NORMANHURST BOYS HIGH SCHOOL

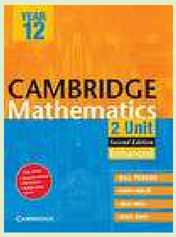
MATHEMATICS ADVANCED
MATHEMATICS EXTENSION 1
(YEAR 11 COURSE)

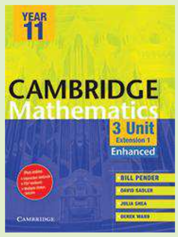
Topic summary and exercises:

Ⓐ **Radians**

(X1) **Further Trigonometric Identities**

With references to





Name: ....

Initial version by H. Lam, February 2014. Last updated June 15, 2023.
Various corrections by students & members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under 🅭 CC BY 2.0.

All textbook references:

- (A) Pender, Sadler, Shea, and Ward (2018)
- (x1) Pender, Sadler, Shea, and Ward (2019, 2019 New Syllabus)
- (x1) Pender, Sadler, Shea, and Ward (1999, 1982 Legacy Syllabus)

Symbols used

⚠ Beware! Heed warning.

(A) Mathematics Advanced content/textbook.

(x1) Mathematics Extension 1 content/textbook.

Literacy: note new word/phrase.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA11-4 uses the concepts and techniques of periodic functions in the solutions of trigonometric equations or proof of trigonometric identities

ME11-3 applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems

Syllabus subtopics

MA-T1 (T1.2) Trigonometry and measure of angles

ME-T2 Further Trigonometric Identities - first half

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from the relevant textbook(s) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Radian measure

Learning Goal(s)

Knowledge
What are *radians* as a measurement for an angle

Skills
Convert between radians and degrees

Understanding
Why radians are needed

By the end of this section am I able to:

8.1 Define and use radian measure and understand its relationship with degree measure

8.2 Know how to set a calculator in radians

8.3 Understand the unit circle definition of $\sin\theta$, $\cos\theta$ and $\tan\theta$ and periodicity using radians

8.4 Solve problems involving trigonometric ratios of angles of any magnitude in both degrees and radians.

8.6 Derive the formula for arc length $l = r\theta$ and for the area of a sector of a circle, $A = \frac{1}{2}r^2\theta$

8.7 Solve problems involving sector areas, arc lengths and combinations of either areas or lengths.

1.1 Curve sketching problem

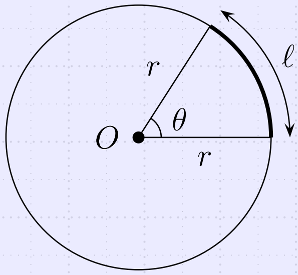
Example 1

How many solutions are there to the graph of $y = \sin x$ and $y = x$?

- If **degrees** is used, how is $y = \sin x$ sketched so that $y = x$ can also be sketched?

1.2 Radian/Degree conversions

Definition 1



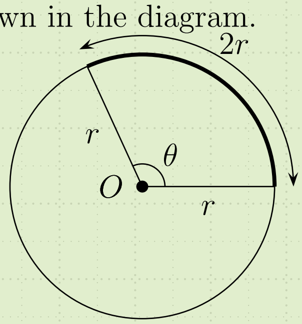
- An angle measuring θ *radians* is the ratio of the ....arc.... ....length.... to the ....radius....

$$\theta = \frac{\ell}{r} \tag{1.1}$$

- One radian arises when the ....arc.... ....length.... is the same length as the ....radius.... of a circle.

Example 2

Find the size of the angle θ shown in the diagram.



Laws/Results

Given $C = 2\pi r$ (circumference), then C can be treated as a special arc. This results in $\theta = 2\pi$, i.e. in 360°, there are 2π radians, or

$$\pi \leftrightarrow 180^\circ$$

Definition 2

From equation (1.1), the length of a circular arc ℓ is

$$\ell = r\theta$$

where θ is in radians.

1.2.1 Examples

Example 3

Convert the following angles in degrees to radians. For the first four, find correct two decimal places and for the remaining, leave your answer in terms of π.

1.	$13°$	**3.**	$136°$	**5.**	$30°$	**7.**	$60°$
2.	$57°$	**4.**	$310°$	**6.**	$45°$	**8.**	$90°$

Answer: 1. 0.23 **2.** 0.99 **3.** 2.37 **4.** 5.41

Example 4

Convert the following angles in radians to degrees, correct to the nearest minute where applicable.

1.	$\frac{\pi}{6}$	**3.**	$\frac{5\pi}{8}$	**5.**	1.56	**7.**	5.76
2.	$\frac{2\pi}{3}$	**4.**	3π	**6.**	2	**8.**	3.77

Important note

Draw picture!

Example 5

The arc of a circle subtends an angle of $100°$ at the centre. If the radius is 12 cm, calculate the exact length of the arc. **Answer:** $\frac{20\pi}{3}$

Example 6

The minute hand of a clock is 20 cm long. Calculate

(a) the angle (in radians) that the minute hand sweeps through in 16 minutes.

(b) the exact arc length along which the tip of the hand travels in 16 minutes.

(c) the shortest distance between the initial and final positions of the tip of the hand, correct to 2 decimal places.

Answer: (a) $\frac{8\pi}{15}$ (b) $\frac{32\pi}{3}$ (c) 29.73 cm

Further exercises

(A) **Ex 9G** (Pender et al., 2018)
- All questions

(X1) **Ex 11G** (Pender et al., 2019)
- All questions

Example 7

[**2012 2U HSC**] What are the solutions of $\sqrt{3}\tan x = -1$ for $0 \leq x \leq 2\pi$?

Further exercises (Legacy Textbooks)

(2) **Ex 4A** (Pender, Sadler, Shea, & Ward, 2009)

- Q1-14

(x1) **Ex 14A** (Pender et al., 1999)

- Q1-13
- Q15-21

Further exercises

(A) **Ex 9H** (Pender et al., 2018)

- All questions

(x1) **Ex 11H** (Pender et al., 2019)

- All questions

Learning Goal(s)

Knowledge
Formulae for areas of sector, arc lengths

Skills
Calculate areas of sectors, arc length and combinations of these

Understanding
Why radians are needed for such calculations

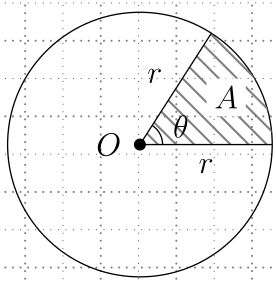
By the end of this section am I able to:

8.6 Derive the formula for arc length $l = r\theta$ and for the area of a sector of a circle, $A = \frac{1}{2}r^2\theta$

8.7 Solve problems involving sector areas, arc lengths and combinations of either areas or lengths.

1.3 Area of sector

- Given the area of a circle is $A = \pi r^2$, then the area of a sector within the circle would be a fraction of the area of the circle.



- The ratio of areas must equal the ratio of the angles subtended at the centre:

$$\frac{A}{A_{\text{circle}}} = \frac{\theta}{2\pi}$$
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$
$$\therefore A = \frac{\cancel{\pi} r^2\theta}{2\cancel{\pi}} = \frac{1}{2}r^2\theta$$

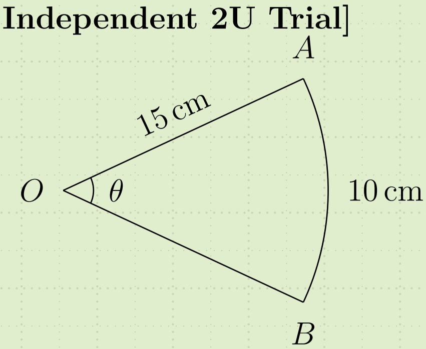
Laws/Results

The area of a sector is

$$A = \frac{1}{2}r^2\theta$$

Example 8

[2009 Independent 2U Trial]



In the diagram, AB is the arc of a circle with centre O. The arc AB is 10 cm and radius OA is 15 cm. Find:

(i) The exact size of $\angle AOB$ in radians **1**

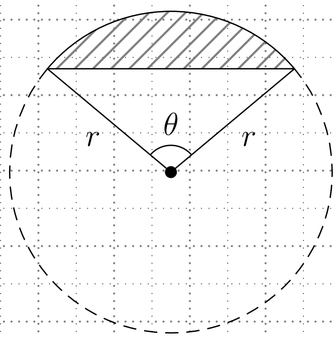
(ii) The exact area of the sector AOB. **1**

Answer: $\theta = \frac{2}{3}$, $A = 75\,\text{cm}^2$

1.4 Area of minor segment

Definition 3

A minor segment is the region of the circle cut off by an arc and the chord that joins the endpoints of the arc.



Steps

1. Area of sector: $A = \frac{1}{2}r^2\theta$
2. Area of triangle (cut off by chord): $A = \frac{1}{2}r^2 \sin\theta$
3. Subtract areas and factorise: $A = \frac{1}{2}r^2(\theta - \sin\theta)$

Definition 4

Area of minor segment:

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

Example 9

A chord AB of a circle with centre O has length 16 cm. If the radius of the circle is 10 cm, calculate

(a) the magnitude of $\angle AOB$.

(b) the length of the minor arc AB.

(c) the area of the minor segment formed by the chord AB.

Example 10

Two circles of radii 3 cm and 4 cm have their centres 5 cm apart. Calculate the area common to both circles correct to 2 decimal places. **Answer:** $6.64\,\text{cm}^2$

Further exercises (Legacy Textbooks)

(2) **Ex 4B**
- Q1-20

(X1) **Ex 14B**
- Q4-26

Further exercises

(A) **Ex 9I** (Pender et al., 2018)
- All questions

(X1) **Ex 11I** (Pender et al., 2019)
- All questions

Section 2

Graphs of trigonometric functions

Learning Goal(s)

Knowledge
Recognise the graphs of the form $y = a\sin(nx + \phi) + k$ and $y = a\cos(nx + \phi) + k$ etc

Skills
Sketch the graphs of the form $y = a\sin(nx + \phi) + k$ and $y = a\cos(nx + \phi) + k$

Understanding
Why radians are needed to sketch the graphs of trigonometric functions on the same set of axes as linear/quadratic graphs

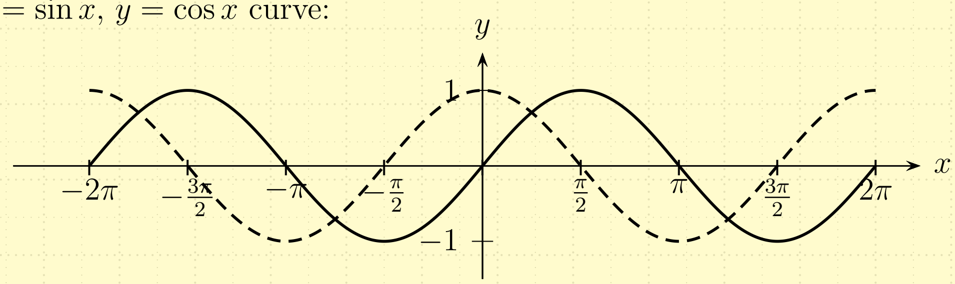
By the end of this section am I able to:

8.5 Recognise the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ and sketch on extended domains in degrees and radians.

2.1 Graphs of $\sin x$, $\cos x$

Laws/Results

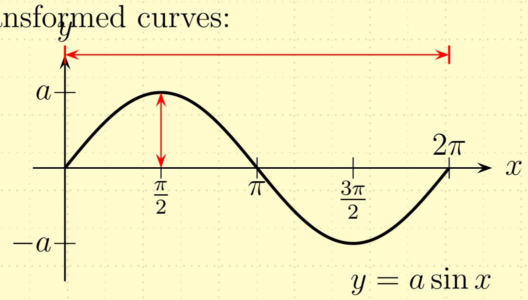
Basic $y = \sin x$, $y = \cos x$ curve:

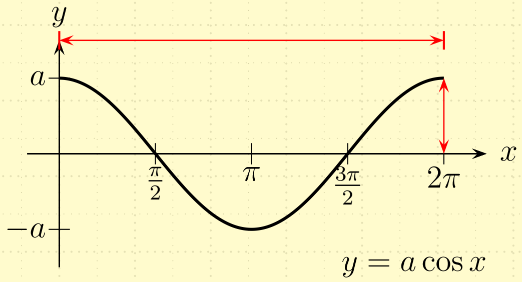


- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : -1 \leq y \leq 1\}$
- Period: $T = 2\pi$
- Property: odd ($\sin x$), even function ($\cos x$)

Laws/Results

Transformed curves:





- General equation: $y = a\sin(nx + \phi)$
- a: amplitude
  (distance between equilibrium position & peak/trough)
- T: period, $\frac{2\pi}{n}$. (Distance between peaks/troughs)
- n: frequency. (The number of times it "appears" from 0 to 2π)
- ϕ: phase shift (left/right shifting)

Steps

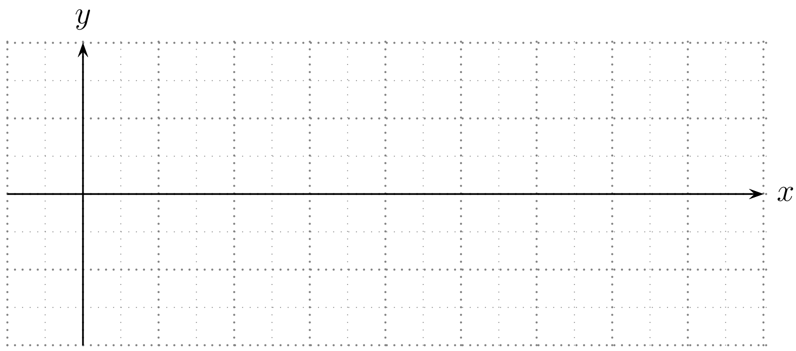
When sketching trigonometric graphs:

- Identify amplitude.
- Identify period.
- Identify further transformations.

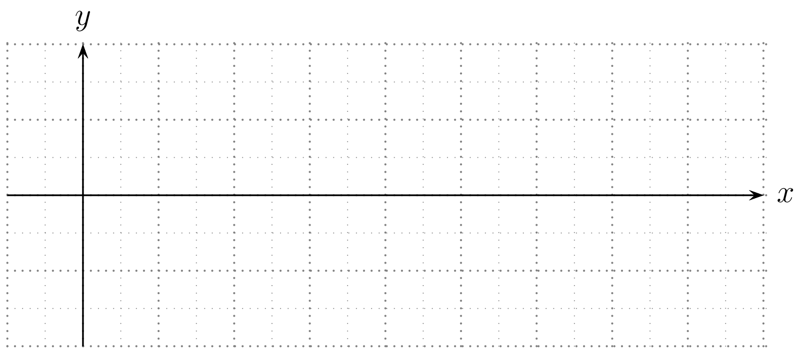
Exercises

Sketch the following graphs:

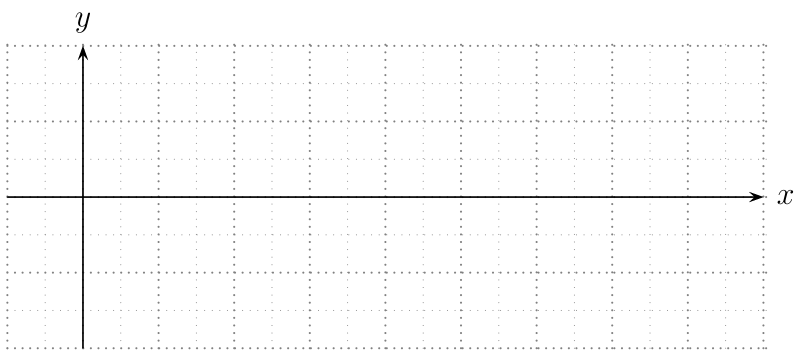
1. $y = \sin x$, $0 \le x \le 2\pi$



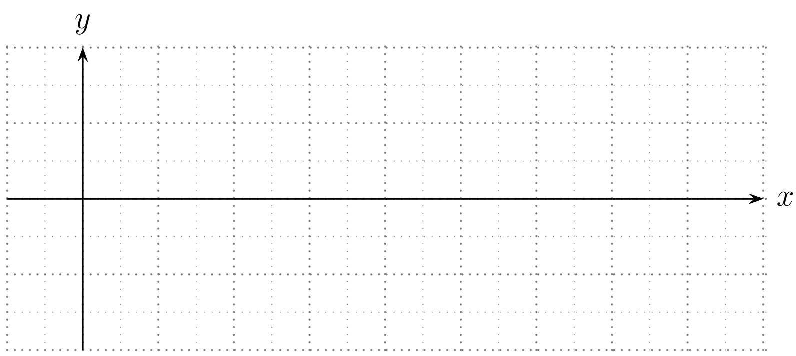
2. $y = 3\cos x$, $0 \le x \le 2\pi$



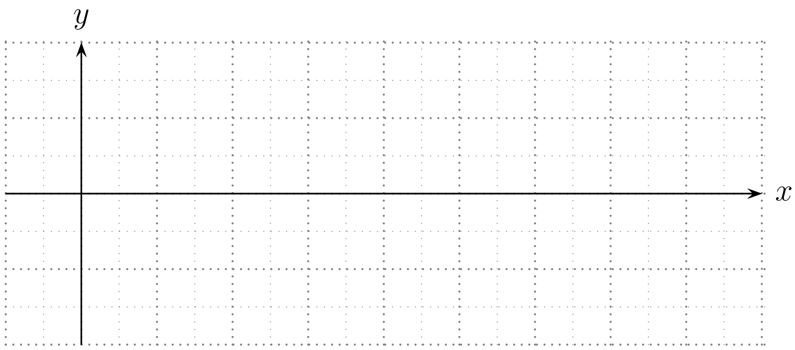
3. $y = \sin 2x$, $0 \le x \le 2\pi$



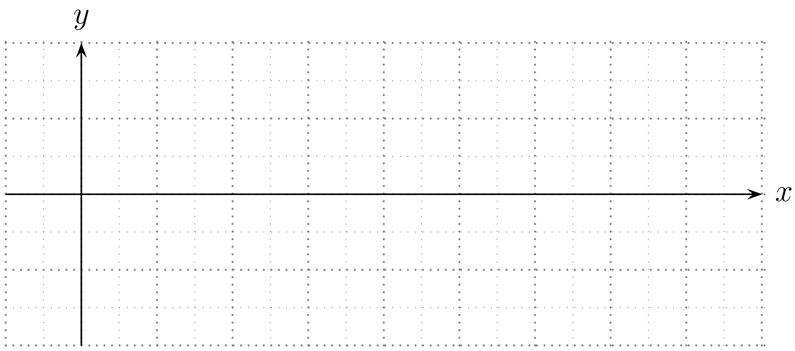
4. $y = 2\cos 2x$, $0 \le x \le 2\pi$



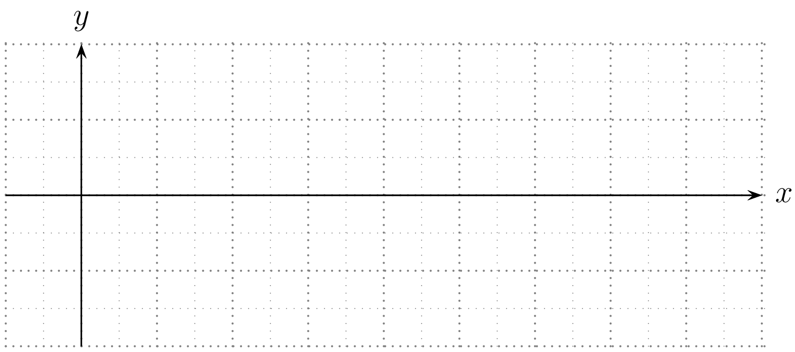
5. $y = -\sin x$, $0 \leq x \leq 2\pi$



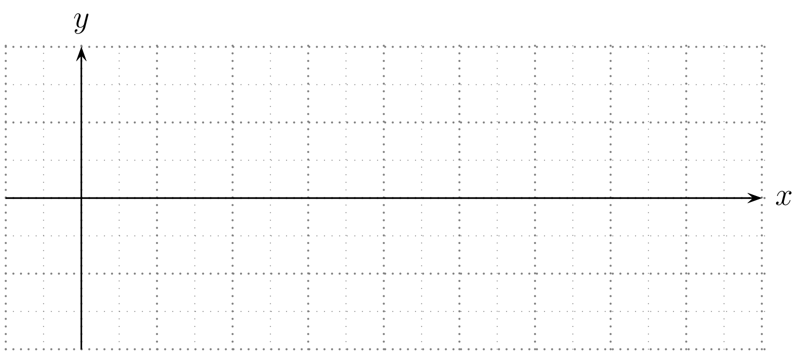
6. $y = -4\cos\frac{1}{2}x$, $0 \leq x \leq 4\pi$



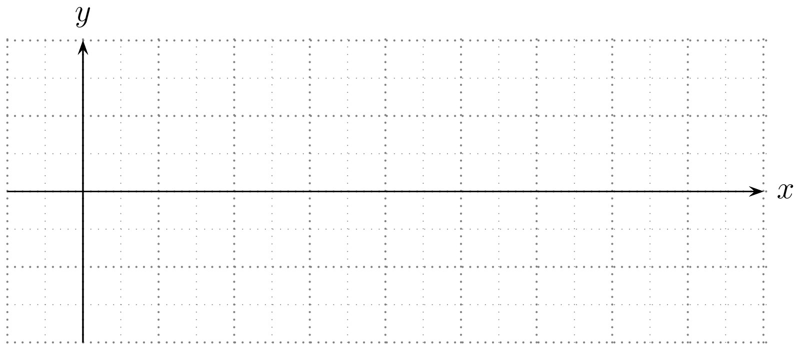
7. $y = -4\cos\frac{1}{2}x$, $0 \leq x \leq 2\pi$



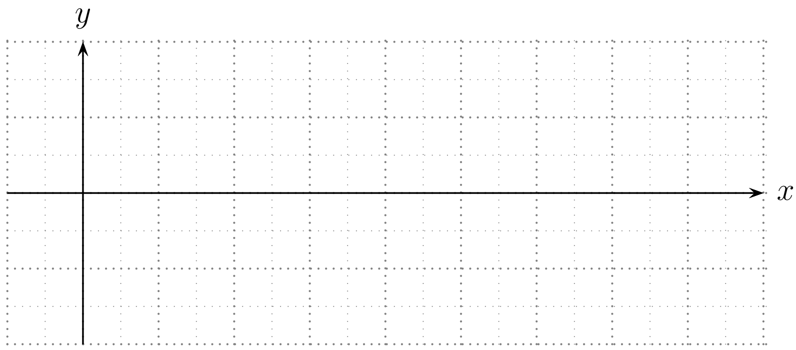
8. $y = 3\sin 3x$, $0 \leq x \leq 2\pi$



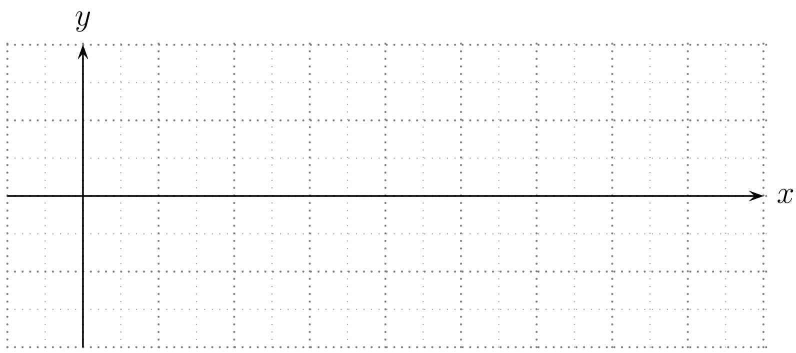
9. $y = 2\sin\frac{1}{3}x$, $0 \le x \le 2\pi$



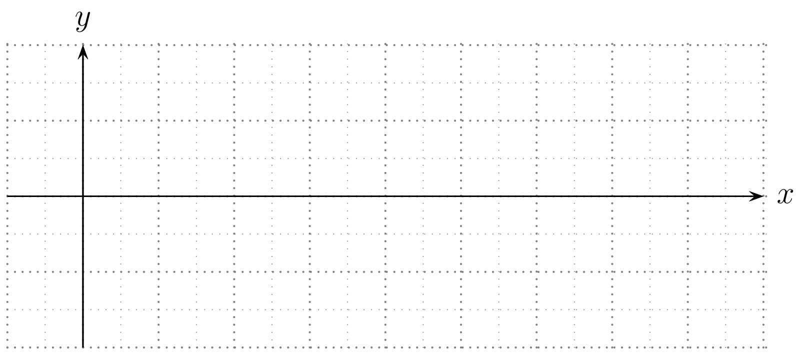
10. $y = 2\cos\frac{3x}{5}$, 1 period.



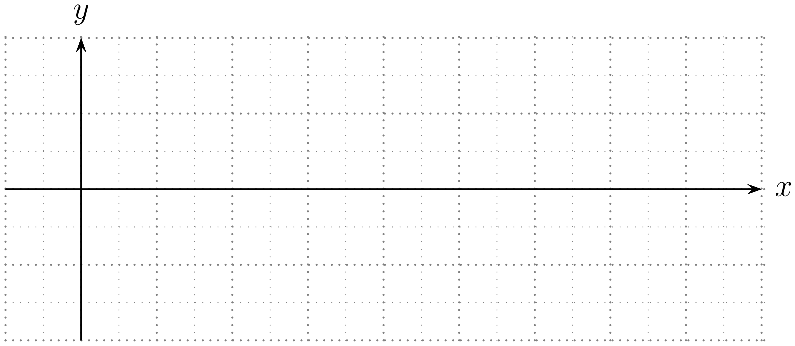
11. $y = 4\cos\left(x - \frac{\pi}{6}\right)$, $0 \le x \le 2\pi$.



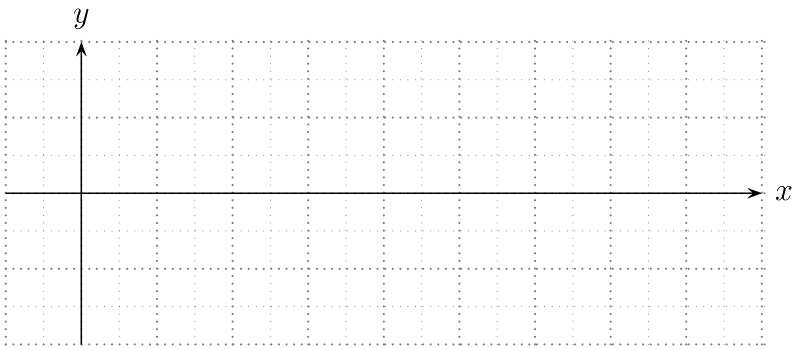
12. $y = \frac{1}{2}\sin(2x - 1)$, $0 \le x \le 2\pi$.



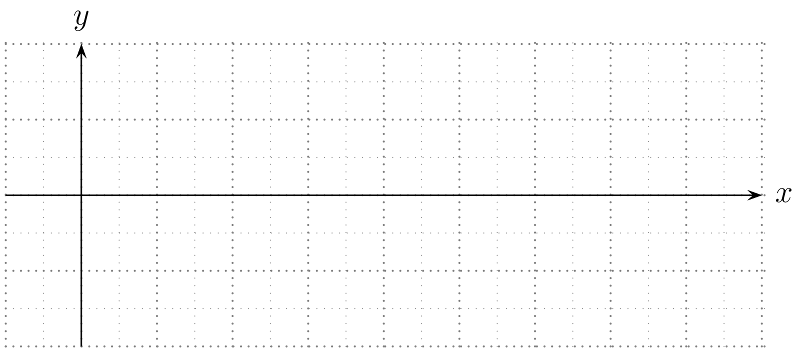
13. $y = 3\cos \pi x$, $0 \le x \le 2$



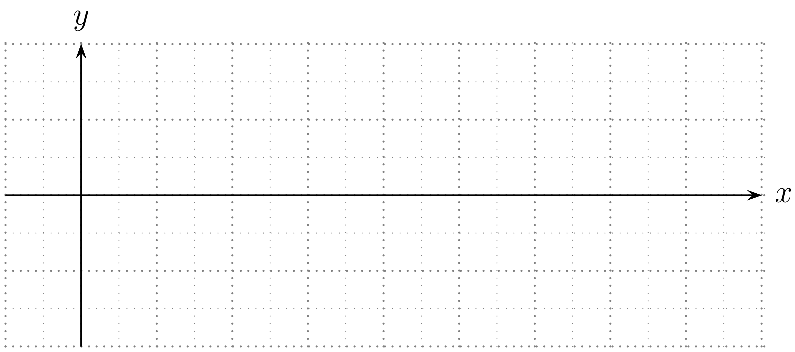
14. $y = \cos(360°x)$, 1 period.



15. $y = \frac{230}{\sqrt{3}} \cos(2\pi \times 50x)$, 1 period.
(Ideal AC wave at 50Hz with max voltage of 230V; Australian Standard AS60038-2000)



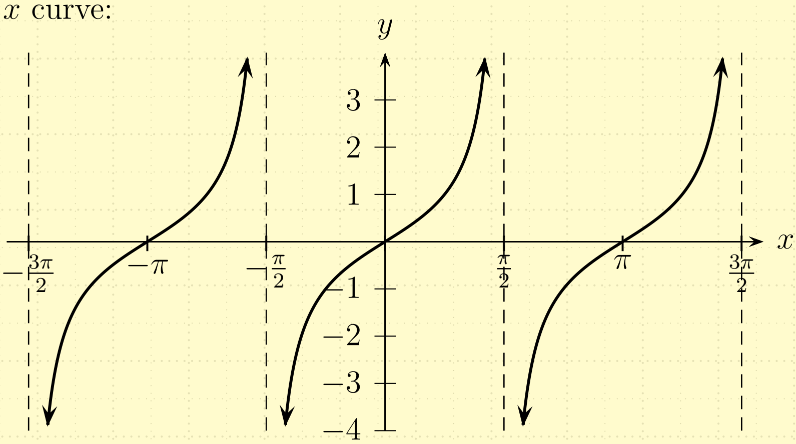
16. $y = (1 \times 10^{-3}) \cos\left(2\pi \times [2.4 \times 10^{9}]\, x\right)$, 1 period. (Ideal 2.4GHz carrier wave)



2.2 Graphs of $\tan x$

Laws/Results

Basic $y = \tan x$ curve:



- Domain: $D = \{x : x \in \mathbb{R}, x \neq \frac{(2k+1)\pi}{2}\}$
- Range: $R = \{y : y \in \mathbb{R}\}$
- Period: $T = \pi$

Laws/Results

Transformed curves:

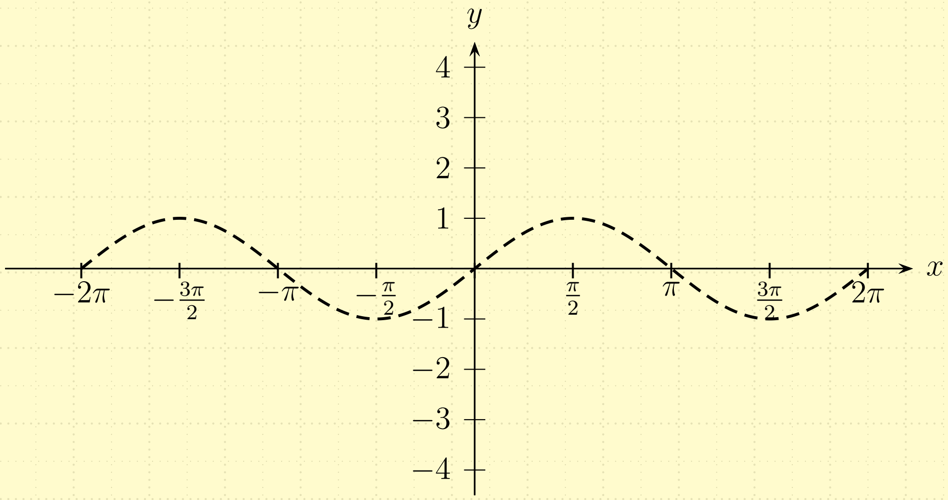
- General equation: $y = a\sin(nx + \phi)$
- a: vertical stretch factor
- T: period, $\dfrac{\pi}{n}$. (Distance before graph repeats)
- n: frequency. (The number of times it "appears" from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$)
- ϕ: phase shift

2.3 Graphs of reciprocal trigonometric functions

Laws/Results

Basic $y = \operatorname{cosec} x$ curve:

- Domain: $D = \{x : x \in \mathbb{R}, x \neq k\pi, k \in \mathbb{Z}\}$
- Range: $R = \{y : y \in]-1, 1[\}$
- Period: $T = \frac{2\pi}{n}$
- Property: odd function



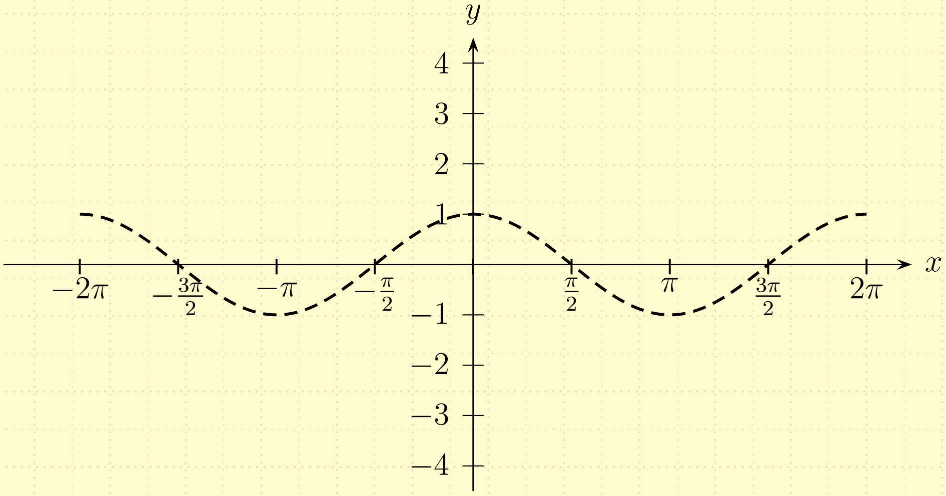
Example 11

Sketch $y = 2\operatorname{cosec}(2x)$ for $-\pi < x < \pi$.

Laws/Results

Basic $y = \sec x$ curve:

- Domain: $D = \{x : x \in \mathbb{R}, x \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}\}$
- Range: $R = \{y :] -1, 1[\}$
- Period: $T = \frac{2\pi}{n}$
- Property: even function



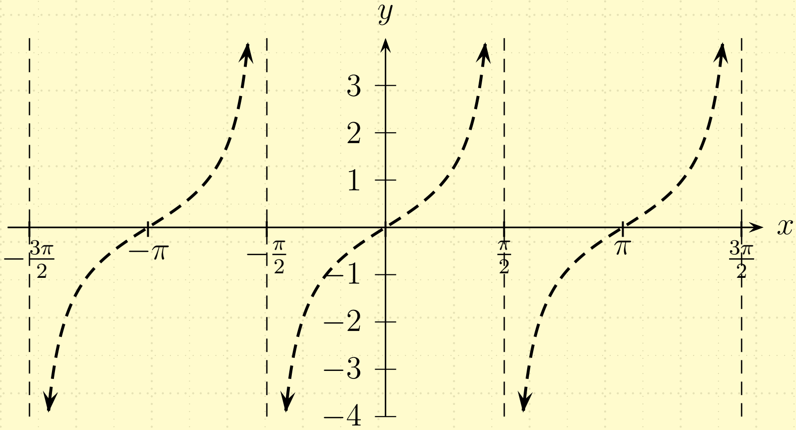
Example 12

Sketch $y = 2\sec\left(x + \frac{\pi}{3}\right)$ for $-\pi < x < \pi$.

Laws/Results

Basic $y = \cot x$ curve:

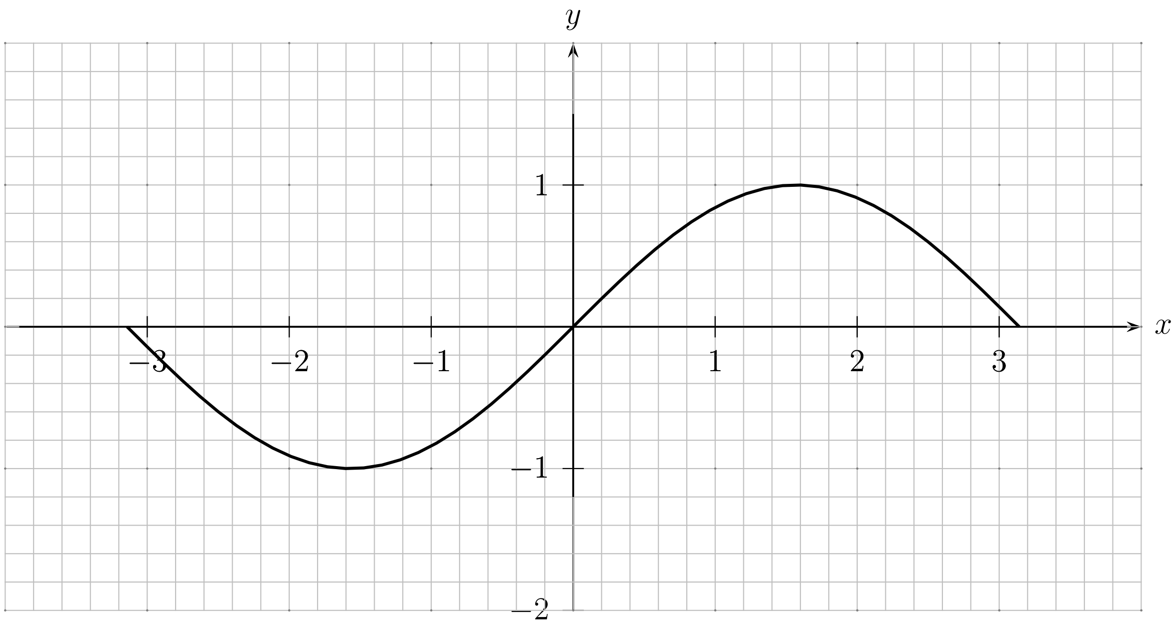
- Domain: $D = \{x : x \in \mathbb{R}, x \neq k\pi\}$
- Range: $R = \{y : y \in \mathbb{R}$
- Period: $T = \frac{\pi}{n}$
- Property: odd function



2.4 Miscellaneous examples

Example 13

[Ex 14C Q9] Given the graph of $y = \sin x$ and the grid paper, find three solutions of the equation $\sin x = \frac{1}{2}x$, giving answers correct to 1 decimal place where necessary.



Example 14

[1995 2U HSC Q10(a)]

(i) Draw the graphs of $y = 4\cos x$ and $y = 2 - x$ on the same set of axes for $-2\pi \leq x \leq 2\pi$. **2**

(ii) Explain why all the solutions of the equation $4\cos x = 2 - x$ must lie between $x = -2$ and $x = 6$. **1**

Example 15

[1999 2U HSC Q10(a)]

(i) Show that $x = \frac{\pi}{3}$ is a solution of $\sin x = \frac{1}{2}\tan x$. **1**

(ii) On the same set of axes, sketch the graphs of the functions $y = \sin x$ and $y = \frac{1}{2}\tan x$ for $-\pi \leq x \leq \pi$. **2**

(iii) Hence find all solutions of $\sin x = \frac{1}{2}\tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **1**

(iv) Use your graphs to solve $\sin x \leq \frac{1}{2}\tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **2**

Further exercises (Legacy Textbooks)

Ex 14C (Pender et al., 1999)

- Q1-8
- Q10-18
- Q21-23

Further exercises

(A) **Ex 9J** (Pender et al., 2018)

- All questions

(X1) **Ex 11J** (Pender et al., 2019)

- All questions

Section 3

(X1) Further trigonometric identities

Learning Goal(s)

Knowledge
Compound and double angle expansions

Skills
Use the compound and double angle formulae

Understanding
When to use the formulae *backwards*

By the end of this section am I able to:

8.8 Derive and use the sum and difference expansions for the trigonometric functions $y = \sin(A \pm B)$, $y = \cos(A \pm B)$ and $y = \tan(A \pm B)$

3.1 Compound angle results

Theorem 1

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B \tag{3.1}$$

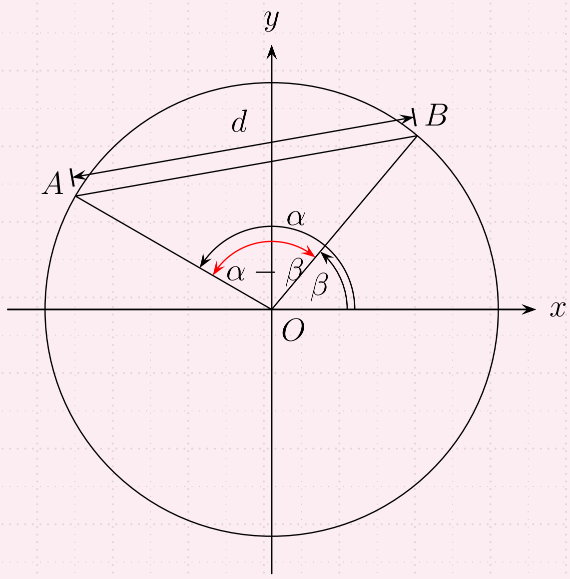
$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B \tag{3.2}$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \tag{3.3}$$

3.1.1 Proof of $\cos(A - B)$

Steps

1. Consider the circle of radius r with a triangle inscribed:



2. Use cosine rule to find d^2 in $\triangle AOB$:

3. Use distance formula to find d^2 in $\triangle AOB$, writing A and B in polar coordinates:

4. Equate:

3.1.2 Proof of $\cos(A + B)$

- Turn B into $-B$ from Section 3.1.1.

3.1.3 Proof of $\sin(A-B)$

- Turn $\sin(A-B)$ into $\cos\left(\frac{\pi}{2}-(A-B)\right)$:

3.1.4 Proof of $\sin(A+B)$

- Turn $\sin(A+B)$ into $\cos\left(\frac{\pi}{2}-(A+B)\right)$:

3.1.5 Proof of $\tan(A \pm B)$

- Use $\tan(A \pm B) = \dfrac{\sin(A \pm B)}{\cos(A \pm B)}$:

3.1.6 Examples

Example 16

Express $\sin\left(x + \frac{2\pi}{3}\right)$ in the form $a\cos x + b\sin x$.

Example 17

Given $\sin\alpha = \frac{1}{3}$ and $\cos\beta = \frac{4}{5}$, where α is acute and $-\frac{\pi}{2} < \beta < 0$, find $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.

Answer: $\frac{2}{15}\left(2 + 3\sqrt{2}\right)$, $\frac{1}{15}\left(8\sqrt{2} + 3\right)$

Example 18

Given $\cos\alpha = \frac{8}{17}$, $0 < \alpha < \frac{\pi}{2}$, and $\tan\beta = \frac{5}{12}$, $0 < \beta < \frac{\pi}{2}$, find exact values for $\sin(\alpha - \beta)$, $\cos(\alpha + \beta)$ and $\tan(\alpha + \beta)$.

Example 19

Find the exact values of

(a) $\sin 75^\circ$ (b) $\cos 105^\circ$ (c) $\tan 15^\circ$

Example 20

Evaluate $\sin\frac{17\pi}{12}$ as an exact value.

Further exercises

Ex 17D

- Q1-9, 13-17

3.2 Double angle results

Learning Goal(s)

Knowledge
Compound and double angle expansions

Skills
Use the compound and double angle formulae

Understanding
When to use the formulae *backwards*

By the end of this section am I able to:

8.9 Derive and use the double angle formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$

Theorem 2

$$\cos(2A) \equiv \cos^2 A - \sin^2 A \quad \text{(basic)}$$

$$= 2\cos^2 A - 1 \quad \text{(variant)} \tag{3.4}$$

$$= 1 - 2\sin^2 A \quad \text{(variant)}$$

$$\sin(2A) \equiv 2\sin A\cos A \tag{3.5}$$

$$\tan(2A) \equiv \frac{2\tan A}{1-\tan^2 A} \tag{3.6}$$

3.2.1 Proof of $\cos 2A$

3.2.2 Proof of $\sin 2A$

3.2.3 Proof of $\tan 2A$

Example 21

Find the exact value for $\sin 2A$, $\cos 2A$ and $\tan 2A$, given $0 < A < \frac{\pi}{2}$ and

(a) $\sin A = \frac{3}{5}$.

(b) $\cos A = \frac{12}{13}$.

(c) $\cos A = \frac{7}{25}$.

(d) $\sin A = \frac{\sqrt{3}}{2}$.

(e) $\tan A = \frac{\ell}{m}$.

3.2.4 Examples

Example 22

Find an expression for $\sin^2 x$ in terms of $\cos 2x$.

Example 23

Sketch $y = \cos^2 x$ for $0 \leq x \leq 2\pi$.

Example 24

Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Further exercises (Legacy Textbooks)

Ex 14D Pender et al. (1999)

- All questions

Further exercises

Ex 17E

- Q1-5, 9-14

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

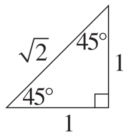
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

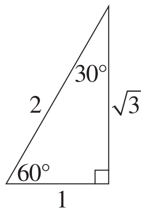
$A = \frac{1}{2}ab\sin C$



$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \frac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

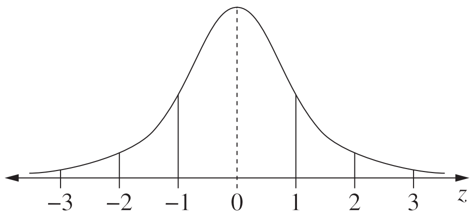
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \quad P(X = x)$

$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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