achieved by replacing x with $x-1$.

From the fact that $x > \dfrac{n}{n-1}$, we have

$$\begin{aligned} nx - x &> n \\ n(x-1) &> x \end{aligned}$$

and hence the required result

$$\begin{aligned} x^n &\geq 1+n(x-1) \\ &> 1+x \end{aligned}$$

Question 14 (b)

Let $z=\alpha$ be any root of $P(z)$. So $\alpha^n+\alpha+1=0$ and hence $\alpha^n=-(\alpha+1)$.

Using the triangle inequality, we have

$$\begin{aligned} |\alpha|^n &= |\alpha+1| \\ &\leq |\alpha|+|1| \\ &= |\alpha|+1 \end{aligned}$$

Suppose by way of contradiction that $|\alpha|>\dfrac{n}{n-1}$. Using (a) (ii), this implies that $|\alpha|^n>|\alpha|+1$.

But this contradicts the earlier result $|\alpha|^n\leq|\alpha|+1$, and hence $|\alpha|\leq\dfrac{n}{n-1}$.

Question 14 (c) (i)

The radius of the cylindrical shell with volume δV_2 is $k-(-x)=k+x$. The height of the cylindrical shell is just $y=f(x)$ and the thickness is δx. Either by finding the volume of the shell as the difference of two cylinders, or by 'unfolding' it into a thin rectangular prism, the result should be the same $\delta V_2=2\pi f(x)(k+x)\delta x$.

Similarly, we deduce that $\delta V_1=2\pi f(-x)(k-x)\delta x$. Since the function is even, we have $f(x)=f(-x)$ and hence $\delta V_1=2\pi f(x)(k-x)\delta x$.

Question 14 (c) (ii)

To find the volume of the left half rotated around $x = k$, we would construct the integral

$$V_2 = \int_0^a 2\pi f(-x)(k+x)\,dx$$

Notice that the limits are not from $x = -a$ to $x = 0$, as one would initially think. This is because the point that defines the location of the shell is $(-x, y)$ as opposed to the usual (x, y). This means that instead of summing the shells from left to right, we sum from right to left. So our first shell is at $-x = 0 \to x = 0$, and our last shell is at $-x = -a \to x = a$.

The volume of the right half rotated around $x = k$ is more standard, and is given by

$$V_2 = \int_0^a 2\pi f(x)(k-x)\,dx$$

Adding the volumes of the shells, we have

$$\begin{aligned}
V &= V_1 + V_2 \\
&= 2\pi\int_0^a f(x)(k-x) + f(-x)(k+x)\,dx \\
&= 2\pi\int_0^a f(x)(k-x) + f(x)(k+x)\,dx \qquad \ldots\left(\text{since } f(x) \text{ even}\right) \\
&= 2\pi\int_0^a 2k \times f(x)\,dx \\
&= 4k\pi\int_0^a f(x)\,dx \qquad \ldots\left(\text{since the domain } 0 \le x \le a \text{ yields only half the area}\right) \\
&= 2k\pi A
\end{aligned}$$

Hence we have the ratio $V/A = 2k\pi$, which is independent of the function.

Question 14 (d) (i)

The division transformation gives us $\left(x^2+1\right)P\left(x\right)+ax+b=x^{4n}\left(1-x\right)^{4n}$. Substitute $x=i$.

$$\begin{aligned} ai+b &= i^{4n}\left(1-i\right)^{4n} \\ &= \left(\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^{4n} \\ &= 2^{2n}\,\text{cis}\left(-n\pi\right) \\ &= 2^{2n}\left(-1\right)^{n} \end{aligned}$$

Equating real and imaginary components, we have $a=0$ and $b=2^{2n}\left(-1\right)^n$. So therefore the remainder is

$$\begin{aligned} ax+b &= 0x+2^{2n}\left(-1\right)^{n} \\ &= 2^{2n}\left(-1\right)^{n} \end{aligned}$$

and hence the result.

Question 14 (d) (ii)

Since $0 \le x \le 1$, we have $0 \le x^{4n} \le 1$ and also $0 \le (1-x)^{4n} \le 1$. Hence $0 \le x^{4n}(1-x)^{4n} \le 1$.

From here, we see that $I_n \to 0$ as $n \to \infty$.

Question 14 (d) (iii)

Integrating the equation in (i), we have

$$\begin{aligned}\int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2}\,dx &= \int_0^1 P(x) + \frac{2^{2n}(-1)^n}{1+x^2}\,dx \\ I_n &= \int_0^1 P(x)\,dx \quad + 2^{2n}(-1)^n \tan^{-1} x\Big|_0^1 \\ &= \int_0^1 P(x)\,dx \quad + 2^{2n-2}(-1)^n \pi\end{aligned}$$

But note that $P(x)$ was constructed by polynomial long division of $x^{4n}(1-x)^{4n}$ with $1+x^2$. That is, $P(x)$ is some polynomial with integer coefficients. When this is integrated from $x=0$ to $x=1$, we obtain some rational number, we which we shall call R_n.

Hence, we have the result

$$\begin{aligned}I_n &= \int_0^1 P(x)\,dx + 2^{2n-2}(-1)^n \pi \\ &= R_n + 2^{2n-2}(-1)^n \pi\end{aligned}$$

Question 14 (d) (iv)

By re-arranging the result in (iii) and taking the absolute value of both sides, we have

$$0 < \left|\frac{R_n}{2^{2n}}\right| = \left|\frac{I_n}{2^{2n}} + (-1)^{n+1}\frac{\pi}{4}\right|$$
$$0 < \left|\frac{R_n}{2^{2n}}\right| \le \left|\frac{I_n}{2^{2n}}\right| + \frac{\pi}{4} \qquad \ldots(\text{triangle inequality})$$

From (ii), $I_n \to 0$ as $n \to \infty$, and so $\left|\frac{R_n}{2^{2n}}\right| \to \frac{\pi}{4}$.

In other words, $\left|\frac{R_n}{2^{2n}}\right|$ is a rational approximation of $\frac{\pi}{4}$, which increases in accuracy with increasing values of *n*.

Question 15 (a)

In the domain $0 < t < \dfrac{\pi}{2}$, we have $\cos t < 1$. Integrating this inequality in the domain, we have

$$\int_0^x \cos t \, dt < \int_0^x 1 \, dt$$

and hence we obtain $\sin x < x$. Arranging this yields $\dfrac{1}{x} < \operatorname{cosec} x$.

The other inequality is obtained similarly by considering the fact that $\sec^2 t > 1$, we have

$$\int_0^x \sec^2 t \, dt > \int_0^x 1 \, dt$$

and hence we obtain $\tan x > x$. Arranging this yields $\dfrac{1}{x} > \cot x$.

Question 15 (b) (i)

Substitute in $\cot\left(\dfrac{(2k-1)\pi}{4n}\right)$ into the LHS.

$$\begin{aligned}
\left(\frac{z+i}{z-i}\right)^{2n} &= \left(\frac{\cot\left(\frac{(2k-1)\pi}{4n}\right)+i}{\cot\left(\frac{(2k-1)\pi}{4n}\right)-i}\right)^{2n} \\
&= \left(\frac{\operatorname{cis}\left(\frac{(2k-1)\pi}{4n}\right)}{\operatorname{cis}\left(-\frac{(2k-1)\pi}{4n}\right)}\right)^{2n} \\
&= \left(\operatorname{cis}\left(\frac{(2k-1)\pi}{2n}\right)\right)^{2n} \\
&= \operatorname{cis}(2k-1)\pi \\
&= -1
\end{aligned}$$

Question 15 (b) (ii)

First, re-arrange the given equation

$$\begin{aligned}\left(\frac{z+i}{z-i}\right)^{2n} &= -1\\ \frac{(z+i)^{2n}}{(z-i)^{2n}} &= -1\\ (z+i)^{2n}+(z-i)^{2n} &= 0\end{aligned}$$

Expand using binomial expansion.

$$\begin{aligned}(z+i)^{2n}+(z-i)^{2n} &= 0\\ \sum_{k=0}^{2n}\binom{2n}{k}z^{2n-k}i^k\left(1+(-1)^k\right) &= 0\end{aligned}$$

Note that terms with odd k cancel. Explicitly writing out the sum, this becomes

$$\begin{aligned}2\left(\binom{2n}{0}z^{2n}i^0+\binom{2n}{2}z^{2n-2}i^2+\binom{2n}{2}z^{2n-4}i^4+\ldots+\binom{2n}{2n}z^0i^{2n}\right) &= 0\\ \binom{2n}{0}z^{2n}-\binom{2n}{2}z^{2n-2}+\binom{2n}{4}z^{2n-4}-\ldots+\binom{2n}{2n}(-1)^n &= 0\end{aligned}$$

Make the substitution $x=z^2$.

$$\binom{2n}{0}x^n-\binom{2n}{2}x^{n-1}+\binom{2n}{4}x^{n-2}-\ldots+\binom{2n}{2n}(-1)^n=0$$

So the solutions of this equations are the squares of the solution of the original equation

$$\left(\frac{z+i}{z-i}\right)^{2n}=-1$$

Hence, the solutions of $P(x)$ are $x=\cot^2\left(\frac{(2k-1)\pi}{4n}\right)$, where $k=1,\ 2,\ 3,\ \ldots,\ n$.

Question 15 (b) (iii)

Consider the sum of the roots of $P(x)$.

$$\sum_{k=1}^{n} \cot^2\left(\frac{(2k-1)\pi}{4n}\right) = \frac{\binom{2n}{2}}{\binom{2n}{0}}$$
$$= 2n^2 - n$$

But recall the familiar trigonometric identity $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

$$\sum_{k=1}^{n} \cot^2\left(\frac{(2k-1)\pi}{4n}\right) = 2n^2 - n$$
$$n + \sum_{k=1}^{n} \cot^2\left(\frac{(2k-1)\pi}{4n}\right) = 2n^2$$
$$\underbrace{1+1+\ldots+1}_{n \text{ times}} + \sum_{k=1}^{n} \cot^2\left(\frac{(2k-1)\pi}{4n}\right) = 2n^2$$
$$\sum_{k=1}^{n} \left[1 + \cot^2\left(\frac{(2k-1)\pi}{4n}\right)\right] = 2n^2$$
$$\sum_{k=1}^{n} \operatorname{cosec}^2\left(\frac{(2k-1)\pi}{4n}\right) = 2n^2$$

Question 15 (b) (iv)

Using the fact that $\frac{1}{\operatorname{cosec}^2 x} > \frac{1}{x^2}$ from (a), we have

$$\sum_{k=1}^{n} \operatorname{cosec}^2\left(\frac{(2k-1)\pi}{4n}\right) > \sum_{k=1}^{n}\left(\frac{4n}{(2k-1)\pi}\right)^2$$

$$2n^2 > \sum_{k=1}^{n}\left(\frac{4n}{(2k-1)\pi}\right)^2$$

Simplify

$$\sum_{k=1}^{n}\left(\frac{4n}{(2k-1)\pi}\right)^2 < 2n^2$$

$$\sum_{k=1}^{n}\frac{16n^2}{(2k-1)^2\pi^2} < 2n^2$$

$$\sum_{k=1}^{n}\frac{1}{(2k-1)^2} < \frac{\pi^2}{8}$$

Similarly, using the fact that $\cot^2 x < \frac{1}{x^2}$ from (a), we have

$$\sum_{k=1}^{n} \cot^2\left(\frac{(2k-1)\pi}{4n}\right) < \sum_{k=1}^{n}\left(\frac{4n}{(2k-1)\pi}\right)^2$$

$$2n^2 - n < \sum_{k=1}^{n}\left(\frac{4n}{(2k-1)\pi}\right)^2$$

Simplify

$$\sum_{k=1}^{n}\frac{16n^2}{(2k-1)^2\pi^2} > 2n^2 - n$$

$$\sum_{k=1}^{n}\frac{1}{(2k-1)^2} > \frac{\pi^2}{16}\left(\frac{2n-1}{n}\right)$$

And we are done proving both bounds for the sum.

Question 15 (b) (v)

We have

$$\frac{\pi^2}{16}\left(\frac{2n-1}{n}\right) < \sum_{k=1}^{n} \frac{1}{(2k-1)^2} < \frac{\pi^2}{8}$$

Taking the limit as $n \to \infty$, we have $\frac{\pi^2}{16}\left(\frac{2n-1}{n}\right) \to \frac{\pi^2}{8}$, which is precisely the upper bound.

Hence, we have $\sum_{k=1}^{n} \frac{1}{(2k-1)^2} \to \frac{\pi^2}{8}$.

Question 15 (c) (i)

Consider $\left(z+\frac{1}{z}\right)^n z^n = \left(z^2+1\right)^n$, where $z = \cos\theta + i\sin\theta$.

$$\begin{aligned}
(2\cos\theta)^n \operatorname{cis}(n\theta) &= \left(\operatorname{cis}(2\theta)+1\right)^n \\
2^n \cos^n\theta \operatorname{cis}(n\theta) &= \left(1+\operatorname{cis}(2\theta)\right)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\operatorname{cis}(2\theta)\right)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \operatorname{cis}(2k\theta)
\end{aligned}$$

Equating the real components, we have

$$2^n \cos^n\theta \cos(n\theta) = \sum_{k=0}^{n} \binom{n}{k} \cos(2k\theta)$$

Question 15 (c) (ii)

Integrate both sides over the interval $-\pi \le x \le \pi$.

$$\int_{-\pi}^{\pi} 2^n \cos^n \theta \cos(n\theta)\, d\theta = \int_{-\pi}^{\pi} \binom{n}{0} \cos 0 + \binom{n}{1} \cos 2\theta + \ldots + \binom{n}{n} \cos 2n\theta \, d\theta$$

Let $I = \int_{-\pi}^{\pi} \cos^n \theta \cos(n\theta)\, d\theta.$ Split the integral on the right hand side.

$$\begin{aligned}
2^n I &= \int_{-\pi}^{\pi} \binom{n}{0} \cos 0 \, d\theta + \underbrace{\int_{-\pi}^{\pi} \binom{n}{1} \cos 2\theta \, d\theta + \ldots + \int_{-\pi}^{\pi} \binom{n}{n} \cos 2n\theta \, d\theta}_{\text{each of these evaluate to zero}} \\
&= \int_{-\pi}^{\pi} \binom{n}{0} \cos 0 \, d\theta \\
&= 2\pi
\end{aligned}$$

Hence $I = \dfrac{\pi}{2^{n-1}}$.

$$\begin{aligned}
\int_{-\pi}^{\pi} \binom{n}{k} \cos(2k\theta)\, d\theta &= \frac{1}{2k} \binom{n}{k} \sin(2k\theta) \Big|_{-\pi}^{\pi} \\
&= \frac{1}{2k} \binom{n}{k} \left[\sin(2k\pi) - \sin(-2k\pi) \right] \\
&= 0
\end{aligned}$$

Question 16 (a)

Let's consider the case where k balls are drawn, and they are all white.

Firstly, we must consider the probability of selecting the number k, which is $\dfrac{1}{n+1}$.

Then, we compute the probability as usual. There are in total $\dbinom{n}{k}$ ways of selecting k balls from a bag containing n balls. We want all the k balls to be white, and this can occur in precisely $\dbinom{m}{k}$ ways. Hence, the probability that the number k is drawn and all k balls are white is

$$\frac{1}{n+1}\times\frac{\binom{m}{k}}{\binom{n}{k}}$$

So the number of balls being white can be $k = 0, 1, 2, 3, \ldots, m$. We stop here because there are only m white balls in total. So the probability of getting $m+1$, $m+2$, $m+3$, … , n white balls is zero; the event simply cannot occur.

Hence, the overall probability is

$$\sum_{k=0}^{m}\frac{1}{n+1}\times\frac{\binom{m}{k}}{\binom{n}{k}}=\frac{1}{n+1}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n}{k}}$$

Question 16 (b) (i)

First observe that the positions of the white balls are completely irrelevant. We focus only on the arrangements of the black and red balls. In total, there are $n-m+1$ balls in a row, including the red ball. There is precisely one position so that the red ball is to the left of the all the black balls, so the probability of this occurring is just $\dfrac{1}{n-m+1}$.

Question 16 (b) (ii)

Consider again the scenario in part (i). Selecting *k* balls from the bag, without replacement, and having them all being white is equivalent to permuting all *n* balls in a row and then observing if the first *k* balls are white. This can be done by introducing a new *red* ball in the $(k+1)^{\text{th}}$ place to terminate the 'string' of *k* consecutive white balls. Note that this red ball is necessarily to the left of all the black balls. This means that the probability of the red ball being to the left of all the black balls, found in (b) (i), is equal to the probability of the first *k* balls being white, found in (a).

Hence, equating the probabilities, we have

$$\frac{1}{n+1}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n}{k}}=\frac{1}{n-m+1}$$

and hence the result

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n}{k}}=\frac{n+1}{n-m+1}.$$

Note:

What we are essentially doing is establishing a one-to-one correspondence between the two scenarios. This one-to-one correspondence ensures that by counting one scenario, we have also counted the other scenario. We construct this correspondence more explicitly below.

For every permutation where there is a string of *k* white balls, where $0 \le k \le m$, it will be followed by the red ball to terminate the string. We can be sure of this because that is how the position of the red ball is defined. Conversely for every positioning of the red ball, there exists some string of white balls to the left of it, again containing any number $0 \le k \le m$ of white balls in it. This establishes a one-to-one correspondence between the probabilistic scenarios where the first *k* balls are all white, and the scenario where the red ball is to the left of all the black balls.

Question 16 (c) (i)

Use integration by parts.

$$\boxed{\begin{array}{ll} u = \left(1 - x^2\right)^n & dv = e^{rx}\,dx \\ du = -2nx\left(1 - x^2\right)^{n-1} dx & v = \dfrac{1}{r}e^{rx} \end{array}}$$

$$\begin{aligned} \int_{-1}^{1} \left(1 - x^2\right)^n e^{rx} dx &= \cancel{\left[\frac{1}{r}\left(1 - x^2\right)^n e^{rx}\right]_{-1}^{1}} + \frac{2n}{r}\int_{-1}^{1} x\left(1 - x^2\right)^n e^{rx} dx \\ &= \frac{2n}{r}\int_{-1}^{1} x\left(1 - x^2\right)^n e^{rx} dx \end{aligned}$$

Use integration by parts again.

$$\boxed{\begin{array}{ll} u = x\left(1 - x^2\right)^{n-1} & dv = e^{rx}\,dx \\ du = \left(1 - x^2\right)^{n-1} - 2(n-1)x^2\left(1 - x^2\right)^{n-2} dx & v = \dfrac{1}{r}e^{rx} \end{array}}$$

$$\int_{-1}^{1}\left(1-x^2\right)^n e^{rx}dx \quad = \frac{2n}{r}\int_{-1}^{1} x\left(1-x^2\right)^n e^{rx}dx$$

$$= \frac{2n}{r}\left(\cancel{\left[\frac{1}{r}x\left(1-x^2\right)^{n-1} e^{rx}\right]_{-1}^{1}} -\frac{1}{r}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx} - 2\left(n-1\right)x^2\left(1-x^2\right)^{n-2} e^{rx}dx \right)$$

$$= -\frac{2n}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx} - 2\left(n-1\right)x^2\left(1-x^2\right)^{n-2} e^{rx}dx$$

$$= -\frac{2n}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx}dx \quad + \frac{4n\left(n-1\right)}{r^2}\int_{-1}^{1} x^2\left(1-x^2\right)^{n-2} e^{rx}dx$$

$$= -\frac{2n}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx}dx \quad - \frac{4n\left(n-1\right)}{r^2}\int_{-1}^{1}\left(\left(1-x^2\right)-1\right)\left(1-x^2\right)^{n-2} e^{rx}dx$$

$$= -\frac{2n}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx}dx \quad - \frac{4n\left(n-1\right)}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx} - \left(1-x^2\right)^{n-2} e^{rx}dx$$

$$= \frac{4n\left(n-1\right)}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-2} e^{rx} \quad - \frac{2n\left(2n-1\right)}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx}dx$$

But recall that $I_n = \frac{a^{2n+1}pq}{n!}\int_{-1}^{1}\left(1-x^2\right)^n e^{rx}\,dx$. So we multiply both sides by $\frac{a^{2n+1}pq}{n!}$.

$$\frac{a^{2n+1}pq}{n!}\int_{-1}^{1}\left(1-x^2\right)^n e^{rx}dx \quad = \frac{a^{2n+1}pq}{n!}\times\frac{4n\left(n-1\right)}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-2} e^{rx}dx \quad - \frac{a^{2n+1}pq}{n!}\times\frac{2n\left(2n-1\right)}{r^2}\int_{-1}^{1}\left(1-x^2\right)^{n-1} e^{rx}dx$$

Now replace everything in terms of I, and use the fact that $r = \frac{a}{b}$.

$$I_n = \frac{a^{2n+1}pq}{(n-2)!} \times \frac{4b^2}{a^2} \int_{-1}^{1} \left(1-x^2\right)^{n-2} e^{rx} \quad - \frac{a^{2n+1}pq}{(n-1)!} \times \frac{2b^2(2n-1)}{a^2} \int_{-1}^{1} \left(1-x^2\right)^{n-1} e^{rx} dx$$

$$= \frac{4a^{2n-1}b^2 pq}{(n-2)!} \int_{-1}^{1} \left(1-x^2\right)^{n-2} e^{rx} \quad - \frac{2a^{2n-1}b^2 pq(2n-1)}{(n-1)!} \int_{-1}^{1} \left(1-x^2\right)^{n-1} e^{rx} dx$$

$$= 4a^2b^2 \times \frac{a^{2n-3}pq}{(n-2)!} \int_{-1}^{1} \left(1-x^2\right)^{n-2} e^{rx} \quad - 2(2n-1)b^2 \times \frac{a^{2n-1}pq}{(n-1)!} \int_{-1}^{1} \left(1-x^2\right)^{n-1} e^{rx} dx$$

$$= 4a^2b^2 I_{n-2} - 2(2n-1)b^2 I_{n-1}$$

Question 16 (c) (ii)

In the domain $-1 \le x \le 1$, we clearly have $\left(1-x^2\right)^n \ge 0$ and $e^{rx} > 0$. The constants in front of I_n are all positive, so we establish the lower bound $I_n > 0$.

Now we construct the upper bound. Notice that in the domain $-1 \le x \le 1$, we have $\left(1-x^2\right)^n \le 1$. Furthermore, we observe that in the same domain, we have $e^{rx} \le e^r$. Combining this, we have

$$\begin{aligned} I_n &= \frac{a^{2n+1}pq}{n!}\int_{-1}^{1}\left(1-x^2\right)^n e^{rx}\,dx \\ &\le \frac{a^{2n+1}pq}{n!}\int_{-1}^{1} e^r\,dx \\ &= \frac{2a^{2n+1}pq}{n!}e^r \end{aligned}$$

Since I_n is positive, we may take the absolute value of both sides.

$$\begin{aligned} |I_n| &\le \left|\frac{2a^{2n+1}pq}{n!}e^r\right| \\ &= \frac{2}{n!}\left|a^{2n+1}pq\right|e^r \\ &= \frac{2pq}{n!}\left|a^{2n+1}\right|e^r \qquad \ldots\left(\text{since } \frac{p}{q} > 0 \Rightarrow pq > 0\right) \\ &= \frac{2pq}{n!}|a|^{2n+1}e^r \\ &\le 2pq\frac{|a|^{2n+1}}{n!}e^{|r|} \end{aligned}$$

And hence the upper bound has been established.

Question 16 (c) (iii)

We prove two contradictory statements.

<u>The integral vanishes:</u>

We first make the following manipulation, so that it is in the form $\dfrac{R^n}{n!}$.

$$\begin{aligned}\left|I_n\right| &\leq 2pq\frac{|a|^{2n+1}}{n!}e^{|r|} \\ &= 2|a|\,pqe^{|r|} \times \frac{\left|a^2\right|^n}{n!}\end{aligned}$$

According to the given result, $\dfrac{\left|a^2\right|^n}{n!} \to 0$ as $n \to \infty$. Since all the other terms are fixed, it follows that $I_n \to 0$ as $n \to \infty$.

<u>The integral yields integers:</u>

First, we must explicitly compute I_0.

$$\begin{aligned}I_0 &= apq\int_{-1}^{1} e^{rx}\,dx \\ &= \frac{apq}{r}\left(e^r - e^{-r}\right) \\ &= \frac{apq}{a/b}\left(\frac{p}{q} - \frac{q}{p}\right) \\ &= b\left(p^2 - q^2\right)\end{aligned}$$

This is an integer since *a*, *b*, *p* and *q* are integers. We also are given that $I_1 = 2ab^2\left(p^2 + q^2\right) - 2b^3\left(p^2 - q^2\right)$, which is clearly an integer. Hence, using the recurrence in (i), we can establish that I_3 is *also* an integer. Continuing this inductively, we deduce that I_n is an integer for all $n = 0, 1, 2, 3, \ldots$

We have a contradiction, as it is not possible for I_n to be an integer that also approaches zero. Hence, have proven that e^r is irrational for all non-zero rational values of *r*.

Question 16 (c) (iv)

Suppose $\log_e(r)$ is rational, where r is a positive rational number other than 1. We can express r as $r = e^{\log_e(r)}$.

But we proved earlier that if k is rational, then e^k is irrational. Therefore $r = e^{\overbrace{\log_e(r)}^{k}}$ must be irrational, which contradicts the rationality of r.

Hence, $\log_e(r)$ is irrational.