2019 Bored of Studies Trial Examinations

Mathematics Extension 2

10th October 2019

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using black or blue pen. Black pen is preferred.
- NESA-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 16.

Total Marks – 100

Section I Pages 2 – 4

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 5 – 17

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 A relation has the implicit derivative $\dfrac{dy}{dx} = xy$. Which of the following is $\dfrac{d^2y}{dx^2}$?

(A) y (B) $y + x$ (C) $y(x^2 + 1)$ (D) $x(y^2 + 1)$

2 An object with mass m falls vertically downwards and experiences a resistance force of kv where k is a constant and v is the particle's velocity at time t. Let g be the acceleration due to gravity. Which of the following is a possible acceleration equation of the object?

(A) $g - kv$ (B) $-g - kv$ (C) $g + \frac{k}{m}v$ (D) $-g + \frac{k}{m}v$

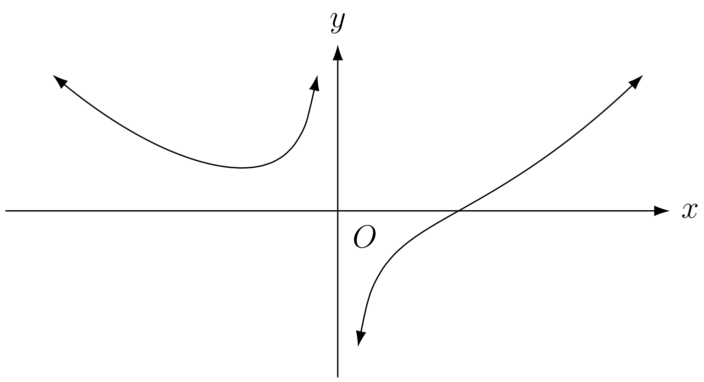
3 For positive integers m and n, consider the complex number

$$z = \cos\left(\frac{(2m-1)\pi}{n}\right) + i\sin\left(\frac{(2m-1)\pi}{n}\right)$$

where $2m - 1$ and n have no common factors other than 1. The complex numbers z^k are plotted on an Argand diagram for all integers $k > 0$. How many distinct points are there?

(A) n (B) $2n$ (C) m (D) $2m$

4 The following shows a sketch of $y = \dfrac{x^k - 1}{x}$.



Which of the following is a possible value for k?

(A) 0 (B) 1 (C) 2 (D) 3

5 A particle moves in circular motion along the unit circle $x^2 + y^2 = 1$ with an angular displacement of θ at time t to the positive x-axis. It is initially at rest at $\theta = 0$, before accelerating such that $\ddot{\theta} = 1$. Which of the following expressions best represents the acceleration of the particle in the y direction?

(A) $\cos\theta - 2\theta\sin\theta$ (B) $-(\cos\theta + 2\theta\sin\theta)$

(C) $\sin\theta - 2\theta\cos\theta$ (D) $-(\sin\theta + 2\theta\cos\theta)$

6 Let z_1 and z_2 be complex numbers such that z_1 is in the first quadrant and z_2 is in the third quadrant on the Argand diagram. Which of the following statements is correct?

(A) $\dfrac{z_1}{z_2}$ is in the first or second quadrant (B) $\dfrac{z_1}{z_2}$ is in the second or third quadrant

(C) $\dfrac{z_1}{z_2}$ is in the third or fourth quadrant (D) $\dfrac{z_1}{z_2}$ is in the fourth or first quadrant

7 How many distinct points of intersection are there between the curves described by the following relations?

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \text{and} \quad (x+1)^2 - y^2 = 4$$

(A) 1 (B) 2 (C) 3 (D) 4

8 Let $P(x) = c_0 + c_1x + c_2x^2$ with non-zero real coefficients c_0, c_1 and c_2.
Consider the following fraction for positive values of m and n

$$\frac{P(x)}{(x+m)^2(x^2-n^2)}.$$

Which of the following could be equivalently expressed as the above fraction for some real numbers a, b and c?

(A) $\dfrac{a}{x+m} + \dfrac{b}{(x+m)^2} + \dfrac{c}{x^2-n^2}$ (B) $\dfrac{a}{(x+m)^2} + \dfrac{b}{x+n} + \dfrac{c}{x-n}$

(C) $\dfrac{a}{(x+m)^2} + \dfrac{b}{x+n} + \dfrac{c}{x^2-n^2}$ (D) $\dfrac{a}{(x+m)^2} + \dfrac{b}{x-n} + \dfrac{c}{x^2-n^2}$

9 Which of the following is equivalent to the definite integral below?

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx$$

(A) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1+\sin x}{1+\cos x}\, dx$

(B) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}\, dx$

(C) $\displaystyle\int_0^{1} \frac{2}{(1+x)^2}\, dx$

(D) $\displaystyle\int_0^{1} \frac{1}{x^2-2}\, dx$

10 What is the general solution to the following system of equations for $k = 1, 2, 3, ..., n$?

$$\begin{aligned}
2x_1 + x_2 + x_3 + \cdots + x_n &= 1 \\
x_1 + 2x_2 + x_3 + \cdots + x_n &= 2 \\
x_1 + x_2 + 2x_3 + \cdots + x_n &= 3 \\
&\vdots \\
x_1 + x_2 + x_3 + \cdots + 2x_n &= n
\end{aligned}$$

(A) $x_k = k - \dfrac{n}{2}$

(B) $x_k = k + \dfrac{n}{2}$

(C) $x_k = k - \dfrac{n(n+1)}{2}$

(D) $x_k = k + \dfrac{n(n+1)}{2}$

Section II

90 marks
Attempt Questions 11—16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available. Throughout Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1}(\sqrt{x}) + \tan^{-1}(x^2)}{1+x^2}\,dx$. **3**

(b) Suppose that a polynomial $P(x)$ with real coefficients has a non-real root α of multiplicity $r \geq 2$.

(i) Express $P(x)$ as a product of real factors and some real polynomial $Q(x)$ in terms of x, α and r. **1**

(ii) Hence, show that $P'(\alpha) = 0$. **2**

(c) (i) Sketch the graphs of $y = x^3 + 1$ and $y = x^2$ on the same set of axes. **1**

(ii) Hence, sketch the graph of $y = x^3 + x^2 + 1$. **2**

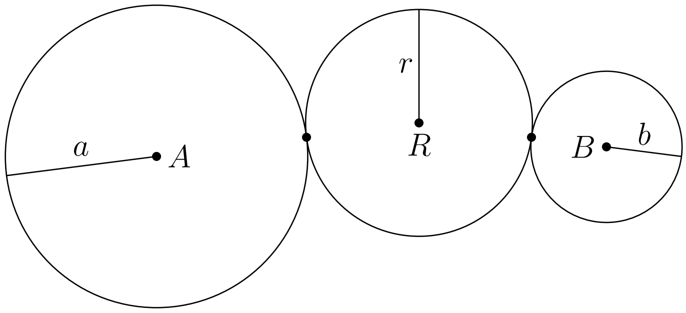
(iii) The region enclosed by the curve $y = x^3 + x^2 + 1$, the line $x = -1$ and the coordinate axes is rotated about the y-axis. Find the volume of the solid of revolution formed. **2**

(d) Let $P(x) = x^4 - 3x^3 - 4x^2 + 9x + 2$ have real roots α, β, γ and δ. By considering the factorisation of $x^2 + 1$, or otherwise, find $(\alpha^2 + 1)(\beta^2 + 1)(\gamma^2 + 1)(\delta^2 + 1)$. **2**

Question 11 continues on page 6

Question 11 (continued)

(e) Consider two distinct non-overlapping circles with fixed centres A and B and fixed radii a and b respectively. Suppose that another circle with centre R and radius r is drawn such that it touches both circles. **2**



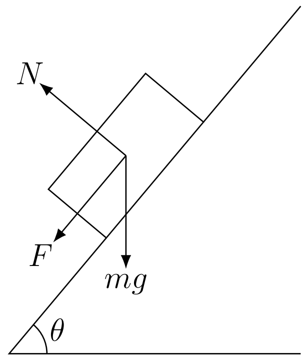
What is the geometric shape of the locus of the point R as r varies? Justify your answer. You do NOT need to provide any equations of this locus.

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) An object of mass m moves along a circular track of radius r with constant speed v without slipping at time t. The track is banked at a positive acute angle of θ to the horizontal.

The object experiences a normal force of N and a frictional force of F which is positive when directed down the banked track. Suppose that the friction force is such that $0 \leq F \leq \mu N$ for some positive constant μ. Let g be the acceleration due to gravity.



(i) By resolving the forces horizontally and vertically, show that **3**

$$v^2 \geq rg\tan\theta.$$

(ii) Find the range of values of μ in terms of θ, such that **2**

$$v^2 \leq \frac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}.$$

(b) Let $P_k\left(ct_k, \dfrac{c}{t_k}\right)$ for $k = 1, 2, 3, 4$ be four points on the hyperbola $xy = c^2$. Suppose that the normals of P_1, P_2, P_3 and P_4 are concurrent and all intersect at the point $R(a, b)$. The equation of the normal at P_k is given by

$$t_k^3 x - t_k y - c(t_k^4 - 1) = 0 \quad \text{(Do NOT prove this).}$$

(i) Show that $t_1 t_2 t_3 t_4 = -1$. **1**

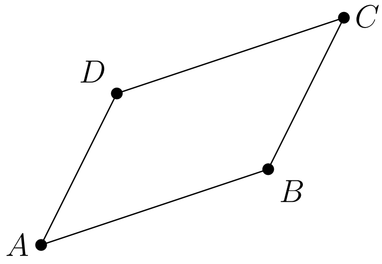
(ii) Show that $\dfrac{1}{t_1^2} + \dfrac{1}{t_2^2} + \dfrac{1}{t_3^2} + \dfrac{1}{t_4^2} = \dfrac{b^2}{c^2}$. **2**

(iii) Hence show that $\dfrac{(P_1R)^2 + (P_2R)^2 + (P_3R)^2 + (P_4R)^2}{3} = a^2 + b^2$. **3**

Question 12 continues on page 8

Question 12 (continued)

(c) Consider a parallelogram with vertices A, B, C and D.



(i) Prove that $AC^2 + BD^2 = 2(AB^2 + AD^2)$. **2**

(ii) Suppose that on the Argand diagram, the points A, B, C and D represent the complex numbers $-z, 1, z$ and -1 respectively. **2**

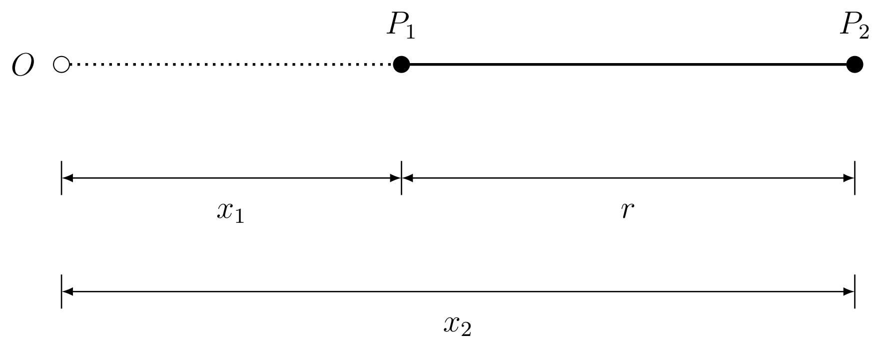
Use the result in (i) to describe the locus traced by z if $\left| z - \frac{1}{z} \right| = 2$.

You do NOT need to specify the Cartesian equation of this locus.

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Two particles, P_1 and P_2 have masses m_1 and m_2 respectively, and are both initially at rest. At time t, they are r units apart. P_1 and P_2 exert forces on each other, so that P_1 moves to the right towards P_2, and P_2 moves to the left towards P_1. Let x_1 and x_2 be the displacements of P_1 and P_2 from a fixed point O.



The forces that P_1 and P_2 experience are given by the following equations

$$F_1 = \frac{Gm_1m_2}{r^2} \quad \text{and} \quad F_2 = -\frac{Gm_1m_2}{r^2} \quad \text{(Do NOT prove this),}$$

where $G > 0$ is constant. Let r_0 be the initial distance between the two particles.

(i) Explain why $\ddot{x}_1 = \dfrac{Gm_2}{r^2}$ and $\ddot{x}_2 = -\dfrac{Gm_1}{r^2}$. **1**

(ii) Let M be the combined masses of the two particles. Show that $\ddot{r} = -\dfrac{GM}{r^2}$. **1**

(iii) Show that $\dot{r} = -\sqrt{2GM\left(\dfrac{1}{r} - \dfrac{1}{r_0}\right)}$. **2**

(iv) Hence show that for $0 < r < r_0$ **3**

$$t = \frac{r_0\sqrt{r_0}}{\sqrt{2GM}}\left(\frac{\pi}{2} + \frac{1}{2}\sin 2\theta - \theta\right)$$

where $\theta = \sin^{-1}\sqrt{\dfrac{r}{r_0}}$.

(v) Using the fact that $\sin x < x$ for positive values of x (do NOT prove this) show that **2**

$$t < \frac{\pi}{\sqrt{GM}}\left(\frac{r_0}{2}\right)^{\frac{3}{2}}.$$

Explain the physical significance of this result.

Question 13 continues on page 10

(b) By considering the expression **2**

$$1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^{n-1}$$

show that for $x \geq -1$ and positive integer n

$$(1 + x)^n \geq 1 + nx.$$

(c) Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. Let

$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

(i) Using the result in part (b), show that $\left(\dfrac{a_{n+1}}{a_n}\right)^{n+1} \geq \dfrac{x_{n+1}}{a_n}$. **1**

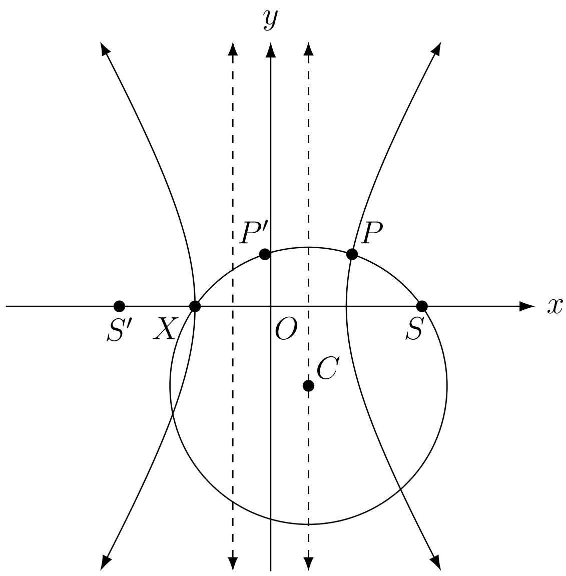
(ii) Hence prove by mathematical induction that for positive integers of n **3**

$$x_1 x_2 \ldots x_n \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^n.$$

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{3a^2} = 1$ with foci S and S' is shown in the diagram below. Let X be the x-intercept of the hyperbola on the negative x-axis.



Let C be a point on the positive directrix and in the fourth quadrant such that it is the centre of a circle passing through the points S and X. Let P be the point of intersection of the circle and the hyperbola in the first quadrant. Let P' be the reflection of P about the positive directrix.

(i) Find the eccentricity of the hyperbola. **1**

(ii) Prove that $\triangle SCP \equiv \triangle XCP'$. **2**

(iii) Hence, show that $\angle SCX = 3 \times \angle PCP'$. **3**

Question 14 continues on page 12

(b) Let $P(x)$ be a monic polynomial of the form

$$P(x) = x^4 - ax^3 - bx^2 - cx - d$$

where $a \geq b \geq c \geq d > 0$ are integers. Let $P(x)$ have non-zero roots α, β, γ and δ.

(i) Show that an equation with the roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$ is **1**

$$dx^4 + cx^3 + bx^2 + ax - 1 = 0.$$

(ii) Suppose $P(x)$ has a positive integer root $x = n$. Using part (i), show that $n > a$. **1**

(iii) Hence, by finding a contradiction, show that $P(x)$ cannot have a positive integer root. **2**

(c) Let ω be a root of $z^n = -1$.

(i) Show that **1**

$$\frac{1}{1-\omega} + \frac{3}{1-\omega^3} + \frac{5}{1-\omega^5} + ... + \frac{2n-1}{1-\omega^{2n-1}} = \sum_{k=1}^{2n-1} \frac{k}{1-\omega^k} - \sum_{k=1}^{n-1} \frac{2k}{1-\omega^{2k}}.$$

(ii) Hence show that **1**

$$\sum_{k=1}^{2n-1} \frac{k}{1-\omega^k} - \sum_{k=1}^{n-1} \frac{2k}{1-\omega^{2k}} = \sum_{k=n}^{2n-1} \frac{k}{1-\omega^k} - \sum_{k=1}^{n-1} \frac{k}{1+\omega^k}.$$

(iii) Deduce that **3**

$$\frac{1}{n}\left[\frac{1}{1-\omega} + \frac{3}{1-\omega^3} + ... + \frac{2n-1}{1-\omega^{2n-1}}\right] = 1 + \frac{1}{1+\omega} + \frac{1}{1+\omega^2} + ... + \frac{1}{1+\omega^{n-1}}.$$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) A triangle has side lengths a, b and c where $a \geq b \geq c$. Show that **3**

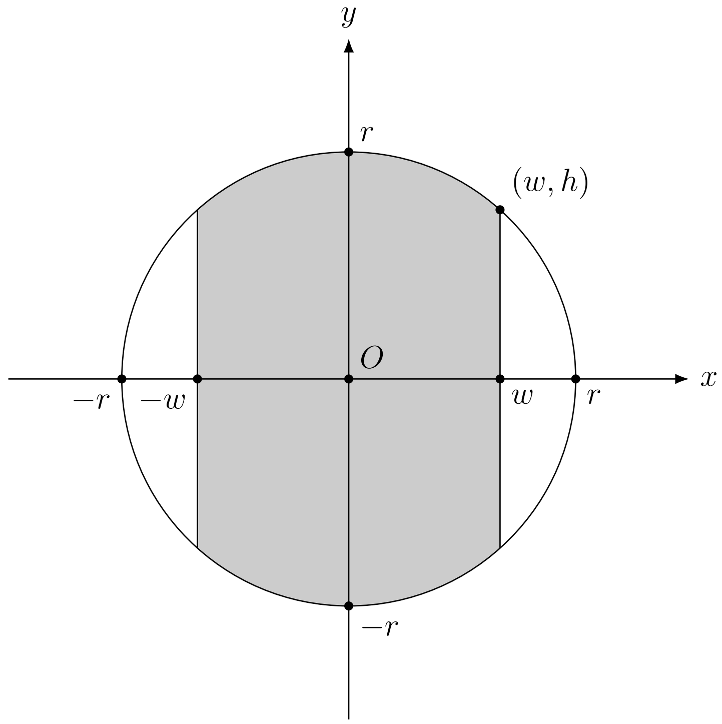
$$\sqrt[3]{\frac{a^3 + b^3 + c^3 + 3abc}{2}} > a.$$

(b) Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x \, dx$ for integers $n \geq 0$.

(i) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$ for $n \geq 2$. **2**

(ii) Hence, evaluate I_6. **1**

(c) The diagram below shows the area bounded by the circle $x^2 + y^2 = r^2$ and the lines $x = w$ and $x = -w$ for some positive values of r and w. **2**



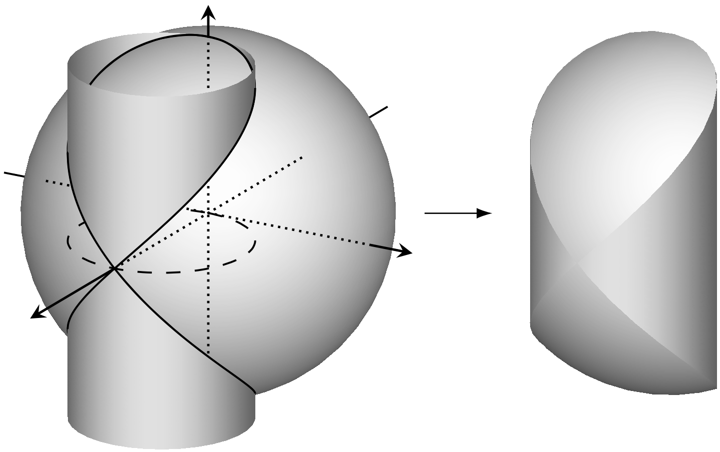
Let (w, h) be a point on the circle as shown in the diagram for some positive value of h. By considering, areas of sectors, or otherwise, show that the shaded area is given by

$$A = 2(w^2 + h^2)\tan^{-1}\left(\frac{w}{h}\right) + 2hw.$$

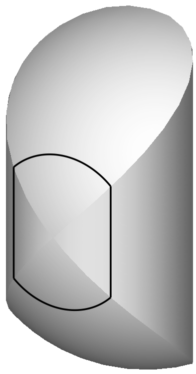
Question 15 continues on page 14

Question 15 (continued)

(d) Consider the solid formed by the intersection of a cylinder of radius $\frac{1}{2}$ units and a sphere of radius 1 unit as shown in the diagram below.



Vertical slices are taken as shown in the diagram below.



Let s be distance between the centre of the slice to the centre of the sphere.
Let h, w and r be defined similar to that in part (c).

(i) Show that $h^2 + s^2 + w^2 = 1$. **1**

(ii) Explain why $\left(s - \dfrac{1}{2}\right)^2 + w^2 = \dfrac{1}{4}$ **1**

(iii) Hence, show that the area of the slice is given by **2**

$$A = 2(1 - s^2)\tan^{-1}\sqrt{s} + 2\sqrt{s}(1 - s)$$

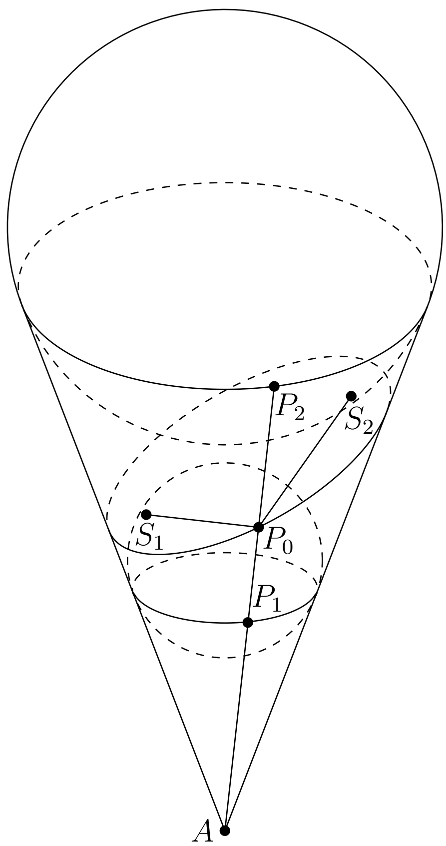
(iv) Using the substitution $\theta = \tan^{-1}\sqrt{s}$ and the result in question (b) (ii), or otherwise, find the volume of the solid. **3**

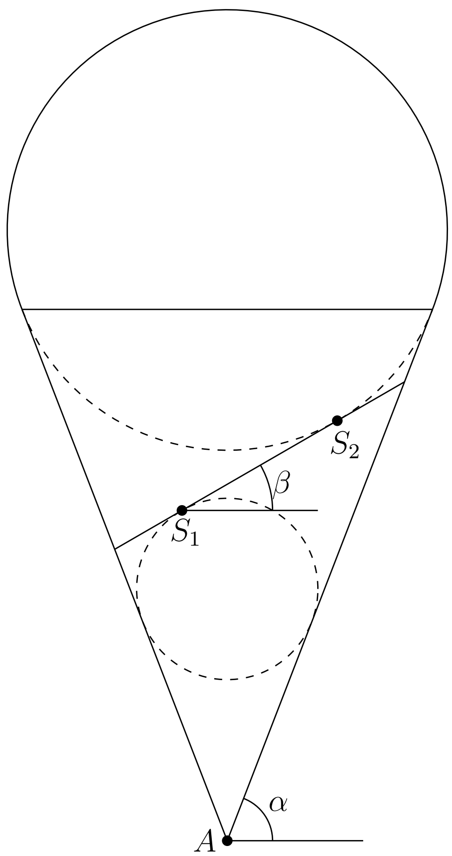
End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) Consider a hollow right circular cone with vertex A which makes an angle of α to the horizontal. A small sphere is inserted towards the vertex of the cone such that it touches the cone on the circle $\mathscr{C}_1$.

A flat ellipse is then inserted into the cone at an angle of β to the horizontal such that its edges touch the sides of the cone and one side of its surface touches the small sphere at S_1. Following that, a large sphere is then inserted to touch the ellipse at S_2. This sphere touches the cone on the circle $\mathscr{C}_2$.





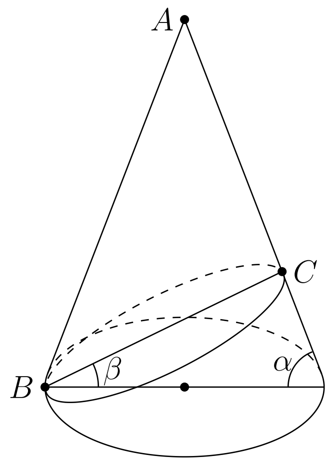
Let P_0 be any point on the perimeter of the ellipse. The line AP_0 is produced such that it touches the circle $\mathscr{C}_1$ at P_1 and $\mathscr{C}_2$ at P_2.

(i) Explain why $P_0P_1 = P_0S_1$. **1**

(ii) Hence deduce that S_1 and S_2 are the foci of the ellipse. **1**

Question 16 continues on page 16

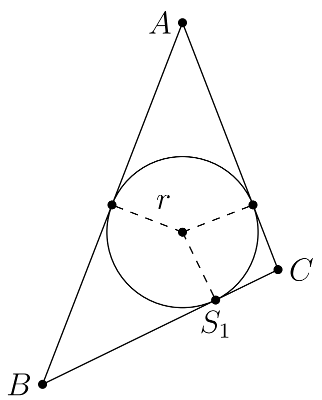
(b) Consider a right circular cone with vertex A which makes an angle of α to the horizontal. The cone is then sliced at point C to point B, on the base of the cone, at an angle of β to the horizontal, where $\beta < \alpha$. The cross-section of the slice is an ellipse with a semi-major axis of a and an eccentricity of e.



(i) Show that $AB = \dfrac{2a\sin(\alpha+\beta)}{\sin 2\alpha}$. **1**

(ii) Show that $AC = \dfrac{2a\sin(\alpha-\beta)}{\sin 2\alpha}$. **1**

(iii) Consider a circle inscribed in $\triangle ABC$ with radius r such that each side is tangent to the circle. Let the point of contact of BC with the circle be S_1. **1**



Show that

$$r = \frac{AC \times AB \sin 2\alpha}{AB + BC + AC}.$$

Question 16 continues on page 17

Question 16 (continued)

(iv) Using the result in part (a), show that **3**

$$e = 1 - \frac{r}{a}\cot\left(\frac{\alpha+\beta}{2}\right).$$

(v) Hence, show that **3**

$$e = \frac{\sin\beta}{\sin\alpha}.$$

You may assume the following results without proof:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

(c) Student A and student B play a game where a fair coin is tossed repeatedly until one of them wins. Suppose that the rules of the game are as follows

- If there are 3 more heads than tails, student A wins the game.
- If there are 3 more tails than heads, student B wins the game.

(i) What is the probability that student A wins the game? **1**

(ii) Let p be the probability that student A wins the game, if the first toss was a head. Explain why **2**

$$p = \frac{1}{2} + \frac{1}{4} \times p.$$

(iii) Suppose that the first toss was a tail instead. What is the probability that student A wins the game? **1**

End of paper

2019 Bored of Studies Trial Examinations

Mathematics Extension 2

Solutions

Section I

Answers

1	(C)	**6**	(B)
2	(D)	**7**	(C)
3	(B)	**8**	(B)
4	(D)	**9**	(C)
5	(A)	**10**	(A)

Brief explanations

1 Using the product rule $\dfrac{d^2x}{dy^2} = y + x\dfrac{dy}{dx}$. Substituting $\dfrac{dy}{dx} = xy$ gives the answer (C).

2 Since gravitational force and resistance force are in opposing directions, the net force must be $mg - kv$ or $-mg + kv$ depending on where the positive direction is defined. The equivalent acceleration equation which satisfies any of these is (D).

3 By De-Moivre's theorem the argument of z^k is $\frac{(2m-1)k\pi}{n}$. When $k = 1$, the argument is $\frac{(2m-1)\pi}{n}$. Therefore, the argument between adjacent points must be $\frac{(2m-1)\pi}{n}$ as k is increases. In order to have a full revolution in rotation (before crossing over to the same arguments) k must increase up to $2n$ (i.e. the argument passes an even multiple of $/pi$). Hence, the answer is (B).

4 Options (A) and (C) are not correct because they result in odd functions and the graph has no symmetry. Choosing (B) results in $y = 1 - \frac{1}{x}$, which is a shifted hyperbola and does not match the shape of the graph. Note that when $x \to -\infty$ then $y \to \infty$, which matches the answer (D).

5 Since $\ddot{\theta} = 1$ then $\dot{\theta} = t$ after applying the initial conditions. Integrating again with respect to t gives $\theta = \frac{1}{2}t^2$ after applying initial conditions. Since $r = 1$, the equations in y are given by

$$\begin{aligned} y &= \sin\theta \\ &= \sin\left(\frac{t^2}{2}\right) \\ \dot{y} &= t\cos\left(\frac{t^2}{2}\right) \\ \ddot{y} &= -t^2\sin\left(\frac{t^2}{2}\right) + \cos\left(\frac{t^2}{2}\right) \\ &= \cos\theta - 2\theta\sin\theta \end{aligned}$$

Hence, the answer is (A).

6 Since $0 < \arg z_1 < \frac{\pi}{2}$ and $-\pi < \arg z_2 < -\frac{\pi}{2}$ then $\frac{\pi}{2} < \arg z_1 - \arg z_2 < \frac{3\pi}{2}$. Hence, $\arg\left(\frac{z_1}{z_2}\right)$ must lie in the second and third quadrants, so the answer is (B).

7 This can be visualised with a sketch of the ellipse and shifted rectangular hyperbola. The right branch of the hyperbola intersects the ellipse twice whereas the left branch touches the ellipse at $(-3, 0)$. Hence, there are three points of intersection, so the answer is (C).

8 Creating a common denominator and equating the numerators for each of the options gives

For option (A) $P(x) = a(x+m)(x^2-n^2) + b(x^2-n^2) + c(x+m)^2$
For option (B) $P(x) = a(x^2-n^2) + b(x+m)^2(x-n) + c(x+m)^2(x+n)$
For option (C) $P(x) = a(x^2-n^2) + b(x+m)^2(x-n) + c(x+m)^2$
For option (D) $P(x) = a(x^2-n^2) + b(x+m)^2(x+n) + c(x+m)^2$

Since $P(x)$ is of degree 2 or less, the coefficient of x^3 on the right hand side must be zero. However, for all options except (B) this leads to $a = 0$ or $b = 0$, which contradicts the requirement that they need to be non-zero. For option (B) the required condition is $b + c = 0$ which is possible for non-zero values of b and c. Hence, the answer is (B).

9 The substitution $x = \frac{\pi}{2} - u$ converts the integral to

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\, dx$$

But since $\dfrac{1}{1+\cos x} < \dfrac{1+\sin x}{1+\cos x}$ for $0 < x < \frac{\pi}{2}$ then option (A) cannot be correct.

Expanding the integrand in option (B) gives

$$\begin{aligned} \frac{1}{1+\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2} &= \frac{1}{2+2\sin\frac{x}{2}\cos\frac{x}{2}} \\ &= \frac{1}{2+\sin x} \end{aligned}$$

Since $\dfrac{1}{2+\sin x} < \dfrac{1}{1+\sin x}$ then option (B) cannot be correct.

Note that the integral is positive in the domain $0 < x < \frac{\pi}{2}$ so option (D) cannot be correct because that integrand is actually negative in its domain $0 < x < 1$.

For option (C), note that the integral can be obtained via t-formula substitution $t = \tan\frac{x}{2}$.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\, dx &= \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\, dt \\ &= \int_0^1 \frac{2}{(1+t)^2}\, dt \\ &= \int_0^1 \frac{2}{(1+x)^2}\, dx \end{aligned}$$

Hence, the answer is (C).

10 By summing all n equations the result is

$$\begin{aligned} (n+1)x_1 + (n+1)x_2 + (n+1)x_3 + \ldots + (n+1)x_n &= 1+2+3+\ldots+n \\ (n+1)(x_1+x_2+x_3+\ldots+x_n) &= \frac{n(n+1)}{2} \\ x_1+x_2+x_3+\ldots+x_n &= \frac{n}{2} \end{aligned}$$

But

$$\begin{aligned} x_1+x_2+x_3+\ldots+x_{k-1}+2x_k+x_{k+1}+\ldots+x_n &= k \\ x_1+x_2+x_3+\ldots+x_n &= k - x_k \end{aligned}$$

Hence the general solution is $x_k = k - \dfrac{n}{2}$, which is (A).

Question 11

(a) Let $u = \sqrt{x} \Rightarrow dx = 2u\,du$. When $x = 0$ then $u = 1$ and when $x = 1$ then $u = 1$. Apply this substitution to the first fraction and use integration by parts.

$$\begin{aligned} \int_0^1 \frac{\tan^{-1}(\sqrt{x})}{1+x^2}\,dx &= \int_0^1 \tan^{-1} u \times \frac{2u}{1+u^4}\,du \\ &= \int_0^1 \tan^{-1} u \times \frac{d}{du}(\tan^{-1}(u^2))\,du \\ &= [\tan^{-1} u \tan^{-1}(u^2)]_0^1 - \int_0^1 \frac{\tan^{-1}(u^2)}{1+u^2}\,du \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{\tan^{-1}(\sqrt{x})}{1+x^2}\,dx + \int_0^1 \frac{\tan^{-1}(u^2)}{1+u^2}\,du &= \frac{\pi^2}{16} \\ \therefore \int_0^1 \frac{\tan^{-1}(\sqrt{x}) + \tan^{-1}(x^2)}{1+x^2}\,dx &= \frac{\pi^2}{16} \end{aligned}$$

Alternative solution:

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$. When $x = 0$ then $\theta = 0$ and when $x = 1$ then $\theta = \frac{\pi}{4}$. Applying this substitution to the whole integral.

$$\begin{aligned} \int_0^1 \frac{\tan^{-1}(\sqrt{x}) + \tan^{-1}(x^2)}{1+x^2}\,dx &= \int_0^{\frac{\pi}{4}} \frac{\tan^{-1}(\sqrt{\tan\theta}) + \tan^{-1}(\tan^2\theta)}{1+\tan^2\theta} \times \sec^2\theta\,d\theta \\ &= \int_0^{\frac{\pi}{4}} \Big(\tan^{-1}(\sqrt{\tan\theta}) + \tan^{-1}(\tan^2\theta)\Big)\,d\theta \end{aligned}$$

Let $y = \tan^{-1}(\sqrt{\tan x})$ then $x = \tan^{-1}(\tan^2 x)$. The area bounded by the curve $y = \tan^{-1}(\sqrt{\tan x})$, the x-axis and the line $x = \frac{\pi}{4}$ is represented by

$$\int_0^{\frac{\pi}{4}} \tan^{-1}(\sqrt{\tan x})\,dx$$

The area bounded by the curve $y = \tan^{-1}(\sqrt{\tan x})$, the y-axis and the line $y = \frac{\pi}{4}$ (which occurs at $x = \frac{\pi}{4}$) is represented by

$$\int_0^{\frac{\pi}{4}} \tan^{-1}(\tan^2 x)\,dx$$

The sum of the two integrals is the area of the square covering the domain $0 \le x \le \frac{\pi}{4}$ and range $0 \le y \le \frac{\pi}{4}$. Hence

$$\int_0^1 \frac{\tan^{-1}(\sqrt{x}) + \tan^{-1}(x^2)}{1+x^2}\,dx = \frac{\pi^2}{16}$$

(b) (i) Since $P(x)$ has real coefficients then non-real roots occur in conjugate pairs, so $\bar{\alpha}$ is also a root of multiplicity r. Hence, $P(x)$ can be expressed as

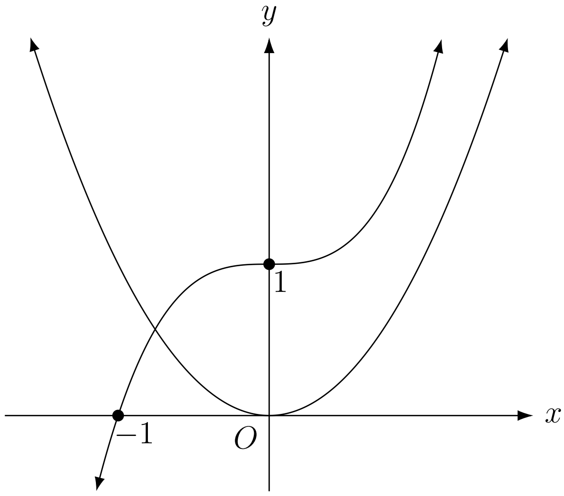
$$\begin{aligned}P(x) &= (x-\alpha)^r(x-\bar{\alpha})^r Q(x)\\ &= (x^2-(\alpha+\bar{\alpha})x+\alpha\bar{\alpha})^r Q(x)\end{aligned}$$

Noting that $(\alpha+\bar{\alpha})$ and $\alpha\bar{\alpha}$ are real.

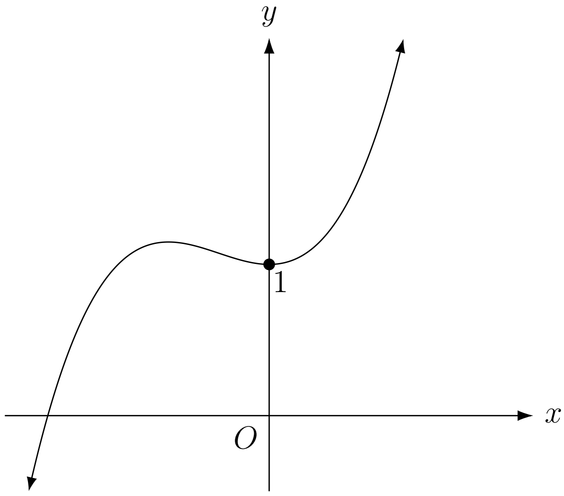
(ii) Using the product rule

$$\begin{aligned}P'(x) &= r(x^2-(\alpha+\bar{\alpha})x+\alpha\bar{\alpha})^{r-1}(2x-(\alpha+\bar{\alpha}))Q(x)+(x^2-(\alpha+\bar{\alpha})x+\alpha\bar{\alpha})^r Q'(x)\\ P'(\alpha) &= r(\alpha^2-(\alpha+\bar{\alpha})\alpha+\alpha\bar{\alpha})^{r-1}(2\alpha-(\alpha+\bar{\alpha}))Q(\alpha)+(\alpha^2-(\alpha+\bar{\alpha})\alpha+\alpha\bar{\alpha})^r Q'(\alpha)\\ &= r(\alpha^2-\alpha^2-\bar{\alpha}\alpha+\alpha\bar{\alpha})^{r-1}(\alpha-\bar{\alpha})Q(\alpha)+(\alpha^2-\alpha^2-\bar{\alpha}\alpha+\alpha\bar{\alpha})^r Q'(\alpha)\\ &= 0\end{aligned}$$

(c) (i) Sketching the graphs $y = x^3+1$ and $y = x^2$



(ii) Using addition of ordinates to obtain the graph



(iii) From the sketch, the radius of the cylindrical shell is $-x$ because $x < 0$ and the height is y. Hence the volume is

$$\begin{aligned}
V &= 2\pi \int_{-1}^{0} (-x)y\, dx \\
&= 2\pi \int_{-1}^{0} (-x^4 - x^3 - x)\, dx \\
&= -2\pi \left[\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^2}{2} \right]_{-1}^{0} \\
&= -2\pi \left(0 - \left(-\frac{1}{5} + \frac{1}{4} + \frac{1}{2} \right) \right) \\
&= \frac{11\pi}{10} \quad \text{cubic units}
\end{aligned}$$

(d) Since α, β, γ and δ are the roots of $P(x)$ then by the factor theorem

$$\begin{aligned}
P(x) &= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \\
P(i) &= (i-\alpha)(i-\beta)(i-\gamma)(i-\delta) \\
&= (\alpha-i)(\beta-i)(\gamma-i)(\delta-i) \\
\text{similarly} \quad P(-i) &= (-i-\alpha)(-i-\beta)(-i-\gamma)(-i-\delta) \\
&= (\alpha+i)(\beta+i)(\gamma+i)(\delta+i) \\
P(i)P(-i) &= (\alpha+i)(\beta+i)(\gamma+i)(\delta+i)(\alpha-i)(\beta-i)(\gamma-i)(\delta-i) \\
&= (\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1) \\
\text{but} \quad P(i) &= i^4 - 3i^3 - 4i^2 + 9i + 2 \\
&= 7 + 12i \\
\text{and} \quad P(-i) &= 7 - 12i \quad \text{as } P \text{ has real coefficients.} \\
\therefore (\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1) &= (7+12i)(7-12i) \\
&= 193
\end{aligned}$$

(e) When circles touch, the line of centres passes through the point of contact, hence

$$\begin{aligned}
AR = a + r \quad &\text{and} \quad BR = b + r \\
|AR - BR| &= |a - b|
\end{aligned}$$

The points A and B are fixed. The radii a and b are also fixed. As r varies and R moves, the difference in distances AR and BR remains constant. This is analagous to the relationship $|PS - PS'|$ being constant for a variable point P and fixed foci S and S'. So the locus has the shape of a hyperbola where A and B are the foci.

Question 12

(a) (i) Resolving the forces horizontally and vertically

$$\begin{aligned}
N\sin\theta + F\cos\theta &= \frac{mv^2}{r} \\
N\cos\theta - F\sin\theta &= mg \\
N(\sin^2\theta + \cos^2\theta) + F\sin\theta\cos\theta - F\sin\theta\cos\theta &= \frac{mv^2\sin\theta}{r} + mg\cos\theta \\
\Rightarrow N &= \frac{mv^2\sin\theta}{r} + mg\cos\theta \\
N\sin\theta\cos\theta - N\cos\theta\sin\theta + F(\cos^2\theta + \sin^2\theta) &= \frac{mv^2\cos\theta}{r} - mg\sin\theta \\
\Rightarrow F &= \frac{mv^2\cos\theta}{r} - mg\sin\theta
\end{aligned}$$

Since $F \geq 0$, $m > 0$ and $\cos\theta > 0$ then

$$\begin{aligned}
\frac{mv^2\cos\theta}{r} - mg\sin\theta &\geq 0 \\
v^2 &\geq rg\tan\theta
\end{aligned}$$

(ii) Since $F \leq \mu N$, $m > 0$ and $\cos\theta > 0$ then

$$\begin{aligned}
\frac{mv^2\cos\theta}{r} - mg\sin\theta &\leq \mu\left(\frac{mv^2\sin\theta}{r} + mg\cos\theta\right) \\
\frac{v^2}{r}(\cos\theta - \mu\sin\theta) &\leq g(\sin\theta + \mu\cos\theta) \\
v^2(1 - \mu\tan\theta) &\leq rg(\tan\theta + \mu)
\end{aligned}$$

For $v^2 \leq \dfrac{rg(\tan\theta + \mu)}{1 - \mu\tan\theta}$ to hold, then it must be that $1 - \mu\tan\theta > 0$, otherwise the inequality sign will be switched. Thus, the range of values allowed for μ are those where $\mu < \cot\theta$.

(b) (i) The equation of a general normal at $\left(ct, \frac{c}{t}\right)$ is

$$\begin{aligned} t^3x - ty - c(t^4 - 1) &= 0 \\ ct^4 - t^3x + ty - c &= 0 \end{aligned}$$

This is a quartic polynomial in t which has the roots t_1, t_2, t_3 and t_4. Since $R(a, b)$ lies on all four normals then the polynomial equation is

$$ct^4 - t^3a + tb - c = 0$$

By product of the roots

$$\begin{aligned} t_1t_2t_3t_4 &= \frac{-c}{c} \\ &= -1 \end{aligned}$$

(ii) First consider

$$\begin{aligned} \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} &= \frac{t_2t_3t_4 + t_1t_3t_4 + t_1t_2t_4 + t_1t_2t_3}{t_1t_2t_3t_4} \\ &= \frac{-\frac{b}{c}}{-1} \\ &= \frac{b}{c} \end{aligned}$$

Also, using the result in (i)

$$\begin{aligned} \frac{1}{t_1t_2} + \frac{1}{t_1t_3} + \frac{1}{t_1t_4} + \frac{1}{t_2t_3} + \frac{1}{t_2t_4} + \frac{1}{t_3t_4} &= -t_3t_4 - t_2t_4 - t_2t_3 - t_1t_4 - t_1t_3 - t_1t_2 \\ &= 0 \end{aligned}$$

But

$$\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)^2 = \frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} + \frac{1}{t_4^2} + 2\left(\frac{1}{t_1t_2} + \frac{1}{t_1t_3} + \frac{1}{t_1t_4} + \frac{1}{t_2t_3} + \frac{1}{t_2t_4} + \frac{1}{t_3t_4}\right)$$

$$\begin{aligned} \frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} + \frac{1}{t_4^2} &= \left(\frac{b}{c}\right)^2 - 2 \times 0 \\ &= \frac{b^2}{c^2} \end{aligned}$$

(iii) Consider

$$
\begin{aligned}
(P_kR)^2 &= (ct_k - a)^2 - \left(\frac{c}{t_k} - b\right)^2 \\
&= c^2t_k^2 - 2act_k + a^2 + \frac{c^2}{t_k^2} - \frac{2bc}{t_k} + b^2
\end{aligned}
$$

Using the result in (ii)

$$
\begin{aligned}
\sum_{k=1}^{4}(P_kR)^2 &= c^2\sum_{k=1}^{4} t_k^2 - 2ac\sum_{k=1}^{4} t_k + 4a^2 + c^2\sum_{k=1}^{4}\frac{1}{t_k^2} - 2bc\sum_{k=1}^{4}\frac{1}{t_k} + 4b^2 \\
&= c^2\sum_{k=1}^{4} t_k^2 - 2ac \times \frac{a}{c} + 4a^2 + c^2 \times \frac{b^2}{c^2} - 2bc \times \frac{b}{c} + 4b^2 \\
&= c^2\sum_{k=1}^{4} t_k^2 + 2a^2 + 3b^2
\end{aligned}
$$

But

$$
\begin{aligned}
\sum_{k=1}^{4} t_k^2 &= \left(\sum_{k=1}^{4} t_k\right)^2 - 2(t_3t_4 + t_2t_4 + t_2t_3 + t_1t_4 + t_1t_3 + t_1t_2) \\
&= \left(\frac{a}{c}\right)^2 - 2 \times 0 \\
&= \frac{a^2}{c^2}
\end{aligned}
$$

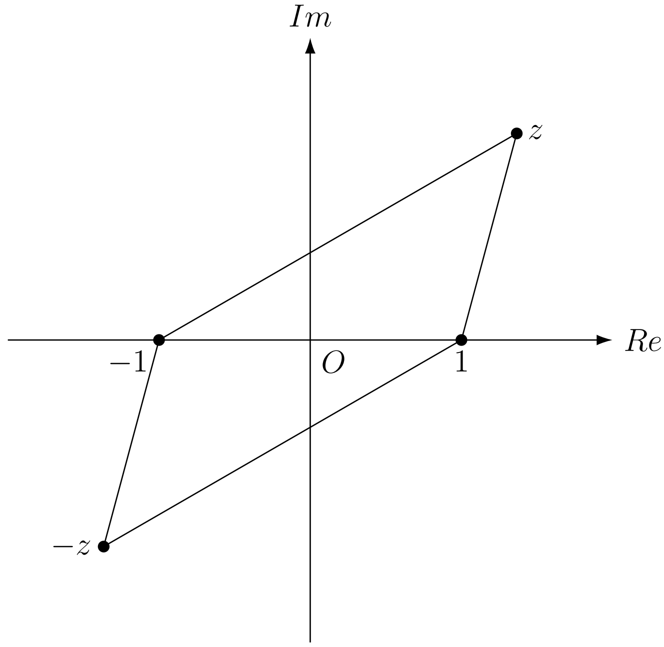
Hence

$$
\begin{aligned}
\sum_{k=1}^{4}(P_kR)^2 &= 3a^2 + 3b^2 \\
\therefore \frac{(P_1R)^2 + (P_2R)^2 + (P_3R)^2 + (P_4R)^2}{3} &= a^2 + b^2
\end{aligned}
$$

(c) (i) Using the cosine rule on $\triangle ABC$ and $\triangle BCD$

$$
\begin{aligned}
AC^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos\angle ABC \\
BD^2 &= CD^2 + BC^2 - 2(CD)(BC)\cos\angle BCD \\
\text{But}\quad AB &= CD \quad \text{(opposite sides of parallelogram)} \\
\angle BCD + \angle ABC &= \pi \quad \text{(adjacent angles of parallelogram)} \\
\Rightarrow BD^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos(\pi - \angle ABC) \\
&= AB^2 + BC^2 + 2(AB)(BC)\cos\angle ABC \\
\therefore AC^2 + BD^2 &= 2(AB^2 + AD^2)
\end{aligned}
$$

(ii) Labelling the points on the Argand diagram



By addition and subtraction of vectors it can be seen that

$$\begin{aligned} |\overrightarrow{AB}| &= |z+1| \\ |\overrightarrow{AD}| &= |z-1| \\ |\overrightarrow{AC}| &= 2|z| \\ |\overrightarrow{BD}| &= 2 \end{aligned}$$

Substitute the lengths into the result in (i)

$$4|z|^2 + 4 = 2(|z+1|^2 + |z-1|^2)$$

But the locus is given by

$$\begin{aligned} \left| z - \frac{1}{z} \right| &= 2 \\ \Rightarrow |z^2 - 1| &= 2|z| \quad \text{noting that } z \neq 0 \end{aligned}$$

Substitute this into the result

$$\begin{aligned} |z^2-1|^2 - 2|z+1|^2 - 2|z-1|^2 + 4 &= 0 \\ |z+1|^2(|z-1|^2 - 2) - 2(|z-1|^2 - 2) &= 0 \\ (|z+1)^2 - 2)(|z-1|^2 - 2) &= 0 \end{aligned}$$

The locus of z is the set of points lying on $|z+1|^2 = 2$ or $|z-1|^2 = 2$. This represents two circles of equal radii $\sqrt{2}$ with centres $(-1, 0)$ and $(1, 0)$.

Question 13

(a) (i) Since $F_1 = m_1\ddot{x}_1$ and $F_2 = m_2\ddot{x}_2$ then

$$\begin{aligned} m_1\ddot{x}_1 &= \frac{Gm_1m_2}{r^2} \\ \Rightarrow \ddot{x}_1 &= \frac{Gm_2}{r^2} \\ m_2\ddot{x}_2 &= -\frac{Gm_1m_2}{r^2} \\ \Rightarrow \ddot{x}_2 &= \frac{Gm_1}{r^2} \end{aligned}$$

(ii) Since $r = x_2 - x_1$ and $M = m_1 + m_2$ then

$$\begin{aligned} \ddot{r} &= \ddot{x}_2 - \ddot{x}_1 \\ &= -\frac{Gm_2}{r^2} - \frac{Gm_1}{r^2} \\ &= -\frac{G(m_1+m_2)}{r^2} \\ &= -\frac{GM}{r^2} \end{aligned}$$

(iii) Using $\ddot{r} = \dfrac{d}{dr}\left(\dfrac{\dot{r}^2}{2}\right)$

$$\begin{aligned} \frac{\dot{r}^2}{2} &= -\frac{GM}{r} + c \quad \text{when} \quad r = r_0, \quad \dot{r} = \dot{x}_2 - \dot{x}_1 = 0 \Rightarrow c = -\frac{GM}{r_0} \\ \dot{r}^2 &= 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right) \\ \dot{r} &= -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} \quad \text{since } r \text{ is decreasing over time so } \dot{r} < 0 \end{aligned}$$

(iv) Use the substitution $\theta = \sin^{-1}\sqrt{\dfrac{r}{r_0}} \Rightarrow r = r_0 \sin^2\theta \Rightarrow dr = 2r_0 \sin\theta\cos\theta\, d\theta$

$$\begin{aligned}
\frac{dr}{dt} &= -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} \\
\frac{dt}{dr} &= -\sqrt{\frac{r_0}{2GM}} \times \sqrt{\frac{r}{r_0 - r}} \\
t &= -\sqrt{\frac{r_0}{2GM}} \int \sqrt{\frac{r}{r_0 - r}}\, dr \\
&= -\sqrt{\frac{r_0}{2GM}} \int \sqrt{\frac{r_0 \sin^2\theta}{r_0(1 - \sin^2\theta)}} \times 2r_0 \sin\theta\cos\theta\, d\theta
\end{aligned}$$

But the range implied by the substitution for $0 < r < r_0$ is $0 < \theta < \frac{\pi}{2}$ so $\sin\theta$ and $\cos\theta$ are all positive. Hence

$$\begin{aligned}
t &= -\sqrt{\frac{r_0}{2GM}} \int \frac{\sin\theta}{\cos\theta} \times 2r_0 \sin\theta\cos\theta\, d\theta \\
&= \frac{r_0\sqrt{r_0}}{\sqrt{2GM}} \int (-2\sin^2\theta)\, d\theta \\
&= \frac{r_0\sqrt{r_0}}{\sqrt{2GM}} \int (\cos 2\theta - 1)\, d\theta \\
&= \frac{r_0\sqrt{r_0}}{\sqrt{2GM}} \left(\frac{1}{2}\sin 2\theta - \theta\right) + C
\end{aligned}$$

When $t \to 0^+$ and $r \to r_0^-$ then $\theta \to \frac{\pi}{2}^-$ so $C = \dfrac{r_0\sqrt{r_0}}{\sqrt{2GM}} \times \dfrac{\pi}{2}$

$$\begin{aligned}
t &= \frac{r_0\sqrt{r_0}}{\sqrt{2GM}} \left(\frac{1}{2}\sin 2\theta - \theta\right) + \frac{r_0\sqrt{r_0}}{\sqrt{2GM}} \times \frac{\pi}{2} \\
&= \frac{r_0\sqrt{r_0}}{\sqrt{2GM}} \left(\frac{\pi}{2} + \frac{1}{2}\sin 2\theta - \theta\right)
\end{aligned}$$

(v) Since $\sin x < x$ for $x > 0$ then $\sin 2\theta < 2\theta$ for $0 < \theta < \frac{\pi}{2}$ as defined in part (iv). This implies that

$$\begin{aligned}
t &< \frac{r_0\sqrt{r_0}}{\sqrt{2GM}} \left(\frac{\pi}{2} + \frac{1}{2} \times 2\theta - \theta\right) \\
&= \frac{\pi r_0\sqrt{r_0}}{2\sqrt{2GM}} \\
&= \frac{\pi}{\sqrt{GM}} \left(\frac{r_0}{2}\right)^{\frac{3}{2}}
\end{aligned}$$

The upper bound of t suggests that this system of P_1 and P_2 moving towards each other cannot be sustained beyond time $t = \dfrac{\pi}{\sqrt{GM}} \left(\dfrac{r_0}{2}\right)^{\frac{3}{2}}$ (in fact, they collide at that time).

(b) From the geometric series

$$\begin{aligned}
\frac{(1+x)^n-1}{(1+x)-1} &= 1+(1+x)+(1+x)^2+...+(1+x)^{n-1}\\
\Rightarrow (1+x)^n-1 &= x\left[1+(1+x)+(1+x)^2+...+(1+x)^{n-1}\right]\\
&= \sum_{k=1}^{n} x(1+x)^{k-1}
\end{aligned}$$

When $x>0$ then

$$\begin{aligned}
(1+x)^{k-1} &> 1\\
\Rightarrow x(1+x)^{k-1} &> x
\end{aligned}$$

When $-1 \leq x \leq 0$ then

$$\begin{aligned}
(1+x)^{k-1} &\leq 1\\
\Rightarrow x(1+x)^{k-1} &\geq x
\end{aligned}$$

Hence

$$\begin{aligned}
(1+x)^n - 1 &\geq \sum_{k=1}^{n} x\\
\therefore (1+x)^n &\geq 1+nx
\end{aligned}$$

(c) (i) Let $x = \dfrac{a_{n+1}}{a_n} - 1$ and replace n with $n+1$ on the result in part (b)

$$\begin{aligned}
\left(\frac{a_{n+1}}{a_n}\right)^{n+1} &\geq 1+(n+1)\left(\frac{a_{n+1}}{a_n}-1\right)\\
&= 1+(n+1)\times\frac{x_1+x_2+...+x_{n+1}}{n+1}\times\frac{n}{x_1+x_2+...+x_n}-(n+1)\\
&= \frac{n(x_1+x_2+...+x_n)+nx_{n+1}}{x_1+x_2+...+x_n}-n\\
&= \frac{nx_{n+1}}{x_1+x_2+...+x_n}\\
&= \frac{x_{n+1}}{a_n}
\end{aligned}$$

(ii) For $n = 1$, LHS $= x_1$ and RHS $= \left(\frac{x_1}{1}\right)^1 = x_1$ $\quad\therefore$ Statement true for $n = 1$
Assume the statement is true for $n = k$

$$x_1 x_2 ... x_k \leq \left(\frac{x_1 + x_2 + ... + x_k}{k}\right)^k$$

Required to prove the statement is true for $n = k + 1$

$$x_1 x_2 ... x_{k+1} \leq \left(\frac{x_1 + x_2 + ... + x_{k+1}}{k+1}\right)^{k+1}$$

$$\begin{aligned}
\text{LHS} &= x_1 x_2 ... x_k x_{k+1} \\
&\leq \left(\frac{x_1 + x_2 + ... + x_k}{k}\right)^k x_{k+1} \quad \text{by assumption} \\
&= \left(\frac{x_1 + x_2 + ... + x_k}{k}\right)^k x_{k+1} \times \frac{a_k}{a_k} \\
&= \frac{x_{k+1}}{a_k} \times a_k^{k+1} \\
&\leq \left(\frac{a_{k+1}}{a_k}\right)^{k+1} \times a_k^{k+1} \quad \text{from part (i)} \\
&= a_{k+1}^{k+1} \\
&= \left(\frac{x_1 + x_2 + ... + x_{k+1}}{k+1}\right)^{k+1} \\
&= \text{RHS}
\end{aligned}$$

Hence the statement is true by induction for positive integers n.

Question 14

(a) (i) From the relationship between the axes and the eccentricity, noting that $e > 1$ for the hyperbola

$$\begin{aligned} 3a^2 &= a^2(e^2 - 1) \\ e^2 &= 4 \\ \therefore e &= 2 \end{aligned}$$

(ii) Since $e = 2$ then the coordinates of S are $(2a, 0)$ and the equation of the directrix is $x = \frac{a}{2}$.

The coordinates of X are $(-a, 0)$, this means that the midpoint of XS lies on the directrix. Hence, X and S are in fact reflections of each other about the positive directrix.

Since P' and P are also reflections of each other about the positive directrix then $XP' = SP$.

Also, $CP = CP'$ and $XC = SC$ due to equal radii.

$$\therefore \triangle SCP \equiv \triangle XCP' \quad (SSS)$$

(iii) Let M be a point on the directrix where it is the midpoint of PP'. Since P' is a reflection of P about the positive directrix then PM is perpendicular to the directrix. Using the locus definition of the hyperbola and the result in part (i)

$$\begin{aligned} \frac{PS}{PM} &= 2 \\ PS &= 2PM \qquad \text{but } PP' = 2PM \\ \Rightarrow PS &= PP' \\ PC & \text{ is common} \\ P'C &= PC \quad \text{(equal radii)} \\ \therefore \triangle PCP' &\equiv \triangle SCP \quad (SSS) \end{aligned}$$

This means that $\triangle XCP'$, $\triangle P'CP$ and $\triangle PCS$ are all congruent, which implies $\angle XCP' = \angle P'CP = \angle PCS$. But

$$\begin{aligned} \angle SCX &= \angle PCX' + \angle PCP' + \angle SCP \\ \therefore \angle SCX &= 3 \times \angle PCP' \end{aligned}$$

(b) (i) Let $y = \dfrac{1}{x} \Rightarrow x = \dfrac{1}{y}$ and substitute into the polynomial equation

$$\left(\frac{1}{y}\right)^4 - a\left(\frac{1}{y}\right)^3 - b\left(\frac{1}{y}\right)^2 - c\left(\frac{1}{y}\right) - d = 0$$
$$dy^4 + cy^3 + by^2 + ay - 1 = 0$$

Hence equation with the roots $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ and $\dfrac{1}{\delta}$ is

$$dx^4 + cx^3 + bx^2 + ax - 1 = 0$$

(ii) If n is a root of $P(x)$ then $\dfrac{1}{n}$ is a root of the equation in (i) so

$$d\left(\frac{1}{n}\right)^4 + c\left(\frac{1}{n}\right)^3 + b\left(\frac{1}{n}\right)^2 + a\left(\frac{1}{n}\right) - 1 = 0$$

But since the coefficients are all positive integers and n is also a positive integer then

$$d\left(\frac{1}{n}\right)^4 + c\left(\frac{1}{n}\right)^3 + b\left(\frac{1}{n}\right)^2 > 0$$
$$d\left(\frac{1}{n}\right)^4 + c\left(\frac{1}{n}\right)^3 + b\left(\frac{1}{n}\right)^2 + a\left(\frac{1}{n}\right) - 1 > a\left(\frac{1}{n}\right) - 1$$
$$\frac{a}{n} - 1 < 0$$
$$n > a$$

(iii) Since n is a positive integer root and $P(x)$ has integer coefficients then n must be a divisor of the constant term $-d$. Since d is a positive integer then

$$\frac{d}{n} \geq 1$$
$$n \leq d \quad \text{but } a \geq d$$
$$\Rightarrow n \leq a$$

This contradicts the inequality in part (ii) and n cannot satisfy both inequalities at the same time. Hence, the initial assumption that $P(x)$ has a positive integer root cannot be be true.

Alternative solution:

Since $a \geq b \geq c \geq d$ then

$$
\begin{aligned}
d\left(\frac{1}{n}\right)^4 + c\left(\frac{1}{n}\right)^3 + b\left(\frac{1}{n}\right)^2 + a\left(\frac{1}{n}\right) &\leq a\left(\frac{1}{n}\right)^4 + a\left(\frac{1}{n}\right)^3 + a\left(\frac{1}{n}\right)^2 + a\left(\frac{1}{n}\right) \\
1 &\leq \frac{a}{n}\left(1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}\right) \\
&= \frac{a}{n}\left(\frac{1 - \frac{1}{n^4}}{1 - \frac{1}{n}}\right) \\
&< \frac{a}{n}\left(\frac{1}{1 - \frac{1}{n}}\right) \\
\Rightarrow n - 1 &< a \\
\Rightarrow n &< a + 1
\end{aligned}
$$

This inequality and the previous inequality imply that n is an integer lying between two consecutive integers. This is a contradiction, so this means the initial assumption that $P(x)$ has a positive integer root cannot be be true.

(c) (i) Writing the sums in full

$$
\sum_{k=1}^{2n-1} \frac{k}{1-\omega^k} = \frac{1}{1-\omega} + \frac{2}{2-\omega^2} + \frac{3}{1-\omega^3} + \frac{4}{1-\omega^4} + \ldots \frac{2n-2}{1-\omega^{2n-2}} + \frac{2n-1}{1-\omega^{2n-1}}
$$

$$
\sum_{k=1}^{n-1} \frac{2k}{1-\omega^{2k}} = \frac{2}{1-\omega^2} + \frac{4}{1-\omega^4} + \frac{6}{1-\omega^6} + \frac{8}{1-\omega^8} + \ldots + \frac{2n-4}{1-\omega^{2n-4}} + \frac{2n-2}{1-\omega^{2n-2}}
$$

$$
\begin{aligned}
\text{Hence RHS} &= \sum_{k=1}^{2n-1} \frac{k}{1-\omega^k} - \sum_{k=1}^{n-1} \frac{2k}{1-\omega^{2k}} \\
&= \frac{1}{1-\omega} + \frac{3}{1-\omega^3} + \frac{5}{1-\omega^5} + \ldots + \frac{2n-1}{1-\omega^{2n-1}} \\
&= \text{LHS}
\end{aligned}
$$

(ii) Consider the general term of the second sum, which can be written as

$$\begin{aligned}\frac{2k}{1-\omega^{2k}} &= \frac{2k}{(1-\omega^k)(1+\omega^k)}\\ &= \frac{k(1-\omega^k+1+\omega^k)}{(1-\omega^k)(1+\omega^k)}\\ &= \frac{k}{1+\omega^k}+\frac{k}{1-\omega^k}\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^{2n-1}\frac{k}{1-\omega^k}-\sum_{k=1}^{n-1}\frac{2k}{1-\omega^{2k}} &= \sum_{k=1}^{2n-1}\frac{k}{1-\omega^k}-\sum_{k=1}^{n-1}\frac{k}{1+\omega^k}-\sum_{k=1}^{n-1}\frac{k}{1-\omega^k}\\ &= \sum_{k=n}^{2n-1}\frac{k}{1-\omega^k}-\sum_{k=1}^{n-1}\frac{k}{1+\omega^k}\end{aligned}$$

(iii) Writing the first sum out in full and noting that $\omega^n=-1$

$$\begin{aligned}\sum_{k=n}^{2n-1}\frac{k}{1-\omega^k} &= \frac{n}{1-\omega^n}+\frac{n+1}{1-\omega^{n+1}}+\frac{n+2}{1-\omega^{n+2}}+...+\frac{2n-1}{1-\omega^{2n-1}}\\ &= \frac{n}{1+\omega^0}+\frac{n+1}{1+\omega^1}+\frac{n+2}{1+\omega^2}+...+\frac{2n-1}{1+\omega^{n-1}}\\ &= \sum_{k=0}^{n-1}\frac{n+k}{1+\omega^k}\end{aligned}$$

$$\begin{aligned}\sum_{k=n}^{2n-1}\frac{k}{1-\omega^k}-\sum_{k=1}^{n-1}\frac{k}{1+\omega^k} &= \sum_{k=0}^{n-1}\frac{n+k}{1+\omega^k}-\sum_{k=1}^{n-1}\frac{k}{1+\omega^k}+\frac{0}{1+\omega^0}\\ &= \sum_{k=0}^{n-1}\frac{n+k}{1+\omega^k}-\sum_{k=0}^{n-1}\frac{k}{1+\omega^k}\\ &= \sum_{k=0}^{n-1}\frac{n+k-k}{1+\omega^k}\\ &= \sum_{k=0}^{n-1}\frac{n}{1+\omega^k}\end{aligned}$$

Hence, using the result from (i)

$$\frac{1}{n}\left[\frac{1}{1-\omega}+\frac{3}{1-\omega^3}+...+\frac{2n-1}{1-\omega^{2n-1}}\right]=1+\frac{1}{1+\omega}+\frac{1}{1+\omega^2}+...+\frac{1}{1+\omega^{n-1}}$$

Question 15

(a) Consider the fact that

$$\begin{aligned} b^3 + c^3 &= (b+c)(b^2 - bc + c^2) \\ &= (b+c)((b+c)^2 - 3bc) \end{aligned}$$

But since a, b and c are the sides of a triangle then $b + c > a$ so

$$\begin{aligned} b^3 + c^3 &> a(a^2 - 3bc) \\ \Rightarrow a^3 + b^3 + c^3 + 3abc &> 2a^3 \\ \therefore \sqrt[3]{\frac{a^3 + b^3 + c^3 + 3abc}{2}} &> a \end{aligned}$$

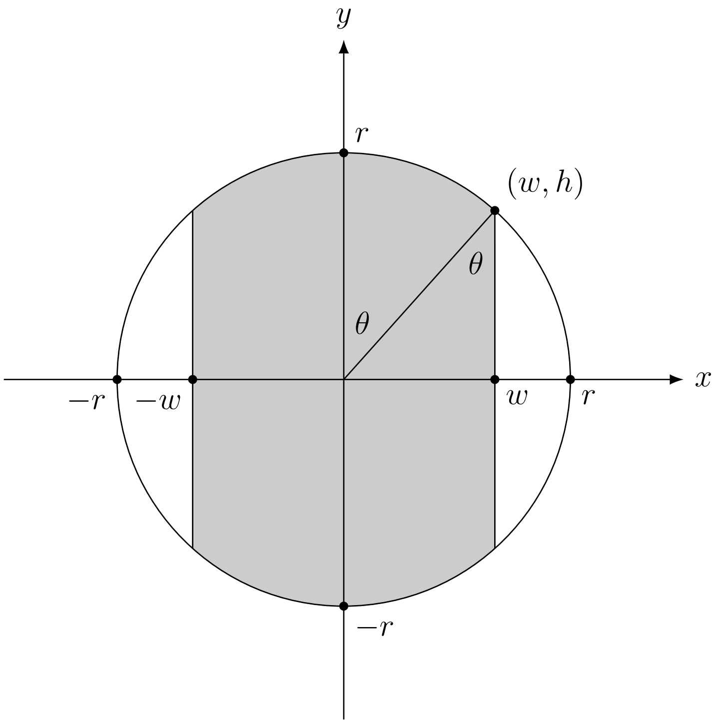
(b) (i) For $n \geq 2$

$$\begin{aligned} I_n + I_{n-2} &= \int_0^{\frac{\pi}{4}} \tan^n x \, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x(\tan^2 x + 1) \, dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx \\ &= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{n-1} \end{aligned}$$

(ii) Using the result in (i), note that

$$\begin{aligned} I_6 + I_4 - (I_4 + I_2) + (I_2 + I_0) &= \frac{1}{5} - \frac{1}{3} + 1 \\ I_6 + I_0 &= \frac{13}{15} \\ I_6 &= \frac{13}{15} - \int_0^{\frac{\pi}{4}} dx \\ &= \frac{13}{15} - \frac{\pi}{4} \end{aligned}$$

(c) Let θ be the acute angle as shown below in the sector



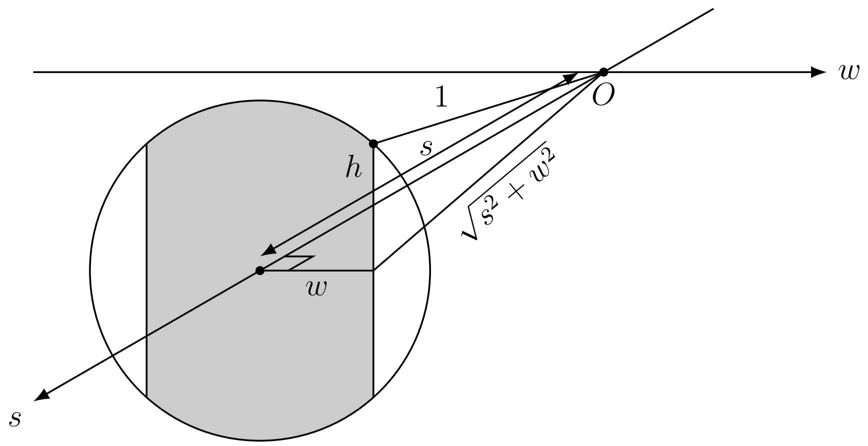
The area of the sector in the first quadrant subtending the angle θ at the centre is $\frac{1}{2}r^2\theta$. The area of the right-angled triangle below it is $\frac{1}{2}wh$. By symmmetry across the four quadrants, the shaded area is given by

$$\begin{aligned} A &= 4 \times \left(\frac{1}{2}r^2\theta + \frac{1}{2}wh \right) \\ &= 2r^2\theta + 2wh \end{aligned}$$

But $\tan\theta = \frac{w}{h}$, or equivalently $\theta = \tan^{-1}\left(\frac{w}{h}\right)$ and $w^2 + h^2 = r^2$ hence

$$A = 2(w^2 + h^2)\tan^{-1}\left(\frac{w}{h}\right) + 2hw$$

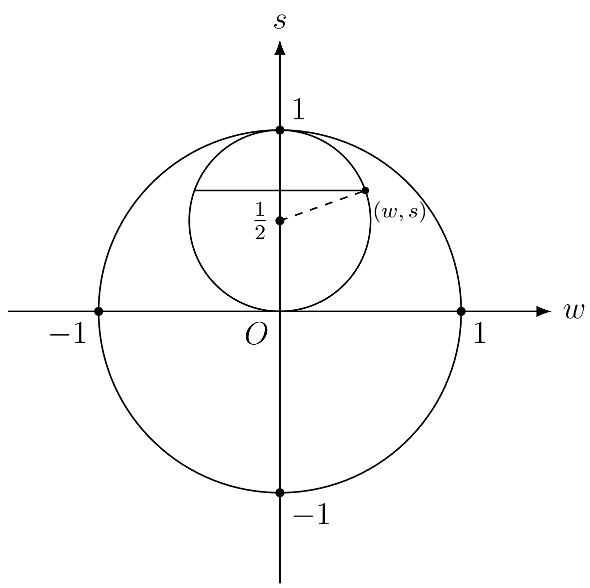
(d) (i) Viewing the slice from the front, it can be seen that s is the distance from the centre of the slice to the centre of the sphere.



The distances h and w are defined similar to part (c). By Pythagoras' theorem, the length of the hypotenuse shown in the diagram to the centre of sphere is $\sqrt{s^2+w^2}$. Since the radius of the sphere is 1 then

$$(\sqrt{s^2+w^2})^2 + h^2 = 1$$
$$\therefore h^2 + s^2 + w^2 = 1$$

(ii) The circular cross-section of the cylinder intersects circular cross-section of the sphere when viewed from above.



Since the radius of the cylinder is $\frac{1}{2}$ and w is half the width of the slice then by Pythagoras' theorem in the cylindrical cross section

$$\left(s - \frac{1}{2}\right)^2 + w^2 = \frac{1}{4}$$

(iii) Making w the subject from part (ii), noting that $w > 0$

$$\begin{aligned} w^2 &= \frac{1}{4} - \left(s - \frac{1}{2}\right)^2 \\ &= \frac{1}{4} - s^2 + s - \frac{1}{4} \\ &= s(1-s) \\ w &= \sqrt{s(1-s)} \end{aligned}$$

Making h the subject from part (i), noting that $h > 0$

$$\begin{aligned} h^2 &= 1 - s^2 - w^2 \\ &= 1 - s^2 - s(1-s) \\ &= 1 - s \\ h &= \sqrt{1-s} \end{aligned}$$

From part (c), the area of the slice is given by

$$\begin{aligned} A &= 2(w^2 + h^2)\tan^{-1}\left(\frac{w}{h}\right) + 2hw \\ &= 2(1-s^2)\tan^{-1}\left(\frac{\sqrt{s(1-s)}}{\sqrt{1-s}}\right) + 2 \times \sqrt{1-s} \times \sqrt{s(1-s)} \\ &= 2(1-s^2)\tan^{-1}\sqrt{s} + 2\sqrt{s}(1-s) \end{aligned}$$

(iv) Let $\theta = \tan^{-1}\sqrt{s} \Rightarrow s = \tan^2\theta \Rightarrow ds = 2\tan\theta\sec^2\theta\, d\theta$.
When $s = 0$ then $\theta = 0$ and when $s = 1$ then $\theta = \frac{\pi}{4}$.

$$\begin{aligned} V &= \int_0^1 A\, ds \\ &= 4\int_0^{\frac{\pi}{4}} \theta(1 - \tan^4\theta)\tan\theta\sec^2\theta\, d\theta + 2\int_0^1 \sqrt{s}(1-s)\, ds \\ &= 4I + 2J \end{aligned}$$

where

$$I = \int_0^{\frac{\pi}{4}} \theta(\tan\theta - \tan^5\theta)\sec^2\theta\, d\theta \quad \text{and} \quad J = \int_0^1 \sqrt{s}(1-s)\, ds$$

Evaluating J first

$$\begin{aligned} J &= \int_0^1 (\sqrt{s} - s\sqrt{s})\, ds \\ &= \left[\frac{2s^{\frac{3}{2}}}{3} - \frac{2s^{\frac{5}{2}}}{5}\right]_0^1 \\ &= \frac{4}{15} \end{aligned}$$

Using integration by parts on I

$$
\begin{aligned}
I &= \left[\theta\left(\frac{\tan^2\theta}{2}-\frac{\tan^6\theta}{6}\right)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \left(\frac{\tan^2\theta}{2}-\frac{\tan^6\theta}{6}\right)\,d\theta \\
&= \frac{\pi}{12}-\frac{1}{2}\int_0^{\frac{\pi}{4}}(\sec^2\theta-1)\,d\theta+\frac{1}{6}\int_0^{\frac{\pi}{4}}\tan^6\theta\,d\theta \\
&= \frac{\pi}{12}-\frac{1}{2}\left[\tan\theta-\theta\right]_0^{\frac{\pi}{4}}+\frac{1}{6}\left(\frac{13}{15}-\frac{\pi}{4}\right) \quad \text{from (b)(ii)} \\
&= \frac{\pi}{12}-\frac{1}{2}+\frac{\pi}{8}+\frac{13}{90}-\frac{\pi}{24} \\
&= \frac{\pi}{6}-\frac{32}{90}
\end{aligned}
$$

Hence

$$
\begin{aligned}
V &= 4\times\left(\frac{\pi}{6}-\frac{32}{90}\right)+2\times\frac{4}{15} \\
&= \frac{2(3\pi-4)}{9} \quad \text{cubic units}
\end{aligned}
$$

Question 16

(a) (i) A circular cross section can be drawn along the surface of the smaller sphere such that S_1 and P_1 are tangents to this circle which intersect at P_0. Since tangents to an external point are equal in length then $P_0P_1 = P_0S_1$.

(ii) Using a similar argument to part (i), $P_0P_2 = P_0S_2$. This means that

$$P_0P_1 + P_0P_2 = P_0S_1 + P_0S_2$$

However, $P_0P_1 + P_0P_2 = P_1P_2$ which is a fixed length independent of the position of P_0 because it is the distance between the fixed circles $\mathscr{C}_1$ and $\mathscr{C}_2$ along the cone.

This means that $P_0S_1 + P_0S_2$ is a constant. Since S_1 and S_2 are fixed points, then the locus of P_0 is precisely the flat ellipse with foci S_1 and S_2.

(b) (i) The angles of $\triangle ABC$ can be found as follows.

$$\begin{aligned}
&\angle ACB = \alpha + \beta \quad \text{(exterior angle of triangle)}\\
&\angle ABC = \alpha - \beta \quad \text{(cone is right so angle to the horizontal is } \alpha\text{)}\\
&\angle BAC = \pi - 2\alpha \quad \text{(angle sum of triangle)}
\end{aligned}$$

Using the sine rule and noting that BC is the major axis of the ellipse so it has length $2a$

$$\begin{aligned}
\frac{AB}{\sin \angle ACB} &= \frac{BC}{\sin \angle BAC}\\
AB &= \frac{2a\sin(\alpha+\beta)}{\sin(\pi - 2\alpha)}\\
&= \frac{2a\sin(\alpha+\beta)}{\sin 2\alpha}
\end{aligned}$$

(ii) Using the sine rule again

$$\begin{aligned}
\frac{AC}{\sin \angle ABC} &= \frac{BC}{\sin \angle BAC}\\
AC &= \frac{2a\sin(\alpha-\beta)}{\sin(\pi - 2\alpha)}\\
&= \frac{2a\sin(\alpha-\beta)}{\sin 2\alpha}
\end{aligned}$$

(iii) Note that the radii are perpendicular to the tangent at the points of contact. The area of $\triangle ABC$ can be found using the radii so

$$\text{Area} = \frac{1}{2} \times r \times AB + \frac{1}{2} \times r \times BC + \frac{1}{2} \times r \times AC$$

However, the area of $\triangle ABC$ can also be expressed as

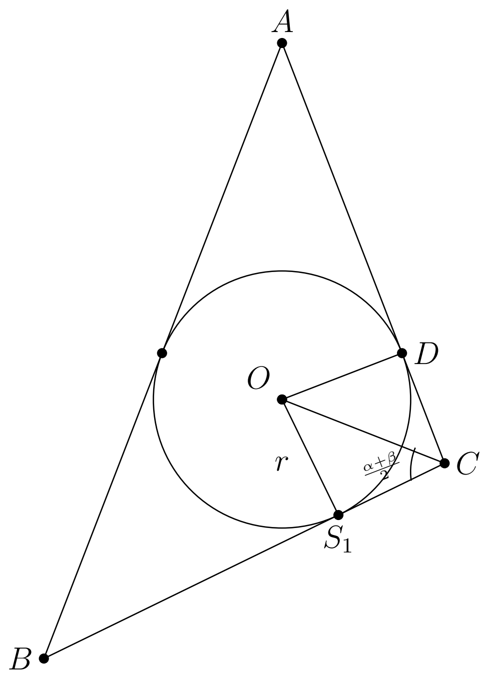
$$\text{Area} = \frac{1}{2} \times AB \times AC \times \sin(\pi - 2\alpha)$$

Equating the two expressions for the area gives

$$\frac{1}{2} \times r \times (AB + BC + AC) = \frac{1}{2} \times AC \times AC \times \sin 2\alpha$$

$$\therefore r = \frac{AC \times AB \sin 2\alpha}{AB + BC + AC}$$

(iv) Let O be the centre of the circle inscribed in $\triangle ABC$ and D be the point of contact of the circle to AC.



$$OS_1 = OD \quad \text{(equal radii)}$$

$$OC \text{ is common}$$

$$S_1C = DC \quad \text{(tangents from an external point are equal)}$$

$$\therefore \triangle OS_1C \equiv \triangle ODC \quad (SSS)$$

$$\Rightarrow \angle OCD = \angle OCS_1 \quad \text{(corresponding angles of congruent triangles)}$$

$$\Rightarrow \angle OCS_1 = \frac{\alpha + \beta}{2}$$

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{r}{CS_1}$$

$$CS_1 = r \cot\left(\frac{\alpha+\beta}{2}\right)$$

But from part (a), it was shown that the smaller sphere touching the sides of the cone touches a focus of the ellipse. This means that circle touching the sides of $\triangle ABC$ touches BC at a focus of the ellipse, which is S_1. Hence

$$\begin{aligned} CS_1 &= BC - BS_1 \\ &= 2a - (a + ae) \\ &= a - ae \end{aligned}$$

Equating the two expressions for CS_1

$$\begin{aligned} a - ae &= r\cot\left(\frac{\alpha+\beta}{2}\right) \\ e &= 1 - \frac{r}{a}\cot\left(\frac{\alpha+\beta}{2}\right) \end{aligned}$$

(v) Starting with the results from (i) and (ii)

$$\begin{aligned} AC \times AB\sin 2\alpha &= \frac{4a^2\sin(\alpha+\beta)\sin(\alpha-\beta)}{\sin 2\alpha} \\ &= \frac{4a^2(\sin^2\alpha\cos^2\beta - \sin^2\beta\cos^2\alpha)}{2\sin\alpha\cos\beta} \\ &= \frac{2a^2(\sin^2\alpha(1-\sin^2\beta) - \sin^2\beta(1-\sin^2\alpha))}{\sin\alpha\cos\alpha} \\ &= \frac{2a^2(\sin^2\alpha - \sin^2\beta)}{\sin\alpha\cos\alpha} \\ &= \frac{2a^2(\sin\alpha+\sin\beta)(\sin\alpha-\sin\beta)}{\sin\alpha\cos\alpha} \\ AB + BC + AC &= \frac{2a\sin(\alpha+\beta)}{\sin 2\alpha} + 2a + \frac{2a\sin(\alpha-\beta)}{\sin 2\alpha} \\ &= \frac{2a(\sin 2\alpha + \sin(\alpha+\beta) + \sin(\alpha-\beta))}{\sin 2\alpha} \\ &= \frac{2a(2\sin\alpha\cos\alpha + 2\sin\alpha\cos\beta)}{2\sin\alpha\cos\alpha} \\ &= \frac{2a\sin\alpha(\cos\alpha+\cos\beta)}{\sin\alpha\cos\alpha} \end{aligned}$$

Substitute this into the result in (iii)

$$\begin{aligned}
r &= \frac{AC \times AB \sin 2\alpha}{AB + BC + AC} \\
&= \frac{a(\sin\alpha + \sin\beta)(\sin\alpha - \sin\beta)}{\sin\alpha(\cos\alpha + \cos\beta)} \\
\frac{r}{a} &= \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} \times \frac{\sin\alpha - \sin\beta}{\sin\alpha} \\
&= \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)} \times \frac{\sin\alpha - \sin\beta}{\sin\alpha} \\
&= \tan\left(\frac{\alpha+\beta}{2}\right) \times \frac{\sin\alpha - \sin\beta}{\sin\alpha}
\end{aligned}$$

Substitute this into the expression in (iv)

$$\begin{aligned}
e &= 1 - \frac{r}{a}\cot\left(\frac{\alpha+\beta}{2}\right) \\
&= 1 - \tan\left(\frac{\alpha+\beta}{2}\right) \times \frac{\sin\alpha - \sin\beta}{\sin\alpha} \times \cot\left(\frac{\alpha+\beta}{2}\right) \\
&= 1 - \frac{\sin\alpha - \sin\beta}{\sin\alpha} \\
&= \frac{\sin\alpha - \sin\alpha + \sin\beta}{\sin\alpha} \\
&= \frac{\sin\beta}{\sin\alpha}
\end{aligned}$$

(c) (i) Since the game ends with either student A or student B winning then

$$P(A \text{ wins}) + P(B \text{ wins}) = 1$$

Since the game is perfectly symmetric then $P(A \text{ wins}) = P(B \text{ wins})$ so $P(A \text{ wins}) = \frac{1}{2}$.

(ii) If the first toss is a heads then student A is ahead of student B by one head. Consider the two cases going forward.

Case 1 - Second toss is a heads

One way for student A to win is for the third toss to also be heads (which gives HH as the next two outcomes). The probability that the second and third toss is heads is $\frac{1}{2} \times \frac{1}{2}$, or equivalently $\frac{1}{4}$.

If the third toss was tails instead, this 'cancels out' the second toss of heads and student A is back to being ahead of student B by one head. The probability of student A eventually winning after that is p by definition. Hence, the probability of student A to win after tossing HT as the next two outcomes is $\frac{1}{2} \times \frac{1}{2} \times p$, or equivalently $\frac{1}{4} \times p$.

Putting the two possibilities together, the probability that student A eventually wins if the first two tosses was heads is $\frac{1}{4} + \frac{1}{4} \times p$.

Case 2 - Second toss is a tails

This 'cancels out' the first toss of heads and student A is back to being tied with student B as if the game was to reset. The probability of student A eventually winning is $\frac{1}{2}$ as explained in part (i).

Hence, the probability that student A eventually wins after tossing T as the next outcome is $\frac{1}{2} \times \frac{1}{2}$, or equivalently $\frac{1}{4}$.

Putting all two cases together, the probability that student A wins the game after the first toss was a heads is given by

$$p = \frac{1}{2} + \frac{1}{4} \times p$$

(iii) Let q be the probability that student B wins if the first toss was a tails. By symmetry of the game, q must also satisfy the condition in (ii)

$$q = \frac{1}{2} + \frac{1}{4} \times q$$

This gives $q = \frac{2}{3}$.

The event that student A wins instead of student B, if the first toss was a tails is the complementary event. This has the probability of $\frac{1}{3}$.