2022 **BORED OF STUDIES TRIAL EXAMINATION**

4th October

Mathematics Extension 2

General instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5–10)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 Which of the following is a unit vector that is perpendicular to $\frac{1}{4}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$?

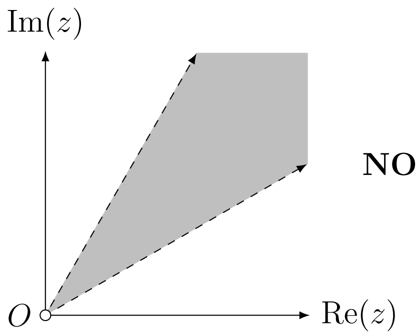
(A) $\frac{2}{9}\underset{\sim}{i} + \frac{1}{9}\underset{\sim}{j} - \frac{2}{9}\underset{\sim}{k}$

(B) $\frac{2}{9}\underset{\sim}{i} - \frac{1}{9}\underset{\sim}{j} - \frac{2}{9}\underset{\sim}{k}$

(C) $\frac{2}{3}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$

(D) $\frac{2}{3}\underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$

2 Which of the following best represents the shaded region on the complex plane below?



NOT TO SCALE

(A) $\frac{\pi}{6} < \arg(z) < \frac{\pi}{3}$

(B) $\frac{\pi}{3} < \arg(z) < \frac{2\pi}{3}$

(C) $\frac{1}{2} < \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} < 3$

(D) $\frac{1}{2} < \frac{\mathrm{Re}(z)}{\mathrm{Im}(z)} < 3$

3 Let X be binomial distributed random variable representing the number of successes after n trials. Consider the statement

$$\forall k \in [0, n], \exists \delta \in (0, 1) : P(X \leq k) > \delta.$$

Which of the following is a negation of this statement?

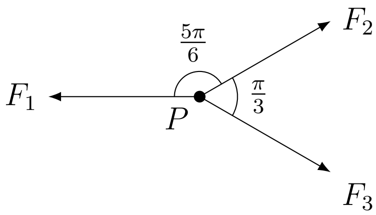
(A) $\forall \delta \in (0, 1), \exists k \in [0, n] : P(X \leq k) \leq \delta$

(B) $\forall k \in [0, n], \exists \delta \in (0, 1) : P(X \leq k) \leq \delta$

(C) $\forall \delta \in (0, 1), \exists k \in [0, n] : P(X \leq k) > \delta$

(D) $\forall k \in [0, n], \exists \delta \in (0, 1) : P(X \leq k) > \delta$

4 Three forces of magnitudes F_1, F_2 and F_3 act on a resting particle P on the same plane as shown in the diagram below.



NOT TO SCALE

The particle continues to remain at rest. What is the ratio $F_1 : F_2 : F_3$?

(A) $3 : 1 : 1$ (B) $1 : 3 : 3$

(C) $\sqrt{3} : 3 : 3$ (D) $3 : \sqrt{3} : \sqrt{3}$

5 Let ω be a non-real root of the equation $z^5 = 1$.

What is the value of $1 + \dfrac{1}{\omega} + \dfrac{1}{\omega^2} + \dfrac{1}{\omega^3} + \dfrac{1}{\omega^4} + \dfrac{1}{\omega^5}$?

(A) -1 (B) 0

(C) 1 (D) 2

6 An object of mass m is attached to a vertical spring hanging from a ceiling. The object has a displacement of y from the origin, with downwards defined as the positive direction.

When the spring is fully stretched and then released, the object is subject to a force of magnitude mky from the spring itself (where k is a positive constant) as well as gravitational force mg (where g is the acceleration due to gravity).

The net force on the object is such that it moves in simple harmonic motion.

What is the centre of this simple harmonic motion?

(A) $\dfrac{g}{k}$ (B) $-\dfrac{g}{k}$

(C) gk (D) $-gk$

7 On the complex plane, the position vector $\overrightarrow{OZ}$ is represented by the complex number z. Which of the following best describes a transformation on $\overrightarrow{OZ}$ to obtain the position vector represented by $i^3\overline{z}$?

(A) Reflect $\overrightarrow{OZ}$ about the x-axis, then rotate anticlockwise by an angle of $\dfrac{\pi}{2}$ about the origin.

(B) Reflect $\overrightarrow{OZ}$ about the y-axis, then rotate anticlockwise by an angle of $\dfrac{\pi}{2}$ about the origin.

(C) Rotate $\overrightarrow{OZ}$ anticlockwise by an angle of $\dfrac{3\pi}{2}$ about the origin.

(D) Reflect $\overrightarrow{OZ}$ about the x-axis, then reflect about the y-axis.

8 Object A is projected upwards from the ground with an initial speed of u. At the same time, object B is released from rest from height h above the ground. Both objects are subjected to gravity and air resistance proportional to the object's own velocity.

It can be shown that the general equation for the velocity of the object over time is $v = ae^{kt} + b$ for some constants a, b and k, where the direction of motion is defined as the positive direction.

Which of the following statements is correct?

(A) $a < 0, b > 0, k > 0$ for object A and $a > 0, b < 0, k > 0$ for object B

(B) $a < 0, b > 0, k < 0$ for object A and $a > 0, b < 0, k < 0$ for object B

(C) $a > 0, b < 0, k > 0$ for object A and $a < 0, b > 0, k > 0$ for object B

(D) $a > 0, b < 0, k < 0$ for object A and $a < 0, b > 0, k < 0$ for object B

9 Suppose that z is any complex number which lies on the unit circle in the complex plane. Which of the following inequalities is always true?

(A) $0 \leq |z^8 - 3| \leq 2$

(B) $1 \leq |z^8 - 3| \leq 3$

(C) $2 \leq |z^8 - 3| \leq 4$

(D) $3 \leq |z^8 - 3| \leq 5$

10 Which of the following integrals has the largest value?

(A) $\displaystyle\int_0^{\pi} \cos^7 x \, dx$

(B) $\displaystyle\int_{-2}^{2} e^{-\frac{1}{2}x^2} \, dx$

(C) $\displaystyle\int_{-\sqrt{3}}^{1} \tan^{-1} x \, dx$

(D) $\displaystyle\int_1^{e} \sqrt{\ln x} \, dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Use the substitution $\sin\theta = \sin x - \cos x$ to evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin 2x})^2}\, dx.$$

(b) Let z be a complex number such that $|z + i|z|| = |z - 2i|z||$. Sketch the locus of z. **2**

(c) A trolley of mass m is initially at rest partway up a frictionless slope at an angle of θ to the horizontal. A student attempts to pull the trolley up the slope with a force of ma for some constant a. Let v and x be the trolley's speed and distance respectively at time t along the slope relative to its initial position. **2**

Show that $v^2 = 2x|a - g\sin\theta|$, where g is the acceleration due to gravity.

(d) Let $f(x)$ be a continuous function for all real x.

(i) Show that $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, for every $a > 0$, if and only if $f(x)$ is an even function. **4**

(ii) Suppose that $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ for some fixed $a > 0$. **1**

Show that $f(x)$ is not necessarily an even function.

(e) It can be shown that, for any three positive numbers a, b and c, that **2**

$$a + b + c \geq 3\sqrt[3]{abc}. \qquad \textbf{(Do NOT prove this)}$$

Use this result to show that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + xz$$

where x, y and z are positive numbers such that $x + y + z = 3$.

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Let a and b be non-zero integers.

(i) Prove that if a^3 is even then a is even. **1**

(ii) Suppose that $(a-b)^3 + a^3 = (a+b)^3$. Prove that both a and b are even. **2**

(b) Using the result $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, find the solutions to **3**

$$\text{cosec } \alpha = \text{cosec } 2\alpha + \text{cosec } 3\alpha$$

for the domain $-\pi < \alpha \leq \pi$.

(c) Let $\underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ be a vector that is perpendicular to a plane $\mathcal{P}$ for some constants a, b and c. Suppose that $R(x, y, z)$ is any point on $\mathcal{P}$ with position vector $\underset{\sim}{r}$.

(i) If d is the shortest distance of the plane $\mathcal{P}$ from the origin, use the relationship between the vectors $\underset{\sim}{n}$ and $\underset{\sim}{r}$ to show that there exists a particular position vector $\underset{\sim}{r}$, which has components that satisfy **2**

$$d = \frac{|ax + by + cz|}{\sqrt{a^2 + b^2 + c^2}}.$$

(ii) Consider the equations of two distinct planes for some constants a, b and c **1**

$$ax + by + cz = m_1$$

$$ax + by + cz = m_2$$

where m_1 and m_2 are distinct non-zero constants.

Explain why the two planes are parallel to each other.

(iii) Hence, express the shortest distance between the two parallel planes in part (ii) in terms of a, b, c, m_1 and m_2, with consideration of different cases. **2**

(d) Let $I_k = \displaystyle\int_0^1 x^k (1-x)^{n-k}\, dx$ for integers n and k, where $0 \leq k \leq n$.

(i) Show that $I_k = \dfrac{k}{n-k+1} I_{k-1}$. **2**

(ii) Hence show that $\displaystyle\int_0^1 \binom{n}{k} x^k (1-x)^{n-k}\, dx = \frac{1}{n+1}$. **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) (i) Express $\dfrac{8}{1+4x^4}$ as a sum of partial fractions over $\mathbb{R}$. **3**

(ii) Hence, show that for some constant c **2**

$$\int \frac{8}{1+4x^4}\,dx = \ln\left(\frac{2x^2+2x+1}{2x^2-2x+1}\right) + 2\tan^{-1}(2x+1) + 2\tan^{-1}(2x-1) + c.$$

(b) An object of mass m moves horizontally in the positive direction with an initial speed of u. At time t, the object has a velocity of v and is subjected to two resistance forces of magnitudes mk and $4mkv^4$, where k is a positive constant. The object decelerates until it comes to rest.

(i) Using the result in part (a)(ii), find the time taken for the object to come to rest in terms of k and u. **2**

(ii) Hence, find the maximum possible time that the object can take to come to rest as the initial speed is increased. Justify that this is a maximum. **2**

(c) Suppose that $x + iy$ is a non-real root of the polynomial **2**

$$P(z) = az^4 + ibz^3 + cz^2 + idz + e$$

where a, b, c, d, e, x and y are real numbers.

By considering $P(iz)$, or otherwise, show that $-x + iy$ is also a root.

(d) Let $\underset{\sim}{r} = \left(\dfrac{\cos 2t}{2}\right)\underset{\sim}{i} + \left(\dfrac{\sin 2t}{2}\right)\underset{\sim}{j} + (\sin t)\,\underset{\sim}{k}$, for some parameter t.

As t varies, $\underset{\sim}{r}$ traces a curve in three-dimensional space.

(i) All points on the curve lie on the surface of a particular sphere with a radius of 1. Find the centre of this sphere. **3**

(ii) All points on the curve also lie on the surface of a cylinder centered about the z-axis. Find the radius of this cylinder. **1**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Two particles A and B move according to the following acceleration equations for some constants b and n, where $n > 0$.

$$\ddot{x}_A = -n^2(x_A - b)$$

$$\ddot{x}_B = -n^2 x_B$$

Initially, particle A has a displacement of b and velocity of n, whilst particle B has a displacement of 1 at rest.

(i) Show by integration that $\dot{x}_A^2 = n^2(1 - (x_A - b)^2)$. **2**

(ii) Hence, or otherwise, find x_A and x_B in terms of b, n and t. **3**

(iii) Find the range of values of b such that particles A and B never collide. **2**

(b) The complex numbers z, z^2 and λz, where $\lambda > 1$, make an equilateral triangle on the complex plane. If $|z| = \sqrt{2}$, show that **3**

$$\lambda = \frac{1+\sqrt{5}}{2}.$$

(c) Suppose that $f(x)$ is a continuous function for all real x with a unique inverse function $f^{-1}(x)$.

(i) Consider the following statement. **1**

All solutions to the equation $f(x) = f^{-1}(x)$ must satisfy $f(x) = x$.

Provide one counterexample to show that the statement is not always true.

(ii) Now suppose that $f(x)$ is an increasing function. Prove that the statement in part (i) is true. **2**

(iii) Hence, find all real solutions of the equation **2**

$$2\sqrt[3]{2x-1} = x^3 + 1.$$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet

(a) The vector equation of line ℓ_1 is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ for some parameter λ. **3**

Suppose that the line ℓ_2 passes through the point $(1, 1, 1)$ and is perpendicular to the line ℓ_1.

Show that ℓ_2 is parallel to the position vector $\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$.

(b) (i) Show that $\sin x < x$, for $x > 0$. **1**

(ii) Hence, show that $\cos(\sin x) > \sin(\cos x)$ for all real x. **3**

(c) Two identical objects A and B, each of mass m, were simultaneously launched from the ground with an initial speed of u and an angle of θ to the horizontal. Let g be the acceleration due to gravity.

Suppose that at time t, object A was subjected to air resistance of magnitudes $mk\dot{x_A}$ and $mk\dot{y_A}$ in its horizontal and vertical components respectively, where k is a positive constant, $\dot{x_A}$ and $\dot{y_A}$ are the horizontal and vertical components of object A's speed respectively.

Object B was not subjected to any air resistance.

(i) By integrating the relevant acceleration equations, show that the displacements equations for object A relative to the launch position are **4**

$$x_A = \frac{u\cos\theta}{k}(1 - e^{-kt}) \qquad \text{and} \qquad y_A = \frac{(g + ku\sin\theta)}{k^2}(1 - e^{-kt}) - \frac{gt}{k}.$$

(ii) Hence, show that object B reaches its maximum height before object A hits the ground, but only after object A has reached its maximum height first. **4**

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet

(a) Prove that $\sqrt{3}$ is irrational. **2**

(b) Let $P(x) = x^4 - 8x^3 + 6x^2 - 8x + 1$ and n be any non-negative integer.

(i) Use the substitution $y = x + x^{-1}$, to express $P(x) = 0$ as a quadratic equation in terms of y. **1**

(ii) Hence, express $P(x)$ as a product of real quadratic polynomials, in terms of x. **1**

(iii) Find the exact value of the non-real root α with positive imaginary part, and show it has a modulus of 1. **2**

(iv) Let $T_n = \omega^{2^n}$ be the general term of a sequence, for some root of unity ω. **1**

For example, $T_0 = \omega, T_3 = \omega^8, T_6 = \omega^{64}$.

Explain why there are only a finite number of possible values for T_n.

(v) It can be shown that $\text{Re}(\alpha^{2^n})$ can be written in the form $a_n - b_n\sqrt{3}$ where a_n and b_n are positive integers. **(Do NOT prove this)** **2**

Find expressions for a_{n+1} and b_{n+1} in terms of a_n and b_n.

(vi) Using mathematical induction, prove that $a_{n+1} > a_n$ and $b_{n+1} > b_n$ for $n \geq 0$. **3**

(vii) By using the previous part and part (a), show that the sequence $S_n = \alpha^{2^n}$ never repeats. **2**

(viii) Hence, explain why the following statement is FALSE. **1**

If α is a non-real root of any monic polynomial $P(x)$ with integer coefficients, and all non-real roots of $P(x)$ have modulus 1, then α is always a root of unity.

End of paper

2022 BORED OF STUDIES TRIAL EXAMINATION

Mathematics Extension 2

Solutions

Section I

Answers

1	C	**6**	A
2	A	**7**	B
3	A	**8**	B
4	D	**9**	C
5	C	**10**	B

Brief explanations

1 First notice that options (A) and (B) each have length of $\frac{1}{9}\sqrt{2^2+1^2+2^2}$, which is not a unit length so the answer cannot be (A) or (B).

Options (C) and (D) each have length of $\frac{1}{3}\sqrt{2^2+1^2+2^2}$ which is a unit length.

Consider the dot product with (D).

$$\begin{aligned}\left(\frac{1}{4}\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}+\frac{1}{2}\underset{\sim}{k}\right)\cdot\left(\frac{2}{3}\underset{\sim}{i}-\frac{1}{3}\underset{\sim}{j}-\frac{2}{3}\underset{\sim}{k}\right)&=\frac{1}{4}\left(\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}\right)\cdot\frac{1}{3}\left(2\underset{\sim}{i}-\underset{\sim}{j}-2\underset{\sim}{k}\right)\\&=\frac{1}{12}(2-2-4)\\&\neq 0\end{aligned}$$

Consider the dot product with (C).

$$\begin{aligned}\left(\frac{1}{4}\underset{\sim}{i}+\frac{1}{2}\underset{\sim}{j}+\frac{1}{2}\underset{\sim}{k}\right)\cdot\left(\frac{2}{3}\underset{\sim}{i}+\frac{1}{3}\underset{\sim}{j}-\frac{2}{3}\underset{\sim}{k}\right)&=\frac{1}{4}\left(\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}\right)\cdot\frac{1}{3}\left(2\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k}\right)\\&=\frac{1}{12}(2+2-4)\\&=0\end{aligned}$$

Hence, the answer is (C).

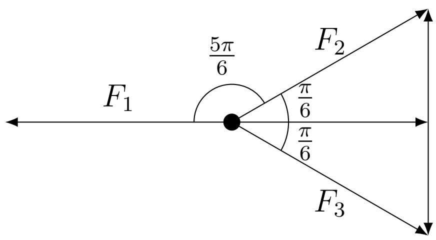
2 The answer cannot be (C) or (D) because $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ can satisfy the inequality with negative values. The answer cannot be (B) because the upper bound extends into the second quadrant. The answer is (A) because the bounds remain in the first quadrant.

3 The statement reads

$$''\text{For every } k \in [0,n] \text{ there exists a } \delta \in (0,1) \text{ such that } P(X \le k) > \delta''.$$

To negate this statement, there needs to be at least one k where there can never be any $\delta \in (0,1)$ such that $P(X \le k) > \delta$. In other words, for every $\delta \in (0,1)$ there is at least one k where $P(X \le k) \le \delta$. Hence, the answer is (A).

4 Resolve the force vectors of magnitude F_2 and F_3 in the directions parallel and perpendicular F_1.



Since the particle remains at rest after the application of the forces then the net forces of the components must be zero, so

$$\begin{aligned}
F_2 \sin\frac{\pi}{6} - F_3 \sin\frac{\pi}{6} &= 0 \\
F_2 &= F_3 \\
F_2 \cos\frac{\pi}{6} + F_3 \cos\frac{\pi}{6} - F_1 &= 0 \\
2F_2 \cos\frac{\pi}{6} &= F_1 \\
F_2 &= \frac{1}{\sqrt{3}} F_1 \\
F_3 &= \frac{1}{\sqrt{3}} F_1
\end{aligned}$$

The ratio $F_1 : F_2 : F_3$ is $1 : \frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}}$, or equivalently $3 : \sqrt{3} : \sqrt{3}$.

Hence, the answer is (D).

5 From the geometric series

$$\begin{aligned}
1 + \frac{1}{\omega} + \frac{1}{\omega^2} + \frac{1}{\omega^3} + \frac{1}{\omega^4} + \frac{1}{\omega^5} &= \frac{(\frac{1}{\omega})^6 - 1}{\frac{1}{\omega} - 1} \qquad \text{but } \omega^5 = 1 \\
&= \frac{\frac{1}{\omega} - 1}{\frac{1}{\omega} - 1} \\
&= 1
\end{aligned}$$

Hence, the answer is (C).

6 Since the net force on the object leads to simple harmonic motion, the force from the spring must be directed opposite to that of the gravitational force. Since downwards is defined as the positive direction, the net force is expressed as

$$\begin{aligned} m\ddot{y} &= mg - mky \\ \ddot{y} &= -k\left(y - \frac{g}{k}\right) \end{aligned}$$

Since $k > 0$, the above acceleration equation is consistent with simple harmonic motion with centre $\frac{g}{k}$. Hence, the answer is (A).

7 Let $z = x + iy$ then

$$\begin{aligned} i^3\overline{z} &= -i(x - iy) \\ &= -y - ix \end{aligned}$$

Considering each of the options:

- (A) is achieved by $i(x - iy)$, or equivalently $y + ix$
- (B) is achieved by $i(-x + iy)$, or equivalently $-y - ix$
- (C) is achieved by $i^3(x + iy)$, or equivalently $y - ix$
- (D) is achieved with $-x - iy$

Hence, the answer is (B).

8 The forces acting on A and B have magnitudes mg and mcv for some positive constant c, noting that m, c and v may be different between A and B. With the direction of motion defined as the positive direction, the net force on A is

$$\begin{aligned} m_A\ddot{x}_A &= -m_Ag - m_Ac_Av_A \\ \frac{dv_A}{dt} &= -c_A\left(v_A + \frac{g}{c_A}\right) \end{aligned}$$

For object B, this is

$$\begin{aligned} m_A\ddot{x}_B &= m_Bg - m_Bc_Bv_B \\ \frac{dv_B}{dt} &= -c_B\left(v_B - \frac{g}{c_B}\right) \end{aligned}$$

These two differential equations are exponential decay equations on the velocity, with general solution $v = a + be^{kt}$ where $k < 0$.

For object A, this implies $a = -\frac{g}{c_A} < 0$ and for object B, it implies $a = \frac{g}{c_B} > 0$.

For object A, when $t = 0$ then $v = u$, which implies $u = a + b > 0$. Since $a < 0$, then $b > 0$ to ensure $u > 0$.

For object B, when $t = 0$ then $v = 0$, which implies $a + b = 0$. Since $a > 0$, then $b < 0$ to ensure $a + b = 0$.

Hence, the answer is (B).

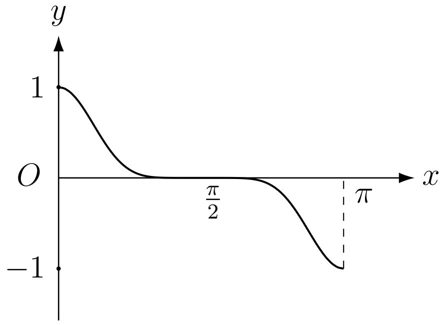
9 Since z lies on the unit circle then $|z| = 1$. From the triangle inequality

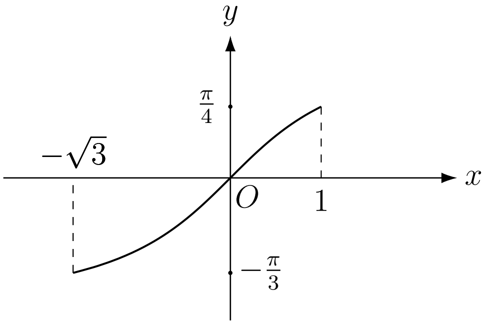
$$\begin{aligned} |z^8 - 3| &\leq |z^8| + |-3| \\ &= |z|^8 + 3 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{Also} \quad |z^8 - 3| + |-z^8| &\geq |z^8 - 3 - z^8| \\ |z^8 - 3| + |z|^8 &\geq |-3| \\ |z^8 - 3| &\geq 2 \end{aligned}$$

Hence, the answer is (C).

10 For options (A) and (C), note that graph of $y = \cos^7 x$ has point symmetry about $x = \dfrac{\pi}{2}$ and $y = \tan^{-1} x$ has point symmetry about the origin.

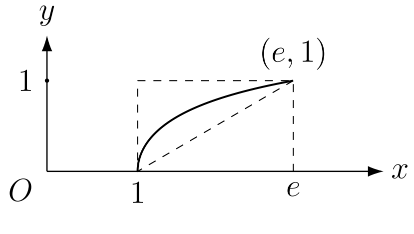




This means that $\displaystyle\int_0^{\pi} \cos^7 x\, dx = 0$ and $\displaystyle\int_{-\sqrt{3}}^{1} \tan^{-1} x\, dx < 0$, so (A) is larger than (C). Option (B) has a known approximate value since it is derived from the area under a normal curve, that is

$$\begin{aligned} \int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx &\approx 0.95 \\ \int_{-2}^{2} e^{-\frac{1}{2}x^2}\, dx &\approx 2.38 \end{aligned}$$

Option (D) cannot be easily determined exactly but we only need to know whether it is above or below option (B) (which is greater than options (A) and (C)). The integral can be approximated using an upper rectangle and a lower triangle.



The graph of $y = \sqrt{\ln x}$ is concave down, so $\dfrac{1}{2}(e - 1) < \displaystyle\int_1^e \sqrt{\ln x}\, dx < e - 1$.

Since the exact area is less than $e - 1 \approx 1.718$, it must be less than 2.38 (the approximate value of option (B)). Hence, the answer is (B).

Question 11

(a) Let $\sin\theta = \sin x - \cos x \Rightarrow \cos\theta\, d\theta = (\cos x + \sin x)\, dx$.

When $x = 0$ then $\theta = -\dfrac{\pi}{2}$ and when $x = \dfrac{\pi}{2}$ then $\theta = \dfrac{\pi}{2}$.

Note that

$$\begin{aligned}
\sin^2\theta &= \sin^2 x + \cos^2 x - 2\sin x\cos x \\
\sin 2x &= \cos^2\theta \\
\sqrt{\sin 2x} &= \cos\theta \quad \text{ noting that } \cos\theta \geq 0 \text{ for } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
\end{aligned}$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin 2x})^2}\, dx &= \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x - \sin x}{(1+\sqrt{\sin 2x})^2}\, dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\cos\theta)^2}\, d\theta - \int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\sqrt{\sin 2x})^2}\, dx
\end{aligned}$$

For the first term, let $t = \tan\frac{\theta}{2} \Rightarrow d\theta = \dfrac{2}{1+t^2}\, dt$.

When $\theta = -\dfrac{\pi}{2}$ then $t = -1$ and when $\theta = \dfrac{\pi}{2}$ then $t = 1$.

For the second term, let $u = \dfrac{\pi}{2} - x \Rightarrow du = -dx$.

When $x = 0$ then $\theta = \dfrac{\pi}{2}$ and when $x = \dfrac{\pi}{2}$ then $\theta = 0$.

Applying these substitutions and noting that u is a dummy variable

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin 2x})^2}\, dx &= \int_{-1}^{1} \frac{\frac{1-t^2}{1+t^2}}{\left(1+\frac{1-t^2}{1+t^2}\right)^2} \times \frac{2}{1+t^2}\, dt + \int_{\frac{\pi}{2}}^{0} \frac{\sin\left(\frac{\pi}{2}-u\right)}{\left(1+\sqrt{\sin 2\left(\frac{\pi}{2}-u\right)}\right)^2}\, du \\
&= \int_{-1}^{1} \frac{1-t^2}{2}\, dt - \int_0^{\frac{\pi}{2}} \frac{\cos u}{(1+\sqrt{\sin(\pi-2u)})^2}\, du \\
&= \frac{1}{2}\int_{-1}^{1} (1-t^2)\, dt - \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin 2x})^2}\, dx \\
2\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin 2x})^2}\, dx &= \frac{1}{2}\left[t - \frac{t^3}{3}\right]_{-1}^{1} \\
&= \frac{2}{3} \\
\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin 2x})^2}\, dx &= \frac{1}{3}
\end{aligned}$$

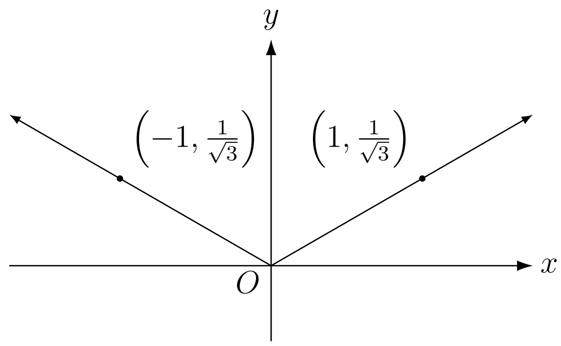
(b) Let $z = x + iy$ and substitute into the locus equation.

$$\begin{aligned}
|z + i|z|| &= |z - 2i|z|| \\
|x + iy + i\sqrt{x^2+y^2}| &= |x + iy - 2i\sqrt{x^2+y^2}| \\
x^2 + \left(y + \sqrt{x^2+y^2}\right)^2 &= x^2 + \left(y - 2\sqrt{x^2+y^2}\right)^2 \\
y^2 + 2y\sqrt{x^2+y^2} + x^2 + y^2 &= y^2 - 4y\sqrt{x^2+y^2} + 4(x^2+y^2) \\
2y\sqrt{x^2+y^2} &= x^2 + y^2 \qquad \text{note that } y \geq 0 \\
\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2} - 2y\right) &= 0
\end{aligned}$$

One solution is $\sqrt{x^2+y^2} = 0$, which is only satisfied by the origin. The other solution is

$$\begin{aligned}
\sqrt{x^2+y^2} - 2y &= 0 \\
x^2 + y^2 &= 4y^2 \\
(x - y\sqrt{3})(x + y\sqrt{3}) &= 0 \\
y &= \pm\frac{x}{\sqrt{3}} \qquad \text{but } y \geq 0 \\
y &= \frac{|x|}{\sqrt{3}}
\end{aligned}$$

The sketch of this locus is below.



Alternative method:

Let $z = re^{i\theta}$ for $-\pi < \theta \leqslant \pi$ and substitute into the locus equation

$$\begin{aligned}
|z + i|z|| &= |z - 2i|z|| \\
|re^{i\theta} + ir| &= |re^{i\theta} - 2ir| \\
r|e^{i\theta} + i| &= r|e^{i\theta} - 2i| \\
r\left(|e^{i\theta} + i| - |e^{i\theta} - 2i|\right) &= 0
\end{aligned}$$

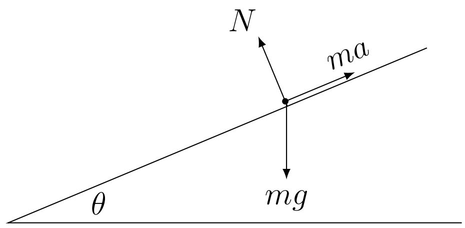
One solution is $r = 0$, which is only satisfied by the origin. The other solution is

$$\begin{aligned}
|e^{i\theta} + i| - |e^{i\theta} - 2i| &= 0 \\
|e^{i\theta} + i|^2 &= |e^{i\theta} - 2i|^2 \\
\cos^2\theta + (\sin\theta + 1)^2 &= \cos^2\theta + (\sin\theta - 2)^2 \\
(\sin\theta + 1)^2 - (\sin\theta - 2)^2 &= 0 \\
3(2\sin\theta - 1) &= 0 \\
\sin\theta &= \frac{1}{2} \\
\theta &= \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}$$

This means that the locus of z is $z = 0$, $\text{Arg}(z) = \frac{\pi}{6}$ or $\text{Arg}(z) = \frac{5\pi}{6}$.

This is equivalent to the Cartesian equation $y = \frac{|x|}{\sqrt{3}}$ and so leads to the same sketch as the other solution.

(c) The two main forces acting on the trolley are the gravitational force and the pulling force from the student up the slope. There is also a normal force, which will be defined to have magnitude N perpendicular to the slope (see remark).



The net force can be directed up the slope or down the slope. This depends on whether the pulling force from the student is enough to overcome the gravity on the slope. Since this is unknown, then absolute values must be considered to ensure the addition/subtraction of the component forces is consistent with the direction of the net force.

Resolving the gravitational force parallel to the slope gives

$$\begin{aligned}
m\ddot{x} &= |ma - mg\cos\theta| \\
\frac{d}{dx}\left(\frac{v^2}{2}\right) &= |a - g\cos\theta| \qquad \text{note the RHS is a constant} \\
v^2 &= 2x|a - g\cos\theta| + c \qquad \text{when } x = 0, v = 0 \Rightarrow c = 0 \\
v^2 &= 2x|a - g\cos\theta|
\end{aligned}$$

Remark: Note that the normal force is not mentioned in the question, but technically does exist and opposes the respective component of the gravitational force. There is no penalty if the normal force is not mentioned, since it is not needed for the main part of the question. It is only provided here for completeness.

(d) (i) Firstly showing that if $f(x)$ is an even function then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Let $u = -x \Rightarrow du = -dx$.

When $x = -a$ then $u = a$ and when $x = 0$ then $u = 0$.

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx \\
&= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \quad \text{but } f(x) \text{ is an even function} \\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \quad \text{but } u \text{ is a dummy variable} \\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

Secondly showing if $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ then $f(x)$ is an even function.

Let $F(x) = \int_{0}^{x} f(u)\,du$, noting that $F'(x) = f(x)$. For every $a > 0$

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= 2\int_{0}^{a} f(x)\,dx \\
\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx &= 2\int_{0}^{a} f(x)\,dx \\
\int_{-a}^{0} f(x)\,dx &= \int_{0}^{a} f(x)\,dx \\
-\int_{0}^{-a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx \\
-F(-a) &= F(a)
\end{aligned}$$

Since this is true for every $a > 0$ then $-F(-x) = F(x)$ for $x > 0$. Differentiating both sides of this equality gives $f(-x) = f(x)$ for $x > 0$, which implies reflection about the y-axis. Hence, $f(x)$ is an even function.

(ii) A counterexample is $f(x) = \sin x$ with $a = 2\pi$. Note that

$$\int_{-2\pi}^{2\pi} \sin x\,dx = 2\int_{0}^{2\pi} \sin x\,dx$$

but $f(x) = \sin x$ is in fact an odd function.

The result in part (i) only holds if it is true for **every** $a > 0$. If the result only holds for a particular choice of a, then it cannot be concluded that $f(x)$ is an even function.

(e) Using the given result, let $a = x^2, b = \sqrt{x}$ and $c = \sqrt{x}$

$$\begin{aligned} x^2 + \sqrt{x} + \sqrt{x} &\geq 3\sqrt[3]{x^2\sqrt{x}\sqrt{x}} \\ x^2 + 2\sqrt{x} &\geq 3x \end{aligned}$$

Applying this similarly to y and z gives

$$\begin{aligned} y^2 + 2\sqrt{y} &\geq 3y \\ z^2 + 2\sqrt{z} &\geq 3z \end{aligned}$$

Summing the results gives

$$\begin{aligned} x^2 + y^2 + z^2 + 2(\sqrt{x} + \sqrt{y} + \sqrt{z}) &\geq 3(x+y+z) \\ 2(\sqrt{x} + \sqrt{y} + \sqrt{z}) &\geq 3(x+y+z) - (x^2+y^2+z^2) \\ &= 3(x+y+z) - (x+y+z)^2 + 2(xy+yz+xz) \\ &= 2(xy+yz+xz) \quad \text{since } x+y+z=3 \\ \therefore \sqrt{x} + \sqrt{y} + \sqrt{z} &\geq xy+yz+xz \end{aligned}$$

Question 12

(a) (i) Suppose that a^3 is even and a is odd then $a = 2M + 1$ for some integer M, so

$$\begin{aligned} a^3 &= (2M+1)^3 \\ &= 8M^3 + 3(2M)^2 + 3(2M) + 1 \\ &= 2(4M^3 + 6M^2 + 3M) + 1 \end{aligned}$$

This suggests that a^3 is odd and contradicts the original assertion that a^3 is even. Hence, it must be that a is even.

(ii) Expanding the equality out gives

$$\begin{aligned} (a-b)^3 + a^3 &= (a+b)^3 \\ a^3 - 3a^2b + 3ab^2 - b^3 + a^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ a^3 &= 2(3a^2b + b^3) \end{aligned}$$

This suggests that a^3 is even, which implies that a must be even from part (i). Let $a = 2N$ where N is some integer. Substitute this into the above result.

$$\begin{aligned} 8N^3 &= 2(12N^2b + b^3) \\ b^3 &= 4N^2(N - 3b) \end{aligned}$$

This suggests that b^3 is even, which implies that b must be even from part (i). Hence, a and b must both be even.

(b) Since $\operatorname{cosec}\theta = \dfrac{2i}{e^{i\theta} - e^{-i\theta}}$ then applying this on $\operatorname{cosec}\alpha = \operatorname{cosec}2\alpha + \operatorname{cosec}3\alpha$ gives

$$\begin{aligned} \frac{2i}{e^{i\alpha} - e^{-i\alpha}} &= \frac{2i}{e^{2i\alpha} - e^{-2i\alpha}} + \frac{2i}{e^{3i\alpha} - e^{-3i\alpha}} \\ \frac{1}{(e^{i\alpha} - e^{-i\alpha})(e^{i\alpha} + e^{-i\alpha})} + \frac{1}{(e^{i\alpha} - e^{-i\alpha})(e^{2i\alpha} + 1 + e^{-2i\alpha})} &= \frac{1}{e^{i\alpha} - e^{-i\alpha}} \\ \frac{1}{e^{i\alpha} + e^{-i\alpha}} + \frac{1}{e^{2i\alpha} + 1 + e^{-2i\alpha}} &= 1 \\ e^{2i\alpha} + 1 + e^{-2i\alpha} + e^{i\alpha} + e^{-i\alpha} &= (e^{i\alpha} + e^{-i\alpha})(e^{2i\alpha} + 1 + e^{-2i\alpha}) \\ e^{3i\alpha} - e^{2i\alpha} + e^{i\alpha} - 1 + e^{-i\alpha} - e^{-2i\alpha} + e^{-3i\alpha} &= 0 \\ e^{6i\alpha} - e^{5i\alpha} + e^{4i\alpha} - e^{3i\alpha} + e^{2i\alpha} - e^{i\alpha} + 1 &= 0 \\ \frac{(-e^{i\alpha})^7 - 1}{-e^{i\alpha} - 1} &= 0 \\ (e^{i\alpha})^7 &= -1 \\ e^{7i\alpha} &= e^{i(2k+1)\pi} \qquad \text{for integer } k \end{aligned}$$

Equating the arguments of both sides gives $\alpha = \dfrac{(2k+1)\pi}{7}$ for integer k.

Since the domain is $-\pi < \alpha \leq \pi$ then $\alpha = \pm\dfrac{\pi}{7}, \pm\dfrac{3\pi}{7}, \pm\dfrac{5\pi}{7}$.

Note that $\pm\pi$ is not a solution as it leads to a zero denominator in the original equation.

(c) (i) There is a particular position vector $\underset{\sim}{r}$ that is perpendicular to the plane such that $|\underset{\sim}{r}|$ gives the shortest distance to the plane. Since $\underset{\sim}{n}$ is also a vector perpendicular to the plane then if $\underset{\sim}{r}$ and $\underset{\sim}{n}$ point in the same direction then

$$\begin{aligned} \underset{\sim}{r} \cdot \underset{\sim}{n} &= |\underset{\sim}{r}||\underset{\sim}{n}| \cos 0 \qquad \text{but } |\underset{\sim}{r}| = d \\ d &= \frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

If $\underset{\sim}{r}$ and $\underset{\sim}{n}$ point in opposite directions then

$$\begin{aligned} \underset{\sim}{r} \cdot \underset{\sim}{n} &= |\underset{\sim}{r}||\underset{\sim}{n}| \cos \pi \qquad \text{but } |\underset{\sim}{r}| = d \\ d &= -\frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Hence, $d = \dfrac{|ax + by + cz|}{\sqrt{a^2 + b^2 + c^2}}$.

(ii) Consider any two points on the plane with equation $ax + by + cz = m_1$. Let these points (x_1, y_1, z_1) and (x_2, y_2, z_2) be represented by the position vectors $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ respectively.

$$\begin{aligned} ax_1 + by_1 + cz_1 &= m_1 \\ ax_2 + by_2 + cz_2 &= m_1 \\ a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) &= 0 \\ \underset{\sim}{n} \cdot (\underset{\sim}{r}_1 - \underset{\sim}{r}_2) &= 0 \quad \text{where } \underset{\sim}{n} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k} \end{aligned}$$

This suggests that $\underset{\sim}{n}$ is perpendicular to the plane.

The above similarly applies to the other plane with equation $ax + by + cz = m_2$.

Since $\underset{\sim}{n}$ is perpendicular to both planes, then the two planes must be parallel by co-interior angles.

(iii) Let d_1 and d_2 be the shortest distances to the origin for the planes with equations $ax + by + cz = m_1$ and $ax + by + cz = m_2$ respectively. This means that

$$d_1 = \frac{|m_1|}{\sqrt{a^2 + b^2 + c^2}} \quad \text{and} \quad d_2 = \frac{|m_2|}{\sqrt{a^2 + b^2 + c^2}}$$

One way to obtain the distance between two parallel planes can be determined using their perpendicular distances to the origin. There are two cases to consider.

Case 1 - the origin is between the parallel planes
The distance between the parallel planes is $d_1 + d_2$, or equivalently $\dfrac{|m_1| + |m_2|}{\sqrt{a^2 + b^2 + c^2}}$.

Case 2 - the origin is not between the parallel planes
The distance between the parallel planes is $|d_1 - d_2|$, or equivalently $\dfrac{||m_1| - |m_2||}{\sqrt{a^2 + b^2 + c^2}}$.

(d) (i) Let $u = (1-x)^{n-k} \Rightarrow du = -(n-k)(1-x)^{n-k-1}\,dx$ and $dv = x^k\,dx \Rightarrow v = \dfrac{x^{k+1}}{k+1}$.

$$\begin{aligned} I_k &= \left[\frac{x^{k+1}(1-x)^{n-k}}{k+1}\right]_0^1 + \frac{n-k}{k+1}\int_0^1 x^{k+1}(1-x)^{n-k-1}\,dx \\ &= \frac{n-k}{k+1} I_{k+1} \end{aligned}$$

Replace k with $k-1$ then $I_k = \dfrac{k}{n-k+1} I_{k-1}$.

(ii) Using the result from part (i)

$$\begin{aligned} I_k &= \frac{k}{n-k+1} I_{k-1} \\ &= \frac{k}{n-k+1} \times \frac{k-1}{n-k+2} I_{k-2} \\ &= \frac{k}{n-k+1} \times \frac{k-1}{n-k+2} \times \frac{k-2}{n-k+3} I_{k-3} \\ &\;\vdots \\ &= \frac{k \times (k-1) \times (k-2) \times \cdots \times 2}{(n-k+1) \times (n-k+2) \times (n-k+3) \times \cdots \times (n-1)} I_1 \\ &= \frac{k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1}{(n-k+1) \times (n-k+2) \times (n-k+3) \times \cdots \times (n-1) \times n} I_0 \end{aligned}$$

Note that

$$\begin{aligned} &(n-k+1) \times (n-k+2) \times (n-k+3) \times \cdots \times (n-1) \times n \\ &= n \times (n-1) \times (n-2) \cdots \times (n-k+1) \times \frac{(n-k)!}{(n-k)!} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

Also

$$\begin{aligned} I_0 &= \int_0^1 (1-x)^n\,dx \\ &= \left[-\frac{(1-x)^{n+1}}{n+1}\right]_0^1 \\ &= \frac{1}{n+1} \end{aligned}$$

Hence

$$I_k = \frac{k!(n-k)!}{n!} \times \frac{1}{n+1}$$

$$\therefore \int_0^1 \binom{n}{k} x^k (1-x)^{n-k}\,dx = \frac{1}{n+1}$$

Question 13

(a) (i) First note that

$$\begin{aligned}1 + 4x^4 &= 1 + 4x^2 + 4x^4 - 4x^2\\ &= (2x^2+1)^2 - 4x^2\\ &= (2x^2+2x+1)(2x^2-2x+1)\end{aligned}$$

Note that in either quadratic factor the discriminant is negative so it is irreducible in $\mathbb{R}$. So the partial fraction decomposition for some constants a, b, c and d is

$$\begin{aligned}\frac{8}{1+4x^4} &= \frac{ax+b}{2x^2+2x+1} + \frac{cx+d}{2x^2-2x+1}\\ 8 &= (ax+b)(2x^2-2x+1) + (cx+d)(2x^2+2x+1)\end{aligned}$$

Equate the coefficients of x^3 and the constant term, which lead to

$$a + c = 0 \tag{1}$$
$$b + d = 8 \tag{2}$$

Substitute $x = \pm 1$, which lead to

$$a + b + 5c + 5d = 8 \tag{3}$$
$$-5a + 5b - c + d = 8 \tag{4}$$

From (1) and (2), substitute $a = -c$ and $b = 8 - d$ into equation (3).

$$\begin{aligned}-c + 8 - d + 5c + 5d &= 8\\ 4c + 4d &= 0\\ c + d &= 0\end{aligned}$$

From (1) and (2), substitute $a = -c$ and $b = 8 - d$ into equation (4).

$$\begin{aligned}5c + 40 - 5d - c + d &= 8\\ 4c - 4d &= -32\\ c - d &= -8\end{aligned}$$

Since $c + d = 0$, this implies that $c = -4$ and $d = 4$. Substitute these back into (1) and (2), gives $a = 4$ and $b = 4$. Hence

$$\frac{8}{1+4x^4} = \frac{4x+4}{2x^2+2x+1} + \frac{-4x+4}{2x^2-2x+1}.$$

(ii) Using the result in part (i)

$$\begin{aligned}
&\int \frac{8}{1+4x^4}\,dx \\
&= \int \left(\frac{4x+4}{2x^2+2x+1} + \frac{-4x+4}{2x^2-2x+1} \right) dx \\
&= \int \left(\frac{4x+2}{2x^2+2x+1} + \frac{-4x+2}{2x^2-2x+1} + \frac{2}{2x^2+2x+1} + \frac{2}{2x^2-2x+1} \right) dx \\
&= \int \left(\frac{4x+2}{2x^2+2x+1} - \frac{4x-2}{2x^2-2x+1} + \frac{1}{x^2+x+\frac{1}{2}} + \frac{1}{x^2-x+\frac{1}{2}} \right) dx \\
&= \int \left(\frac{4x+2}{2x^2+2x+1} - \frac{4x-2}{2x^2-2x+1} + \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}} + \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}} \right) dx \\
&= \ln(2x^2+2x+1) - \ln(2x^2-2x+1) + \frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{2}}\right) + \frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + c \\
&= \ln\left(\frac{2x^2+2x+1}{2x^2-2x+1}\right) + 2\tan^{-1}(2x+1) + 2\tan^{-1}(2x-1) + c
\end{aligned}$$

(b) (i) From the net force equation and using the result from part (a)(ii)

$$\begin{aligned}
m\frac{dv}{dt} &= -mk - 4mkv^4 \\
\frac{dv}{dt} &= -k(1+4v^4) \\
t &= -\frac{1}{k}\int \frac{dv}{1+4v^4} \\
t &= -\frac{1}{8k}\left[\ln\left(\frac{2v^2+2v+1}{2v^2-2v+1}\right) + 2\tan^{-1}(2v+1) + 2\tan^{-1}(2v-1)\right] + c
\end{aligned}$$

When $t = 0, v = u$ so

$$c = \frac{1}{8k}\left[\ln\left(\frac{2u^2+2u+1}{2u^2-2u+1}\right) + 2\tan^{-1}(2u+1) + 2\tan^{-1}(2u-1)\right]$$

The object comes to rest when $v = 0$, so the time it takes is

$$\begin{aligned}
t &= -\frac{1}{8k}\left[\ln 1 + 2\tan^{-1}1 + 2\tan^{-1}(-1)\right] + c \\
&= \frac{1}{8k}\left[\ln\left(\frac{2u^2+2u+1}{2u^2-2u+1}\right) + 2\tan^{-1}(2u+1) + 2\tan^{-1}(2u-1)\right]
\end{aligned}$$

(ii) Note that part (i) can be re-expressed in the form of t as a function of u.

From part (a)(ii)

$$\begin{aligned} t &= \frac{1}{8k}\left[\ln\left(\frac{2u^2+2u+1}{2u^2-2u+1}\right) + 2\tan^{-1}(2u+1) + 2\tan^{-1}(2u-1)\right] \\ \frac{dt}{du} &= \frac{1}{k(1+4u^4)} > 0 \quad \text{given } k > 0 \end{aligned}$$

This means that as u increases, the time taken for the particle to come to rest from the two resistance forces also increases (i.e. there is no stationary point). Hence, the maximum possible time that the object can take to come to rest occurs when $u \to \infty$. This implies that

$$\begin{aligned} \frac{2u^2+2u+1}{2u^2-2u+1} = \frac{2+\frac{2}{u}+\frac{1}{u^2}}{2-\frac{2}{u}+\frac{1}{u^2}} &\to 1 \\ \tan^{-1}(2u+1) &\to \frac{\pi}{2} \\ \tan^{-1}(2u-1) &\to \frac{\pi}{2} \\ t &\to \frac{1}{8k}(\ln 1 + \pi + \pi) = \frac{\pi}{4k} \end{aligned}$$

This means that the maximum possible time that the object can take to come to rest is $\frac{\pi}{4k}$ (technically an infinitesimal amount below that value).

(c) Given $P(z) = az^4 + ibz^3 + cz^2 + idz + e$ then

$$\begin{aligned} P(iz) &= a(iz)^4 + ib(iz)^3 + c(iz)^2 + id(iz) + e \\ &= az^4 + bz^3 - cz^2 - dz + e \end{aligned}$$

If $x + iy$ is a root of $P(z)$ then $i(x+iy)$, or equivalently $-y + ix$, is a root of $P(iz)$.

However, $P(iz)$ has real coefficients so that means that $\overline{-y+ix}$, or equivalently $-y - ix$, is also a root of $P(iz)$.

If $-y - ix$ is a root of $P(iz)$ then $\frac{-y-ix}{i}$, or equivalently $-x + iy$, is a root of $P(z)$.

(d) (i) Let the centre of the sphere have coordinates (a, b, c) and be represented by the position vector $\underset{\sim}{c}_0$. Since the sphere has radius 1 then for all t

$$
\begin{aligned}
|\underset{\sim}{r} - \underset{\sim}{c}_0| &= 1 \\
\left(\frac{\cos 2t}{2} - a\right)^2 + \left(\frac{\sin 2t}{2} - b\right)^2 + (\sin t - c)^2 &= 1 \\
\frac{\cos^2 2t}{4} - a\cos 2t + a^2 + \frac{\sin^2 2t}{4} - b\sin 2t + b^2 + \sin^2 t - 2c\sin t + c^2 &= 1 \\
a^2 + b^2 + c^2 - a\cos 2t - b\sin 2t - 2c\sin t + \sin^2 t &= \frac{3}{4}
\end{aligned}
$$

Since this relation is true for all t, then substitute $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, which leads to

$$
\begin{aligned}
a^2 + b^2 + c^2 - a &= \frac{3}{4} && (5) \\
a^2 + b^2 + c^2 - b - c\sqrt{2} &= \frac{1}{4} && (6) \\
a^2 + b^2 + c^2 + a - 2c &= -\frac{1}{4} && (7) \\
a^2 + b^2 + c^2 + b - c\sqrt{2} &= \frac{1}{4} && (8)
\end{aligned}
$$

Taking (6) $-$ (8), gives $b = 0$.

Taking (5) + (7) gives

$$
\begin{aligned}
2(a^2 + b^2 + c^2) - 2c &= \frac{1}{2} \\
a^2 + b^2 + c^2 - c &= \frac{1}{4}
\end{aligned}
$$

Equate this with (8) gives $c = c\sqrt{2}$ which implies $c = 0$.

Taking these results with (5) $-$ (7) gives $a = -\frac{1}{2}$.

Hence, the centre of the sphere is $\left(-\frac{1}{2}, 0, 0\right)$.

(ii) If the the axis of the cylinder is the z-axis then for a given point $\left(\frac{\cos 2t}{2}, \frac{\sin 2t}{2}, \sin t\right)$, its distance from the corresponding point on the z-axis, $(0, 0, \sin t)$ must be the radius of the cylinder, which will be denoted as r.

$$
\begin{aligned}
\left(\frac{\cos 2t}{2} - 0\right)^2 + \left(\frac{\sin 2t}{2} - 0\right)^2 + (\sin t - \sin t)^2 &= r^2 \\
\frac{\cos^2 2t}{4} + \frac{\sin^2 2t}{4} &= r^2 \\
r &= \frac{1}{2} \quad \text{noting that } r > 0
\end{aligned}
$$

Question 14

(a) (i) From the acceleration equation of A

$$\begin{aligned}\ddot{x}_A &= -n^2(x_A - b)\\ \frac{d}{dx_A}\left(\frac{\dot{x}_A^2}{2}\right) &= -n^2(x_A - b)\\ \frac{\dot{x}_A^2}{2} &= -n^2\left(\frac{x_A^2}{2} - bx_A\right) + c \qquad \text{when } x_A = b, \dot{x}_A = n \Rightarrow c = -\frac{n^2b^2}{2} + \frac{n^2}{2}\\ \dot{x}_A^2 &= -n^2(x_A^2 - 2bx_A + b^2 - 1)\\ &= n^2(1 - (x_A - b)^2)\end{aligned}$$

(ii) From the acceleration equation, the general displacement equation for particle A with centre $x_A = b$ is $x_A = b + a_A\cos(nt + \alpha_A)$ for constants a_A and α_A.

The extremities of the simple harmonic motion occur when $\dot{x}_A = 0$, which implies $x_A = b - 1$ and $x_A = b + 1$ from part (i). This suggests that $a_A = 1$.

When $t = 0, x_A = b$, which implies $\cos\alpha_A = 0$ or $\alpha_A = \frac{\pi}{2}$.

Hence, $x_A = b + \cos\left(nt + \frac{\pi}{2}\right)$.

Particle B has a similar period to particle A, but is centred about the origin according to the acceleration equation.

Since the particle has a displacement of 1 at rest, then the amplitude of particle B's motion must be 1.

This means that the displacement equation for particle B can be reduced to $x_B = \cos(nt + \alpha_B)$.

When $t = 0$, $x_B = 1$, which implies $\cos\alpha_B = 1$ or $\alpha_B = 0$.

Hence, $x_B = \cos(nt)$.

(iii) For the particles to not collide, $|x_A - x_B| > 0$ at all times.

$$\begin{aligned}|x_A - x_B| &= \left|b + \cos\left(nt + \frac{\pi}{2}\right) - \cos(nt)\right|\\ &= |b - \sin(nt) - \cos(nt)|\\ &= \left|b - \sqrt{2}\left(\sin(nt)\cos\frac{\pi}{4} - \cos(nt)\sin\frac{\pi}{4}\right)\right|\\ &= \left|b - \sqrt{2}\sin\left(nt + \frac{\pi}{4}\right)\right|\end{aligned}$$

As t varies, $\sqrt{2}\sin\left(nt + \frac{\pi}{4}\right)$ has a range of $\left[-\sqrt{2}, \sqrt{2}\right]$. To guarantee no collision, the constant b must exceed that range. Hence, $|b| > \sqrt{2}$.

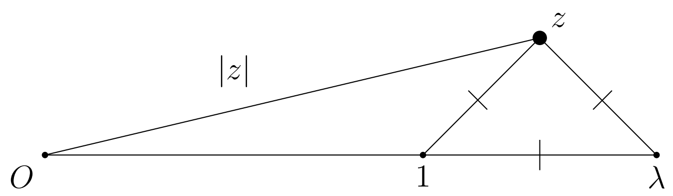
(b) If z, z^2 and λz are in an equilateral triangle then

$$\begin{aligned}|z^2 - z| &= |z^2 - \lambda z|\\ |z||z-1| &= |z||z-\lambda|\\ |z-1| &= |z-\lambda|\end{aligned}$$

Also, given $\lambda > 1$

$$\begin{aligned}|z^2 - z| &= |\lambda z - z|\\ |z||z-1| &= |z||\lambda-1|\\ |z-1| &= \lambda - 1\end{aligned}$$

This implies that $1, \lambda$ and z make an equilateral triangle on the complex plane.



Since $|z| = \sqrt{2}$ then by the cosine rule

$$\begin{aligned}(\sqrt{2})^2 &= (\lambda-1)^2 + \lambda^2 - 2\lambda(\lambda-1)\cos\frac{\pi}{3}\\ \lambda^2 - \lambda - 1 &= 0\\ \lambda &= \frac{1 \pm \sqrt{5}}{2} \qquad \text{but } \lambda > 1\\ \lambda &= \frac{1+\sqrt{5}}{2}\end{aligned}$$

(c) (i) A counterexample is $f(x) = -x$. The inverse function is $f^{-1}(x) = -x$. This only satisfies $f(x) = x$ at $x = 0$, but $f(x) = f^{-1}(x)$ elsewhere.

(ii) The solution to $f(x) = f^{-1}(x)$ is found at the point of intersection of $y = f(x)$ and $y = f^{-1}(x)$.

Suppose that a solution (x_0, y_0) lies above the line $y = x$, that is $y_0 > x_0$.

If $f(x)$ is increasing then this means that $f(y_0) > f(x_0)$.

However, $y_0 = f(x_0)$ and $y_0 = f^{-1}(x_0)$, or equivalently $x_0 = f(y_0)$. Substituting this into the inequality gives $x_0 > y_0$, which is a contradiction.

A similar argument applies for the case where a solution (x_0, y_0) lies below the line $y = x$, that is $y_0 < x_0$.

This means that $y_0 = x_0$, so the solutions of $f(x) = f^{-1}(x)$ must satisfy $f(x) = x$.

(iii) Let $f(x) = \dfrac{x^3+1}{2}$, then its inverse is the solution to

$$\begin{aligned} x &= \frac{y^3+1}{2} \\ y &= \sqrt[3]{2x-1} \end{aligned}$$

This means that the solutions to $\sqrt[3]{2x-1} = \dfrac{x^3+1}{2}$ can be found by solving for $\dfrac{x^3+1}{2} = x$, or equivalently $x^3 - 2x + 1 = 0$.

By inspection, $x = 1$ is a solution to this cubic polynomial. Let the other roots be α and β. By sum and product of the roots

$$\begin{aligned} \alpha + \beta + 1 &= 0 \\ \alpha + \beta &= -1 \\ \alpha\beta(1) &= -1 \\ \alpha\beta &= -1 \end{aligned}$$

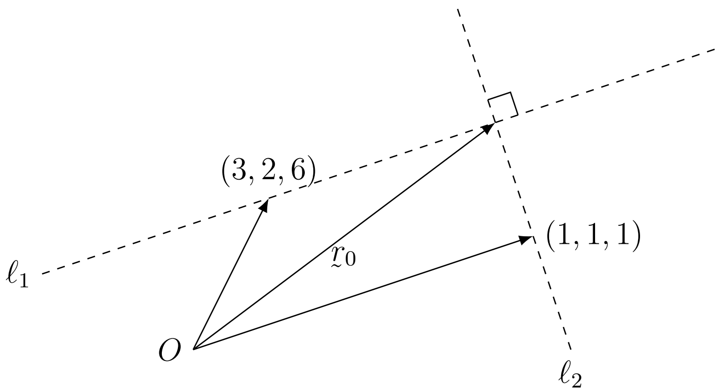
The other two roots are the solutions of $x^2 + x - 1 = 0$.

Hence, the solutions to the equation are $x = 1, \dfrac{-1 \pm \sqrt{5}}{2}$.

Question 15

(a) Let $\underset{\sim}{r}_0$ be the position vector of the point of intersection between ℓ_1 and ℓ_2. This corresponds to a particular value of λ, which needs to be solved for.

Consider the illustration below.



NOT TO SCALE

Since $(3, 2, 6)$ is a point on ℓ_1 (when $\lambda = 0$) then the vector $\underset{\sim}{r}_0 - (3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k})$ is perpendicular to the vector $\underset{\sim}{r}_0 - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$, so

$$\begin{aligned}
[\underset{\sim}{r}_0 - (3\underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k})] \cdot [\underset{\sim}{r}_0 - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})] &= 0 \\
\lambda[2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}] \cdot [\underset{\sim}{r}_0 - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})] &= 0 \\
[2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}] \cdot [(3 + 2\lambda)\underset{\sim}{i} + (2 + \lambda)\underset{\sim}{j} + (6 + \lambda)\underset{\sim}{k} - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})] &= 0 \\
[2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}] \cdot [(2 + 2\lambda)\underset{\sim}{i} + (1 + \lambda)\underset{\sim}{j} + (5 + \lambda)\underset{\sim}{k}] &= 0 \\
2(2 + 2\lambda) + (1 + \lambda) + (5 + \lambda) &= 0 \\
10 + 6\lambda &= 0 \\
\lambda &= -\frac{5}{3}
\end{aligned}$$

This means that

$$\begin{aligned}
\underset{\sim}{r}_0 - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) &= (2 + 2\lambda)\underset{\sim}{i} + (1 + \lambda)\underset{\sim}{j} + (5 + \lambda)\underset{\sim}{k} \\
&= -\frac{4}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} + \frac{10}{3}\underset{\sim}{k} \\
&= -\frac{2}{3}\left(2\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}\right)
\end{aligned}$$

Note that $\underset{\sim}{r}_0 - (\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$ is parallel to ℓ_2 and is a scalar multiple of $2\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$.

Hence, ℓ_2 must also be parallel to the position vector $\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$.

Remark: Unlike in two-dimensions, there are actually infinitely many different direction vectors that can be perpendicular to ℓ_1 in three dimensional space, but not parallel to ℓ_2. Simply showing the dot product is zero is not sufficient enough to specify the particular direction vector for ℓ_2.

(b) (i) Let $f(x) = \sin x - x$ for $x \geq 0$, then

$$f'(x) = \cos x - 1 \leq 0$$

Since $f(x)$ is non-increasing then its maximum value occurs in the minimum value of the domain, which is $x = 0$. Hence for $x > 0$

$$\begin{aligned} f(x) &< f(0) \\ \sin x - x &< 0 \\ \therefore \sin x &< x \end{aligned}$$

(ii) First note that since the trigonometric functions have a period of 2π then considering a selected domain, say $x \in (-\pi, \pi]$, is sufficient to understand the properties across all real x.

However, since $\cos x$ is an even function and $\sin x$ is an odd function

$$\begin{aligned} \cos(\sin(-x)) &= \cos(-\sin x) \\ &= \cos(\sin x) \\ \text{also} \quad \sin(\cos(-x)) &= \sin(\cos x) \end{aligned}$$

This suggests symmetry between the positive and negative domain for either side of the inequality, which means the focus can be further narrowed to the domain $x \in [0, \pi]$ to understand the properties across all real x.

Since $\cos x$ is an even function and $-1 \leq \sin x \leq 1$ then $\cos(\sin x) \geq 0$ for all real x. Also, $-1 \leq \cos x < 0$ in the domain $x \in \left(\frac{\pi}{2}, \pi\right]$, which suggests that $\sin(\cos x) < 0$ in this domain. This implies that $\sin(\cos x) < 0 \leq \cos(\sin x)$ for $x \in \left(\frac{\pi}{2}, \pi\right]$.

For the domain $x \in \left(0, \frac{\pi}{2}\right)$, using the result in part (i)

$$\begin{aligned} \sin x &< x \\ \cos(\sin x) &> \cos x \qquad \text{since } \cos x \text{ is a decreasing function for } x \in \left(0, \frac{\pi}{2}\right) \end{aligned}$$

However, since $\cos x > 0$ in this domain (i.e. a positive real number) then replacing x with $\cos x$ on the result of part (i) gives $\sin(\cos x) < \cos x$. Hence, $\sin(\cos x) < \cos x < \cos(\sin x)$ in the domain $x \in \left(0, \frac{\pi}{2}\right)$.

The remaining parts are the values exactly at $x = 0, \frac{\pi}{2}$.

When $x = 0$, the inequality holds because $\sin 1 < 1$.

When $x = \frac{\pi}{2}$, the inequality holds because $\cos 1 > 0$.

Since the inequality holds for $x \in [0, \pi]$ then by symmetry and periodicity $\cos(\sin x) > \sin(\cos x)$ also holds for all real x.

(c) (i) From the net force on the horizontal component

$$\begin{aligned}
m\ddot{x}_A &= -mk\dot{x}_A \\
\frac{d\dot{x}_A}{dt} &= -k\dot{x}_A \\
\ln \dot{x}_A &= -kt + c \qquad \text{when } t = 0, \dot{x}_A = u\cos\theta \Rightarrow c = \ln(u\cos\theta) \\
\dot{x}_A &= u\cos\theta e^{-kt} \\
x_A &= -\frac{u\cos\theta}{k}e^{-kt} + c \qquad \text{when } t = 0, x_A = 0 \Rightarrow c = \frac{u\cos\theta}{k} \\
x_A &= \frac{u\cos\theta}{k}(1 - e^{-kt})
\end{aligned}$$

From the net force on the vertical component

$$\begin{aligned}
m\ddot{y}_A &= -mg - mk\dot{y}_A \\
\frac{d\dot{y}_A}{dt} &= -(g + k\dot{y}_A) \\
t &= -\frac{1}{k}\ln(g + k\dot{y}_A) + c \qquad \text{when } t = 0, \dot{y}_A = u\sin\theta \Rightarrow c = \frac{1}{k}\ln(g + ku\sin\theta) \\
t &= -\frac{1}{k}\ln\left(\frac{g + k\dot{y}_A}{g + ku\sin\theta}\right) \\
\dot{y}_A &= \frac{(g + ku\sin\theta)}{k}e^{-kt} - \frac{g}{k} \\
y_A &= -\frac{(g + ku\sin\theta)}{k^2}e^{-kt} - \frac{gt}{k} + c \qquad \text{when } t = 0, y_A = 0 \Rightarrow c = \frac{g + ku\sin\theta}{k^2} \\
y_A &= \frac{(g + ku\sin\theta)}{k^2}(1 - e^{-kt}) - \frac{gt}{k}
\end{aligned}$$

(ii) For the vertical component of object B

$$\begin{aligned}
\ddot{y}_B &= -g \\
\dot{y}_B &= -gt + c \qquad \text{when } t = 0, \dot{y}_B = u\sin\theta \Rightarrow c = u\sin\theta \\
\dot{y}_B &= -gt + u\sin\theta
\end{aligned}$$

Let t_B be the time take for object B to reach maximum height, which occurs when $\dot{y}_B = 0$ so $t_B = \dfrac{u\sin\theta}{g}$. Substitute this into the equations of $\dot{y}_A$ and y_A.

$$\begin{aligned}
\dot{y}_A &= \frac{(g + ku\sin\theta)}{k}e^{-kt_B} - \frac{g}{k} \\
&= \frac{g}{k}\left[(1 + kt_B)\, e^{-kt_B} - 1\right] \\
y_A &= \frac{(g + ku\sin\theta)}{k^2}(1 - e^{-kt_B}) - \frac{gt_B}{k} \\
&= \frac{g}{k^2}\left[(1 + kt_B)(1 - e^{-kt_B}) - kt_B\right] \\
&= \frac{g}{k^2}\left[1 - (1 + kt_B)e^{-kt_B}\right]
\end{aligned}$$

Let $f(x) = 1 - (1+x)e^{-x}$.

$$\begin{aligned} f'(x) &= (1+x)e^{-x} - e^{-x} \\ &= xe^{-x} \end{aligned}$$

Stationary points occur at $f'(x) = 0$ which gives $x = 0$. Since $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$, there is a minimum turning point at $x = 0$. Since $f(x)$ is continuous for all real x and there are no other stationary points then this also where the absolute minimum occurs. Hence

$$\begin{aligned} f(x) &\geq f(0) \\ 1 - (1+x)e^{-x} &\geq 0 \end{aligned}$$

Note that in particular, if x is non-zero then $1 - (1+x)e^{-x} > 0$.

Let $x = kt_B$, noting that $k > 0$ and $t_B > 0$ the above inequality suggests that when object B is at its maximum height

- $y_A > 0$ which means object A has yet to hit the ground
- $\dot{y}_A < 0$ which means object A is on the downwards trajectory (i.e. already past its maximum height)

Alternative method:

First derive the time to maximum height for object B in the same way as above to get $t_B = \dfrac{u\sin\theta}{g}$.

Let t_A be the time taken for object A to reach maximum height, which occurs when $\dot{y}_A = 0$.

$$\begin{aligned} \frac{(g + ku\sin\theta)}{k}e^{-kt_A} - \frac{g}{k} &= 0 \\ e^{-kt_A} &= \frac{g}{g + ku\sin\theta} \qquad \text{but } t_B = \frac{u\sin\theta}{g} \\ &= \frac{1}{1 + kt_B} \\ t_A &= \frac{1}{k}\ln(1 + kt_B) \end{aligned}$$

Let t_G be the time taken for object A to reach the ground, which occurs when $y_A = 0$

$$\begin{aligned} \frac{(g + ku\sin\theta)}{k^2}(1 - e^{-kt_G}) - \frac{gt_G}{k} &= 0 \\ e^{-kt_G} &= 1 - \frac{gkt_G}{g + ku\sin\theta} \qquad \text{but } t_B = \frac{u\sin\theta}{g} \\ e^{-kt_G} &= 1 - \frac{kt_G}{1 + kt_B} \\ &= \frac{1 + k(t_B - t_G)}{1 + kt_B} \end{aligned}$$

However, since object A reaches its maximum height before it hits the ground $t_G > t_A \Rightarrow e^{-kt_G} < e^{-kt_A}$. Noting that $k > 0$ and $t_B > 0$

$$\begin{aligned}\frac{1+k(t_B-t_G)}{1+kt_B} &< \frac{1}{1+kt_B}\\ k(t_B-t_G) &< 0\\ \therefore t_B &< t_G\end{aligned}$$

Consider the function $f(x) = x - \ln(1+x)$

$$\begin{aligned}f'(x) &= 1 - \frac{1}{1+x}\\ f''(x) &= \frac{1}{(1+x)^2}\end{aligned}$$

Stationary points occur at $f'(x) = 0$ which gives $x = 0$ and $f''(0) = 1 > 0$, which suggests that a minimum turning point occurs at $x = 0$. Since $f(x)$ is continuous for all real x and there are no other stationary points then this also where the absolute minimum occurs. Hence

$$\begin{aligned}f(x) &\geq f(0)\\ x - \ln(1+x) &\geq 0\end{aligned}$$

Note that in particular, if x is non-zero then $x - \ln(1+x) > 0$.

Let $x = kt_B$, noting that $k > 0$ and $t_B > 0$ the above inequality suggests

$$\begin{aligned}&kt_B > \ln(1+kt_B) \quad \text{but } t_A = \frac{1}{k}\ln(1+kt_B)\\ &\therefore t_B > t_A\end{aligned}$$

Putting the inequalities together gives $t_A < t_B < t_G$. This implies that the time taken for object B to reach its maximum height occurs after object A has reached its maximum height but before object A hits the ground.

Question 16

(a) Suppose that $\sqrt{3} = \dfrac{p}{q}$ where p and q are integers with no common factors.

$$\begin{aligned}\frac{p^2}{q^2} &= 3\\ p^2 &= 3q^2\end{aligned}$$

This suggests that p^2 is divisible by 3. If p was not divisible by 3 then p^2 would not be divisible by 3. Hence, p is divisible by 3. That is, $p = 3M$ for some integer M.

$$\begin{aligned}9M^2 &= 3q^2\\ 3M^2 &= q^2\end{aligned}$$

This suggests that q^2 is divisible by 3. If q was not divisible by 3 then q^2 would not be divisible by 3. Hence, q is divisible by 3. That is, $q = 3N$ for some integer N.

However, this implies that p and q have a common factor of 3 which contradicts the initial assertion that $\sqrt{3} = \dfrac{p}{q}$ where p and q are integers with no common factors.

Hence, $\sqrt{3}$ is irrational.

Alternative method:

Suppose that $\sqrt{3} = \dfrac{p}{q}$ where p and q are integers.

$$\begin{aligned}\sqrt{3} &= \sqrt{3} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}\\ &= \frac{3-\sqrt{3}}{\sqrt{3}-1}\\ \therefore \frac{p}{q} &= \frac{3-\frac{p}{q}}{\frac{p}{q}-1}\\ &= \frac{3q-p}{p-q}\end{aligned}$$

This suggests that $\sqrt{3}$ can also be written in the form $\dfrac{a}{b}$, where $a = 3q - p$ and $b = p - q$.

Since $\sqrt{3} > \dfrac{3}{2}$ then $2p > 3q \Rightarrow 3q - p < p$, or equivalently $a < p$.

Since $\sqrt{3} < 2$ then $p < 2q \Rightarrow p - q < q$, or equivalently $b < q$.

This suggests that it is always possible to construct a new quotient $\dfrac{a}{b}$ from an existing quotient $\dfrac{p}{q}$, where the numerator and denominator have each decreased (i.e. $a < p$ and $b < q$).

Consequently, it is possible to construct another quotient from $\frac{a}{b}$, say $\frac{c}{d}$, where the numerator and denominator have each decreased (i.e. $c < a$ and $d < b$).

Applying this logic sequentially, then there must be an infinite number of quotients possible for $\sqrt{3}$ where the numerator and denominator continue to decrease relative to the original specification of $\frac{p}{q}$.

However, since $\sqrt{3} < 3$ then $p < 3q$, or equivalently $a > 0$.

Similarly, since $\sqrt{3} > 1$ then $p > q$, or equivalently $b > 0$.

This suggests that the numerator and denominator will always be positive. There are only a finite number of positive integers below p and similarly only a finite number of positive integers below q. This is a contradiction, so the original assumption that $\sqrt{3} = \frac{p}{q}$ is false.

Hence, $\sqrt{3}$ is irrational.

(b) (i) Since $x = 0$ is not a root of $P(x)$ then $P(x) = 0 \iff x^{-2}P(x) = 0$.

$$\begin{aligned} x^{-2}(x^2 - 8x + 6 - 8x^{-1} + x^{-2}) &= 0 \\ x^2 + 2 + x^{-2} - 8(x + x^{-1}) + 4 &= 0 \\ y^2 - 8y + 4 &= 0 \end{aligned}$$

(ii) Using the result in part (i)

$$\begin{aligned} y^2 - 8y + 4 &= 0 \\ y^2 - 8y + 16 &= 12 \\ (y-4)^2 &= (2\sqrt{3})^2 \\ (y - 4 - 2\sqrt{3})(y - 4 + 2\sqrt{3}) &= 0 \\ (x + x^{-1} - 4 - 2\sqrt{3})(x + x^{-1} - 4 + 2\sqrt{3}) &= 0 \\ (x^2 - (4 + 2\sqrt{3})x + 1)(x^2 - (4 - 2\sqrt{3})x + 1) &= 0 \end{aligned}$$

$$\therefore P(x) = (x^2 - (4 + 2\sqrt{3})x + 1)(x^2 - (4 - 2\sqrt{3})x + 1)$$

(iii) For the quadratic factor $x^2 - (4+2\sqrt{3})x + 1$, the discriminant is

$$\begin{aligned}\Delta &= 4(2+\sqrt{3})^2 - 4 \\ &= 4(7+4\sqrt{3}) - 4 \\ &= 4(6+4\sqrt{3}) > 0\end{aligned}$$

This suggests that the roots are real, so the non-real root cannot come from this quadratic factor.

For the other quadratic factor $x^2 - (4-2\sqrt{3})x + 1$, the discriminant is

$$\begin{aligned}\Delta &= 4(2-\sqrt{3})^2 - 4 \\ &= 4(7-4\sqrt{3}) - 4 \\ &= 4(6-4\sqrt{3}) < 0\end{aligned}$$

This suggests that the roots are non-real, so their exact values are

$$\begin{aligned}x &= \frac{4-2\sqrt{3} \pm 2\sqrt{6-4\sqrt{3}}}{2} \\ &= 2-\sqrt{3} \pm \sqrt{6-4\sqrt{3}} \\ &= 2-\sqrt{3} \pm i\sqrt{4\sqrt{3}-6}\end{aligned}$$

Since α is defined as the non-real root with the positive imaginary part and the quadratic polynomial has real coefficients, then its roots are

$$\begin{aligned}\alpha &= 2-\sqrt{3} + i\sqrt{4\sqrt{3}-6} \\ \overline{\alpha} &= 2-\sqrt{3} - i\sqrt{4\sqrt{3}-6}\end{aligned}$$

This means that the product of the roots $\alpha\overline{\alpha} = 1 \Rightarrow |\alpha|^2 = 1$. Hence, $|\alpha| = 1$.

(iv) If ω is an m^{th} root of unity, where m is a positive integer, then it satisfies the equation $z^m = 1$. When the roots are plotted on the complex plane, they lie on the unit circle and their arguments are multiples of $\frac{2\pi}{m}$.

When ω is raised to *any* integer power, its argument is still a multiple of $\frac{2\pi}{m}$ by De-Moivre's theorem, which implies that it would coincide with one of the existing roots of unity.

This consequently implies that any powers of an m^{th} root of unity will have at most m distinct values. Therefore, the possible set of values of T_n is finite.

(v) Let $\alpha^{2^n} = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$

$$\begin{aligned}
\alpha^{2^{n+1}} &= (\alpha^{2^n})^2 \\
&= x_n^2 - y_n^2 + 2ix_ny_n \quad \text{but from part (iii), } |\alpha| = 1 \Rightarrow x_n^2 + y_n^2 = 1 \\
&= 2x_n^2 - 1 + 2ix_ny_n \\
\mathrm{Re}(\alpha^{2^{n+1}}) &= 2x_n^2 - 1 \quad \text{but } \mathrm{Re}(\alpha^{2^n}) = a_n - b_n\sqrt{3} \\
a_{n+1} - b_{n+1}\sqrt{3} &= 2(a_n - b_n\sqrt{3})^2 - 1 \\
&= 2a_n^2 + 6b_n^2 - 1 - 4a_nb_n\sqrt{3}
\end{aligned}$$

Hence, $a_{n+1} = 2a_n^2 + 6b_n^2 - 1$ and $b_{n+1} = 4a_nb_n$.

Alternative method:

Let $\alpha^{2^n} = \cos\theta + i\sin\theta$ noting that $|\alpha| = 1$.

$$\begin{aligned}
\alpha^{2^{n+1}} &= (\alpha^{2^n})^2 \\
&= \cos 2\theta + i\sin 2\theta \\
\mathrm{Re}(\alpha^{2^{n+1}}) &= \cos 2\theta \\
&= 2\cos^2\theta - 1 \quad \text{but } \mathrm{Re}(\alpha^{2^n}) = a_n - b_n\sqrt{3} \\
a_{n+1} - b_{n+1}\sqrt{3} &= 2(a_n - b_n\sqrt{3})^2 - 1 \\
&= 2a_n^2 + 6b_n^2 - 1 - 4a_nb_n\sqrt{3}
\end{aligned}$$

Hence, $a_{n+1} = 2a_n^2 + 6b_n^2 - 1$ and $b_{n+1} = 4a_nb_n$.

(vi) When $n = 0$, $\mathrm{Re}(\alpha^{2^0}) = 2 - \sqrt{3}$, which implies that $a_0 = 2$ and $b_0 = 1$.

Also,

$$\begin{aligned}
\mathrm{Re}(\alpha^{2^1}) &= 2(2 - \sqrt{3})^2 - 1 \\
&= 13 - 8\sqrt{3}
\end{aligned}$$

This implies that $a_1 = 13$ and $b_1 = 8$. Since $a_1 > a_0$ and $b_1 > b_0$ then the statement is true for $n = 0$.

Assume the statement is true for some positive integer $n = k$.

$$a_{k+1} > a_k \qquad \text{and} \qquad b_{k+1} > b_k$$

Required to prove that the statement is true for $n = k + 1$.

$$a_{k+2} > a_{k+1} \qquad \text{and} \qquad b_{k+2} > b_{k+1}$$

Using the results in part (v)

$$\begin{aligned}
b_{k+2} &= 4a_{k+1}b_{k+1} \quad \text{but } a_{k+1} \geq 1 \text{ since it is a positive integer} \\
&\geq 4b_{k+1} \\
&> b_{k+1} \\
a_{k+2} &= 2a_{k+1}^2 + 6b_{k+1}^2 - 1 \\
&= a_{k+1}^2 + a_{k+1}^2 + 6b_{k+1}^2 - 1
\end{aligned}$$

For the first term, use the fact that $a_{k+1} \geq 1 \Rightarrow a_{k+1}^2 \geq a_{k+1}$, since a_{k+1} is a positive integer.

For the remaining terms, use the fact that $a_{k+1} \geq 1$ and $b_{k+1} \geq 1$ since a_{k+1} and b_{k+1} are positive integers.

This implies that $a_{k+2} \geq a_{k+1} + 6 > a_{k+1}$.

Since the statement is true for $n = 0$, then by mathematical induction it is true for all integers $n \geq 0$.

(vii) Assume that there exists specific integers m and n where $m > n \geq 0$ such that the sequence repeats, giving $S_m = S_n$. This means that their real parts must also be equal, that is

$$\begin{aligned}
a_m - b_m\sqrt{3} &= a_n - b_n\sqrt{3} \\
\sqrt{3} &= \frac{a_m - a_n}{b_m - b_n}
\end{aligned}$$

From part (vi), $a_m > a_n$ and $b_m > b_n$ so the numerator and denominator are both positive, making it a well-defined rational number.

However, this is a contradiction to part (a) where $\sqrt{3}$ is an irrational number. Hence, such a value of m cannot exist and so the real part of the sequence S_n, and consequently S_n itself, can never repeat.

(viii) Consider the specific case of $P(x) = x^4 - 8x^3 + 6x^2 - 8x + 1$ which is monic, has integer coefficients and non-real roots with modulus of 1. If α were a root of unity, then from part (iv), the sequence $T_n = \alpha^{2^n}$ has only finitely many values.

This implies that there exists some pair of distinct integers m and n where $T_m = T_n$. However, from part (vii) such a pair of distinct integers is not possible, which gives a contradiction. Hence, the initial assertion that α is a root of unity must be false.

This counterexample for $P(x) = x^4 - 8x^3 + 6x^2 - 8x + 1$ proves the more general statement on any monic polynomial $P(x)$ with integer coefficients, is false.