2018 Bored of Studies Trial Examinations

Mathematics Extension 2

8th October 2018

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 6 – 17

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 What is the limiting value of the following product?

$$\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}\right)\ldots$$

(A) -1

(B) $-i$

(C) 1

(D) i

2 Which of the following parametric coordinates could describe the rectangular hyperbola $xy = 1$, for all possible values in its domain?

(A) $(\cos\theta, \sec\theta)$

(B) $(\cos\theta, \cot\theta)$

(C) $(\tan\theta, \sec\theta)$

(D) $(\tan\theta, \cot\theta)$

3 A real-coefficient polynomial $P(x)$ has all roots with even multiplicity. Which of the following statements is always true?

(A) $P(x)$ must be non-negative for all real x

(B) $P(x)$ must have both positive and negative values

(C) $P'(x)$ must be non-negative for all real x

(D) $P'(x)$ must have both positive and negative values

4 Let $f(x)$ be a continuous and differentiable function over all real x. The graph of $y = f(x)$ has n_1 stationary points and n_2 x-intercepts where $n_1 \geq n_2$.

The graph of $y = [f(x)]^2$ has n stationary points. Which of the following statements is always true?

(A) $n = n_2$

(B) $n \geq n_2$

(C) $n = n_1 + n_2$

(D) $n \geq n_1 + n_2$

5 Consider the points A, B and C on the first quadrant of the Argand diagram, which represent the complex numbers z_1, z_2 and z_3 respectively. Suppose that $z_2 = z_1 + z_3$ and

$$\arg\left(\frac{z_1 - z_3}{z_1 + z_3}\right) = \arg\left(\frac{z_1}{z_3}\right) = \frac{\pi}{2}.$$

What is the most accurate description of the quadrilateral $OABC$?

(A) Parallelogram

(B) Rhombus

(C) Rectangle

(D) Square

6 Let P, Q and R be distinct points on the Argand diagram with equal modulus, represented by complex numbers p, q and r respectively.

If $r = 1$, which of the following conditions ensures that ΔPQR is equilateral?

(A) $p + q = -1$

(B) $p + q = 0$

(C) $p + q = 1$

(D) $p + q = 2$

7 Let $P(x)$ and $Q(x)$ be real-coefficient polynomials without any common factors. Suppose that

$$\int \frac{P(x)}{Q(x)}\,dx = x^2 + \ln|(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)| + C,$$

where a_1, a_2, a_3, …, a_n are constants

What is the degree of $P(x)$?

(A) $n-1$

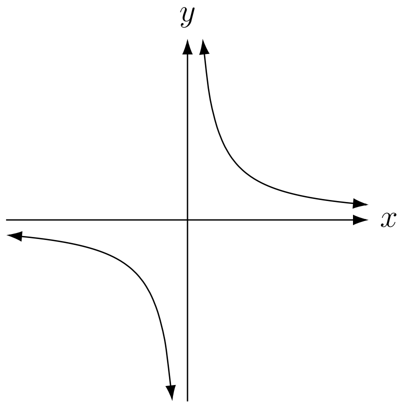
(B) n

(C) $n+1$

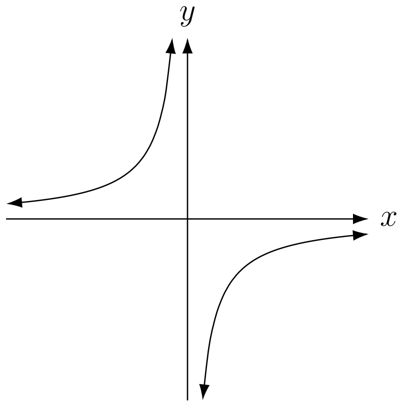
(D) $n+2$

8 Which of the following represents the graph of the locus of $\left||z-\sqrt{i}|-|z+\sqrt{i}|\right| = k$, for some $k > 0$?

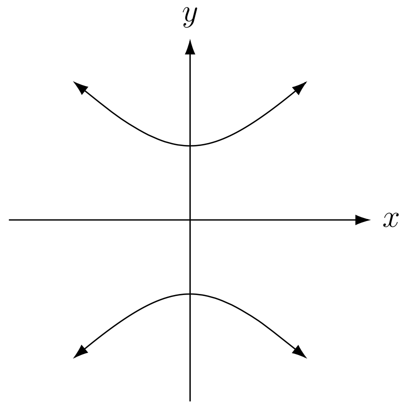
(A)



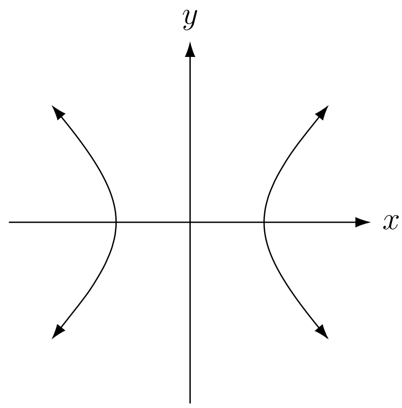
(B)



(C)



(D)



9 Let ω be a cube root of unity. Which of the following is the value of

$$(1+\omega)(1+\omega^2)(1+\omega^3)\ldots(1+\omega^{2018})?$$

(A) 2^{672}

(B) 2^{2016}

(C) $2^{671}(1+i\sqrt{3})$

(D) $2^{671}(1-i\sqrt{3})$

10 Consider n evenly-spaced points on a circle. Every point is connected to all other points, forming a number of intersecting chords. How many times do the chords intersect each other at most?

(A) $\binom{n}{1}$

(B) $\binom{n}{2}$

(C) $\binom{n}{3}$

(D) $\binom{n}{4}$

Section II

90 marks
Attempt Questions 11—16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Evaluate $\displaystyle\int_0^1 \frac{dx}{2+\sqrt{1+x}+\sqrt{1-x}}$. **3**

(b) Consider the curve $\mathcal{C}$ defined parametrically by

$$x = \frac{3t}{1+t^3} \quad \text{and} \quad y = \frac{3t^2}{1+t^3}.$$

(i) Show that the Cartesian equation of $\mathcal{C}$ is $x^3 + y^3 = 3xy$. **1**

(ii) What is the limiting value of t so that $x \to \infty$? **1**

(iii) By considering $x + y$, deduce that $y = -x - 1$ is an asymptote. **1**

(iv) Find the critical points of $\mathcal{C}$. **3**

(v) Hence, sketch the graph of $\mathcal{C}$ showing the critical points and asymptote. **2**

(c) Suppose $f(x)$ is a monotonic function for $a \le x \le b$, where $f(a) = c$ and $f(b) = d$.

(i) Show that **2**

$$\int_c^d f^{-1}(x)\,dx = \int_a^b x f'(x)\,dx.$$

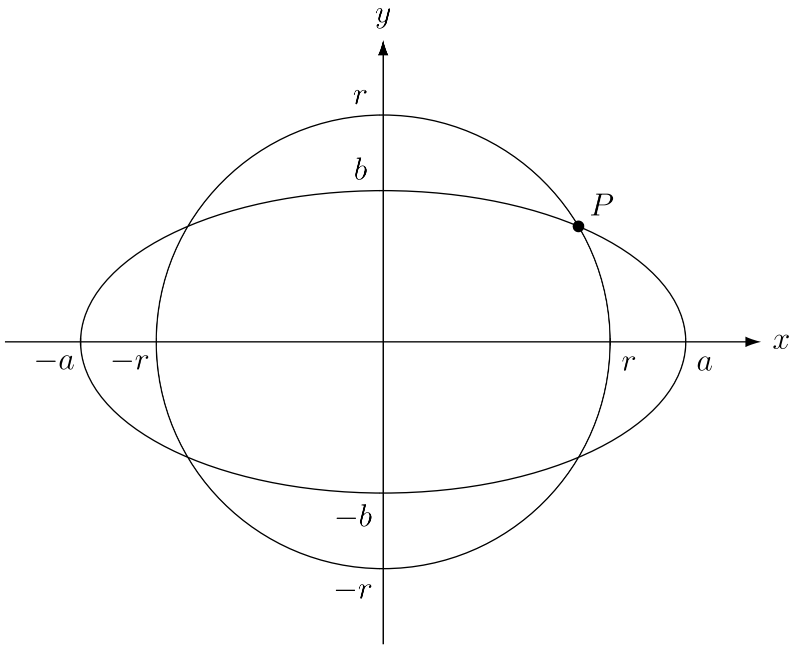
(ii) Hence, or otherwise, show that **2**

$$\int_a^b f(x)\,dx + \int_c^d f^{-1}(x)\,dx = bd - ac.$$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Consider the ellipse and circle with Cartesian equations $x^2+y^2 = r^2$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$, where $b < r < a$ so that the circle intersects the ellipse at 4 points.



(i) Show that the circle intersects the ellipse in the first quadrant at the point **2**

$$P\left(a\sqrt{\frac{r^2-b^2}{a^2-b^2}},\, b\sqrt{\frac{a^2-r^2}{a^2-b^2}}\right).$$

(ii) Let θ be the acute angle between the circle and the ellipse at P. Show that **2**

$$\tan\theta = \frac{\sqrt{(a^2-r^2)(r^2-b^2)}}{ab}.$$

(iii) Suppose the axes of the ellipse are fixed, and the radius of the circle can vary. Show that the maximum value of θ occurs when $r = \sqrt{\dfrac{a^2+b^2}{2}}$. **2**

Question 12 continues on page 8

Question 12 (continued)

(b) Define the polynomial $P(x) = x^3 - x^2 + k$, where k is real, such that $P(x)$ has exactly two non-real roots $a \pm ib$ and one real root γ.

(i) Show that **3**

$$(a^2 + b^2)\left[k - (a^2 + b^2)^2\right] = -k^2.$$

(ii) Explain why $a^2 + b^2$ is the root of the polynomial $Q(t) = t^3 - kt - k^2$. **1**

(iii) Hence, find the value(s) of k such that $P(x)$ has its non-real roots lying on the unit circle. **1**

(iv) For those values of k from (iii), does the real root also lie on the unit circle? Justify your answer. **1**

(c) For any positive integer n, define the integral

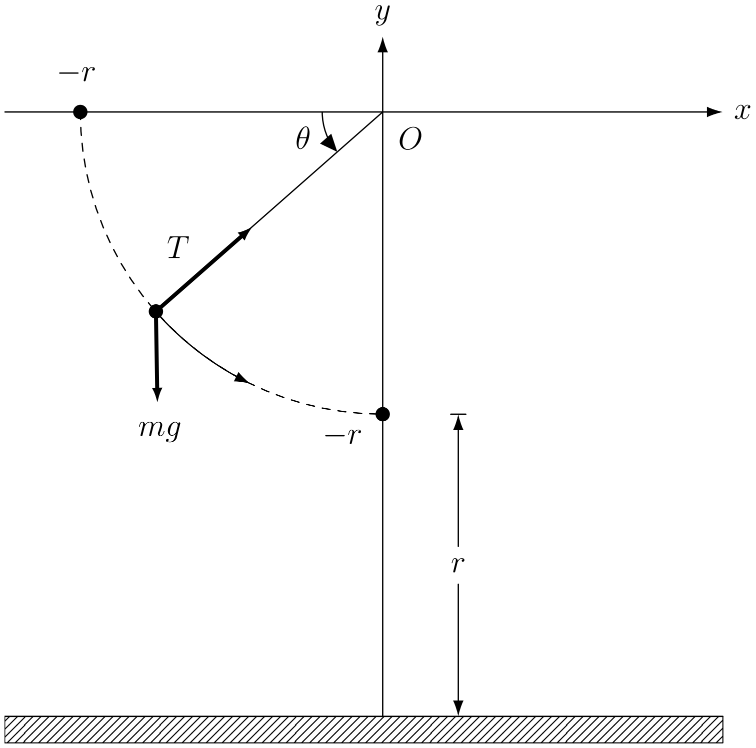
$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\, dx.$$

Show that $I_n = \dfrac{\sqrt{2} - (n-1)I_{n-2}}{n}$ for $n \geq 2$. **3**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) A string of length r is held at fixed point O with an object of mass m attached to the other end. Let O be the origin of the number plane, such that the object is initially at rest at the point $(-r, 0)$ and begins moving anticlockwise along the arc of a circle. Suppose that the fixed point of the string is $2r$ units above the ground.



Let θ be the angular displacement of the object at time t relative to its initial position. At a given time, the object is subjected to a tension force of T and gravitational force mg.

(i) Let v be the tangential velocity of the object at time t. By resolving the forces into tangential and normal components, show that **4**

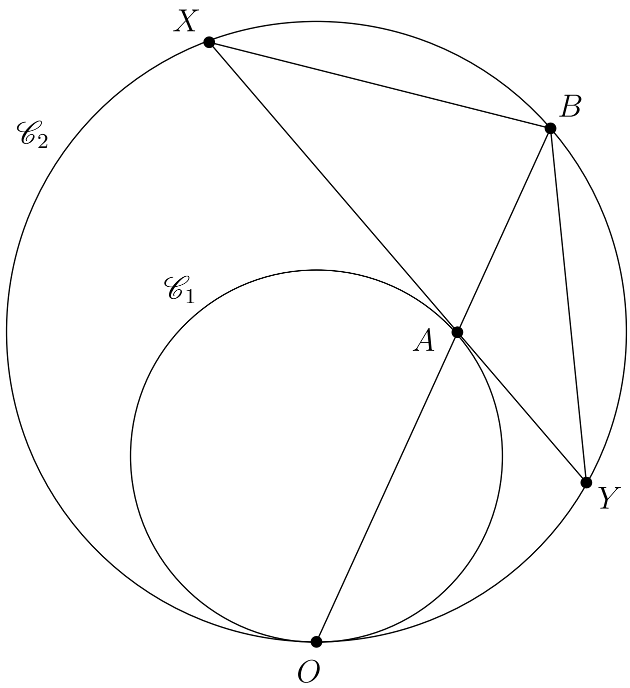
$$v^2 = 2gr\sin\theta.$$

(ii) When the object reaches the point $(0, -r)$, the string is cut so that the object is released at a horizontal velocity to fall to the ground. Deduce that the object has speed $\sqrt{2gr}$ at release. **1**

(iii) Hence, show that the object hits the ground at the point $(2r, -2r)$. **2**

Question 13 continues on page 10

(b) The diagram below shows a smaller circle $\mathscr{C}_1$ inside and tangential to a larger circle $\mathscr{C}_2$ at point O. A line through O meets $\mathscr{C}_1$ and $\mathscr{C}_2$ at A and B respectively. From A, a tangent is drawn to intersect $\mathscr{C}_2$ at X and Y. **4**



Show that $\triangle BXY$ is isosceles.

(c) Let $f(x) = 1 + x^r - (1+x)^r$ for $0 \leq r \leq 1$ and $x \geq 0$.

(i) Show that $f(x) \geq 0$. **2**

(ii) Hence, show that for any real values a and b **2**

$$|a+b|^r \leq |a|^r + |b|^r\,.$$

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) A particle of mass m was originally intended to move in simple harmonic motion at time t according to the displacement equation $x = A\cos(nt)$ for positive constants A and n.

However, once the particle was released from rest it experienced a resistance force of $2mnv$, where v is the velocity of the particle.

(i) Explain why the acceleration equation of the particle with the resistance is **1**

$$\ddot{x} = -n^2 x - 2nv.$$

(ii) Let $y = v + nx$. Show that $\dot{y} + ny = 0$. **1**

(iii) Hence show that the displacement equation of the particle is given by **3**

$$x = A(1+nt)e^{-nt}.$$

Hint: Consider the fact that $\dfrac{d}{dx}\left(e^x f(x)\right) = e^x\left(f(x) + f'(x)\right).$

(iv) Sketch the displacement equation of the particle and describe its behaviour over time. **2**

(b) Consider the hyperbola $\mathcal{H}$ defined by the equation

$$y = \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{x}.$$

It can be shown that $y = \dfrac{x}{\sqrt{3}}$ is an asymptote of $\mathcal{H}$. **(Do NOT prove this)**

(i) Sketch the graph of $\mathcal{H}$. **1**

(ii) Find the points intersection of $\mathcal{H}$ and the line $y = x\sqrt{3}$. **2**

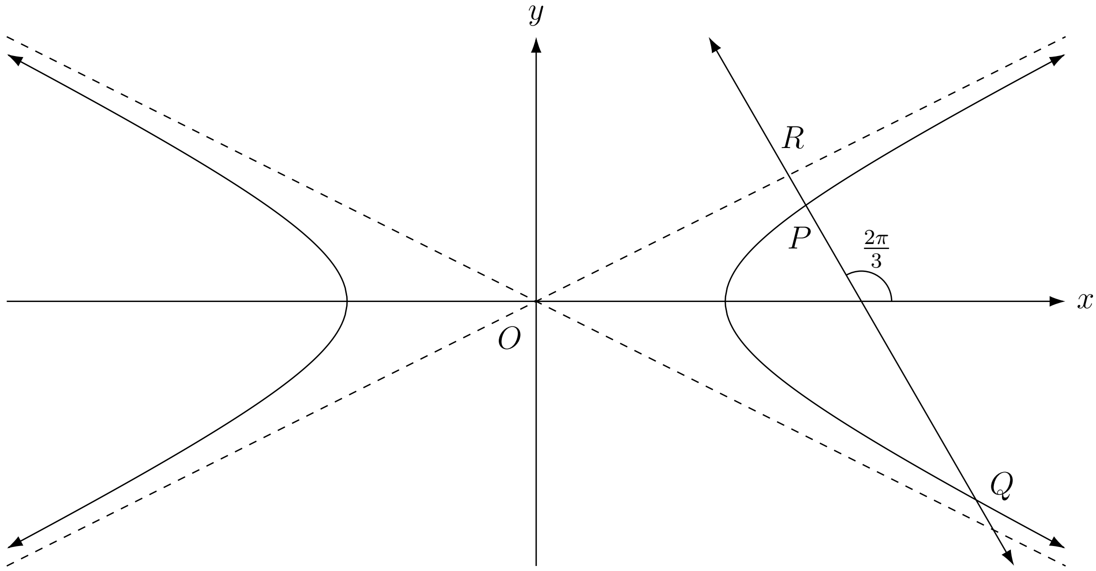
(iii) Suppose that $\mathcal{H}$ is rotated clockwise about the origin by $\dfrac{\pi}{3}$. Using part (ii), or otherwise, show that the equation of the rotated curve is **2**

$$\frac{x^2}{6} - \frac{y^2}{2} = 1.$$

Question 14 continues on page 12

Question 14 (continued)

(c) The diagram below shows a hyperbola with the equation $\dfrac{x^2}{6} - \dfrac{y^2}{2} = 1$.



The chord PQ on the positive branch of the hyperbola intersects the positive x-axis at an angle of $\dfrac{2\pi}{3}$. Chord PQ is produced to intersect the positive asymptote $y = \dfrac{x}{\sqrt{3}}$ at R. Let h be the length of OR.

The region enclosed by the chord and the hyperbola is rotated about the positive asymptote $y = \dfrac{x}{\sqrt{3}}$ to form a solid of revolution with volume V.

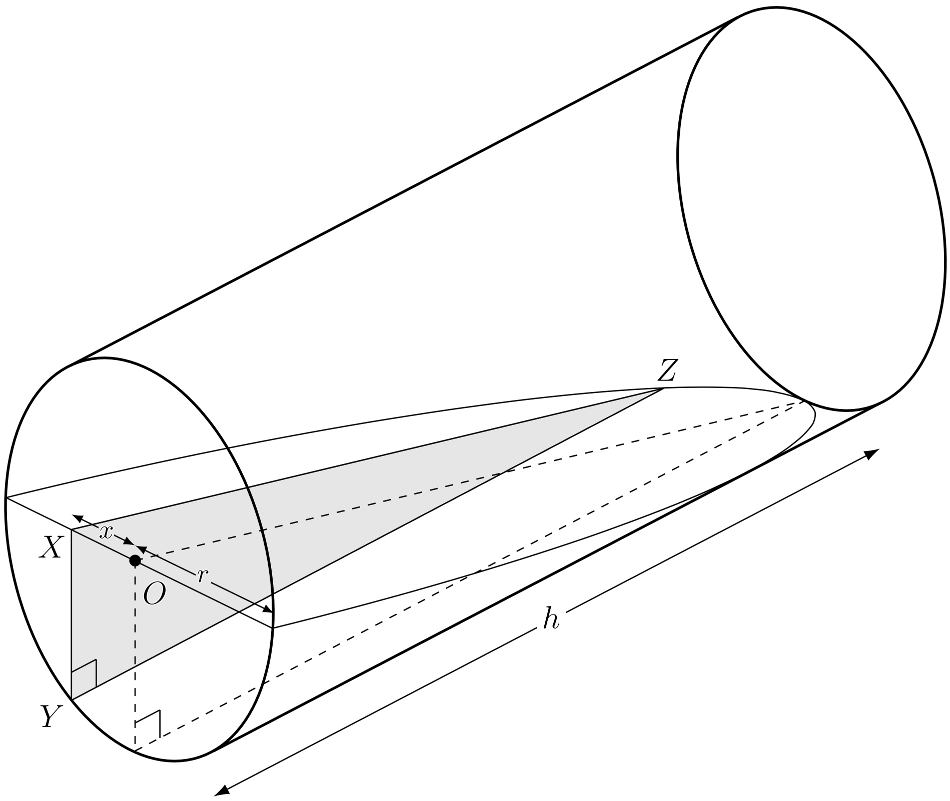
Using the result in part (b)(iii), or otherwise, show that **3**

$$V = \pi(h^2 - 4)^{\frac{3}{2}}.$$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) The diagram below shows a cylinder with height h and radius r, which is filled with water. The cylinder is tilted, spilling any excess water out of it so that the water's surface is level with the centre of cylinder's base.



Cross-sectional slices taken perpendicular to the base are right-angled triangles, and the water surface forms a semi-ellipse. Let $OX = x$.

(i) Find the equation of the semi-ellipse in terms of h and r. **1**

(ii) Show that the area of ΔXYZ is $\dfrac{h}{2r}(r^2 - x^2)$. **3**

(iii) Hence, find the proportion of the cylinder that is filled with water. **2**

Question 15 continues on page 14

(b) Let $f(x)$ and $g(x)$ be polynomials such that

$$(1 + ix)^n = f(x) + ig(x),$$

where n is a positive integer.

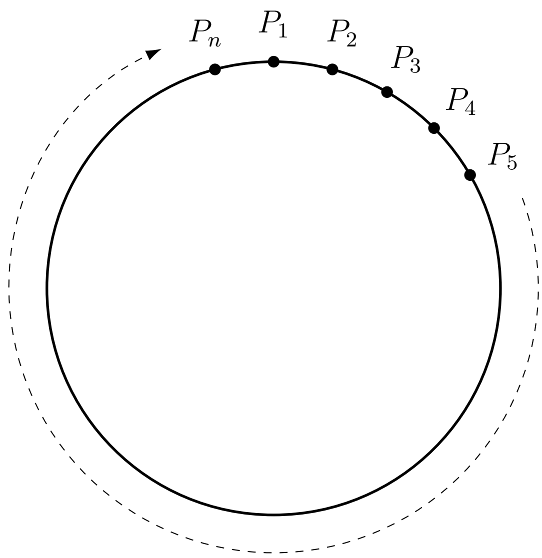
(i) Show that **1**

$$(a - bi)(1 + ix)^n + (a + bi)(1 - ix)^n = 2(af(x) + bg(x)),$$

where a and b are any non-zero real numbers.

(ii) Hence, or otherwise, show that the polynomial $af(x) + bg(x)$ only has real roots. **3**

(c) There are n people sitting in a circle labelled P_1 to P_n as shown in the diagram below.



Every second person is removed from the table in a clockwise direction, initially removing P_2, until eventually there is only one person remaining on the table.

(i) Suppose there are 2^m people in the circle labelled P_1 to P_{2^m}, where m is a positive integer. Use mathematical induction to prove that P_1 always wins. **3**

(ii) Hence, find the labelled position of the remaining person if the circle contains 2018 people. **2**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) Elle draws a card from a bag containing cards labelled 1, 2, 3, …, n. The rules are as follows.

- If she draws the lowest number '1', she wins the game.
- If she draws the highest number 'n' in the bag, she loses the game.
- If she draws any other number, she returns the card that she drew and then adds another card labelled $n+1$ to the bag and redraws.

This process repeats until eventually Elle wins or loses. Suppose the bag begins with only three cards labelled 1, 2 and 3.

(i) Show that the probability of Elle winning the game on her k^{th} turn is **2**

$$\frac{2}{k(k+1)(k+2)}.$$

(ii) Hence, or otherwise, show that **1**

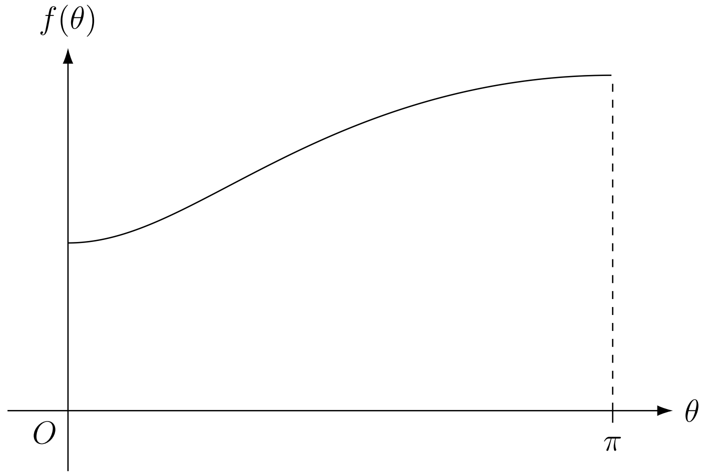
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{4}.$$

(b) (i) Find the roots of $P(z)=z^{2n}-1$ in modulus-argument form. **1**

(ii) Hence, express $\dfrac{z^{2n}-1}{z^2-1}$ as the product of real quadratic factors. **1**

Question 16 continues on page 16

(c) The diagram below shows the graph of $f(\theta) = \ln\left(a^2 - 2a\cos\theta + 1\right)$, where $0 \le \theta \le \pi$ and $a > 1$.



(i) Use part (b) to show that **2**

$$\frac{\pi}{n}\sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) = R + \frac{\pi}{n}\ln\left(\frac{a+1}{a-1}\right),$$

where $R = \dfrac{\pi}{n}\ln\left(a^{2n} - 1\right)$.

(ii) Show that **1**

$$\frac{\pi}{n}\sum_{k=0}^{n-1} f\left(\frac{k\pi}{n}\right) = R + \frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right).$$

(iii) Define the integral

$$I(a) = \int_0^{\pi} \ln\left(a^2 - 2a\cos\theta + 1\right) d\theta.$$

Explain why **1**

$$R + \frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right) < I(a) < R + \frac{\pi}{n}\ln\left(\frac{a+1}{a-1}\right).$$

Question 16 continues on page 17

(iv) It can be shown that for $x > 1$ **2**

$$\ln\left(\frac{x}{x-1}\right) < \frac{1}{x-1} \qquad \textbf{(Do NOT prove this)}$$

Use this result to show that

$$2\pi\ln a - \frac{\pi}{n}\left(\frac{1}{a^{2n}-1}\right) < R < 2\pi\ln a.$$

(v) Show that $I(a) = 2\pi\ln a$. **1**

(d) Define similarly the same integral from part (c)

$$I(b) = \int_0^{\pi} \ln\left(b^2 - 2b\cos\theta + 1\right) d\theta\,,$$

but for all $b > 0$, $b \neq 1$.

(i) Show that $I(b) = 2\pi\ln b + I\left(\frac{1}{b}\right)$. **1**

(ii) Use part (c) to deduce that **2**

$$I(b) = \begin{cases} 2\pi\ln b\,, & \text{for } b > 1 \\ 0\,, & \text{for } 0 < b < 1 \end{cases}$$

End of paper

2018 Bored of Studies Trial Examinations

Mathematics Extension 2

Solutions

Section I

Answers

1	A	**6**	A
2	D	**7**	C
3	D	**8**	A
4	B	**9**	A
5	D	**10**	D

Brief explanations

1 Using De-Moivre's theorem, a limiting sum can be derived in the arguments with first term $\frac{\pi}{2}$ and common ratio of $\frac{1}{2}$. This leads to a limiting value of $\cos\pi + i\sin\pi$ so the answer is (A).

2 Note that (B) and (C) do not even satisfy the Cartesian equation $xy = 1$. Also, note that as θ varies, $\cos\theta$ and $\sec\theta$ give restricted ranges whereas $\tan\theta$ and $\cot\theta$ do not (provided $\tan\theta \neq 0$). Hence, the answer is (D).

3 A simple counterexample for (A) and (B) is $P(x) = -x^2$, which has a root of multiplicity 2 at $x = 0$ with $P(x) \leq 0$. For the remaining options, consider the fact that $P(x)$ must have an even degree if it only has roots of even multiplicity. Therefore, $P'(x)$ must have an odd degree which means that it must have at least one real root, so the answer is (D).

4 If $y = [f(x)]^2$ then $\dfrac{dy}{dx} = 2f(x)f'(x)$. The derivative is zero if $f(x) = 0$ (which has n_2 real solutions) or $f'(x) = 0$ (which has n_1 real solutions). It is possible for values of x to exist such that $f(x) = f'(x) = 0$. Therefore, the number of stationary points of $y = [f(x)]^2$ is at most $n_1 + n_2$ but at least n_2 (since it is less than or equal to n_1). Hence, the answer is (B).

5 Consider the vectors representing OA, OB and OC on the Argand diagram, which form a parallelogram by definition of the complex numbers provided. The statement $\arg\left(\frac{z_1}{z_3}\right) = \frac{\pi}{2}$ suggests that $\angle AOC$ is right angle and the statement $\arg\left(\frac{z_1 - z_3}{z_1 + z_3}\right) = \frac{\pi}{2}$ suggests that the diagonals of $OABC$ intersect at right angles. This suggests that $OABC$ must be a square so the answer is (D).

6 Since the complex numbers have equal modulus of 1 then P and Q must lie on the unit circle. For $\triangle PQR$ to be equilateral and $OR = 1$, it is only possible that P and Q lie on the second and third quadrants as conjugates. This means the real parts must be negative, which suggests (A). Further inspection can be done to verify that indeed $p + q = -1$.

7 The derivative of the right-hand side (which should equal the integrand) is $2x + \frac{A'(x)}{A(x)}$ where $A(x)$ is $(x - a_1)(x - a_2)(x - a_3)...(x - a_n)$. The is equivalent to $\frac{2xA(x) + A'(x)}{A(x)}$. Since $A(x)$ is a monic polynomial of degree n then $A'(x)$ has degree $n-1$. The numerator then must have degree $n + 1$ due to the $2xA(x)$ term, so the answer is (C).

8 The locus has a geometric interpretation of being the difference in distance between two fixed points equalling a constant. This defines a hyperbola where the fixed points are the focii (recall the property $|PS - PS'| = 2a$. From De-Moivre's theorem, notice that $\sqrt{i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$. This means that the focii are points in the first and third quadrants, which defines a hyperbola described by (A).

9 First note that

$$\begin{aligned}
(1+\omega)(1+\omega^2) &= 1 + \omega + \omega^2 + \omega^3 \\
&= \frac{\omega^4 - 1}{\omega - 1} \qquad \text{but } \omega^3 = 1 \\
&= 1
\end{aligned}$$

Use the fact that $\omega^3 = 1$ to reduce any powers higher than 3 so

$$\begin{aligned}
(1+\omega)(1+\omega^2)(1+\omega^3)...((1+\omega^{2018}) &= [(1+\omega)(1+\omega^2)(1+\omega^3)]^{672}(1+\omega)(1+\omega^2) \\
&= 2^{672}
\end{aligned}$$

So the answer is (A).

10 Any intersection point of two chords must occur with at most 4 distinct points on the circle. With n points on the circle, there must be $\binom{n}{4}$ possible intersections. Hence, the answer is (D).

Question 11

(a) Consider the substitution $x = \sin 2\theta \Rightarrow dx = 2\cos 2\theta \, d\theta$

When $x = 1$ then $\theta = \frac{\pi}{4}$ and when $x = 0$ then $\theta = 0$.

Consider the fact that $(\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$ and $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$. Since $0 \leq \theta \leq \frac{\pi}{4}$ then $\cos\theta \geq \sin\theta \geq -\sin\theta$, or equivalently, $\cos\theta \pm \sin\theta \geq 0$, hence

$$\begin{aligned}
\int_0^1 \frac{dx}{2 + \sqrt{1+x} + \sqrt{1-x}} &= \int_0^{\frac{\pi}{4}} \frac{2\cos 2\theta}{2 + \sqrt{1+\sin 2\theta} + \sqrt{1 - \sin 2\theta}} \, d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{2\cos 2\theta}{2 + \cos\theta - \sin\theta + \cos\theta + \sin\theta} \, d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{2\cos^2\theta - 2 + 1}{1 + \cos\theta} d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{2(\cos\theta + 1)(\cos\theta - 1) + 1}{1 + \cos\theta} d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(2(\cos\theta - 1) + \frac{1}{1 + \cos\theta} \right) d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(2(\cos\theta - 1) + \frac{1}{2\cos^2\frac{\theta}{2}} \right) d\theta \\
&= \left[2\sin\theta - 2\theta + \tan\frac{\theta}{2} \right]_0^{\frac{\pi}{4}} \\
&= \sqrt{2} - \frac{\pi}{2} - \tan\frac{\pi}{8}
\end{aligned}$$

(b) (i) Observe that $y = tx$, so

$$\begin{aligned}
x^3 + y^3 &= (1 + t^3)x^3 \\
&= (1 + t^3) \times \frac{27t^3}{(1+t^3)^3} \\
&= 3 \times \frac{3t}{1+t^3} \times \frac{3t^2}{1+t^3} \\
&= 3xy
\end{aligned}$$

Note that this Cartesian Equation is symmetric in x, y.

(ii) From the parametric form of x, it is easy to see that as $t \to -1$ from the negative side, $x \to \infty$

(iii) Using the parametric form

$$\begin{aligned} x+y &= \frac{3t+3t^2}{1+t^3} \\ &= \frac{3t(1+t)}{(1-t+t^2)(1+t)} \\ &= \frac{3t}{1-t+t^2} \end{aligned}$$

From part (ii), $x \to \infty$ when $t \to -1$, so it follows an asymptote occurs when $x+y = \lim_{t\to -1} \dfrac{3t}{1-t+t^2} = -1$

Which gives $y = -x - 1$ as an asymptote.

(iv) Implicitly differentiating with respect to x, we have:

$$\begin{aligned} x^3 + y^3 &= 3xy \\ 3x^2 + 3y^2\frac{dy}{dx} &= 3y + 3x\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{x^2 - y}{x - y^2} \end{aligned}$$

The critical points occur when $y = x^2$ and $x = y^2$.

Substitute $y = x^2$ into the Cartesian equation

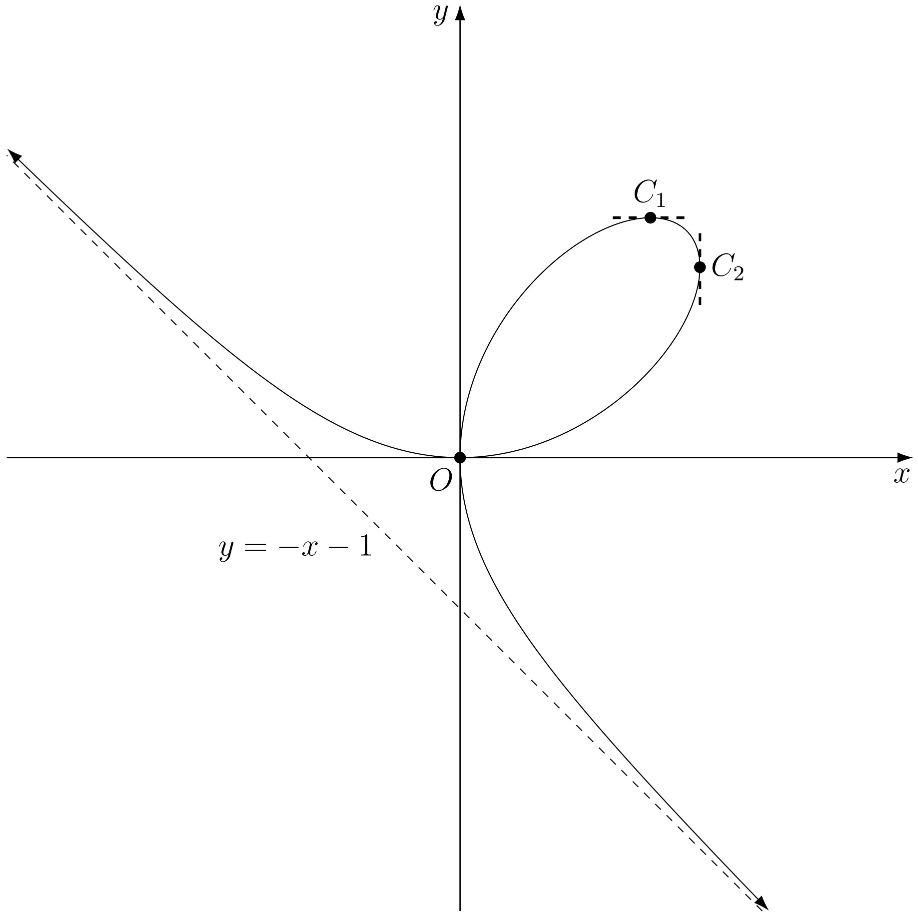
$$\begin{aligned} x^3 + x^6 &= 3x^3 \\ x^3(x^3 - 2) &= 0 \\ x &= 0, \sqrt[3]{2} \\ y &= 0, \sqrt[3]{4} \end{aligned}$$

Similarly, with the condition $x = y^2$ then $x = 0, \sqrt[3]{4}$ and $0, \sqrt[3]{2}$. Thus, the critical points are $(0,0), (\sqrt[3]{4}, \sqrt[3]{2}), (\sqrt[3]{2}, \sqrt[3]{4})$

(v) Let C_1 have the coordinates $(\sqrt[3]{2}, \sqrt[3]{4})$ and C_2 have the coordinates $(\sqrt[3]{4}, \sqrt[3]{2})$ as shown in the graph below.

There is an asymptote of $x + y = -1$ and a horizontal and vertical gradient occuring at the origin.

The graph is also symmetric in x and y, and is therefore symmetric about the line $y = x$.



(c) (i) Let $x = f(u)$, so $dx = f'(u)\,du$. It is given that f is monotonic in the interval $a \leq x \leq b$ so a unique inverse function $f^{-1}(x)$ exists such that $u = f^{-1}(x)$.

When $x = d$ then $u = f^{-1}(d) = b$ and when $x = c$ then $u = f^{-1}(c) = a$

$$\begin{aligned}\int_c^d f^{-1}(x)\,dx &= \int_a^b uf'(u)\,du \\ &= \int_a^b xf'(x)\,dx \quad (u \text{ is a dummy variable})\end{aligned}$$

(ii) Integrating by parts, we have

$$\begin{aligned}\int_c^d f^{-1}(x)\,dx\,dx &= \int_a^b xf'(x)\,dx \\ &= [xf(x)]_a^b - \int_a^b f(x)\,dx \\ &= bf(b) - af(a) - \int_a^b f(x)\,dx\end{aligned}$$

$$\therefore \int_c^d f^{-1}(x)\,dx + \int_a^b f(x)\,dx = bd - ac$$

Question 12

(a) (i) Rearranging the equation for the circle to be $y^2 = r^2 - x^2$ and for the ellipse to be $y^2 = b^2 - \dfrac{b^2x^2}{a^2}$. Equate these to get

$$\begin{aligned}
r^2 - x^2 &= b^2 - \frac{b^2x^2}{a^2} \\
x^2\left(1 - \frac{b^2}{a^2}\right) &= r^2 - b^2 \\
x^2 &= \frac{a^2(r^2 - b^2)}{a^2 - b^2} \\
x &= a\sqrt{\frac{r^2 - b^2}{a^2 - b^2}} \qquad \text{as in the first quadrant } x > 0
\end{aligned}$$

Substituting this into $y^2 = r^2 - x^2$ we obtain

$$\begin{aligned}
y^2 &= r^2 - \frac{a^2(r^2 - b^2)}{a^2 - b^2} \\
y^2 &= \frac{(a^2r^2 - b^2r^2 - a^2r^2 + a^2b^2)}{a^2 - b^2} \\
y^2 &= \frac{b^2(a^2 - r^2)}{a^2 - b^2} \\
y &= b\sqrt{\frac{a^2 - r^2}{a^2 - b^2}} \qquad \text{as in the first quadrant } y > 0
\end{aligned}$$

Thus the point of intersection is

$$\left(a\sqrt{\frac{r^2 - b^2}{a^2 - b^2}}, b\sqrt{\frac{a^2 - r^2}{a^2 - b^2}}\right).$$

(ii) By implicit differentiation

$$\begin{aligned}
\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{b^2x}{a^2y}
\end{aligned}$$

For the circle we can take $a = b = 1$ to get $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

From part (i), $\dfrac{x}{y} = \dfrac{a}{b}\sqrt{\dfrac{r^2 - b^2}{a^2 - r^2}}$. If m_C and m_E are the gradients of the circle and ellipse respective at the point of intersection then

$$
\begin{aligned}
\tan\theta &= \left|\frac{m_E - m_C}{1 + m_C m_E}\right| \\
&= \left|\frac{\frac{x}{y}\left(1 - \frac{b^2}{a^2}\right)}{1 + \frac{b^2x^2}{a^2y^2}}\right| \\
&= \left|\frac{\frac{a}{b}\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}\left(1 - \frac{b^2}{a^2}\right)}{1 + \frac{r^2 - b^2}{a^2 - r^2}}\right| \\
&= \left|\frac{(a^2 - b^2)(a^2 - r^2)\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}}{ab(a^2 - b^2)}\right| \\
&= \left|\frac{1}{ab}\sqrt{(r^2 - b^2)(a^2 - r^2)}\right| \\
&= \frac{\sqrt{(a^2 - r^2)(r^2 - b^2)}}{ab} \qquad \text{(as } a, b > 0\text{)}
\end{aligned}
$$

(iii) As $\tan\theta$ is an increasing function for acute angles of θ, then the maximum value of θ must give the maximum of $\tan\theta$. Thus, it is sufficient to find the maximum value of $\tan\theta$ to find the maximum value of θ. Since a and b are constants, it is sufficient to simply find the maximum value of $(a^2 - r^2)(r^2 - b^2)$ (define this as $S(r)$).

$$
\begin{aligned}
S(r) &= (a^2 - r^2)(r^2 - b^2) \\
S'(r) &= -2r(r^2 - b^2) + 2r(a^2 - r^2) \\
&= 2r(a^2 + b^2 - 2r^2) \\
S''(r) &= 2(a^2 + b^2) - 12r^2
\end{aligned}
$$

When $S'(r) = 0$ then $r = \sqrt{\frac{a^2 + b^2}{2}}$ (as $r > 0$).

We can check that $S''\left(\sqrt{\frac{a^2 + b^2}{2}}\right) = -4(a^2 + b^2) < 0$ so it gives a local maximum. Since there are no other values of r in the domain $r > 0$ that are solutions to $S'(r) = 0$ then the absolute maximum of θ occurs at $r = \sqrt{\frac{a^2 + b^2}{2}}$.

(b) (i) Let $t = a^2 + b^2$. By product of roots

$$\begin{aligned}(a - bi)(a + bi)\gamma &= -k \\ \gamma &= -\frac{k}{t}\end{aligned}$$

Substitute this into $P(x)$

$$\begin{aligned}\gamma^3 - \gamma^2 + k &= 0 \\ k - \gamma^2 &= -\gamma^3 \\ k - \frac{k^2}{t^2} &= \frac{k^3}{t^3} \\ 1 - \frac{k}{t^2} &= \frac{k^2}{t^3} \\ t(t^2 - k) &= k^2 \\ (a^2 + b^2)(k - (a^2 + b^2)^2) &= -k^2\end{aligned}$$

(ii) From (i), $t(k - t^2) = -k^2$, or equivalently, $t^3 - kt - k^2 = 0$ where $t = a^2 + b^2$. Thus $a^2 + b^2$ is a root of $Q(t) = t^3 - kt - k^2$.

(iii) If the non-real roots of $P(x)$ lie on the unit circle then $|a \pm ib| = a^2 + b^2 = 1$. From part (ii), this means that $1 - k - k^2 = 0$. The roots of this quadratic are $k = \dfrac{-1 \pm \sqrt{5}}{2}$.

(iv) From the product of roots, we have $\gamma = -k$, if $a^2 + b^2 = 1$. As $k \neq \pm 1$, then the real root γ does not lie on the unit circle.

(c)

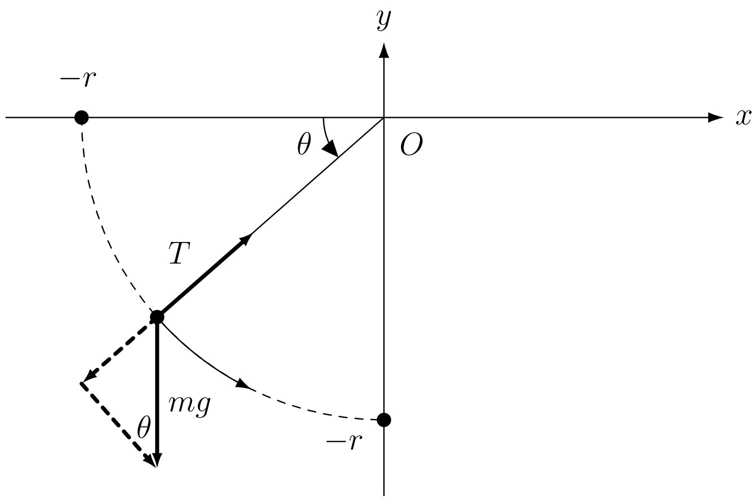
$$
\begin{aligned}
I_n &= \int_0^1 \frac{x^n}{\sqrt{1+x^2}} \quad dx \\
&= \int_0^1 \left(\frac{x^n + x^{n-2}}{\sqrt{1+x^2}} - \frac{x^{n-2}}{\sqrt{1+x^2}} \right) dx \\
&= \int_0^1 \frac{x^{n-2}(1+x^2)}{\sqrt{1+x^2}}\, dx - \int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}}\, dx \\
&= \int_0^1 x^{n-2}\sqrt{1+x^2}\, dx - I_{n-2} \\
&= \left[\frac{x^{n-1}}{n-1}\sqrt{1+x^2} \right]_0^1 - \int_0^1 \frac{x^{n-1}}{n-1} \times \frac{x}{\sqrt{1+x^2}}\, dx - I_{n-2} \quad \text{using integration by parts} \\
&= \frac{\sqrt{2}}{n-1} - \frac{1}{n-1}\int_0^1 \frac{x^n}{\sqrt{1+x^2}}\, dx - I_{n-2} \\
&= \frac{\sqrt{2}}{n-1} - \frac{1}{n-1} I_n - I_{n-2} \\
(n-1)I_n &= \sqrt{2} - I_n - (n-1)I_{n-2} \\
nI_n &= \sqrt{2} - (n-1)I_{n-2} \\
I_n &= \frac{\sqrt{2} - (n-1)I_{n-2}}{n}
\end{aligned}
$$

Question 13

(a) (i) Let s be the arc length displacement of the object. The following relationship holds between the arc length motion and the angular motion.

$$\begin{aligned} s &= r\theta \\ \frac{ds}{dt} &= r\frac{d\theta}{dt} \\ v &= r\dot{\theta} \end{aligned}$$

Resolve the gravitational force into tangential and normal components as shown



In the tangential direction

$$\begin{aligned} m\frac{dv}{dt} &= mg\cos\theta \qquad \text{but } v = r\dot{\theta} \\ mr\frac{d\dot{\theta}}{dt} &= mg\cos\theta \\ r\frac{d}{d\theta}\left(\frac{(\dot{\theta})^2}{2}\right) &= g\cos\theta \\ \frac{r(\dot{\theta})^2}{2} &= g\sin\theta + C \qquad \text{but when } \theta = 0,\ \dot{\theta} = 0 \text{ so } C = 0 \\ r(\dot{\theta})^2 &= 2g\sin\theta \\ v^2 &= 2gr\sin\theta \end{aligned}$$

(ii) When the object reaches the point $(0, -r)$ the angular displacement is $\dfrac{\pi}{2}$ so $v^2 = 2gr$, hence the speed of release is $\sqrt{2gr}$.

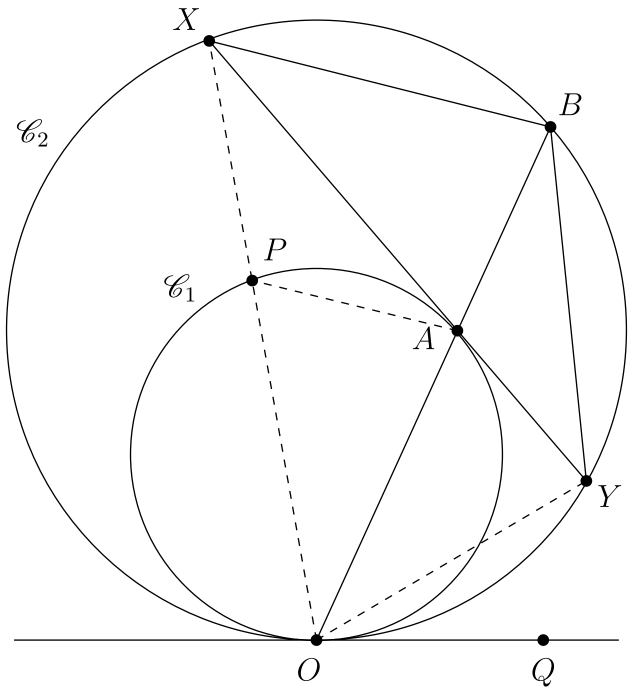
(iii) When the object is released it undergoes projectile motion with an initial speed of $\sqrt{2gr}$ at the point $(0, -r)$. If t is now defined as the time since the object is released, the equations of motion are

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$
$$\dot{x} = \sqrt{2gr} \qquad \dot{y} = -gt$$
$$x = t\sqrt{2gr} \qquad y = -\frac{1}{2}gt^2 - r$$

The Cartesian equation is therefore $y = -\dfrac{x^2}{4r} - r$.

The object hits the ground when $y = -2r$ which means $x^2 = 4r^2$. Since $x > 0$ for the projectile motion then $x = 2r$, hence the object hits the ground at the coordinates $(2r, -2r)$.

(b) Construct the tangent at O and define the points P and Q as shown in the diagram below



Let $\angle YOQ = x$ and $\angle AOY = y$

$$\begin{aligned}
&\angle YOQ = \angle PXA = x \quad \text{(angle in alternate segment)}\\
&\angle AOQ = \angle APO = x + y \quad \text{(angle in alternate segment)}\\
&\angle APO = \angle PXA + \angle XAP \quad \text{(exterior angle of triangle)}\\
\therefore\ &\angle XAP = y\\
\text{but } &\angle XAP = \angle POA = y \quad \text{(angle in alternate segment)}\\
\text{and } &\angle POA = \angle AYB = y \quad \text{(angles in the same segment)}\\
\text{also } &\angle AOY = \angle AXB = y \quad \text{(angles in the same segment)}\\
\therefore\ &\angle AXB = \angle AYB \quad \text{which means } \triangle BXY \text{ is isosceles}
\end{aligned}$$

(c) (i) When $r = 0$, $f(x) = 1$ and when $r = 1$, $f(x) = 0$ so $f(x) \geq 0$ for $r = 0, 1$. For $0 < r < 1$, investigate the behaviour of $f(x)$ by looking at its derivative $f'(x)$.

$$\begin{aligned} f(x) &= 1 + x^r - (1+x)^r \\ f'(x) &= rx^{r-1} - r(1+x)^{r-1} \\ &= r\left(\frac{1}{x^{1-r}} - \frac{1}{(1+x)^{1-r}}\right) \qquad \text{for } 0 < r < 1 \end{aligned}$$

If $x > 0$, then $\dfrac{1}{x} > \dfrac{1}{1+x}$ which means that $\dfrac{1}{x^{1-r}} - \dfrac{1}{(1+x)^{1-r}} > 0$. Hence, $f'(x) > 0$ for $x > 0$.

Since the domain is $x \geq 0$ and $f(x)$ is increasing for $x > 0$, then the minimum value of $f(x)$ is $f(0)$. This means that $f(x) \geq 0$.

(ii) Let $x = \dfrac{|a|}{|b|}$ for the inequality $f(x) \geq 0$.

$$\begin{aligned} 1 + \frac{|a|^r}{|b|^r} - \left(1 + \frac{|a|}{|b|}\right)^r &\geq 0 \\ |a|^r + |b|^r &\geq (|a| + |b|)^r \\ \therefore |a|^r + |b|^r &\geq |a+b|^r \qquad \text{using the triangle inequality } |a| + |b| \geq |a+b| \end{aligned}$$

Question 14

(a) (i) The acceleration equation for the particle in simple harmonic motion is $-n^2x$. With the resistance force, the net force is given by

$$\begin{aligned} m\ddot{x} &= -mn^2x - 2mnv \\ \ddot{x} &= -n^2x - 2nv \end{aligned}$$

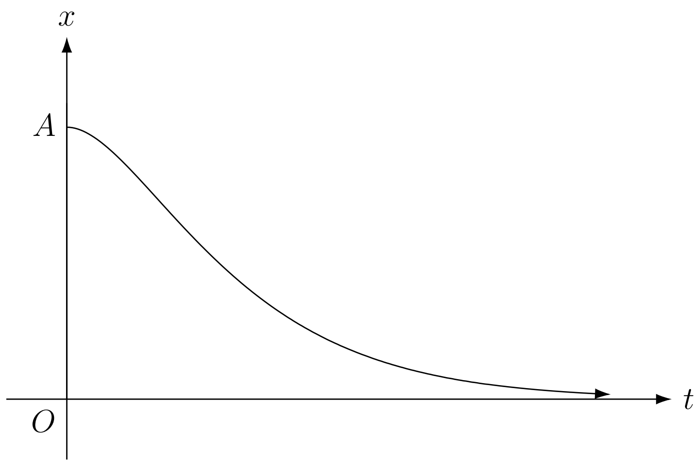
(ii) Using the definition of y

$$\begin{aligned} \dot{y} + ny &= \frac{dv}{dt} + n\frac{dx}{dt} + nv + n^2x \\ &= \ddot{x} + 2nv + n^2x \\ &= 0 \quad \text{from (i)} \end{aligned}$$

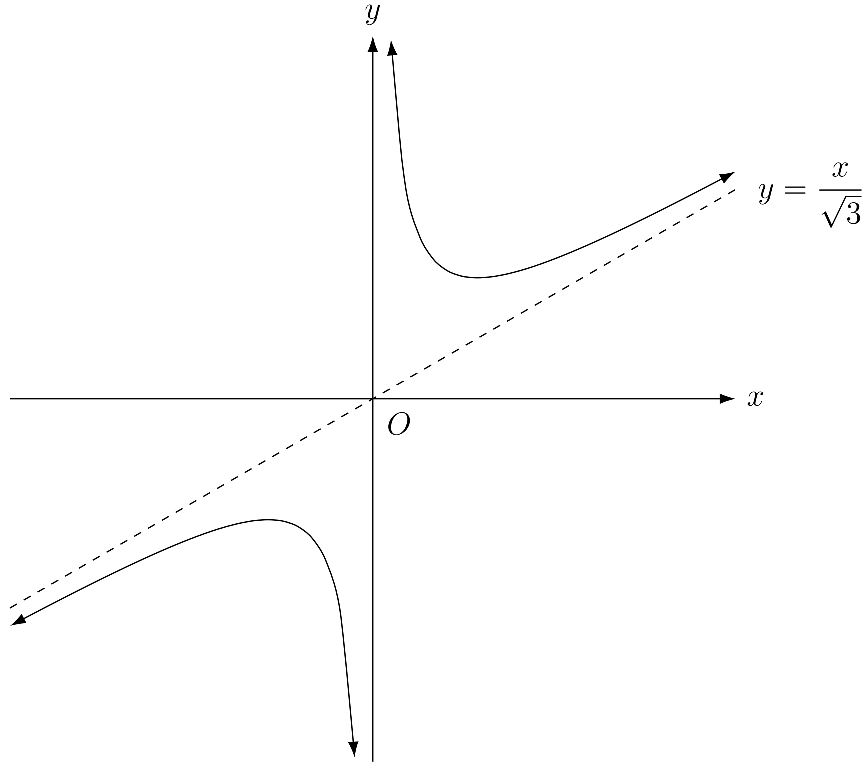
(iii) The general solution to $\dot{y}+ny = 0$, or equivalently $\dfrac{dy}{dt} = -ny$, is $y = Be^{-nt}$ for some constant B. When $t = 0$, $v = 0$, $x = A$ and so $y = nA$. This means that $B = nA$, hence

$$\begin{aligned} y &= nAe^{-nt} \\ v + nx &= nAe^{-nt} \\ \frac{dx}{dt}e^{nt} + nxe^{nt} &= nA \\ \frac{d}{dt}\left(xe^{nt}\right) &= nA \\ xe^{nt} &= nAt + C \qquad \text{when } t = 0,\ x = A \text{ so } C = A \\ xe^{nt} &= nAt + A \\ x &= A(1 + nt)e^{-nt} \end{aligned}$$

(iv) When $t = 0$, $x = A$ and when $t \to \infty$ then $x \to 0$. This means that the particle starts from rest at the displacement $x = A$ accelerates in speed towards $x = 0$ to a point and then slows down in speed approaching $x = 0$.



(b) (i) By addition of ordinates of the hyperbola $y = \dfrac{\sqrt{3}}{x}$ and the line $y = \dfrac{x}{\sqrt{3}}$ the following graph is obtained.



(ii) Solve simultaneously for x.

$$
\begin{aligned}
x\sqrt{3} &= \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{x} \\
3x &= x + \frac{3}{x} \\
2x^2 - 3 &= 0 \\
x &= \pm\sqrt{\frac{3}{2}} \\
y &= \pm\frac{3}{\sqrt{2}}
\end{aligned}
$$

Hence the points of intersection are $\left(\sqrt{\dfrac{3}{2}}, \dfrac{3}{\sqrt{2}}\right)$ and $\left(-\sqrt{\dfrac{3}{2}}, -\dfrac{3}{\sqrt{2}}\right)$.

(iii) Note that the asymptote $y = \frac{x}{\sqrt{3}}$ has an angle of inclination of $\frac{\pi}{6}$ to the positive x-axis and $\frac{\pi}{3}$ to the positive y-axis.

This means that a clockwise rotation about the origin by $\frac{\pi}{3}$ leads to the following changes in the asymptotes:

- y-axis asymptote becomes $y = \frac{x}{\sqrt{3}}$ asymptote.
- $y = \frac{x}{\sqrt{3}}$ asymptote becomes $y = -\frac{x}{\sqrt{3}}$ asymptote.

This means that $\frac{b}{a} = \frac{1}{\sqrt{3}}$ for a hyperbola with general equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

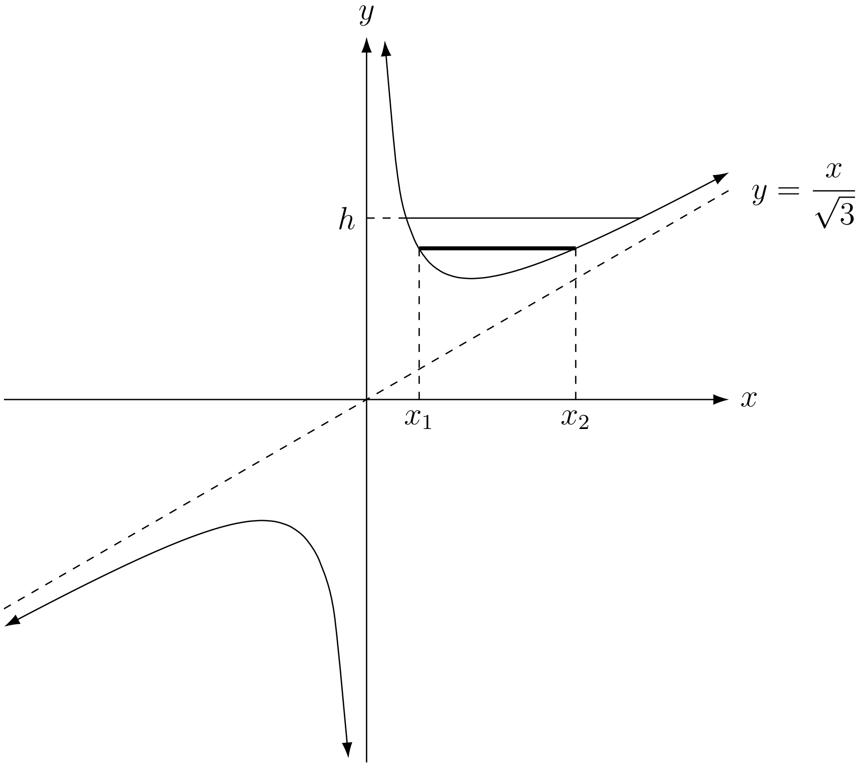
From part (ii), note that the line $y = x\sqrt{3}$ has an angle of inclination of $\frac{\pi}{3}$ to the positive x-axis. Upon rotation, this becomes the x-axis.

This means that the points of intersection found in (ii) will become the x-intercepts of the rotated curve. By considering the distance of this point to the origin, this allows us to find the value of a.

$$\begin{aligned}
a^2 &= \left(\sqrt{\frac{3}{2}}\right)^2 + \left(\frac{3}{\sqrt{2}}\right)^2 \\
&= 6 \\
\frac{b^2}{a^2} &= \frac{1}{3} \\
b^2 &= 2
\end{aligned}$$

Hence, the equation of the hyperbola arising from the rotation is $\frac{x^2}{6} - \frac{y^2}{2} = 1$

(c) From part (b)(iii), it has been established that the curves $y = \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{x}$ and $\frac{x^2}{6} - \frac{y^2}{2} = 1$ are equivalent by rotation. The region described is in fact equivalent to the region bounded by the curve $y = \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{x}$ and the line $y = h$ (which lies above the minimum turning point). This region is rotated about the y-axis to create the equivalent solid of revolution.



Consider a slice with inner and outer radius being x_1 and x_2 where $x_2 \geq x_1$. These are the roots of $y = \dfrac{x}{\sqrt{3}} + \dfrac{\sqrt{3}}{x}$, or equivalently, the roots of $x^2 - y\sqrt{3} + 3 = 0$.

$$\begin{aligned}
\delta V &= \pi(x_2^2 - x_1^2)\,\delta y \\
&= \pi(x_2 + x_1)(x_2 - x_1)\,\delta y \\
&= \pi(x_2 + x_1)\sqrt{(x_2 + x_1)^2 - 4x_1x_2}\,\delta y \qquad \text{since } (x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_1x_2 \\
&= \pi y\sqrt{3} \times \sqrt{3y^2 - 12}\,\delta y \qquad \text{using sum and product of roots} \\
&= 3\pi y\sqrt{y^2 - 4}\,\delta y
\end{aligned}$$

To find the limits of integration, the minimum turning point is needed.

$$\begin{aligned}
y &= \frac{x}{\sqrt{3}} + \frac{\sqrt{3}}{x} \\
\frac{dy}{dx} &= \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{x^2}
\end{aligned}$$

Stationary points occur when $\dfrac{dy}{dx} = 0$ which gives $x = \pm\sqrt{3}$. Since, we are dealing with the positive branch, then the minimum turning point occurs at $x = \sqrt{3}$ and $y = 2$. Hence

$$\begin{aligned}
V &= 3\pi \int_2^h y\sqrt{y^2 - 4}\,dy \\
&= \pi \left[(y^2 - 4)^{\frac{3}{2}} \right]_2^h \\
&= \pi(h^2 - 4)^{\frac{3}{2}}
\end{aligned}$$

Question 15

(a) (i) As one of the half-axes has length $\sqrt{r^2+h^2}$ and the other half-axis has length r, an equation is $\dfrac{x^2}{r^2}+\dfrac{y^2}{r^2+h^2}=1$.

(ii) As the vertical base of the cylinder is a circle and $OX = x$ then $XY = \sqrt{r^2-x^2}$.

Since the water surface is a semi-ellipse, then $XZ = \sqrt{(r^2+h^2)\left(1-\dfrac{x^2}{r^2}\right)}$.

This means that by Pythagoras' theorem

$$\begin{aligned} YZ &= \sqrt{(r^2+h^2)\left(1-\frac{x^2}{r^2}\right)-(r^2-x^2)} \\ &= \sqrt{r^2-x^2+h^2\left(1-\frac{x^2}{r^2}\right)-(r^2-x^2)} \\ &= h\sqrt{1-\frac{x^2}{r^2}} \\ &= \frac{h}{r}\sqrt{r^2-x^2} \end{aligned}$$

Let A be the area of the triangular cross section $\triangle XYZ$ then

$$\begin{aligned} A &= \frac{1}{2}\times\sqrt{r^2-x^2}\times\frac{h}{r}\sqrt{r^2-x^2} \\ &= \frac{h}{2r}(r^2-x^2) \end{aligned}$$

(iii) To find the volume of the water in the cylinder, integrate over x as follows:

$$\begin{aligned} V &= \int_{-r}^{r}\frac{h}{2r}(r^2-x^2)\,dx \\ &= \frac{h}{r}\int_0^r (r^2-x^2)\,dx \quad \text{(as integrand is an even function)} \\ &= \frac{h}{r}\left[r^2x-\frac{x^3}{3}\right]_0^r \\ &= \frac{h}{r}\left(r^3-\frac{1}{3}r^3\right) \\ &= \frac{2}{3}r^2h \end{aligned}$$

The total volume of a cylinder full of water is $\pi r^2 h$, hence the proportion that is filled is $\dfrac{2}{3\pi}$.

(b) (i) Using the fact that $\overline{z}^n = \overline{z^n}$

$$\begin{aligned}(a-bi)(1+ix)^n+(a+bi)(1-ix)^n &= (a-bi)(1+ix)^n+(a+bi)\overline{(1+ix)^n}\\ &= (a-bi)(f(x)+ig(x))+(a+bi)(f(x)-ig(x))\\ &= (a-bi+a+bi)f(x)+i(a-bi-a-bi)g(x)\\ &= 2af(x)-2bi^2g(x)\\ &= 2(af(x)+bg(x))\end{aligned}$$

(ii) From part (i), the roots of the polynomial $af(x)+bg(x)$ must also be the solutions to

$$\begin{aligned}(a-bi)(1+ix)^n+(a+bi)(1-ix)^n &= 0\\ (a-bi)(1+ix)^n &= -(a+bi)(1-ix)^n\\ |(a-bi)||(1+ix)^n| &= |-(a+bi)||(1-ix)^n\\ |1+ix|^n &= |1-ix|^n\\ |1+ix| &= |1-ix|\end{aligned}$$

using the fact that $|(a-bi)| = |-(a+bi)| = a^2+b^2$ and the fact that $|1+ix|$ and $|1-ix|$ are non-negative real numbers.

Now consider $|1+z| = |1-z|$ as a locus. This is a line equidistant from 1 and -1, and perpendicular to the connecting line, the real axis. Thus z must be purely imaginary. So ix must be purely imaginary, which implies x is purely real. Hence, $af(x)+bg(x)$ only has real roots.

(c) (i) When $m=1$, there are only 2 people playing, and so as the first person removed is P_2 then P_1 must win. Now assume that for $m=k$ for some $k>0$, that P_1 wins when 2^k people are playing.

Consider the case where $m=k+1$ so 2^{k+1} people are playing. According to the rules of the game, we remove every person with an even number, so $P_2, P_4, \ldots, P_{2^{k+1}}$. After removing these 2^k people, there will be half as many people left (i.e. another 2^k people). The first person to be removed will be the person clockwise next to P_1. However, notice that this is the exact same game as playing with 2^k people (but with relabelling). Hence, by the induction hypothesis, the person who wins will be P_1. Since it is true for $m=1$, it is true for all positive integers of m.

(ii) The idea will be to remove people until the number of people remaining is a power of 2. As 1024 is the closest power of 2 below 2018, this will require 994 people removed. The 994th person to be removed will be the $2\times 994 = 1988$th person (labelled P_{1988}). After this, with 1024 people, the next person to be removed will be the 1990th person, and so the 1989th person effectively takes the role of the 1st person in part (i). Hence, the final person remaining will be the 1989th person.

Question 16

(a) (i) For Elle to win the game on her kth turn, she must neither win or lose all the games before the kth turn, and then win. When there are n cards, note that $n-2$ possibilities that allow her to redraw. That is, all cards except for 1 and n. Thus there is a $\frac{n-2}{n}$ chance to neither win or lose. On the kth round there is a $\frac{1}{k+2}$ chance of winning, as there are $k+2$ cards in total, but only 1 can be drawn to win. Hence the probability will be

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{k-2}{k}\cdot\frac{k-1}{k+1}\cdot\frac{1}{k+2}$$

Observe that in this expression, each term from 3 through to $k-1$ appears in both the numerator and the denominator, and so we can cancel these terms out to arrive with our final expression

$$\frac{2}{k(k+1)(k+2)}$$

(ii) At every stage, there is an equal chance that Elle picks the highest number or the lowest number. Since the process repeats until eventually Elle wins or loses, we can conclude that the probability she wins is exactly $\frac{1}{2}$.

However, we can also express this as the probability she wins the first round, added to the probability she wins the second round, and so on. There is no limit on the total rounds she can play. Hence the probability she wins is also equal to the limiting sum

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{k(k+1)(k+2)}$$

Equating these we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{k(k+1)(k+2)}=\frac{1}{2}$$
$$\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{4}$$

(b) (i) The roots of $z^{2n}=1$ are simply the $2n$th roots of unity, which has solutions

$$\pm 1, \operatorname{cis}\left(\pm\frac{\pi}{n}\right), \operatorname{cis}\left(\pm\frac{2\pi}{n}\right), \ldots, \operatorname{cis}\left(\pm\frac{(n-1)\pi}{n}\right)$$

(ii) Since the coefficients of $P(z)$ are real, the non-real roots of $z^{2n}-1$ come in conjugate pairs. For ease of notation, let $\theta_k = \dfrac{k\pi}{n}$. Since $(z+1)$ and $(z-1)$ are real factors then

$$\frac{z^{2n}-1}{z^2-1} = (z-\operatorname{cis}\theta_1)\,(z-\operatorname{cis}(-\theta_1))\ldots(z-\operatorname{cis}\theta_{n-1})\,(z-\operatorname{cis}(-\theta_{n-1}))$$

To obtain the real quadratic factors, consider the product of the linear factors

$$\begin{aligned}(z-\operatorname{cis}\theta_1)\,(z-\operatorname{cis}(-\theta_1)) &= z^2-(\operatorname{cis}\theta_1+\operatorname{cis}(-\theta_1))z+\operatorname{cis}\theta_1\operatorname{cis}(-\theta_1)\\ &= z^2-2z\cos\theta_1+1\end{aligned}$$

Apply this similarly to the other linear factors to obtain

$$\begin{aligned}\frac{z^{2n}-1}{z^2-1} &= \left(z^2-2z\cos\theta_1+1\right)\left(z^2-2z\cos\theta_2+1\right)\ldots\left(z^2-2z\cos\theta_{n-1}+1\right)\\ &= \left(z^2-2z\cos\left(\frac{\pi}{n}\right)+1\right)\ldots\left(z^2-2z\cos\left(\frac{(n-1)\pi}{n}\right)+1\right)\end{aligned}$$

(c) (i)

$$\begin{aligned}\sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) &= \sum_{k=1}^{n}\ln\left(a^2-2a\cos\left(\frac{k\pi}{n}\right)+1\right)\\ &= \ln(a^2+2a+1)+\sum_{k=1}^{n-1} f\left(\frac{k\pi}{n}\right)\end{aligned}$$

But note that

$$\begin{aligned}\sum_{k=1}^{n-1} f\left(\frac{k\pi}{n}\right) &= \ln\left[\left(a^2-2a\cos\left(\frac{\pi}{n}\right)+1\right)\ldots\left(a^2-2a\cos\left(\frac{(n-1)\pi}{n}\right)+1\right)\right]\\ &= \ln\left(\frac{a^{2n}-1}{a^2-1}\right)\qquad\text{from part (b)(ii)}\end{aligned}$$

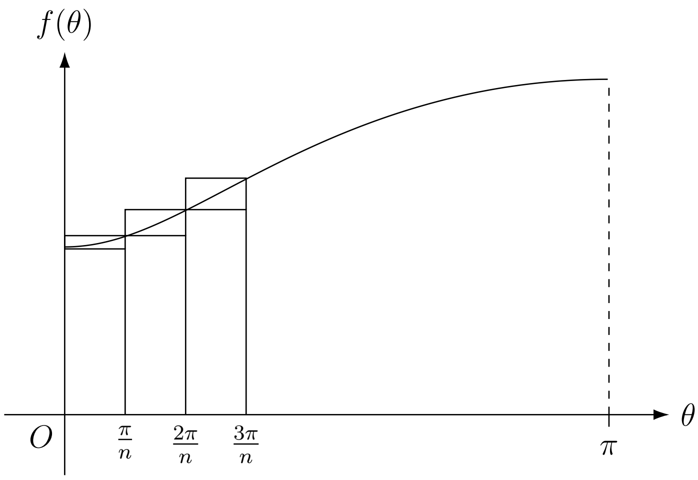
Hence

$$\begin{aligned}\sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) &= \ln(a^2+2a+1)+\ln\left(\frac{a^{2n}-1}{a^2-1}\right)\\ &= \ln(a+1)^2+\ln(a^{2n}-1)-\ln(a^2-1)\\ &= 2\ln(a+1)+\ln(a^{2n}-1)-\ln(a-1)-\ln(a+1)\\ &= \ln(a^{2n}-1)+\ln\left(\frac{a+1}{a-1}\right)\\ \therefore \frac{\pi}{n}\sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) &= R+\frac{\pi}{n}\ln\left(\frac{a+1}{a-1}\right)\end{aligned}$$

(ii) Observe that the sum can be manipulated as follows:

$$
\begin{aligned}
\sum_{k=0}^{n-1} f\left(\frac{k\pi}{n}\right) &= \sum_{k=0}^{n} f\left(\frac{k\pi}{n}\right) - f(\pi) \\
&= \sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) + f(0) - f(\pi) \\
&= \sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) + \ln(a^2 - 2a + 1) - \ln(a^2 + 2a + 1) \\
&= \sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) + 2\ln\left(\frac{a-1}{a+1}\right) \\
\frac{\pi}{n}\sum_{k=0}^{n-1} f\left(\frac{k\pi}{n}\right) &= \frac{\pi}{n}\sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) + \frac{2\pi}{n}\ln\left(\frac{a-1}{a+1}\right) \\
&= R + \frac{\pi}{n}\ln\left(\frac{a+1}{a-1}\right) + \frac{2\pi}{n}\ln\left(\frac{a-1}{a+1}\right) \quad \text{(from part (i))} \\
&= R - \frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right) + \frac{2\pi}{n}\ln\left(\frac{a-1}{a+1}\right) \\
&= R + \frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right)
\end{aligned}
$$

(iii) Consider rectangles of width $\dfrac{\pi}{n}$ as shown in the diagram below.



Note that $f(\theta)$ is an increasing function in the domain given. Let A_L be the sum of the areas of the lower rectangles, which lie under the curve.

$$
\begin{aligned}
A_L &= \frac{\pi}{n}f(0) + \frac{\pi}{n}f\left(\frac{\pi}{n}\right) + \frac{\pi}{n}f\left(\frac{2\pi}{n}\right) + \dots + \frac{\pi}{n}f\left(\frac{(n-1)\pi}{n}\right) \\
&= \frac{\pi}{n}\sum_{k=0}^{n-1} f\left(\frac{k\pi}{n}\right) \\
&= R + \frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right) \qquad \text{from part (ii)}
\end{aligned}
$$

Let A_U be the sum of the areas of the upper rectangles, which are above the curve.

$$\begin{aligned}
A_U &= \frac{\pi}{n} f\left(\frac{\pi}{n}\right) + \frac{\pi}{n} f\left(\frac{2\pi}{n}\right) + ... + \frac{\pi}{n} f\left(\frac{n\pi}{n}\right) \\
&= \frac{\pi}{n} \sum_{k=1}^{n} f\left(\frac{k\pi}{n}\right) \\
&= R + \frac{\pi}{n} \ln\left(\frac{a+1}{a-1}\right) \qquad \text{from part (i)}
\end{aligned}$$

Since $I(a)$ gives the area under the curve, we can see that $A_L < I(a) < A_U$, or equivalently

$$R + \frac{\pi}{n} \ln\left(\frac{a-1}{a+1}\right) < I(a) < R + \frac{\pi}{n} \ln\left(\frac{a+1}{a-1}\right)$$

(iv) Using the given inequality we have $-\ln\left(\frac{x}{x-1}\right) > -\frac{1}{x-1}$ and hence

$$\begin{aligned}
R &= \frac{\pi}{n} \ln\left(a^{2n}-1\right) \\
&= -\frac{\pi}{n} \ln\left(\frac{a^{2n}}{a^{2n}(a^{2n}-1)}\right) \\
&= \frac{\pi}{n} \ln\left(a^{2n}\right) - \frac{\pi}{n} \ln\left(\frac{a^{2n}}{a^{2n}-1}\right) \\
&= 2\pi \ln a - \frac{\pi}{n} \ln\left(\frac{a^{2n}}{a^{2n}-1}\right) \\
&> 2\pi \ln a - \frac{\pi}{n} \left(\frac{1}{a^{2n}-1}\right)
\end{aligned}$$

To obtain the right side inequality notice that

$$\begin{aligned}
a^{2n} - 1 &< a^{2n} \\
\ln\left(a^{2n}-1\right) &< \ln\left(a^{2n}\right) \qquad \text{since ln is an increasing function} \\
\frac{\pi}{n} \ln\left(a^{2n}-1\right) &< \frac{\pi}{n} \ln\left(a^{2n}\right) \\
R &< 2\pi \ln a
\end{aligned}$$

Hence

$$2\pi \ln a - \frac{\pi}{n} \left(\frac{1}{a^{2n}-1}\right) < R < 2\pi \ln a$$

(v) From part (iv) we can deduce that

$$2\pi \ln a - \frac{\pi}{n}\left(\frac{1}{a^{2n}-1}\right) + \frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right) < R + \ln\left(\frac{a-1}{a+1}\right)$$

and

$$R + \ln\left(\frac{a+1}{a-1}\right) < 2\pi \ln a + \frac{\pi}{n}\ln\left(\frac{a+1}{a-1}\right)$$

Hence, using part (iii)

$$2\pi \ln a - \frac{\pi}{n}\left(\frac{1}{a^{2n}-1}\right) + \frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right) < I(a) < 2\pi \ln a + \frac{\pi}{n}\ln\left(\frac{a+1}{a-1}\right).$$

When $n \to \infty$, we can see that since $a > 1$

$$\frac{\pi}{n}\left(\frac{1}{a^{2n}-1}\right) \to 0$$
$$\frac{\pi}{n}\ln\left(\frac{a-1}{a+1}\right) \to 0$$
$$\frac{\pi}{n}\ln\left(\frac{a+1}{a-1}\right) \to 0$$

So $I(a) \to 2\pi \ln a$, which is in fact the exact value of the integral as it has no dependence on n. Hence $I(a) = 2\pi \ln a$.

(d) (i)

$$\begin{aligned} I(b) &= \int_0^\pi \ln\left(b^2 - 2b\cos\theta + 1\right) d\theta \\ &= \int_0^\pi \ln(b^2)\, d\theta + \int_0^\pi \ln\left(\frac{1}{b^2} - \frac{2}{b}\cos\theta + 1\right) d\theta \\ &= \int_0^\pi 2\ln b\, d\theta + I\left(\frac{1}{b}\right) \\ &= 2\pi \ln b + I\left(\frac{1}{b}\right) \end{aligned}$$

(ii) From part (c)(v), if $b > 1$ we have $I(b) = 2\pi \ln b$.

If $0 < b < 1$ then it can be expressed as $b = \dfrac{1}{B}$ for some $B > 1$ and so

$$\begin{aligned} I(B) &= 2\pi \ln B \quad \text{from part (c)(v)} \\ 2\pi \ln B + I\left(\frac{1}{B}\right) &= 2\pi \ln B \quad \text{from part (i)} \\ \therefore I(b) &= 0 \end{aligned}$$

Hence

$$I(b) = \begin{cases} 2\pi \ln b\,, & \text{for } b > 1 \\ 0\,, & \text{for } 0 < b < 1 \end{cases}$$