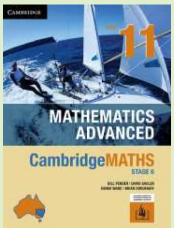
NORMANHURST BOYS HIGH SCHOOL

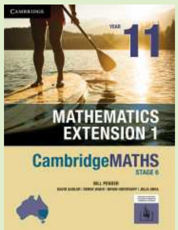
MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 11 COURSE

Topic summary and exercises:

Ⓐ ⓍⅠ **The Exponential Function**

With references to





Name: ....

Initial version by H. Lam, October 2013, Revised June 2019 for latest syllabus. Last updated June 1, 2021.
Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under ⓒ CC BY 2.0.

Symbols used

Beware! Heed warning.

Ⓐ Mathematics Advanced content.

(X1) Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA11-6 manipulates and solves expressions using the logarithmic and index laws, and uses logarithms and exponential functions to solve practical problems

Syllabus subtopics

MA-E1 (1.3, 1.4) Logarithms and Exponentials

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Mathematics Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b) or *CambridgeMATHS Mathematics Advanced* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Differentiation of the exponential function

Learning Goal(s)

Knowledge
Establish the value of e

Skills
Differentiate exponential functions

Understanding
The derivative of e^x is itself.

By the end of this section am I able to:

9.2 Consider the gradient function of $y = ax$ by graphical methods

9.3 Establish and use the formula $\frac{d(e^x)}{dx} = e^x$

9.7 Apply the differentiation rules to functions involving the exponential function, $f(x) = ke^{ax}$, where k and a are constants

1.1 From first principles

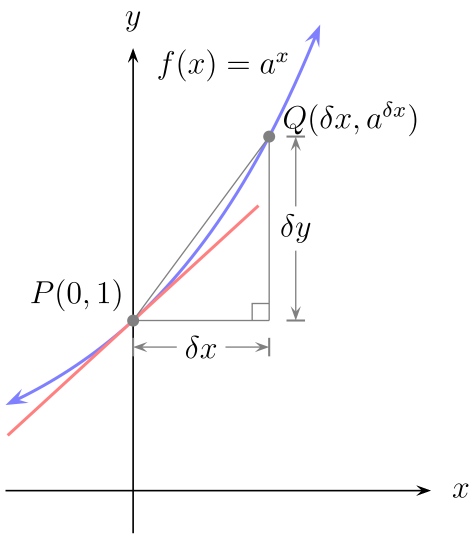
- **Goal:** find the gradient of the tangent to $f(x) = a^x$.
  - First by finding gradient of the tangent to $f(x) = a^x$ at $(0, 1)$.
- Draw ...secant... from $P(0, 1)$ to $Q\left(\delta x, a^{\delta x}\right)$ on the curve to approximate tangent.
- Gradient of ...secant... :

  ....

- Gradient (m) of tangent at $(0, 1)$

  $$m = \lim_{\delta x \to 0} \frac{a^{\delta x} - 1}{\delta x}$$

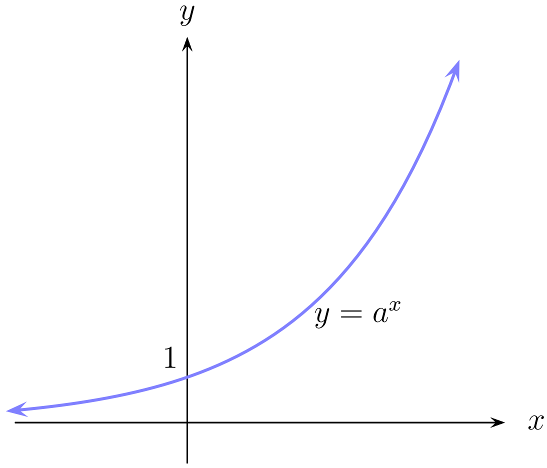
- m only exists as a limit, but will need this again (see next slide).



Gradient of tangent

Differentiating $f(x) = a^x$ from first principles, using index laws:

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{a^x(a^{\delta x} - 1)}{\delta x} \\ &= a^x \times \underbrace{\lim_{\delta x \to 0} \frac{a^{\delta x} - 1}{\delta x}}_{=m} = ma^x \\ &= my \end{aligned}$$



Laws/Results

Derivative of $y = a^x$ at *any* x value is the function itself, multiplied by the gradient of the function at $(0, 1)$.

What is the the gradient of the tangent at $x = 0$?

- Numerical value of m is still unknown.
- m will change as the value of a changes.
- Testing a few values of a,

$$\begin{aligned} a = 2 \quad & m = \frac{2^{0.0001} - 1}{0.0001} \approx 0.69314\cdots \\ a = 2.5 \quad & m = \frac{2.5^{0.0001} - 1}{0.0001} \approx 0.91629\cdots \\ a = 2.7 \quad & m = \frac{2.7^{0.0001} - 1}{0.0001} \approx 0.993256\cdots \\ a = 2.8 \quad & m = \frac{2.8^{0.0001} - 1}{0.0001} \approx 1.029624\cdots \end{aligned}$$

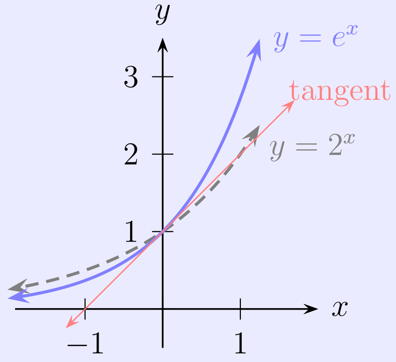
there will be a value (denote to be e) somewhere between 2.7 and 2.8 that makes $m = 1$ and $\frac{dy}{dx} = e^x = y$, i.e. the derivative is the function itself.

Definition 1

The Exponential Function is

$f(x) = e^x$

with property $\frac{d}{dx}(e^x) = e^x$



1.2 Miscellaneous properties

1.2.1 e^x

$f(x) = e^x$:

- $e \approx$ 2.718 281 828 459 to 12 d.p., and is transcendental like π.
- $f'(0) = e^0 = 1$. Notice $f'(0) = ma^0 = m \neq 1$ for other values of $a > 1$.
- $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

1.2.2 Derivative of a^x

- Change base to e^x.
- Derivation of change of base rule:

Steps

1. Let $y = a^x$.
2. Take logarithm of base e on both sides, resulting in $\log_e y = x \log_e a$.
3. Exponentiate both sides to obtain explicit equation in y: $y = e^{x \log_e a}$

Further exercises

(A) Ex 9A
- Q4-6

(X1) Ex 11A
- Q5-6

1.3 Curve transformations

Further exercises

Ex 9B
- Q4-9

Ex 11B
- Q4-9

1.4 Derivatives of e^x

Important note

The ....chain.... ....rule.... is almost always required.

1.4.1 Standard questions

Example 1

Differentiate e^{4x+1}.

Answer: $4e^{4x+1}$

Example 2

Differentiate $e^{2x} - 4x^3$.

Answer: $2e^{2x} - 12x^2$

Example 3

Find the derivative to e^{1-x^2}.

Answer: $-2xe^{1-x^2}$

Example 4

Find the derivative to $y = \left(e^{2x} - 3\right)^4$.

Answer: $8e^{2x}\left(e^{2x} - 3\right)^3$

Example 5

Find the derivative to $y = x^3 e^x$. **Answer:** $x^2 e^x (3 + x)$

Example 6

Find the derivative to $y = xe^{5x-2}$ **Answer:** $e^{5x-2}(1 + 5x)$

Example 7

Find the derivative to $y = \dfrac{e^{5x}}{x}$ **Answer:** $\dfrac{e^{5x}(5x-1)}{x^2}$

1.4.2 Harder derivatives

Example 8

Find the derivative to $y = \dfrac{e^x}{1-x^2}$

Answer: $\dfrac{e^x\left(1+2x-x^2\right)}{\left(1-x^2\right)^2}$

Example 9

Evaluate $\dfrac{d}{dx}\left(e^{-\frac{1}{x}}\right)$.

Answer: $x^{-2}e^{-\frac{1}{x}}$

Example 10

Evaluate $\dfrac{d}{dx}\sqrt{x}e^{3-7x^2}$.

Answer: $x^{-\frac{1}{2}}e^{3-7x^2}\left(\frac{1}{2}-14x^2\right)$

Example 11

Evaluate $\dfrac{d}{dx}\left(\dfrac{e^{4x}+3}{e^{-4x}-3}\right)$.

Answer: $\frac{8e^{8x}+12e^{4x}-12e^{12x}}{(1-3e^{4x})^2}$

Example 12

Find $f'(x)$ if $f(x) = ee^x e^{x^2}$. **Answer:** $(2x+1)e^{x^2+x+1}$

Example 13

[2003 CSSA 2U] Given that $y = 3e^{-2x}$, show that $2\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 2y = 0$.

Example 14

For what values of λ does the function $y = e^{\lambda x}$ satisfy the equation $y'' + y' - 6y = 0$?

Answer: $\lambda = -3, 2$

Supplementary Exercises

1. Differentiate the following with respect to x.

(a) $y = 7e^x$
(b) $y = \dfrac{2e^x}{5}$
(c) $y = e^{x^2+1}$
(d) $y = e^{2x^2+5}$
(e) $y = e^{3-5x}$
(f) $y = e^{-x^3+6x-1}$
(g) $y = e^{3x^2+4x+4}$
(h) $y = e^{9x^2+5x^3-6}$
(i) $y = xe^x$
(j) $y = x^2e^{-x}$
(k) $y = x^2e^{-x^2}$
(l) $y = xe^{2x}$
(m) $y = \dfrac{e^x + e^{-x}}{2}$
(n) $y = \dfrac{e^x - e^{-x}}{2}$
(o) $y = \dfrac{e^{2x}}{x}$
(p) $y = e^{x-\sqrt{x}}$
(q) $y = e^{1+\sqrt{x}}$
(r) $y = \left(e^{3x}+1\right)^4$
(s) $y = e^{\frac{1}{x}}$
(t) $y = \dfrac{e^x - 1}{e^x + 1}$
(u) $y = 4^{3x^2}$
(v) $y = 2^x x^2$
(w) $y = x^3 - 3^x$

2. If $f(x) = 5^{x^2 \ln x}$, find $f'(1)$.

3. Determine the value of the positive constant a if $\dfrac{d}{dx}\left(a^x - x^a\right) = 0$ when $x = 1$.

Answers

1. (a) $7e^x$ (b) $\frac{2}{5}e^x$ (c) $2xe^{x^2+1}$ (d) $4xe^{2x^2+5}$ (e) $-5e^{3-5x}$ (f) $\left(-3x^2+6\right)e^{-x^3+6x-1}$ (g) $(6x+4)e^{3x^2+4x+4}$ (h) $\left(15x^2+18x\right)e^{9x^2+5x^3-6}$ (i) $e^x(x+1)$ (j) $xe^{-x}(2-x)$ (k) $2xe^{-x^2}\left(1-x^2\right)$ (l) $e^{2x}(2x+1)$ (m) $\frac{1}{2}\left(e^x - e^{-x}\right)$ (n) $\frac{1}{2}\left(e^x + e^{-x}\right)$ (o) $\frac{e^{2x}(2x-1)}{x^2}$ (p) $\left(1 - \frac{1}{2\sqrt{x}}\right)e^{x-\sqrt{x}}$ (q) $\frac{1}{2\sqrt{x}}e^{1+\sqrt{x}}$ (r) $12e^{3x}\left(e^{3x}+1\right)^3$ (s) $-\frac{e^{\frac{1}{x}}}{x^2}$ (t) $\frac{2e^x}{(e^x+1)^2}$ (u) $6x4^{3x^2}\ln 4$ (v) $2^{x+1}x + x^2 2^x \ln 2$ (w) $3x^2 - 3^x \ln 3$ **2.** $\ln 5$ **3.** e

Further exercises

(A) Ex 9C
- All questions

(X1) Ex 11C
- All questions

Further exercises (Legacy Textbooks)

(2) Ex 2C
- Q1, 2 last 3 columns
- Q3-5, 8-11 last 2 columns
- Q12 last column
- Q15
- Q17-19 last 2 columns
- Q20-21 last 2 columns

(X1) Ex 13A
- Q2-4, 6
- Q7 last row

Section 2

Applications of differentiation

Learning Goal(s)

Knowledge
Identify the features of exponential functions and their transformations

Skills
Apply concepts from calculus to exponential functions

Understanding
The inverse relationship of exponential and logarithmic functions

By the end of this section am I able to:

9.1 Graph an exponential function of the form $y = ax$ for $a > 0$ and its transformations $y = kax + c$ and $y = kax + c$ where k and c are constants.

9.4 Explore the transformations of $y = e^x$

Important note

All knowledge from *Introduction to differentiation* topic prerequisite to this section.

2.1 Tangents and normals

Example 15

Find the equation of the tangent to the curve $y = e^{3x}$ at $x = 1$. **Answer:** $y = 3e^3x - 2e^3$

Example 16

Find the equation of the tangent to the curve $y = e^{x^2}$ at $(1, e)$. **Answer:** $y = 2ex - e$

Example 17

[2005 HSC Q5] Find the coordinates of the point P on the curve $y = 2e^x + 3x$ at which the tangent to the curve is parallel to the line $y = 5x - 3$.

Example 18

Find the equation of the tangent to the curve at $y = x^e e^x$ at $(1, e)$.

Answer: $y = (e + e^2)x - e^2$

2.2 Curve sketching

Example 19

[2007 NSB Task 3]

(i) Sketch the curve $y = e^x - 2$.

(ii) On the same sketch, find graphically the number of solutions of the equation $e^x - x - 2 = 0$.

Example 20

[**2012 Sydney Boys' Trial**] For the curve $y = xe^{-x}$

(i) Prove that $\dfrac{dy}{dx} = -e^{-x}(x-1)$. **1**

(ii) Find any stationary points and determine their nature. **2**

(iii) Prove that $\dfrac{d^2y}{dx^2} = e^{-x}(x-2)$. **1**

(iv) Show that there is a point of inflexion on this curve and find the coordinates of this point. **2**

(v) Sketch the curve, showing the coordinates of the point of inflexion and any stationary points. **1**

Example 21

[1999 S&GPCA Trial] The function $g(x) = xe^{x+1}$ has first derivative $g'(x) = (1+x)e^{x+1}$ and second derivative $g''(x) = (2+x)e^{x+1}$.

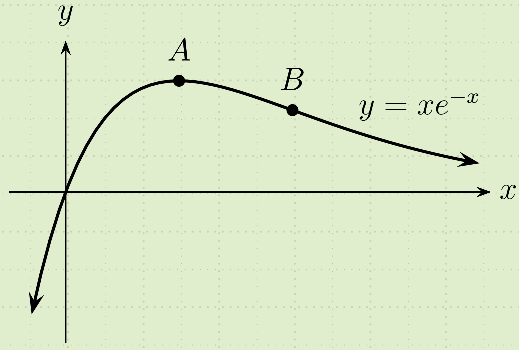
(i) Find the coordinates of the stationary point and determine its nature. **2**

(ii) Find the coordinates of the point of inflexion. **1**

(iii) By using an appropriate scale, neatly sketch the $y = g(x)$ for $-\frac{5}{2} \le x \le \frac{1}{2}$, showing the stationary point and the point of inflexion. **3**

Example 22

[**2002 S&GPCA Trial**] The diagram shows a sketch of the curve $y = xe^{-x}$. The curve has a maximum turning point at A and has a point of inflexion at B.



(i) Find the coordinates of A. **2**

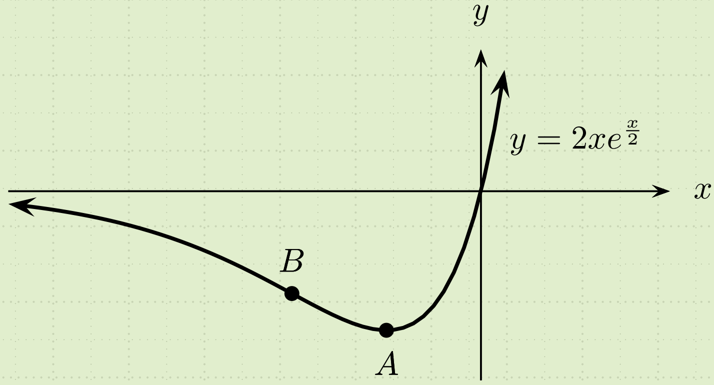
(ii) Find the coordinates of B. **2**

(iii) For what values of x does the curve concave up? **1**

(iv) For what values of p (except $p = \frac{1}{e}$) does the equation $xe^{-x} = p$ have exactly one real root? **1**

Example 23

[2003 S&GPCA Trial] The diagram shows a sketch of the curve $y = 2xe^{\frac{x}{2}}$.



The curve has a minimum turning point at A and has a point of inflexion at B.

(i) Find the coordinates of A. **2**

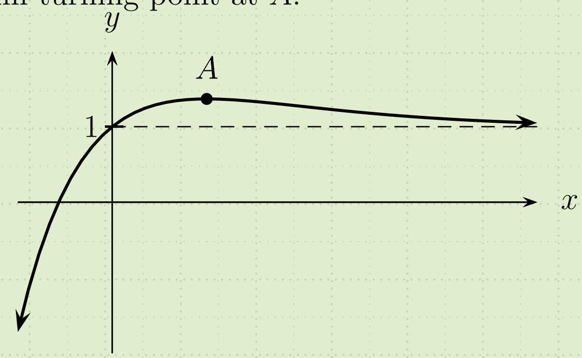
(ii) Find the coordinates of B. **2**

(iii) For what values of x does the curve concave down? **1**

(iv) For what values of k does the equation $xe^{\frac{x}{2}} = k$ have two unequal real roots? **1**

Example 24

[2004 S&GPCA Trial] The diagram shows a sketch of the curve $y = xe^{-x} + 1$. The curve has a maximum turning point at A.



(i) Find the coordinates of A. **2**

(ii) For what values of x does the curve concave up? **2**

Example 25

[2008 NSB Task 3]

(a) What are the coordinates and the nature of the stationary point(s) of the curve $y = \dfrac{e^x}{x^2+1}$? **4**

(b) What is the range of this function? **1**

(c) Sketch this curve, showing any intercepts. **2**

Example 26

[1995 3U HSC Q4] X1 Consider the function $f(x) = \dfrac{e^x}{3 + e^x}$. Note that e^x is always positive, and that $f(x)$ is defined for all real x.

(a) Show that $f(x)$ has no stationary points. **2**

(b) Find the coordinates of the points of inflexion, given that **1**

$$f''(x) = \frac{3e^x\left(3 - e^x\right)}{\left(3 + e^x\right)^3}$$

(c) Show that $0 < f(x) < 1$ for all x. **1**

(d) Describe the behaviour of $f(x)$ for very large positive and very large negative values of x, i.e. as $x \to \infty$ and $x \to -\infty$. **2**

(e) Sketch the curve $y = f(x)$. **2**

Further exercises

Ⓐ Ex 9D
- All questions

(X1) Ex 11D
- All questions

Further exercises (Legacy Textbooks)

② Ex 2D
- All questions

(X1) Ex 13B
- Q6-9
- Q12-22

2.3 (x1) Inverse functions

Example 27

[1999 3U HSC] (x1) Consider the function $f(x) = e^x - 1 - x$.

(i) Show that the minimum of $f(x)$ occurs at $x = 0$. **2**

(ii) Deduce that $f(x) \geq 0$ for all x. **1**

(iii) On the same set of axes, sketch $y = e^x - 1$ and $y = x$. **2**

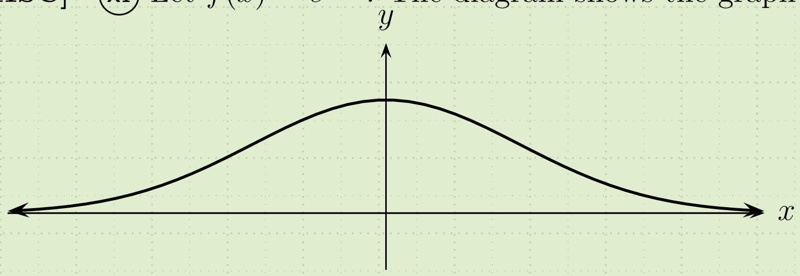
(iv) Find the inverse function of $g(x) = e^x - 1$. **1**

(v) State the domain of $g^{-1}(x)$. **1**

(vi) For what values of x is $\log_e(1 + x) \leq x$? Justify your answer. **2**

Example 28

[2010 Ext 1 HSC] (x1) Let $f(x) = e^{-x^2}$. The diagram shows the graph $y = f(x)$.



(a) The graph has two points of inflexion. Find the x coordinates of these points. **3**

(b) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function. **1**

(c) Find a formula for $f^{-1}(x)$ if the domain of $f(x)$ is restricted to $x \geq 0$. **2**

(d) State the domain of $f^{-1}(x)$. **1**

(e) Sketch of the curve $f^{-1}(x)$. **1**

Example 29

[2002 Ext 1 HSC Q7] X1 Let $g(x) = e^x + \dfrac{1}{e^x}$ for all real values of x and let $f(x) = e^x + \dfrac{1}{e^x}$ for $x \leq 0$.

i. Sketch the graph $y = g(x)$ and explain why $g(x)$ does not have an inverse function. **2**

ii. On a separate diagram, sketch the graph of the inverse function $f^{-1}(x)$. **1**

iii. Find an expression for $y = f^{-1}(x)$ in terms of x. **3**

2.4 Taylor Series expansion

Important note

Not in the syllabus, but provided here as an enrichment exercise.

Steps

To find the Taylor Series expansion to e^x about $x = 0$: If e^x can be approximated by

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

then find a_0, a_1, a_2, a_3 etc to obtain the polynomial approximation.

1. To obtain a_0, set $x = 0$:

$$f(0) = e^0 = 1$$

2. To obtain a_1, differentiate $f(x)$ and let $x = 0$:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

$$f'(0) = a_1 = e^0 = 1$$

3. Differentiate $f'(x)$ and let $x = 0$ to obtain a_2:

$$f''(x) = e^x$$

$$f''(0) = 2a_2 + 3(2)a_3x + 4(3)a_4x^2 + \cdots$$

$$= e^0 = 1$$

$$\therefore a_2 = \frac{1}{2}$$

4. Differentiate $f''(x)$ and let $x = 0$ to obtain a_3:

$$f^{(3)}(x) = e^x$$

$$f^{(3)}(0) = 3(2)a_3 + 4(3)(2)a_4x + 5(4)(3)a_5x^2 + \cdots$$

$$= e^0 = 1$$

$$\therefore a_3 = \frac{1}{3 \times 2 \times 1} = \frac{1}{3!}$$

Steps

(continued from previous page...)

5. Differentiate $f^{(3)}(x)$ and let $x = 0$ to obtain a_4:

$$f^{(4)}(x) = e^x$$

$$f^{(4)}(0) = 4(3)(2)a_4 + 5(4)(3)(2)a_5x + 6(5)(4)(3)a_6x^2\cdots$$

$$= e^0 = 1$$

$$\therefore a_4 = \frac{1}{4 \times 3 \times 2 \times 1} = \frac{1}{4!}$$

6. Generalise:

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

What happens when you differentiate the Taylor polynomial for $f(x) = e^x$?

GeoGebra

Try the `TaylorPolynomial` function with parameter `exp(x)` about $x = 0$ and a degree of your choice!

See also: How does a calculator calculate $\sin x$? (University of Melbourne)

Section 3

Rates of change

Learning Goal(s)

Knowledge
Examine the rate of change in exponential functions

Skills
Simplify and solve exponential and logarithmic functions algebraically

Understanding
Applications of exponential and logarithmic functions in real world contexts

By the end of this section am I able to:

9.5 Establish and use the algebraic properties of exponential functions to simplify and solve problems

9.6 Solve problems involving exponential functions in a variety of practical and abstract contexts, using technology, and algebraically in simple cases

9.8 Work with natural logarithms in a variety of practical and abstract contexts

9.9 Graph a logarithmic function $y = \log_a x$ for $a > 0$ and its transformations $y = k\log_a x + c$, using technology or otherwise, where k and c are constants.

9.10 Solve problems involving indices using (natural) logarithms

Important note

- Use exactly the same rules for the previous work done on *Rates of change* from the *Introduction to Differentiation* topic
- Replace the polynomial-like functions with the exponential function
- Be prepared to use some logarithms base e to finish off the problem.

3.1 Logarithms base e

Important note

Ensure you review work from **Topic 6 - Indices and Logarithms**!

Example 30

Sketch the following graphs, labelling all important points:

(a) $y = \log_e(-x)$ (b) $y = \log_e x - 2$ (c) $y = \log_e(x+3)$

Further exercises

Ⓐ **Ex 9C**

- All questions

(x1) **Ex 11E**

- All questions

3.2 Exponential rates of change

Example 31

(Pender et al., 2019b, p.528) The number B of bacteria in a laboratory culture is growing according to the formula $B = 4\,000e^{0.1t}$, where t is the time in hours after the experiment was started. Answer these questions correct to three significant figures.

(a) How many bacteria are there after 5 hours?

(b) Find the rate $\dfrac{dB}{dt}$ at which the number of bacteria is increasing, and show that

$$\frac{dB}{dt} = 0.1 \times B$$

(c) At what rate are the bacteria increasing after 5 hours?

(d) What is the average rate of increase over the first five hours?

(e) When are there 10 000 bacteria?

(f) When are the bacteria increasing by 10 000 per hour?

Answer: (a) 6 590 (b) Show (c) 659 bacteria/hr (d) 518 bacteria/hr (e) $\approx$ 9.16 hrs (f) $\approx$ 32.2 hrs

Important note

Part (b) is asking you to ....mask........ the time variable with the dependent variable. More coming in the actual X1 *Rates of Change* topic.

Example 32

[2008 2U HSC] Light intensity is measured in lux. The light intensity at the surface of a lake is 6 000 lux. The light intensity, I lux, a distance s metres below the surface of the lake is given by

$$I = Ae^{-ks}$$

where A and k are constants.

(i) Write down the value of A. **1**

(ii) The light intensity 6 metres below the surface of the lake is 1000 lux. **2**

Find the value of k.

(iii) At what rate, in lux per metre, is the light intensity decreasing 6 metres below the surface of the lake? **2**

Example 33

[2009 2U HSC] Radium decays at a rate proportional to the amount of radium present. That is, if $Q(t)$ is the amount of radium present at time t, then $Q = Ae^{-kt}$, where k is a positive constant and A is the amount present at $t = 0$. It takes 1 600 years for the amount of radium to reduce by half.

(i) Find the value of k. **2**

(ii) A factory site is contaminated with radium. The amount of radium on the site is currently three times the safe level. **2**

How many years will it be before the amount of radium reaches the safe level?

Further exercises

(A) Ex 9F

- All questions

(X1) Ex 11F

- All questions

References

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019a). *CambridgeMATHS Stage 6 Mathematics Advanced Year 11* (1st ed.). Cambridge Education.

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