2021 **BORED OF STUDIES TRIAL EXAMINATION**

2nd November

Mathematics Extension 2

General instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5–13)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The complex number z lies on the fourth quadrant of the complex plane. If $\text{Arg}(z) = \theta$, which of the following represents an argument of $-z$?

(A) $\pi - \theta$

(B) $\pi + \theta$

(C) θ

(D) $-\theta$

2 The line ℓ has the vector equation

$$\underset{\sim}{r} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Consider an interval on the line ℓ where $0 \leq \mu \leq 1$.

What is the shortest distance from this interval to the point $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$?

(A) $2\sqrt{2}$

(B) $2\sqrt{3}$

(C) 3

(D) 8

3 Which of the following statements is true?

(A) $\forall x > 0, \exists y < 0 : y > \dfrac{1}{x}$

(B) $\forall x > 0, \exists y < 0 : xy > 1$

(C) $\forall x < 0, \exists y > 0 : y < \dfrac{1}{x}$

(D) $\forall x < 0, \exists y > 0 : xy < 1$

4 Suppose that $z = -1 + i\sqrt{3}$. Which of the following is $\text{Arg}(z^5)$?

(A) $-\dfrac{2\pi}{3}$

(B) $\dfrac{2\pi}{3}$

(C) $\dfrac{10\pi}{3}$

(D) $\dfrac{20\pi}{3}$

5 A particle moves in simple harmonic motion according to the equation $v^2 = f(x)$, where v is particle's velocity at a given displacement x. Suppose that $f(1) = 0, f(5) = 0$ and $f'(2) = 4$. What is the period of the particle's motion?

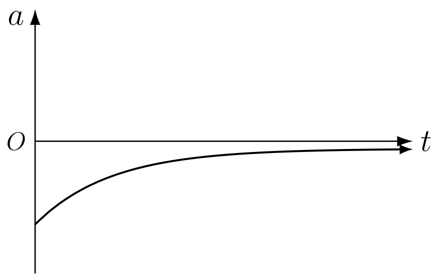
(A) π (B) $\sqrt{2}\pi$

(C) 2π (D) $2\sqrt{2}\pi$

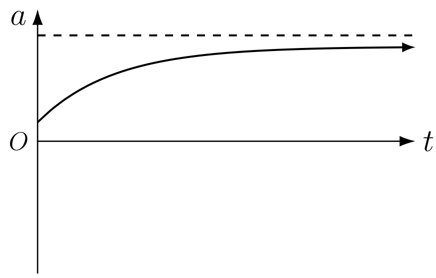
6 An object of mass m moves in a straight line and is subjected to a resistance force that is proportional to its speed. Define the direction of the object as the positive direction.

Let a be the object's acceleration at time t. Which of the following acceleration-time graphs best represents the motion of the object?

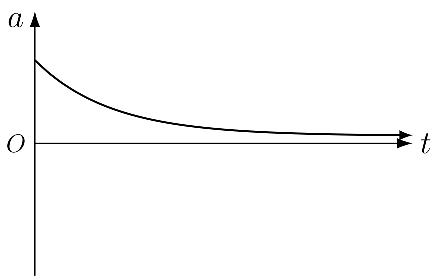
(A)



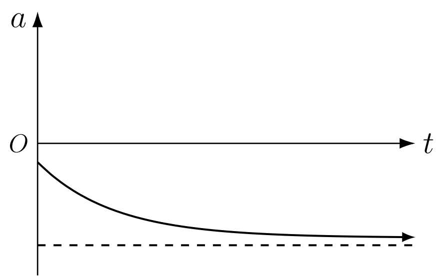
(B)



(C)



(D)



7 Let z be a complex number where $0 < \text{Arg}(z) < \dfrac{\pi}{4}$. Which of the following is correct?

(A) iz lies on the second quadrant and $z - iz$ lies on the first quadrant

(B) iz lies on the second quadrant and $z - iz$ lies on the fourth quadrant

(C) iz lies on the fourth quadrant and $z - iz$ lies on the first quadrant

(D) iz lies on the fourth quadrant and $z - iz$ lies on the second quadrant

8 Suppose that $f(x)$ is a continuous and differentiable function with only one turning point at $x = \epsilon$ over the interval (a, b), where $a < b$.

If $f(a) > f(b)$ and $f'(a) > f'(b)$, which of the following inequalities is always true?

(A) $f(\epsilon) \leq \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \leq f(a)$

(B) $f(a) \leq \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \leq f(\epsilon)$

(C) $f(b) \leq \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \leq f(\epsilon)$

(D) $f(\epsilon) \leq \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \leq f(b)$

9 Consider the proposition

"If p divides ab, then p divides a or p divides b".

Which of the following examples disproves the contrapositive of the proposition?

(A) $p = 31,\ a = 402,\ b = 134$

(B) $p = 61,\ a = 194,\ b = 792$

(C) $p = 91,\ a = 539,\ b = 338$

(D) $p = 49,\ a = 199,\ b = 687$

10 Four points lie on a sphere, represented by the following vectors.

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \underset{\sim}{b} = \frac{1}{3}\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \quad \underset{\sim}{c} = \frac{1}{3}\begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \quad \underset{\sim}{d} = \frac{1}{5}\begin{pmatrix} 2 \\ 11 \\ 0 \end{pmatrix}$$

Which of following vectors represents the centre of the sphere?

(A) $0\underset{\sim}{i} + 0\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$

(B) $0\underset{\sim}{i} + 0\underset{\sim}{j} - 2\underset{\sim}{k}$

(C) $0\underset{\sim}{i} + 0\underset{\sim}{j} + \frac{2}{3}\underset{\sim}{k}$

(D) $0\underset{\sim}{i} + 0\underset{\sim}{j} + 2\underset{\sim}{k}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Use the substitution $u = \dfrac{1}{x}$ to evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{\ln x}{(x^2 - x + 1)^2}\, dx$. **4**

(b) Let a, b, x and y be non-zero real numbers such that

$$(x + iy)^2 = a + ib.$$

(i) Show that **3**

$$x^2 = \frac{\sqrt{a^2 + b^2} + a}{2} \quad \text{and} \quad y^2 = \frac{\sqrt{a^2 + b^2} - a}{2}.$$

(ii) Hence, explain why there are always two *distinct pairs* of values of (x, y). **1**

(c) An object of mass m is dropped from rest and is subjected to a gravitational force mg and a resistance force proportional to v^n, where g is the acceleration due to gravity, v is the velocity of the object at time t and n is a positive integer. Let V_T be the object's terminal velocity, for the given value of n.

(i) Show that when $n = 1$, then $v = V_T\left(1 - e^{-\frac{gt}{V_T}}\right)$. **2**

(ii) Find the velocity of the object when $n = 2$, in terms of g, t and V_T. **3**

(iii) Which scenario ($n = 1$ or $n = 2$) leads to the object approaching its own terminal velocity at a faster rate? Justify your answer using your results in parts (i) and (ii). **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) A child slides down a spiral shaped slide from height h to the ground. Let $\underset{\sim}{r}$ be the position vector of the child at time t seconds, relative to the centre of the slide at ground level. For a given positive constant a, the position vector is given by

$$\underset{\sim}{r} = (a \sin t)\underset{\sim}{i} + (a \cos t)\underset{\sim}{j} + \left(h - \frac{t^2}{4}\right)\underset{\sim}{k}$$

where $\underset{\sim}{i}$, $\underset{\sim}{j}$ are horizontal unit vectors and $\underset{\sim}{k}$ is a unit vector in the vertical direction.

(i) Find the time taken for the child to reach the ground in terms of h. **1**

(ii) When viewed directly from above, the child appears to be moving in a circular motion. Find the exact number of revolutions that the child appears to complete during the slide, in terms of h. **1**

(iii) Find the maximum speed of the child's motion in terms of a and h. **2**

(b) Suppose that for integer $n > 1$ **2**

$$P(z) = z^n \sin\alpha - z \sin n\alpha + \sin(n-1)\alpha,$$

where α is a real number not equal to a multiple of π.

Show that $z^2 - 2z\cos\alpha + 1$ is a factor of $P(z)$.

(c) Let P be a point represented by the position vector **2**

$$\underset{\sim}{p} = r\cos\theta\underset{\sim}{i} + r\sin\theta\underset{\sim}{j} + r\underset{\sim}{k},$$

where r and θ are parameters.

The point P also lies on a particular plane. This plane is perpendicular to the vector $r\cos\theta\underset{\sim}{i} + r\sin\theta\underset{\sim}{j} - r\underset{\sim}{k}$. Show that this plane also passes through the origin.

Question 12 continues on page 7

(d) Let a, b, c and d be positive real numbers. By considering the expansion of **2**

$$(a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2,$$

or otherwise, show that

$$abcd \leq \left(\frac{a+b+c+d}{4}\right)^4.$$

(e) Let $P(x) = x^4 + px^3 + qx^2 + rx + s$ where p, q, r and s are real constants. $P(x)$ has real roots α, β, γ and δ. Consider the following statement.

If α, β, γ and δ are all positive, then $p^4 \geq 256s$.

(i) Using the result from part (d), or otherwise, show that this statement is always true. **1**

(ii) Write down the converse of this statement and explain (using a proof or counterexample) whether this converse is always true. **2**

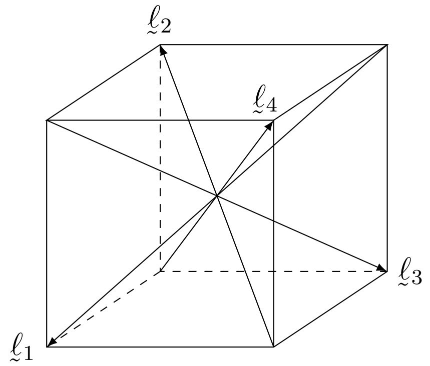
(iii) Write down the contrapositive of the statement. **1**

(iv) Hence, deduce that $x^4 - x^3 - 11x^2 + 9x + 18$ has at least one root that is not positive, given that all of its roots are real. **1**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Let $\underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2, \underset{\sim}{\ell}_3$ and $\underset{\sim}{\ell}_4$ be vectors representing the diagonals that lie inside a cube, as shown in the diagram below. The vectors are placed such that the tips are equally spaced from each other.

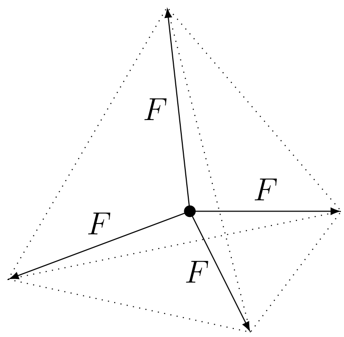


(i) Let $\hat{\underset{\sim}{x}}$ be a unit vector which makes an angle of α, β, γ and δ to the vectors $\underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2, \underset{\sim}{\ell}_3$ and $\underset{\sim}{\ell}_4$ respectively. By considering dot products, show that **3**

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

(ii) Hence, show that the obtuse angle between $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ is $\cos^{-1}\left(-\frac{1}{3}\right)$. **2**

(iii) A particle is subjected to four forces, each of magnitude F, in three dimensional space. The forces are directed such that the tips of their vectors form a regular triangular pyramid. **3**



By first considering projections in the direction of one of the force vectors, and using the result from part (ii), show that the net force on the particle is zero.

Question 13 continues on page 9

(b) An object of mass m is expected to move in simple harmonic motion about the origin with period $\frac{2\pi}{n}$ and amplitude a. It is initially at rest with a displacement of $-a$.

However, the object is instead subjected to both a resistance force of magnitude $mb\dot{x}$ and a positive driving force of magnitude $mc\sin(\omega t)$, where b, c and ω are non-negative constants, and $\dot{x}$ is the velocity of the object at time t.

(i) Find the object's displacement equation in terms of a and n, for $b = c = 0$. **1**

(ii) Now suppose that $b > 0$ and $c > 0$ instead. Explain why the object's acceleration equation is **1**

$$\ddot{x} = -n^2 x - b\dot{x} + c\sin(\omega t).$$

(iii) When $t \geq \frac{2\pi}{\omega}$, the object reaches a steady state such that it moves in simple harmonic motion about the origin with period $\frac{2\pi}{\omega}$ and amplitude A. **3**

If the object is at $x = -A$ when $t = \frac{2\pi}{\omega}$, show that $\omega = n$.

(iv) Hence, deduce that $A = \dfrac{c}{bn}$. **1**

(v) Describe the effect on the object's motion if the magnitude of the resistance force is close to zero. **1**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Given two distinct unit vectors $\hat{\underset{\sim}{m}}$ and $\hat{\underset{\sim}{n}}$, the two lines ℓ_1 and ℓ_2 have the vector equations $\underset{\sim}{r} = \lambda\hat{\underset{\sim}{m}}$ and $\underset{\sim}{r} = \mu\hat{\underset{\sim}{n}}$ respectively, for some parameters λ and μ.

Suppose that for some parameter θ,

$$\hat{\underset{\sim}{m}} = \left(\frac{\cos\theta + 1}{2}\right)\underset{\sim}{i} + \left(\frac{\sin\theta}{\sqrt{2}}\right)\underset{\sim}{j} + \left(\frac{\cos\theta - 1}{2}\right)\underset{\sim}{k}.$$

(i) Find a unit vector $\hat{\underset{\sim}{n}}$ that is independent of θ, such that the angle between ℓ_1 and ℓ_2 is always $\frac{\pi}{4}$. **2**

(ii) The line ℓ_2 intersects the plane $x - z = 4$ at the point C. Find the coordinates of C. **1**

(iii) The line ℓ_1 intersects the plane $x - z = 4$ at the point P. Show that as θ varies, P describes a circle of centre C and radius $2\sqrt{2}$. **3**

(b) Consider the polynomial

$$P(z) = z^3\left(\left(z + \frac{1}{z}\right)^3 + \left(z + \frac{1}{z}\right)^2 - 2\left(z + \frac{1}{z}\right) - 1\right).$$

(i) Show that **3**

$$P(z) = \left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right).$$

(ii) Suppose that the roots of $x^3 + x^2 - 2x - 1$ are α, β and γ. Let **3**

$$A = \sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma}$$
$$B = \sqrt[3]{\alpha\beta} + \sqrt[3]{\beta\gamma} + \sqrt[3]{\gamma\alpha}.$$

Show that $A^3 = 3AB - 4$ and $B^3 = 3AB - 5$.

You may assume the following result without proof:

$$(u + v + w)^3 \equiv u^3 + v^3 + w^3 + 3(u + v)(v + w)(w + u).$$

(iii) Deduce that $AB = 3 - \sqrt[3]{7}$. **1**

(iv) Hence, find the exact value of $\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{6\pi}{7}}$. **2**

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) Suppose that for $0 < \alpha < \frac{\pi}{2}$, $0 < \beta < \frac{\pi}{2}$ and for integer $n \geq 0$

$$s_n = \sum_{k=0}^{n} e^{i(\alpha + k\beta)}.$$

Let P_n be a point on the complex plane represented by the complex number s_n.

(i) Using vectors, show that $\angle OP_0P_1 = \angle P_0P_1P_2$. **3**

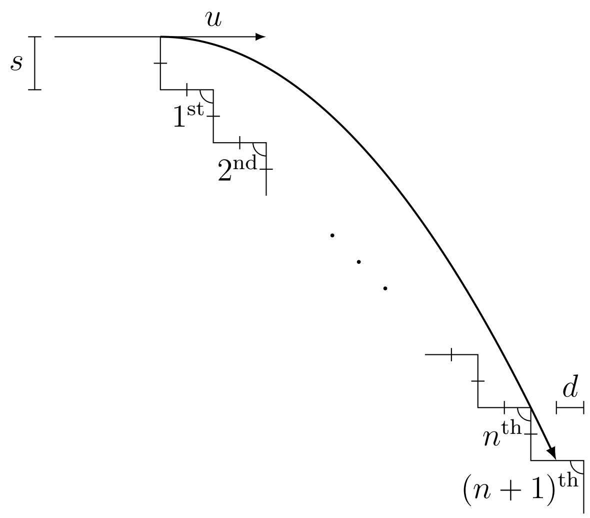
(ii) Suppose that the point P_{n+1} coincides with the origin. Find the maximum number of sides needed for the polygon $OP_0P_1P_2...P_n$, such there is only one possible value of the acute angle β. **2**

(b) Show that for $x \geq 0$ **3**

$$0 \leq \frac{x}{2} + 1 - \sqrt{1+x} \leq \frac{x^2}{8}.$$

Question 15 continues on page 12

(c) A particle is launched horizontally with an initial speed of u at the top of a flight of stairs. Each step has both a width and a height of s.



The particle just clears the edge of the nth step before landing on the $(n+1)$th step. Let d be the horizontal distance from the landing point to the edge of that step.

Let the initial position of the particle be the origin and $\underset{\sim}{r}$ be the displacement vector of the particle at time t given by

$$\underset{\sim}{r} = ut\underset{\sim}{i} - \frac{gt^2}{2}\underset{\sim}{j} \quad \text{(Do NOT prove this)}.$$

(i) Show that $u = \sqrt{\dfrac{nsg}{2}}$. **2**

(ii) Show that $\dfrac{d}{s} = n + 1 - \sqrt{n(n+1)}$. **2**

(iii) Hence, use the inequality in part (b) to show that **1**

$$\frac{1}{2} \leq \frac{d}{s} \leq \frac{1}{2} + \frac{1}{8n}.$$

(iv) Show that when the particle is at the edge of the nth step, the vertical speed is twice the horizontal speed. **1**

(v) Describe the nature of the particle's *path* from the edge of the nth step to its landing point, when $n \to \infty$. **1**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) Let $I_n = \displaystyle\int_0^{\pi} \frac{\sin(nx)}{3 - 2\cos x}\, dx$ for some integer $n \geq 0$. Show that **2**

$$\sum_{k=1}^{2n+1} \left(\frac{3}{2} I_k - \frac{1}{2} I_{k+1} - \frac{1}{2} I_{k-1} \right) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}.$$

(b) Suppose a, b, c and d are real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Show that **2**

$$(1-a)(1-b) \geq cd.$$

(c) Let w be a fixed non-zero complex number and λ be a fixed real number such that $0 < \lambda < 1$. The complex number z satisfies the inequality

$$\left| \frac{z - \lambda w}{\lambda z - w} \right| < 1$$

where $z \neq \lambda w$ and $\lambda z \neq w$.

(i) Show that the complex number $\dfrac{z}{w}$ lies inside the unit circle. **2**

(ii) Hence, sketch the locus of z when $|w| = 1$. **1**

(d) Let n be a positive integer and λ be a real number such that $0 < \lambda < 1$. Let

$$P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \lambda^{k(n-k)} z^k.$$

(i) Show that **2**

$$P_{n+1}(z) = z\lambda^n P_n\left(\frac{z}{\lambda}\right) + P_n(\lambda z).$$

(ii) Suppose that all the roots of $P_n(z)$ have a modulus of one and that $P_{n+1}(z)$ has a root of modulus less than one. **3**

Using the results from part (c), show that there exists a contradiction.

(iii) Hence, prove by mathematical induction that for all positive integers n, all the roots of $P_n(z)$ have a modulus of one. **3**

End of paper

2021 BORED OF STUDIES TRIAL EXAMINATION

Mathematics Extension 2

Solutions

Section I

Answers

1	B	**6**	A
2	C	**7**	B
3	D	**8**	C
4	A	**9**	C
5	B	**10**	B

Brief explanations

1 If the principal argument of z is in the fourth quadrant, then the argument of $-z$ must be in the second quadrant. Since θ is a negative acute angle, then the only option which lies in the second quadrant is $\pi + \theta$, which is (B).

2 Let d be the distance, then

$$\begin{aligned} d^2 &= (-1 + \mu - 1)^2 + (4 - 2)^2 + (3 - 5)^2 \\ &= (\mu - 2)^2 + 8 \end{aligned}$$

As μ varies, this gives a parabola with a minimum turning point at $(2, 8)$.

However, there is a domain restriction that $0 \leq \mu \leq 1$. Note that d^2 is strictly decreasing in that domain, so the minimum of d^2 (and consequently the minimum of d) occurs at the endpoint $\mu = 1$. This gives $d = 3$, hence the answer is (C).

3 (A) implies that a negative value is greater than a positive value, which is false.

(B) implies that a negative value (xy) is greater than 1, which is false.

(C) implies that a positive value is less than a negative value, which is false.

(D) implies that a negative value (xy) is less than 1, which is true.

Hence, the answer is (D).

4 The principal argument is $\text{Arg}(z) = \dfrac{2\pi}{3}$, hence $\arg(z^5) = \dfrac{10\pi}{3}$.

To get the principal argument, $\text{Arg}(z^5) = -\dfrac{2\pi}{3}$, so the answer is (A).

5 The particle is at rest at $x = 1$ and $x = 5$, which suggests that the centre of motion is at $x = 3$. Hence, the acceleration equation has the form

$$\begin{aligned} \frac{d}{dx}\left(\frac{v^2}{2}\right) &= -n^2(x-3) \\ \frac{f'(x)}{2} &= -n^2(x-3) \\ \frac{f'(2)}{2} &= n^2 \\ n &= \sqrt{2} \quad \text{for } n > 0 \end{aligned}$$

The period is $\frac{2\pi}{n}$, hence the answer is (B).

6 The net force on the object is just the resistance force against the motion of the object. Hence, its acceleration a must always be negative.

Since the resistance force is proportional to the speed then the acceleration equation of the object is given by

$$\frac{dv}{dt} = -kv$$

for some positive constant k.

This differential equation represents an exponential decay with general solution $v = Ae^{-kt}$ for some positive constant A (defining the direction of motion as the positive direction). This means that $a = -kAe^{-kt}$, so as $t \to \infty$ then $a \to 0$. Hence, the answer is (A).

7 Since $0 < \text{Arg}(z) < \dfrac{\pi}{4}$ and $\text{Arg}(iz) = \dfrac{\pi}{2} + \text{Arg}(z)$ then iz must lie in the second quadrant.

Since $z - iz = (1-i)z$, then $\text{Arg}(z - iz) = \text{Arg}(1-i) + \text{Arg}(z) = \text{Arg}(z) - \dfrac{\pi}{4}$.

This means that $-\dfrac{\pi}{4} < \text{Arg}(z - iz) < 0$, which is the fourth quadrant. Hence, the answer is (B).

Remark: This can also be done by subtraction of vectors, but a distinction needs to be made between the vector *representing* the complex number (which has no defined position, just a direction and magnitude) versus the *position* vector which indicates the location of the complex number.

8 Since $a < \epsilon < b$ and there is only one turning point at $x = \epsilon$ then $f'(a)$ and $f'(b)$ must be opposite signs. Since $f'(a) > f'(b)$ then the turning point must be going from positive to negative gradient, implying a maximum point. This implies that options (A) and (D) cannot be possible.

(B) can be shown as not always true using a counterexample, such as $f(x) = -x^2$ with $a = -1$ and $b = 3$.

The result in (C) can in fact be proven. Since $f(a) > f(b)$ then $f(b)$ is the minimum value and $f(\epsilon)$ is the maximum value in the interval $[a, b]$, hence

$$\begin{aligned} f(b) < f(x) &< f(\epsilon) \\ \int_a^b f(b)\,dx < \int_a^b f(x)\,dx &< \int_a^b f(\epsilon)\,dx \\ (b-a)f(b) < \int_a^b f(x)\,dx &< (b-a)f(\epsilon) \\ f(b) < \frac{1}{b-a}\int_a^b f(x)\,dx &< f(\epsilon) \quad \text{noting that } a < b \end{aligned}$$

Hence, the answer is (C).

9 The contrapositive of the proposition is

'If p does not divide both a and b, then p does not divide ab.'

For an example to disprove the contrapositive of the proposition, it must not divide both a and b, but divides ab. The only choice this occurs is in option (C).

10 From the given options available, it can deduced that the centre of the sphere is given by the position vector $0\underset{\sim}{i} + 0\underset{\sim}{j} + z_0\underset{\sim}{k}$, where z_0 is based on the choices. The equation of the sphere is therefore $x^2 + y^2 + (z - z_0)^2 = r^2$ where z_0 and r are unknown.

Substitute $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j}$

$$\begin{aligned} 1^2 + 2^2 + (0 - z_0)^2 &= r^2 \\ 5 + z_0^2 &= r^2 \end{aligned}$$

Substitute $\underset{\sim}{c} = \frac{4}{3}\underset{\sim}{i} + \frac{4}{3}\underset{\sim}{j} + \frac{1}{3}\underset{\sim}{k}$

$$\begin{aligned} 4^2 + 4^2 + (1 - 3z_0)^2 &= 9r^2 \\ 33 - 6z_0 + 9z_0^2 &= 9r^2 \\ 11 - 2z_0 + 3z_0^2 &= 3(5 + z_0^2) \\ -2z_0 &= 4 \\ z_0 &= -2 \end{aligned}$$

Hence, the centre is (B).

Question 11

(a) Let $u = \dfrac{1}{x} \Rightarrow dx = -\dfrac{1}{u^2}\,du$. When $x = \frac{1}{2}$ then $u = 2$ and when $x = 2$ then $u = \frac{1}{2}$.

$$\begin{aligned}\int_{\frac{1}{2}}^{2} \frac{\ln x}{(x^2 - x + 1)^2}\,dx &= -\int_{2}^{\frac{1}{2}} \frac{\ln\left(\frac{1}{u}\right)}{\left(\frac{1}{u^2} - \frac{1}{u} + 1\right)^2} \times \frac{1}{u^2}\,du \\ &= -\int_{\frac{1}{2}}^{2} \frac{u^2 \ln u}{(1 - u + u^2)^2}\,du \\ &= -\int_{\frac{1}{2}}^{2} \frac{x^2 \ln x}{(1 - x + x^2)^2}\,dx \qquad \text{as } u \text{ is a dummy variable}\end{aligned}$$

Hence

$$\begin{aligned}2\int_{\frac{1}{2}}^{2} \frac{\ln x}{(x^2 - x + 1)^2}\,dx &= \int_{\frac{1}{2}}^{2} \frac{\ln x}{(x^2 - x + 1)^2}\,dx - \int_{\frac{1}{2}}^{2} \frac{x^2 \ln x}{(1 - x + x^2)^2}\,dx \\ &= \int_{\frac{1}{2}}^{2} \frac{(1 - x^2)\ln x}{(x^2 - x + 1)^2} \times \frac{\frac{1}{x^2}}{\frac{1}{x^2}}\,dx \\ &= \int_{\frac{1}{2}}^{2} \frac{\left(\frac{1}{x^2} - 1\right)\ln x}{\left(x - 1 + \frac{1}{x}\right)^2}\,dx\end{aligned}$$

Using integration by parts, let

$$u = \ln x \Rightarrow du = \frac{1}{x}\,dx \text{ and } dv = \frac{\frac{1}{x^2} - 1}{\left(x - 1 + \frac{1}{x}\right)^2}\,dx \Rightarrow v = \frac{1}{x - 1 + \frac{1}{x}}$$

$$\begin{aligned}2\int_{\frac{1}{2}}^{2} \frac{\ln x}{(x^2 - x + 1)^2}\,dx &= \left[\frac{\ln x}{x - 1 + \frac{1}{x}}\right]_{\frac{1}{2}}^{2} - \int_{\frac{1}{2}}^{2} \frac{1}{x} \times \frac{1}{x - 1 + \frac{1}{x}}\,dx \\ &= \frac{4\ln 2}{3} - \int_{\frac{1}{2}}^{2} \frac{1}{x^2 - x + 1}\,dx \\ &= \frac{4\ln 2}{3} - \int_{\frac{1}{2}}^{2} \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\,dx \\ &= \frac{4\ln 2}{3} - \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2(x - \frac{1}{2})}{\sqrt{3}}\right)\right]_{\frac{1}{2}}^{2} \\ &= \frac{4\ln 2}{3} - \frac{2\pi}{3\sqrt{3}}\end{aligned}$$

$$\therefore \int_{\frac{1}{2}}^{2} \frac{\ln x}{(x^2 - x + 1)^2}\,dx = \frac{2\ln 2}{3} - \frac{\pi}{3\sqrt{3}}$$

(b) (i) Equating imaginary parts gives $2xy = b \Rightarrow y = \dfrac{b}{2x}$.

Equating real parts gives

$$\begin{aligned} x^2 - y^2 &= a \\ x^2 - \frac{b^2}{4x^2} &= a \\ 4x^4 - 4ax^2 - b^2 &= 0 \\ (2x^2 - a)^2 &= a^2 + b^2 \\ x^2 &= \frac{\pm\sqrt{a^2+b^2} + a}{2} \end{aligned}$$

Note that

$$\begin{aligned} \sqrt{a^2+b^2} &> \sqrt{a^2} \\ &= |a| \\ &\geq a \\ \Rightarrow a - \sqrt{a^2+b^2} &< 0 \end{aligned}$$

This means that taking the negative root gives a negative value for x^2 which is not possible as x is real. Hence, only the positive root is valid so

$$x^2 = \frac{\sqrt{a^2+b^2} + a}{2}$$

and

$$\begin{aligned} y^2 &= x^2 - a \\ &= \frac{\sqrt{a^2+b^2} - a}{2} \end{aligned}$$

(ii) The values of x and y are

$$x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} \quad \text{and} \quad y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}.$$

This initially suggests that there may be four possible combinations of (x, y). However, note that $2xy = b$, where the sign of b is *fixed*. That is, b cannot be positive and negative at the same time.

If $b > 0$ then x and y are either both negative or both positive. If $b < 0$ then x and y must have opposite signs. These two scenarios cannot hold simultaneously, so there must always be two distinct pairs of (x, y).

(c) (i) Defining the direction of motion as positive, let the constant of proportionality be mk where m is the mass of the object and k is a positive constant. For $n = 1$, the net force is given by

$$\begin{aligned}
m\ddot{x} &= mg - mkv \\
\frac{dv}{dt} &= g - kv \\
t &= \int \frac{dv}{g - kv} \\
&= -\frac{1}{k}\ln(g - kv) + c \quad \text{when } t = 0, v = 0 \text{ so } c = \frac{1}{k}\ln g \\
&= -\frac{1}{k}\ln\left(\frac{g - kv}{g}\right) \\
e^{-kt} &= \frac{g - kv}{g} \\
v &= \frac{g}{k}\left(1 - e^{-kt}\right)
\end{aligned}$$

The terminal velocity occurs when $\ddot{x} = 0$ so $V_T = \frac{g}{k}$, hence $v = V_T\left(1 - e^{-\frac{gt}{V_T}}\right)$.

(ii) Applying the same approach for $n = 2$

$$\begin{aligned}
m\ddot{x} &= mg - mkv^2 \\
\frac{dv}{dt} &= g - kv^2 \\
t &= \int \frac{dv}{g - kv^2} \\
&= \frac{1}{2\sqrt{g}} \int \frac{2\sqrt{g}}{(\sqrt{g} + \sqrt{k}v)(\sqrt{g} - \sqrt{k}v)}\, dv \\
&= \frac{1}{2\sqrt{g}} \int \frac{\sqrt{g} - \sqrt{k}v + \sqrt{g} + \sqrt{k}v}{(\sqrt{g} + \sqrt{k}v)(\sqrt{g} - \sqrt{k}v)}\, dv \\
&= \frac{1}{2\sqrt{g}} \int \left(\frac{1}{\sqrt{g} + \sqrt{k}v} + \frac{1}{\sqrt{g} - \sqrt{k}v}\right) dv \\
&= \frac{1}{2\sqrt{kg}}\left(\ln(\sqrt{g} + \sqrt{k}v) - \ln(\sqrt{g} - \sqrt{k}v)\right) + c \quad \text{when } t = 0, v = 0 \text{ so } c = 0 \\
&= \frac{1}{2\sqrt{kg}} \ln\left(\frac{\sqrt{g} + \sqrt{k}v}{\sqrt{g} - \sqrt{k}v}\right) \\
&= -\frac{1}{2\sqrt{kg}} \ln\left(\frac{\sqrt{g} - \sqrt{k}v}{\sqrt{g} + \sqrt{k}v}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
e^{-2t\sqrt{kg}} &= \frac{\sqrt{g}-\sqrt{k}v}{\sqrt{g}+\sqrt{k}v} \\
\sqrt{g}e^{-2t\sqrt{kg}} + \sqrt{k}ve^{-2t\sqrt{kg}} &= \sqrt{g}-\sqrt{k}v \\
\sqrt{k}v(1+e^{-2t\sqrt{kg}}) &= \sqrt{g}(1-e^{-2t\sqrt{kg}}) \\
v &= \sqrt{\frac{g}{k}}\left(\frac{1-e^{-2t\sqrt{kg}}}{1+e^{-2t\sqrt{kg}}}\right)
\end{aligned}$$

The terminal velocity occurs when $\ddot{x}=0$ so $V_T=\sqrt{\frac{g}{k}}$, hence $v=V_T\left(\frac{1-e^{-\frac{2gt}{V_T}}}{1+e^{-\frac{2gt}{V_T}}}\right)$

(iii) Factorising the result for $n=2$

$$\begin{aligned}
v &= V_T\left(\frac{1-e^{-\frac{2gt}{V_T}}}{1+e^{-\frac{2gt}{V_T}}}\right) \\
&= V_T\left(1-e^{-\frac{gt}{V_T}}\right)\times\left(\frac{1+e^{-\frac{gt}{V_T}}}{1+e^{-\frac{2gt}{V_T}}}\right)
\end{aligned}$$

However, for $t \geq 0$ note that $\frac{1+e^{-\frac{gt}{V_T}}}{1+e^{-\frac{2gt}{V_T}}} > 1$ since $e^{-\frac{gt}{V_T}} > e^{-\frac{2gt}{V_T}}$.

This means that the result for $n=2$, has a similar 'structure' to its velocity equation as the $n=1$ case (relative to their own terminal velocity), except there is another multiplicative factor which 'inflates' the velocity. This suggests that the scenario $n=2$ will approach its own terminal velocity at a faster rate.

Question 12

(a) (i) At $t = 0$, the child is initially at height h so the child reaches the ground when the z-component, which represents the vertical height, is zero.

$$\begin{aligned} h - \frac{t^2}{4} &= 0 \\ t &= 2\sqrt{h} \quad \text{noting that } t \geq 0 \end{aligned}$$

(ii) The x and y components represent the parametric equation of a circle with radius a. These components have a period of 2π, which represents the time taken to complete a full revolution of the circle (i.e. the time taken for (x, y) to return back to the same pair of values).

Hence, the exact number of revolutions is $\dfrac{2\sqrt{h}}{2\pi}$, or equivalently $\dfrac{\sqrt{h}}{\pi}$.

(iii) The velocity vector is given by $\dot{\underset{\sim}{r}} = (a\cos t)\underset{\sim}{i} - (a\sin t)\underset{\sim}{j} - \dfrac{t}{2}\underset{\sim}{k}$.

The speed is the length of the velocity vector, so consider the squared length

$$\begin{aligned} |\dot{\underset{\sim}{r}}|^2 &= a^2\cos^2 t + a^2\sin^2 t + \frac{t^2}{4} \\ &= a^2 + \frac{t^2}{4} \end{aligned}$$

As t varies the curve of $|\dot{\underset{\sim}{r}}|^2$ describes a parabola with minimum turning point at $(0, a^2)$. Since $0 \leq t \leq 2\sqrt{h}$, then $|\dot{\underset{\sim}{r}}|^2$ is non-decreasing in this domain.

Hence, the largest value of $|\dot{\underset{\sim}{r}}|^2$ occurs at the endpoint $t = 2\sqrt{h}$. This means that the maximum speed is $\sqrt{a^2 + h}$.

(b) Note that $z^2 - 2z\cos\alpha + 1 = (z - e^{i\alpha})(z - e^{-i\alpha})$ so it is sufficient to show that $z = e^{i\alpha}$ and $z = e^{-i\alpha}$ are roots of $P(z)$.

$$\begin{aligned} P(e^{i\alpha}) &= e^{in\alpha}\sin\alpha - e^{i\alpha}\sin n\alpha + \sin(n-1)\alpha \\ &= \cos n\alpha\sin\alpha + i\sin n\alpha\sin\alpha - \cos\alpha\sin n\alpha - i\sin\alpha\sin n\alpha + \sin(n-1)\alpha \\ &= \sin(n-1)\alpha - \sin(n-1)\alpha \\ &= 0 \end{aligned}$$

Therefore, $z = e^{i\alpha}$ is a root of $P(z)$. Since $P(z)$ has real coefficients, then the conjugate $z = e^{-i\alpha}$ is also a root of $P(z)$. Hence, $z = e^{i\alpha}$ and $z = e^{-i\alpha}$ are the roots of $P(z)$ which implies that $z^2 - 2z\cos\alpha + 1$ is a factor of $P(z)$.

(c) Let $\underset{\sim}{q} = r\cos\theta\underset{\sim}{i} + r\sin\theta\underset{\sim}{j} - r\underset{\sim}{k}$ and let $\underset{\sim}{s} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ be a point on the plane. The equation of the plane can be found given the vector $\underset{\sim}{p} - \underset{\sim}{s}$ is perpendicular to $\underset{\sim}{q} - \underset{\sim}{s}$.

$$\begin{aligned}(\underset{\sim}{p} - \underset{\sim}{s}) \cdot (\underset{\sim}{q} - \underset{\sim}{s}) &= 0\\ (r\cos\theta - x)(r\cos\theta - x) + (r\sin\theta - y)(r\sin\theta - y) + (r - z)(-r - z) &= 0\\ (r\cos\theta - x)^2 + (r\sin\theta - y)^2 - (r^2 - z^2) &= 0\end{aligned}$$

To test whether the origin lies on this plane

$$\begin{aligned}LHS &= (r\cos\theta - x)^2 + (r\sin\theta - y)^2 - (r^2 - z^2) \quad \text{substitute } (0,0,0)\\ &= r^2\cos^2\theta + r^2\sin^2\theta - r^2\\ &= 0\\ &= RHS\end{aligned}$$

Hence, the plane also passes through the origin.

(d) Use the fact that

$$\begin{aligned}(a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2 &\geq 0\\ a^4 - 2a^2b^2 + b^4 + c^4 - 2c^2d^2 + d^4 + 2(a^2b^2 - 2abcd + c^2d^2) &\geq 0\\ a^4 + b^4 + c^4 + d^4 - 4abcd &\geq 0\\ \left(\frac{a^4 + b^4 + c^4 + d^4}{4}\right)^4 &\geq (abcd)^4\end{aligned}$$

Replace a^4 with a, b^4 with b, c^4 with c and d^4 with d which gives

$$abcd \leq \left(\frac{a + b + c + d}{4}\right)^4.$$

(e) (i) From the result in part (d), let $a = \alpha, b = \beta, c = \gamma$ and $d = \delta$. If α, β, γ and δ are positive then

$$\begin{aligned}\alpha\beta\gamma\delta &\leq \left(\frac{\alpha + \beta + \gamma + \delta}{4}\right)^4\\ s &\leq \left(\frac{-p}{4}\right)^4\\ p^4 &\geq 256s\end{aligned}$$

Hence, the statement is always true.

(ii) The converse statement is

$$\text{If } p^4 \geq 256s, \text{ then } \alpha, \beta, \gamma \text{ and } \delta \text{ are all positive.}$$

A counterexample is $\alpha = 1, \beta = 2, \gamma = 3, \delta = -1$, then $p = -5$ and $s = -6$. Clearly $p^4 \geq 256s$, but not all the roots are positive. Hence, the converse statement is not always true.

Remark: Counterexamples can be generated for any $s < 0$, because this means $p^4 \geq 256s$ will always be true, since $p^4 \geq 0$. Since s is the product of roots, there must be either one or three negative roots to give $s < 0$.

(iii) The contrapositive statement is

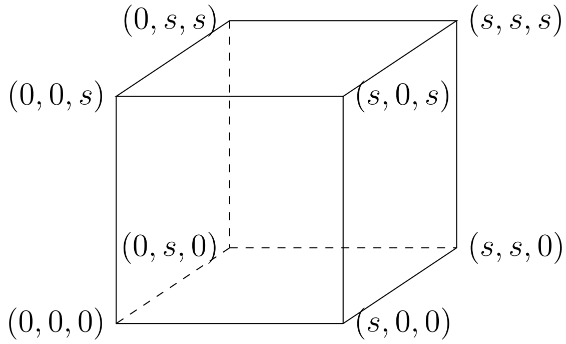
$$\text{If } p^4 < 256s, \text{ then not all of } \alpha, \beta, \gamma \text{ and } \delta \text{ are positive.}$$

(iv) The polynomial $x^4 - x^3 - 11x^2 + 9x + 18$ has the values $p = -1$ and $s = 18$. In this case, $p^4 < 256s$. Since the original statement is always true as proven in part (i), then the contrapositive must also be always true. Hence, there must be at least one root that is not positive for $x^4 - x^3 - 11x^2 + 9x + 18$.

Question 13

(a) (i) Let s be the side length of the cube. Suppose that the tip of the vector $\underset{\sim}{\ell}_1$ in the diagram is the origin of the three dimensional space and the edges of the cube lie on the positive direction of the axes.

The results still hold without loss of generality (i.e. where the origin is (x_0, y_0, z_0) more generally) because the vectors have no defined position, only a direction and magnitude. The coordinates of the vertices of the cube are shown below.



This means that the vectors of the respective diagonals are represented as

$$\underset{\sim}{\ell}_1 = -s\underset{\sim}{i} - s\underset{\sim}{j} - s\underset{\sim}{k} \qquad \underset{\sim}{\ell}_2 = -s\underset{\sim}{i} + s\underset{\sim}{j} + s\underset{\sim}{k}$$
$$\underset{\sim}{\ell}_3 = s\underset{\sim}{i} + s\underset{\sim}{j} - s\underset{\sim}{k} \qquad \underset{\sim}{\ell}_4 = s\underset{\sim}{i} - s\underset{\sim}{j} + s\underset{\sim}{k}$$

Since $\hat{\underset{\sim}{x}}$ is an arbitrary unit vector, let $\hat{\underset{\sim}{x}} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ where $\sqrt{a^2+b^2+c^2} = 1$.

$$\begin{aligned}
\cos^2\alpha &= \left(\frac{\hat{\underset{\sim}{x}} \cdot \underset{\sim}{\ell}_1}{|\hat{\underset{\sim}{x}}||\underset{\sim}{\ell}_1|}\right)^2 \\
&= \frac{(-as - bs - cs)^2}{s^2+s^2+s^2} \\
&= \frac{(-a-b-c)^2}{3} \\
&= \frac{a^2+b^2+c^2+2(ab+bc+ac)}{3} \qquad \text{but } \sqrt{a^2+b^2+c^2} = 1 \\
&= \frac{1+2(ab+bc+ac)}{3}
\end{aligned}$$

Similarly

$$\begin{aligned}
\cos^2\beta &= \frac{1+2(-ab+bc-ac)}{3} \\
\cos^2\gamma &= \frac{1+2(ab-bc-ac)}{3} \\
\cos^2\delta &= \frac{1+2(-ab-bc+ac)}{3}
\end{aligned}$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \dfrac{4}{3}$.

(ii) Let $\underset{\sim}{\hat{x}}$ be a specific unit vector that is parallel to $\underset{\sim}{\ell}_1$. This means that $\alpha = 0$. By symmetry, the angle between $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ is equal to angle between $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_3$ as well as $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_4$. This means that $\gamma = \beta$ and $\delta = \beta$.

Substitute these into the result from part (i)

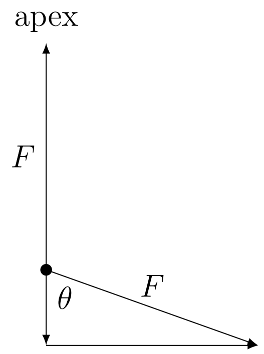
$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta &= \frac{4}{3}\\ 1 + 3\cos^2\beta &= \frac{4}{3}\\ \cos^2\beta &= \frac{1}{9}\\ \cos\beta &= \pm\frac{1}{3}\end{aligned}$$

Since β is an obtuse angle, then the negative root is taken.

Hence, the obtuse angle between $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ is $\cos^{-1}\left(-\frac{1}{3}\right)$.

(iii) Notice that the tips of the vectors $\underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2, \underset{\sim}{\ell}_3$ and $\underset{\sim}{\ell}_4$ make up the vertices of the regular triangular pyramid.

Consider the projection of three of the force vectors onto the direction of the fourth force vector (say the apex of the triangular pyramid). For each of those three force vectors, let θ be the acute angle as shown in the diagram below.

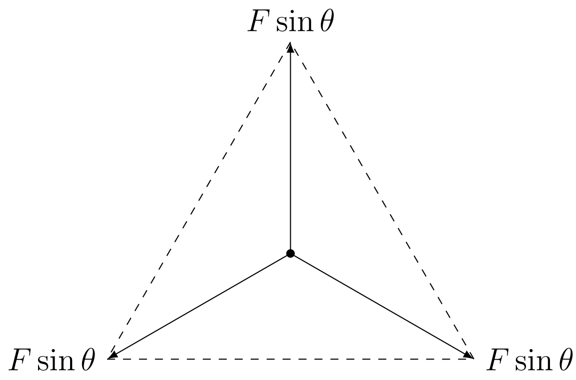


Note that the obtuse angle between two adjacent force vectors is $\cos^{-1}\left(-\frac{1}{3}\right)$ from part (ii).

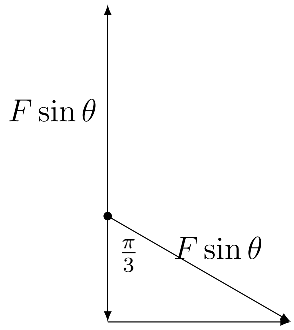
The net force in the direction of the apex (call this 'z' direction) is given by

$$\begin{aligned}F_z &= F - 3F\cos\theta\\ &= F - 3F\cos\left(\pi - \cos^{-1}\left(-\frac{1}{3}\right)\right)\\ &= F + 3F\cos\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)\\ &= 0\end{aligned}$$

Now consider the remaining components which are projected onto the base of the triangular pyramid.



Applying a similar approach, projecting two force vectors onto the third force vector (say the 'vertical' vector in above diagram)



The net force in the direction of the 'vertical' force vector (call this 'y' direction) is given by

$$\begin{aligned} F_y &= F\sin\theta - 2F\sin\theta\cos\frac{\pi}{3} \\ &= 0 \end{aligned}$$

The net force in the remaining direction (call this 'x' direction) is given by

$$\begin{aligned} F_x &= F\sin\theta\sin\frac{\pi}{3} - F\sin\theta\sin\frac{\pi}{3} \\ &= 0 \end{aligned}$$

Since all perpendicular components of the force on the particle are zero, then the overall net force on the particle must be zero.

(b) (i) If $b = c = 0$ then the object moves in simple harmonic motion about the origin with the initial displacement at $x = -a$. For some constant α, the displacement equation has the form

$$\begin{aligned} x &= a\cos(nt+\alpha) \quad \text{when } t = 0, x = -a \\ -a &= a\cos\alpha \\ \alpha &= \pi \\ x &= a\cos(nt+\pi) \end{aligned}$$

(ii) The force on the object causing the simple harmonic motion is $-mn^2x$, with resistance force in the negative direction of magnitude $mb\dot{x}$ and a driving force in the positive direction magnitude $mc\sin(\omega t)$. The net force is given by

$$\begin{aligned} m\ddot{x} &= -mn^2x - mb\dot{x} + mc\sin(\omega t) \\ \ddot{x} &= -n^2x - b\dot{x} + c\sin(\omega t) \end{aligned}$$

(iii) When $t \geq \frac{2\pi}{\omega}$, the object stabilises into a simple harmonic motion with the general displacement equation

$$\begin{aligned} x &= A\cos(\omega t + \beta) \quad \text{when } t = \frac{2\pi}{\omega}, x = -A \\ -A &= A\cos\beta \\ \beta &= \pi \\ x &= A\cos(\omega t + \pi) \end{aligned}$$

This displacement equation must be a solution to the differential equation in part (ii) for $t \geq \frac{2\pi}{\omega}$, so substituting this in gives

$$-A\omega^2\cos(\omega t + \pi) = -An^2\cos(\omega t + \pi) + Ab\omega\sin(\omega t + \pi) + c\sin(\omega t)$$

This is true for all $t \geq \frac{2\pi}{\omega}$. Substitute $t = \frac{2\pi}{\omega}$ which gives

$$\begin{aligned} -A\omega^2\cos(3\pi) &= -An^2\cos(3\pi) + Ab\omega\sin(3\pi) + c\sin(2\pi) \\ \omega^2 &= n^2 \\ \omega &= n \quad \text{noting that } \omega \geq 0 \end{aligned}$$

(iv) Substituting the displacement equation into the differential equation for $t \geq \frac{2\pi}{\omega}$

$$\begin{aligned} -A\omega^2\cos(\omega t + \pi) &= -An^2\cos(\omega t + \pi) + Ab\omega\sin(\omega t + \pi) + c\sin(\omega t) \quad \text{but } \omega = n \\ -An^2\cos(nt + \pi) &= -An^2\cos(nt + \pi) + Abn\sin(nt + \pi) + c\sin(nt) \\ 0 &= -Abn\sin(nt) + c\sin(nt) \\ A &= \frac{c}{bn} \end{aligned}$$

(v) When the magnitude of the resistance force is close to zero then b is a very small value (mass and velocity are given). From part (iv), very small values of b cause the value of A to be very large. This suggests that whilst the object reaches a steady state in simple harmonic motion, it has a very large amplitude.

Remark: This is related to a physical phenomenon known as resonance. The $\omega = n$ result means that the period of the driving force matches the 'natural' period of the original simple harmonic motion. When there is a very small (almost negligible) resistance force to counterbalance this, the resultant amplitude of the system can get disproportionately large (all else equal).

Question 14

(a) (i) Let $\underset{\sim}{\hat{n}} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ for some constants a, b and c where $\sqrt{a^2+b^2+c^2} = 1$.

For the line between ℓ_1 and ℓ_2 be always $\dfrac{\pi}{4}$ then

$$\begin{aligned}\underset{\sim}{\hat{m}} \cdot \underset{\sim}{\hat{n}} &= \cos\frac{\pi}{4}\\ a\left(\frac{\cos\theta+1}{2}\right) + b\left(\frac{\sin\theta}{\sqrt{2}}\right) + c\left(\frac{\cos\theta-1}{2}\right) &= \frac{1}{\sqrt{2}}\\ \sqrt{2}(a+c)\cos\theta + 2b\sin\theta + \sqrt{2}(a-c) - 2 = 0\end{aligned}$$

Since $\underset{\sim}{\hat{n}}$ must be a unit vector regardless of the value of θ then the above result holds as θ varies if $a = -c$ and $b = 0$. This implies that

$$\begin{aligned}\sqrt{2}(a-c) &= 2\\ a &= \frac{1}{\sqrt{2}}\end{aligned}$$

Hence, a unit vector $\underset{\sim}{\hat{n}}$ which ensures that the angle between ℓ_1 and ℓ_2 is $\dfrac{\pi}{4}$ is

$$\underset{\sim}{\hat{n}} = \frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{\sqrt{2}}\underset{\sim}{k}$$

(ii) Solve simultaneously $x - z = 4$ and $\underset{\sim}{r} = \mu\underset{\sim}{\hat{n}}$

$$\begin{aligned}\frac{\mu}{\sqrt{2}} - \left(-\frac{\mu}{\sqrt{2}}\right) &= 4\\ \mu &= 2\sqrt{2}\end{aligned}$$

Substitute this back into the equation of ℓ_2 gives the coordinates of C as $(2, 0, -2)$.

(iii) Solve simultaneously $x - z = 4$ and $\underset{\sim}{r} = \lambda\underset{\sim}{\hat{m}}$

$$\begin{aligned}\lambda\left(\frac{\cos\theta+1}{2}\right) - \lambda\left(\frac{\cos\theta-1}{2}\right) &= 4\\ \lambda &= 4\end{aligned}$$

Substitute this back into the equation of ℓ_1 gives the coordinates of P as $\left(2\cos\theta+2, 2\sqrt{2}\sin\theta, 2\cos\theta-2\right)$.

Therefore, the vector $\overrightarrow{CP}$ is represented by $2\cos\theta\underset{\sim}{i} + 2\sqrt{2}\sin\theta\underset{\sim}{j} + 2\cos\theta\underset{\sim}{k}$, so

$$\begin{aligned}|\overrightarrow{CP}| &= \sqrt{4\cos^2\theta + 8\sin^2\theta + 4\cos^2\theta}\\ &= 2\sqrt{2} \quad \text{which is independent of } \theta\end{aligned}$$

Since CP is a constant length, then as θ varies P will describe a circle of centre C and radius $2\sqrt{2}$.

(b) (i) Expanding $P(z)$

$$\begin{aligned}
P(z) &= z^3\left(\left(z+\frac{1}{z}\right)^3+\left(z+\frac{1}{z}\right)^2-2\left(z+\frac{1}{z}\right)-1\right)\\
&= (z^2+1)^3+z(z^2+1)^2-2z^2(z^2+1)-z^3\\
&= z^6+3z^4+3z^2+1+z^5+2z^3+z-2z^4-2z^2-z^3\\
&= z^6+z^5+z^4+z^3+z^2+z+1\\
&= \frac{z^7-1}{z-1}
\end{aligned}$$

Find the roots of $z^7-1=0$. Let $z=re^{i\theta}$. For integer k

$$\begin{aligned}
r^7e^{i7\theta} &= e^{i2k\pi} \quad \text{equating modulus and argument}\\
r=1,\ \theta &= \frac{2k\pi}{7} \quad \text{where } k=0,\pm1,\pm2,\pm3\\
z^7-1 &= (z-e^{i0})(z-e^{i\frac{2\pi}{7}})(z-e^{-i\frac{2\pi}{7}})(z-e^{i\frac{4\pi}{7}})(z-e^{-i\frac{4\pi}{7}})(z-e^{i\frac{6\pi}{7}})(z-e^{-i\frac{6\pi}{7}})
\end{aligned}$$

Note that for a given root $z=\omega$, where $\omega=e^{i\alpha}$

$$\begin{aligned}
(z-\omega)(z-\bar{\omega}) &= z^2-(\omega+\bar{\omega})z+\omega\bar{\omega}\\
&= z^2-2z\cos\alpha+1
\end{aligned}$$

This means that

$$z^7-1=(z-1)\left(z^2-2z\cos\frac{2\pi}{7}+1\right)\left(z^2-2z\cos\frac{4\pi}{7}+1\right)\left(z^2-2z\cos\frac{6\pi}{7}+1\right)$$

Hence

$$P(z)=\left(z^2-2z\cos\frac{2\pi}{7}+1\right)\left(z^2-2z\cos\frac{4\pi}{7}+1\right)\left(z^2-2z\cos\frac{6\pi}{7}+1\right)$$

(ii) For ease of notation, let $a=\sqrt[3]{\alpha}, b=\sqrt[3]{\beta}$ and $c=\sqrt[3]{\gamma}$

$$\begin{aligned}
A^3 &= (a+b+c)^3\\
&= a^3+b^3+c^3+3(a+b)(b+c)(a+c)\\
&= a^3+b^3+c^3+3(ab+ac+b^2+bc)(a+c)\\
&= a^3+b^3+c^3+3(ab+ac+bc)(a+c)+3b^2(a+c)\\
&= a^3+b^3+c^3+3(ab+bc+ac)(a+b+c-b)+3b^2(a+c)\\
&= a^3+b^3+c^3+3(ab+bc+ac)(a+b+c)-3b(ab+ac+bc)+3b^2(a+c)\\
&= a^3+b^3+c^3+3(ab+bc+ac)(a+b+c)-3abc
\end{aligned}$$

Note that $A=a+b+c$ and $B=ab+bc+ac$. Also,

$$\begin{aligned}
a^3+b^3+c^3 &= \alpha+\beta+\gamma\\
&= -1\\
abc &= \sqrt[3]{\alpha\beta\gamma}\\
&= 1
\end{aligned}$$

Substitute these in gives

$$\begin{aligned} A^3 &= -1 + 3AB - 3 \\ &= 3AB - 4 \end{aligned}$$

Applying a similar approach for B^3 and using the results above

$$\begin{aligned} B^3 &= (ab + bc + ac)^3 \\ &= (ab)^3 + (bc)^3 + (ac)^3 + 3(ab + bc)(bc + ac)(ab + ac) \\ &= (ab)^3 + (bc)^3 + (ac)^3 + 3abc(a + c)(a + b)(b + c) \\ &= (ab)^3 + (bc)^3 + (ac)^3 + 3abc(ab + bc + ac)(a + b + c) - 3(abc)^2 \\ &= -2 + 3AB - 3 \\ &= 3AB - 5 \end{aligned}$$

Noting that $(ab)^3 + (bc)^3 + (ac)^3 = \alpha\beta + \beta\gamma + \alpha\gamma = -2$.

Alternative Method:

Let

$$\begin{aligned} Q(x) &= (x - a)(x - b)(x - c) \\ &= x^3 - (a + b + c)x^2 + (ab + bc + ac)x - abc \\ &= x^3 - Ax^2 + Bx - 1 \end{aligned}$$

$$\begin{aligned} A^3 &= (a + b + c)^3 \\ &= a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a) \\ &= \alpha + \beta + \gamma + 3(A - c)(A - a)(A - b) \\ &= -1 + 3Q(A) \\ &= -1 + 3(A^3 - A(A^2) + BA - 1) \\ &= 3AB - 4 \end{aligned}$$

Similarly, for B^3

$$\begin{aligned} B^3 &= (ab + bc + ac)^3 \\ &= a^3b^3 + b^3c^3 + a^3c^3 + 3(ab + bc)(ac + bc)(ab + ac) \\ &= \alpha\beta + \beta\gamma + \gamma\alpha + 3abc(a + c)(a + b)(b + c) \\ &= -2 + 3abc(A - b)(A - c)(A - a) \\ &= -2 + 3Q(A) \\ &= -2 + 3(A^3 - A(A^2) + BA - 1) \\ &= 3AB - 5 \end{aligned}$$

(iii) Consider A^3B^3

$$\begin{aligned}
A^3B^3 &= (3AB-4)(3AB-5) \\
&= 9A^2B^2 - 27AB + 20 \\
A^3B^3 - 9A^2B^2 + 27AB &= 20 \\
A^3B^3 - 3(A^2B^2)(3) + 3(AB)(3^2) - 3^3 &= -7 \\
(AB-3)^3 &= -7 \\
AB &= 3 + \sqrt[3]{-7} \\
&= 3 - \sqrt[3]{7}
\end{aligned}$$

(iv) Note that $P(z) = 0$ when $\left(z + \dfrac{1}{z}\right)^3 + \left(z + \dfrac{1}{z}\right)^2 - 2\left(z + \dfrac{1}{z}\right) - 1 = 0$

From part (i), the roots of $P(z)$ are $e^{-i\frac{2\pi}{7}}, e^{i\frac{2\pi}{7}}, e^{-i\frac{4\pi}{7}}, e^{i\frac{4\pi}{7}}, e^{-i\frac{6\pi}{7}}$, and $e^{i\frac{6\pi}{7}}$.

Let $x = z + \dfrac{1}{z}$, so the roots of the equation $x^3 + x^2 - 2x - 1 = 0$ are

$$x = 2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7}.$$

From part (iii), $AB = 3 - \sqrt[3]{7}$ and from part (ii) $A^3 = 3AB - 4$, so

$$\begin{aligned}
A &= \sqrt[3]{3AB-4} \\
&= \sqrt[3]{5 - 3\sqrt[3]{7}}
\end{aligned}$$

Hence

$$\begin{aligned}
\sqrt[3]{2\cos\frac{2\pi}{7}} + \sqrt[3]{2\cos\frac{4\pi}{7}} + \sqrt[3]{2\cos\frac{6\pi}{7}} &= \sqrt[3]{5 - 3\sqrt[3]{7}} \\
\therefore \sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{6\pi}{7}} &= \sqrt[3]{\frac{5 - 3\sqrt[3]{7}}{2}}
\end{aligned}$$

Question 15

(a) (i) The complex numbers representing the points P_0, P_1 and P_2 are respectively

$$s_0 = e^{i\alpha}, \quad s_1 = e^{i\alpha} + e^{i(\alpha+\beta)}, \quad s_2 = e^{i\alpha} + e^{i(\alpha+\beta)} + e^{i(\alpha+2\beta)}.$$

As n increases, the position of P_n moves in a clockwise direction due to the parallelogram rule in adding complex numbers. Also, $|s_0| = |s_1 - s_0| = |s_2 - s_1| = 1$, which suggests that OP_0, P_0P_1 and P_1P_2 have a unit length.

Let $\angle OP_0P_1 = \theta$. The vector $\overrightarrow{P_0P_1}$ is obtained by rotating the vector $\overrightarrow{P_0O}$ clockwise by θ.

$$\begin{aligned} s_1 - s_0 &= (0 - s_0)e^{-i\theta} \\ e^{i(\alpha+\beta)} &= -e^{i\alpha}e^{-i\theta} \\ e^{i\theta} &= -e^{-i\beta} \end{aligned}$$

Let $\angle P_0P_1P_2 = \epsilon$. The vector $\overrightarrow{P_1P_2}$ is obtained by rotating the vector $\overrightarrow{P_1P_0}$ clockwise by ϵ.

$$\begin{aligned} s_2 - s_1 &= (s_0 - s_1)e^{-i\epsilon} \\ e^{i(\alpha+2\beta)} &= -e^{i(\alpha+\beta)}e^{-i\epsilon} \\ e^{i\epsilon} &= -e^{-i\beta} \end{aligned}$$

This means that $e^{i\theta} = e^{i\epsilon}$ so $\theta = \epsilon$. Note that the other solutions are where they differ by multiples of 2π, which is not possible for the given acute values of α and β. Hence, $\angle OP_0P_1 = \angle P_0P_1P_2$.

(ii) If P_{n+1} coincides with the origin, then $s_{n+1} = 0$ so

$$\begin{aligned} \sum_{k=0}^{n+1} e^{i(\alpha+k\beta)} &= 0 \\ \frac{e^{i\alpha}\left(e^{i\beta(n+2)} - 1\right)}{e^{i\beta} - 1} &= 0 \\ e^{i\beta(n+2)} &= 1 \end{aligned}$$

This suggests that $\beta = \dfrac{2k\pi}{n+2}$ for integer k.

To retrieve a unique acute angle of β, there cannot be consecutive integer values of k that allow β to be acute. In other words

$$\begin{aligned} \frac{2\pi}{n+2} < \frac{\pi}{2} &< \frac{4\pi}{n+2} \\ 4 < n+2 &< 8 \\ 2 < n &< 6 \end{aligned}$$

The largest value of n allowed is $n = 5$, which gives a heptagon $OP_0P_1P_2P_3P_4P_5$. Hence, the maximum number of sides needed such there is only one possible value of the acute angle β is 7.

(b) Let $f(x) = \dfrac{x}{2} + 1 - \sqrt{1+x}$

$$\begin{aligned} f'(x) &= \frac{1}{2} - \frac{1}{2\sqrt{1+x}} \\ f''(x) &= \frac{1}{4(1+x)^{\frac{3}{2}}} > 0 \quad \text{for } x \geq 0 \end{aligned}$$

Stationary points occur when $f'(x) = 0$

$$\begin{aligned} \frac{1}{2} - \frac{1}{2\sqrt{1+x}} &= 0 \\ \sqrt{1+x} &= 1 \\ x &= 0 \\ f(0) &= 0 \end{aligned}$$

Since there are no other stationary points and $f(x)$ is concave up for $x \geq 0$ then $(0, 0)$ is the absolute minimum point of $f(x)$. Hence, $f(x) \geq 0$ for $x \geq 0$.

Let $g(x) = f(x) - \dfrac{x^2}{8}$

$$\begin{aligned} g'(x) &= \frac{1}{2} - \frac{1}{2\sqrt{1+x}} - \frac{x}{4} \\ g''(x) &= \frac{1}{4(1+x)^{\frac{3}{2}}} - \frac{1}{4} \leq 0 \quad \text{for } x \geq 0 \end{aligned}$$

Stationary points occur when $g'(x) = 0$

$$\begin{aligned} \frac{1}{2} - \frac{1}{2\sqrt{1+x}} - \frac{x}{4} &= 0 \\ 2 - x &= \frac{2}{\sqrt{1+x}} \qquad (*) \\ (2-x)^2 &= \frac{4}{1+x} \\ (x+1)(x^2 - 4x + 4) &= 4 \\ x^3 - 4x^2 + 4x + x^2 - 4x + 4 &= 4 \\ x^2(x-3) &= 0 \\ x &= 0 \text{ or } x = 3 \end{aligned}$$

However, $x = 3$ cannot be a solution since it does not satisfy $(*)$. The squaring of both sides introduced a solution that was not valid. So only one stationary point occurs at $x = 0$. Since $g(x)$ is concave down for $x > 0$ and $g(0) = 0$ then $(0, 0)$ is the absolute maximum point of $g(x)$ for $x \geq 0$. Hence, $g(x) \leq 0$ for $x \geq 0$.

Putting these results together gives

$$0 \leq \frac{x}{2} + 1 - \sqrt{1+x} \leq \frac{x^2}{8}.$$

Alternative Method:

Let $u = \sqrt{1+x}$.

$$\begin{aligned}\frac{x}{2} + 1 - \sqrt{1+x} &= \frac{u^2-1}{2} + 1 - u \\ &= \frac{u^2 - 2u + 1}{2} \\ &= \frac{(u-1)^2}{2} \\ &\geq 0\end{aligned}$$

Hence, $\dfrac{x}{2} + 1 - \sqrt{1+x} \geq 0$. Now consider

$$\begin{aligned}\frac{x^2}{8} - \left(\frac{x}{2} + 1 - \sqrt{1+x}\right) &= \frac{(u^2-1)^2}{8} - \frac{(u-1)^2}{2} \\ &= \frac{(u-1)^2}{2}\left(\frac{(u+1)^2}{4} - 1\right) \\ &= \frac{(u-1)^2}{2}\left(\frac{u+1}{2} - 1\right)\left(\frac{u+1}{2} + 1\right) \\ &= \frac{(u-1)^3(u+3)}{8} \\ &\geq 0 \quad (\text{since } u \geq 1 \text{ for } x \geq 0)\end{aligned}$$

Hence, $\dfrac{x}{2} + 1 - \sqrt{1+x} \leq \dfrac{x^2}{8}$ and putting the two results together gives

$$0 \leq \frac{x}{2} + 1 - \sqrt{1+x} \leq \frac{x^2}{8}.$$

(c) (i) When the particle clears the nth step, it has travelled ns units vertically and ns units horizontally based on the configuration of the stairs. Hence, it has the coordinates of $(ns, -ns)$ with the initial position as the origin.

Equating vertical displacement

$$\begin{aligned}-\frac{gt^2}{2} &= -ns \\ t &= \sqrt{\frac{2ns}{g}} \quad \text{noting that } t \geq 0\end{aligned}$$

Equating horizontal displacement and substituting for t

$$\begin{aligned}ut &= ns \\ u &= ns\sqrt{\frac{g}{2ns}} \\ &= \sqrt{\frac{nsg}{2}}\end{aligned}$$

(ii) When the particle lands on the $(n+1)$th step, it has travelled $(n+1)s$ units vertically and $(n+1)s - d$ units horizontally based on the configuration of the stairs. Hence, it has the coordinates of $((n+1)s - d, -(n+1)s)$ with the initial position as the origin.

Equating vertical displacement

$$\begin{aligned}
-\frac{gt^2}{2} &= -(n+1)s \\
t &= \sqrt{\frac{2(n+1)s}{g}} \quad \text{noting that } t \geq 0
\end{aligned}$$

Equating horizontal displacement and substituting for t and u from part (i)

$$\begin{aligned}
ut &= (n+1)s - d \\
d &= (n+1)s - \sqrt{\frac{nsg}{2}} \times \sqrt{\frac{2(n+1)s}{g}} \\
\frac{d}{s} &= n + 1 - \sqrt{n(n+1)}
\end{aligned}$$

(iii) Let $x = \dfrac{1}{n}$. From the inequality in part (b)

$$\begin{aligned}
0 \leq \frac{1}{2n} + 1 - \sqrt{\frac{1}{n} + 1} &\leq \frac{1}{8n^2} \\
0 \leq n\left(\frac{1}{2n} + 1 - \sqrt{\frac{1}{n} + 1}\right) &\leq \frac{1}{8n} \\
0 \leq \frac{1}{2} + n - \sqrt{n(n+1)} &\leq \frac{1}{8n} \\
0 \leq \frac{d}{s} - \frac{1}{2} &\leq \frac{1}{8n} \\
\frac{1}{2} \leq \frac{d}{s} &\leq \frac{1}{2} + \frac{1}{8n}
\end{aligned}$$

(iv) The velocity vector is $\dot{\underset{\sim}{r}} = u\underset{\sim}{i} - gt\underset{\sim}{j}$. Consider the ratio of the vertical speed and horizontal speed at $(ns, -ns)$. Using the result in part (i)

$$\begin{aligned}
\frac{|\dot{r}_y|}{|\dot{r}_x|} &= \frac{gt}{u} \\
&= g \times \sqrt{\frac{2ns}{g}} \times \sqrt{\frac{2}{nsg}} \\
&= 2
\end{aligned}$$

Hence, vertical speed is twice the horizontal speed at the edge of the nth step.

(v) When $n \to \infty$ then $\dfrac{1}{8n} \to 0$, so $\dfrac{d}{s} \to \dfrac{1}{2}$ or equivalently $d \to \dfrac{s}{2}$.

This means that for large n, the particle will land on the midpoint of the step. From part (iv), the vertical speed is twice the horizontal speed when it touches the nth step. From this point to the landing point on the $(n+1)$th step, the particle covers a vertical distance of s, which is twice the horizontal distance it covers at the limiting value of $\frac{s}{2}$.

This equal proportionality relationship between distance and speed implies that the particle's motion approaches a *linear* path (in particular, one with a gradient of -2) from the edge of the nth step to its landing point, when $n \to \infty$.

Question 16

(a) Use the definition of I_n to evaluate the term

$$
\begin{aligned}
\frac{3}{2}I_k - \frac{1}{2}I_{k+1} - \frac{1}{2}I_{k-1} &= \frac{1}{2}\int_0^{\pi} \frac{3\sin kx - \sin(k+1)x - \sin(k-1)x}{3-2\cos x}\,dx \\
&= \frac{1}{2}\int_0^{\pi} \frac{3\sin kx - 2\sin kx\cos x}{3-2\cos x}\,dx \\
&= \frac{1}{2}\int_0^{\pi} \frac{(3-2\cos x)\sin kx}{3-2\cos x}\,dx \\
&= \frac{1}{2}\int_0^{\pi} \sin kx\,dx \\
&= -\frac{1}{2k}[\cos kx]_0^{\pi} \quad \text{but } \cos k\pi = (-1)^k \\
&= \frac{1-(-1)^k}{2k} \\
&= \begin{cases} 0 \text{ if } k \text{ is even.} \\ \frac{1}{k} \text{ if } k \text{ is odd.} \end{cases}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{k=1}^{2n+1}\left(\frac{3}{2}I_k - \frac{1}{2}I_{k+1} - \frac{1}{2}I_{k-1}\right) &= \frac{1}{1} + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots + 0 + \frac{1}{2n+1} \\
&= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}.
\end{aligned}
$$

(b) Use the fact that

$$
\begin{aligned}
(a+b-1)^2 &\geq 0 \\
a^2+b^2+1^2+2(-a-b+ab) &\geq 0 \quad \text{but } a^2+b^2+c^2+d^2=1 \\
1-(c^2+d^2)+1-2a-2b+2ab &\geq 0 \\
2-2a-2b+2ab-c^2-d^2 &\geq 0 \\
2(1-a)-2b(1-a) &\geq c^2+d^2 \\
2(1-a)(1-b) &\geq c^2+d^2 \quad \text{but } c^2+d^2 \geq 2cd \\
2(1-a)(1-b) &\geq 2cd \\
(1-a)(1-b) &\geq cd
\end{aligned}
$$

(c) (i) From the given inequality

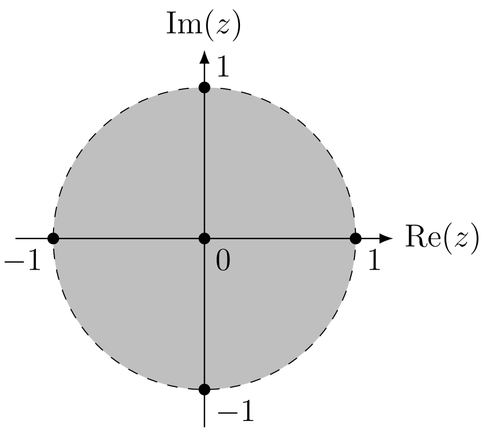
$$\begin{aligned}
\left|\frac{z-\lambda w}{\lambda z - w}\right| &< 1 \\
|z-\lambda w| &< |\lambda z - w| \\
|z-\lambda w|^2 &< |\lambda z - w|^2 \\
(z-\lambda w)(\overline{z-\lambda w}) &< (\lambda z - w)(\overline{\lambda z - w}) \\
(z-\lambda w)(\overline{z}-\lambda \overline{w}) &< (\lambda z - w)(\lambda \overline{z} - \overline{w}) \\
z\overline{z} - \lambda(z\overline{w} + \overline{w}z) + \lambda^2 w\overline{w} &< \lambda^2 z\overline{z} - \lambda(z\overline{w} + \overline{w}z) + w\overline{w} \\
|z|^2 + \lambda^2|w|^2 - \lambda^2|z|^2 - |w|^2 &< 0 \\
(|z|^2 - |w|^2)(1-\lambda^2) &< 0
\end{aligned}$$

Since $0 < \lambda < 1$, then $1 - \lambda^2 > 0$, which implies that

$$\begin{aligned}
|z|^2 - |w|^2 &< 0 \\
\left|\frac{z}{w}\right|^2 &< 1 \\
\left|\frac{z}{w}\right| &< 1
\end{aligned}$$

Hence, $\dfrac{z}{w}$ lies inside the unit circle.

(ii) When $|w| = 1$ then from part (i), the locus of z is $|z| < 1$. This represents the region inside the unit circle.



(d) (i) Applying the definition of $P_n(z)$

$$\begin{aligned}
\text{RHS} &= z\lambda^n P_n\left(\frac{z}{\lambda}\right) + P_n(\lambda z) \\
&= z\lambda^n \sum_{k=0}^{n} \binom{n}{k} \lambda^{k(n-k)} \left(\frac{z}{\lambda}\right)^k + \sum_{k=0}^{n} \binom{n}{k} \lambda^{k(n-k)} (\lambda z)^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \lambda^{(k+1)(n-k)} z^{k+1} + \sum_{k=0}^{n} \binom{n}{k} \lambda^{k(n+1-k)} z^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \lambda^{(k+1)(n-k)} z^{k+1} + \binom{n}{0} \lambda^0 z^0 + \sum_{k=1}^{n} \binom{n}{k} \lambda^{k(n+1-k)} z^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \lambda^{(k+1)(n-k)} z^{k+1} + 1 + \sum_{k=1}^{n} \binom{n}{k} \lambda^{k(n+1-k)} z^k
\end{aligned}$$

Rewriting the sum

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \lambda^{(k+1)(n-k)} z^{k+1} &= \binom{n}{0} \lambda^{(1)(n)} z^1 + \binom{n}{1} \lambda^{(2)(n-1)} z^2 + \cdots + \binom{n}{n} \lambda^{(n+1)(0)} z^{n+1} \\
&= \sum_{k=1}^{n} \binom{n}{k-1} \lambda^{k(n+1-k)} z^k + \binom{n}{n} \lambda^{(n+1)(0)} z^{n+1} \\
&= \sum_{k=1}^{n} \binom{n}{k-1} \lambda^{k(n+1-k)} z^k + z^{n+1}
\end{aligned}$$

Substitute this back into the expression and using Pascal's triangle relation

$$\begin{aligned}
\text{RHS} &= \sum_{k=1}^{n} \binom{n}{k-1} \lambda^{k(n+1-k)} z^k + z^{n+1} + 1 + \sum_{k=1}^{n} \binom{n}{k} \lambda^{k(n+1-k)} z^k \\
&= 1 + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] \lambda^{k(n+1-k)} z^k + z^{n+1} \\
&= 1 + \sum_{k=1}^{n} \binom{n+1}{k} \lambda^{k(n+1-k)} z^k + z^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} \lambda^{k(n+1-k)} z^k \\
&= P_{n+1}(z) \\
&= \text{LHS}
\end{aligned}$$

Noting that $\binom{n+1}{0} \lambda^{(0)(n+1)} z^0 = 1$ and $\binom{n+1}{n+1} \lambda^{(n+1)(0)} z^{n+1} = z^{n+1}$.

(ii) Let the root of $P_{n+1}(z)$ which has a modulus less than one be w, and let the roots of $P_n(z)$ be $z_1, z_2, \cdots, z_{n-1}$ and z_n, which each have a modulus of one.

Noting that $P(z)$ is a monic polynomial

$$P_n(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$$

Using the result from the part (i)

$$\begin{aligned}
P_{n+1}(w) &= 0 \\
w\lambda^n P_n\left(\frac{w}{\lambda}\right) + P_n(\lambda w) &= 0 \\
w &= -\frac{P_n(\lambda w)}{\lambda^n P_n\left(\frac{w}{\lambda}\right)} \\
&= -\frac{(\lambda w - z_1)(\lambda w - z_2)\cdots(\lambda w - z_n)}{\lambda^n(\frac{w}{\lambda} - z_1)(\frac{w}{\lambda} - z_2)\cdots(\frac{w}{\lambda} - z_n)} \\
&= -\frac{(\lambda w - z_1)(\lambda w - z_2)\cdots(\lambda w - z_n)}{(w - \lambda z_1)(w - \lambda z_2)\cdots(w - \lambda z_n)} \\
|w| &= \left|\frac{\lambda w - z_1}{w - \lambda z_1}\right|\left|\frac{\lambda w - z_2}{w - \lambda z_2}\right|\cdots\left|\frac{\lambda w - z_n}{w - \lambda z_n}\right| \\
&= \left|\frac{z_1 - \lambda w}{\lambda z_1 - w}\right|\left|\frac{z_2 - \lambda w}{\lambda z_2 - w}\right|\cdots\left|\frac{z_n - \lambda w}{\lambda z_n - w}\right|
\end{aligned}$$

If $|w| < 1$ then there must exist at least one z_k such that $\left|\dfrac{z_k - \lambda w}{\lambda z_k - w}\right| < 1$.

From part (c)(i), this implies that

$$\begin{aligned}
\left|\frac{z_k}{w}\right| &< 1 \quad \text{but } |z_k| = 1 \\
|w| &> 1
\end{aligned}$$

Hence, there is a contradiction.

Remark: Note that $P_n\left(\frac{w}{\lambda}\right)$ and $P_n(\lambda w)$ cannot be zero, so the division is defined. This can be shown as follows.

If $P_n\left(\frac{w}{\lambda}\right) = 0$ then this implies that $P_n(\lambda w) = 0$ (and vice versa). Since the roots of $P_n(z)$ all have modulus of one then

$$\begin{aligned}
\left|\frac{w}{\lambda}\right| &= |\lambda w| \\
|\lambda| &= 1
\end{aligned}$$

But $0 < \lambda < 1$, which is a contradiction. Hence, it is not possible for either $P_n\left(\frac{w}{\lambda}\right) = 0$ or $P_n(\lambda w) = 0$.

(iii) When $n = 1$, $P_1(z) = 1 + z$. This only has one root $z = -1$ which has a modulus of one. Hence, the statement is true for $n = 1$.

Assume the statement is true for some $n = k$. That is, all the roots of $P_k(z)$ have a modulus of one.

Required to prove that the roots of $P_{k+1}(z)$ all have a modulus of one.

If the roots of $P_k(z)$ are assumed to have modulus one, then from the contradiction in part (ii), $P_{k+1}(z)$ cannot have also have a root of modulus less than one.

A similar contradiction argument can constructed for the case where it assumed that $P_{k+1}(z)$ has a root of modulus greater than 1. From part (ii), if $|w| > 1$ then there must exist at least one z_k such that

$$\left|\frac{z_k - \lambda w}{\lambda z_k - w}\right| > 1 \quad \text{but} \quad \left|\frac{z_k - \lambda w}{\lambda z_k - w}\right| = \left|\frac{\lambda w - z_k}{w - \lambda z_k}\right|$$
$$\left|\frac{w - \lambda z_k}{\lambda w - z_k}\right| < 1$$

From part (c)(i), this implies that

$$\left|\frac{w}{z_k}\right| < 1 \quad \text{but } |z_k| = 1$$
$$|w| < 1$$

Hence, there is a contradiction.

This means that if the roots of $P_k(z)$ are assumed to have modulus one, then from the contradictions, $P_{k+1}(z)$ cannot also have a root of modulus less than one or greater than one. The only possibility remaining is that all of the roots of $P_{k+1}(z)$ are of modulus one.

Since the statement is true for $n = 1$, then by mathematical induction it is also true all positive integers n.