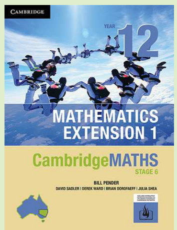
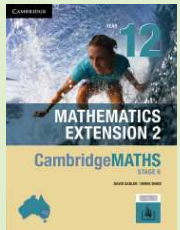
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1/2 (YEAR 12 COURSE)

Topic summary and exercises:

With references to





X1 X2 **Applications of Calculus: Mechanics**

Name: ....

Initial version by H. Lam, 2014. Last updated September 30, 2023.
Major revisions April 2020 for Mathematics Extension 1 & Extension 2, with assistance from I. Ham. Additional contributions from I. Ham, with assistance from M. Ho.
Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under (cc) CC BY 2.0.

Symbols used

Beware! Heed warning.

(x1) Mathematics Extension 1 content.

(x2) Mathematics Extension 2 content.

Literacy: note new word/phrase.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME12-2 applies concepts and techniques involving vectors and projectiles to solve problems

MEX12-6 uses mechanics to model and solve practical problems

Syllabus subtopics

ME-V1 Introduction to Vectors

MEX-M1 Applications of Calculus to Mechanics

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019) or *CambridgeMATHS Year 12 Extension 2* (Sadler & Ward, 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Part I

X2 Further mechanics in one dimension

Section 1

X2 Motion in terms of displacement

Learning Goal(s)

Knowledge
Motion in terms of displacement has time factor masked

Skills
Solving problems by forming and solving differential equations

Understanding
Differences between motion in terms of *time* and motion in terms of *displacement*

By the end of this section am I able to:

30.1 Motion in one dimension, in terms of time.

30.2 Derive and use the equations of motion of a particle travelling in a straight line with both constant and variable acceleration

30.3 Consider and solve problems involving motion in a straight line with both constant and non-constant acceleration and derive and use the expressions $\frac{dv}{dt}$, $v\frac{dv}{dx}$ and $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ for acceleration.

30.4 Describe mathematically the motion of particles in situations other than projectile motion and simple harmonic motion

30.5 Derive $v^2 = g(x)$ and the equations for velocity and displacement in terms of time when given $\ddot{x} = f(x)$ and initial conditions, and describe the resulting motion.

1.1 Velocity in terms of displacement

Fill in the spaces

- Velocity/acceleration in prior work dependent entirely on ....time.... .
- Work on velocity/acceleration in Extension 2 may be dependent on ....displacement.... (x):

$$v \longrightarrow x$$

⚠ ....Displacement.... x is always dependent on ....time.... , i.e. $x \longrightarrow t$

- Consequently, velocity that is dependent on ....displacement.... is also dependent on ....time.... .

Chain diagram:

- To find displacement-time equation from velocity-displacement function, gather similar variables and use ....separation.... ....of.... ....variables.... (See **Topic 29 - X1 Differential Equations**).

- Speed limits on roads an example of velocity dependent on displacement.
- "40 km/h school zones": velocity dependent on displacement as well as time.

1.1.1 Displacement by integration (separation of variables)

Example 1

If the velocity of a particle is given by $v = \sqrt{4x + 25}$, where x is the displacement from the origin and initially, the particle is at the origin. Find and expression for

Answer: $x = \frac{1}{4}(2t+5)^2 - \frac{25}{4}$, $v = (2t+5)$

(a) x in terms of t.

(b) v in terms of t.

Example 2

A particle's velocity in metres per second is given by $v = 2(1 + 3x)$. Initially, the particle is at the origin.

(a) Find an expression for the position of the particle at any time t seconds.

(b) Hence find expressions for the velocity and acceleration in terms of time.

(c) Show that the acceleration $\ddot{x}$ of the particle at any position x is given by

$$\ddot{x} = 12(1 + 3x)$$

(d) Where is the particle at rest, and what is its acceleration at that position?

Example 3

[2004 CSSA Ext 1 Trial Q7] A particle is moving in a straight line. At time t seconds it has displacement x metres to the right of a fixed point O and velocity v metres per second is given by

$$v = \sin x \cos x$$

The particle starts $\dfrac{\pi}{4}$ metres to the right of O.

i. Show that $\dfrac{d}{dx}(\ln \tan x) = \dfrac{1}{\sin x \cos x}$. **1**

ii. Hence show that the displacement of the particle is given by **3**

$$x = \tan^{-1}\left(e^t\right)$$

iii. Find the limiting position of the particle and sketch the graph of x against t. **2**

Answer: $x \to \frac{\pi}{2}^-$

Example 4

[2006 CSSA Ext 1 Trial Q6] A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line and velocity v metres per second given by

$$v = \frac{x(2-x)}{2}$$

The particle starts 1 metre to the right of O.

Answer: $x = \frac{2}{1+e^{-t}}$

i. Show that $\dfrac{2}{x(2-x)} = \dfrac{1}{x} + \dfrac{1}{2-x}$ **1**

ii. Find an expression for x in terms of t. **3**

1.2 Acceleration in terms of displacement

1.2.1 Chain rule

- Evaluate $\dfrac{d}{dx}\left(\dfrac{1}{2}y^2\right)$. Note the differential!

Theorem 1

If velocity is defined in terms of displacement, i.e. $v \longrightarrow x$, then

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Proof – potentially to be reproduced in examinations.

Steps

If $v \longrightarrow x$, then evaluate $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$:

1. $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{d}{dv}\left(\dfrac{1}{2}v^2\right)\dfrac{dv}{dx}$
2. $= v\dfrac{dv}{dx}$
3. $= \dfrac{dx}{dt} \times \dfrac{dv}{dx}$
4. $= \dfrac{dv}{dt}$
5. $= \ddot{x}$

- Verify with chain diagram. Fill in differentials:

$$\begin{tikzcd} v \arrow[r, "\frac{dv}{dx}"] \arrow[rr, bend right, "\frac{dv}{dt}"'] & x \arrow[r, dashed, "\frac{dx}{dt}"] & t \end{tikzcd}$$

1.2.2 Acceleration by differentiation

Example 5

(c.f. Example 2) A particle's velocity in metres per second is given by $v = 2(1+3x)$. Show that the acceleration of the particle in terms of x is $\ddot{x} = 12(1+3x)$.

Example 6

[1996 3U HSC Q5] A particle is moving along the x axis. Its velocity v at position x is given by

$$v = \sqrt{10x - x^2}$$

Find the acceleration of the particle when $x = 4$. **Answer:** $\ddot{x} = 1$

Example 7

[1997 3U HSC Q5] A particle moves along the x axis, starting at $x = 0.1$ at time $t = 0$. The velocity of the particle is described by

$$v = \sqrt{2x}e^{-x^2} \qquad x \geq 0.1$$

where x is the displacement of the particle from the origin.

i. Show that the particle has acceleration given by **2**

$$a = e^{-2x^2}\left(1 - 4x^2\right) \qquad x \geq 0.1$$

ii. Hence find the fastest speed attained by the particle **1**

iii. Show that T, the time taken to travel from $x = 1$ to $x = 2$, can be expressed as **2**

$$T = \int_1^2 \frac{1}{\sqrt{2x}}e^{x^2}\,dx$$

iv. Use the trapezoidal rule with 3 function values to obtain an approximate value for T. **2**

Example 8

[**2008 Independent Ext 1 Trial Q6**] A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity v ms^{-1} given by $v = 2 - x$ and acceleration a ms^{-2}. Initially the particle is 4 metres to the left of O.

i. Find an expression for a in terms of x. **1**

ii. Use integration to show that $x = 2 - 6e^{-t}$. **2**

iii. Find the exact time taken by the particle to travel 4 metres from its starting point. **1**

iv. Sketch the graph of x against t showing the intercepts on the axes and the equations of any asymptotes. **2**

Answer: $\ddot{x} = x - 2$, $t = \ln 3$

Example 9

[2003 CSSA Ext 1 Trial Q3] A particle is moving in a straight line. At time t seconds it has displacement x metres (where $0 \leq x < \frac{\pi}{2}$) from a fixed point O on the line, velocity v ms^{-1} given by $v = \cos^2 x$ and acceleration a ms^{-2}. The particle starts at O.

i. Find expressions for a in terms of x and for x in terms of t. **2**

ii. Sketch the graph of x against t. **1**

iii. Describe the motion of the particle from its initial position to its limiting position. **2**

1.2.3 Velocity by integration

Theorem 2

If $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, by integrating both sides w.r.t. x,

$$\frac{1}{2}v^2 = \int \ddot{x}\, dx$$

Important note

- Constant of integration will be found from initial conditions supplied.
- To continue further to find $x \longrightarrow t$, beware of potential negative sign when taking square roots.

Example 10

The acceleration of a particle moving along the x axis is given by

$$\ddot{x} = 9(x-2)$$

where x is the displacement in metres from the origin after t seconds. The particle is initially 4 metres to the left of O and moving towards the origin at $18\,\mathrm{ms}^{-1}$.

(a) Find the velocity of the particle in terms of x.

(b) Find the displacement of the particle in terms of time.

(c) Find the velocity and acceleration when $t = 1$.

Answer: (a) $v = -3(x-2)$ (b) $x = 2 - 6e^{-3t}$ (c) $\dot{x} = 18e^{-3}$, $\ddot{x} = -54e^{-3}$

Example 11

[2001 Ext 1 HSC Q7] A particle moves in a straight line so that its acceleration is given by

$$\frac{dv}{dt} = x - 1$$

where v is its velocity and x is its displacement from the origin. Initially, the particle is at the origin and has velocity $v = 1$.

i. Show that $v^2 = (x - 1)^2$. **2**

ii. By finding an expression for $\dfrac{dt}{dx}$, or otherwise, find x as a function of t. **2**

Answer: $x = 1 - e^{-t}$

Example 12

A particle is moving along the x axis, starting from a position 2 metres to the right of the origin (that is, $x = 2$ when $t = 0$) with an initial velocity of $5\,\text{ms}^{-1}$ and an acceleration given by

$$\ddot{x} = 2x^3 + 2x$$

i. Show that $\dot{x} = x^2 + 1$. **2**

ii. Hence find an expression for x in terms of t. **3**

Answer: $x = \tan\left(t + \tan^{-1} 2\right)$

Example 13

[2003 Ext 1 HSC Q6] The acceleration of a particle P is given by the equation

$$\frac{d^2x}{dt^2} = 8x\left(x^2+4\right)$$

where x metres is the displacement of P from a fixed point O after t seconds.

Initially the particle is at O and has velocity $8\,\text{ms}^{-1}$ in the positive direction.

i. Show that the speed at any position x is given by $2\left(x^2+4\right)\,\text{ms}^{-1}$. **3**

ii. Hence find the time taken for the particle to travel 2 metres from O. **2**

Answer: $t = \frac{\pi}{16}$

Example 14

[**2008 CSSA Ext 1 Q4**] The acceleration of a particle P, moving in a straight line, is given by $\ddot{x} = 2x - 3$ where x metres is the displacement from the origin O. Initially the particle is at O and its velocity v is 2 metres per second.

i. Show that the velocity of the particle is $v^2 = 2x^2 - 6x + 4$. **2**

ii. Calculate the velocity and acceleration of P at $x = 1$ and briefly describe the motion of P after it moves from $x = 1$. **2**

Note property of v^2: $v^2 \geq 0$.

Example 15

[1998 3U HSC Q6] The acceleration a ms^{-2} of a particle P moving in a straight line is given by

$$a = 3\left(1 - x^2\right)$$

where x metres is the displacement of the particle to the right of the origin. Initially the particle is at the origin and is moving with a velocity of 4 ms^{-1}.

i. Show that the velocity v ms^{-1} of the particle is given by **2**

$$v^2 = 16 + 6x - 2x^3$$

ii. Will the particle ever return to the origin? Justify your answer. **2**

1.3 Summary of one dimensional motion

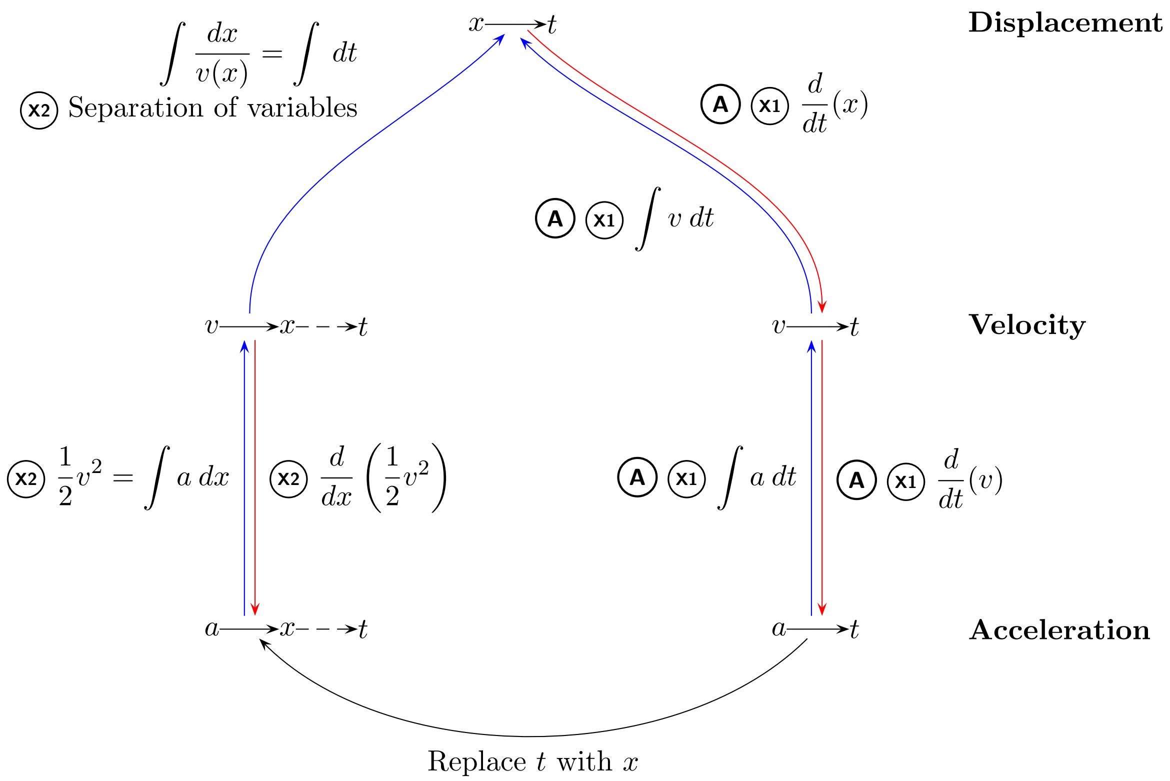


Figure 1.1: Relationship between displacement, velocity, acceleration and its derivatives/primitives w.r.t t and x

Further exercises (Legacy Textbooks)

Ex 3E (Pender, Sadler, Shea, & Ward, 2000)

- Q3-6
- Q10-23

Further exercises

Ex 6A (Sadler & Ward, 2019)

- Q1-5
- Q10-16, 19

Section 2

Simple Harmonic Motion

Learning Goal(s)

Knowledge
What is SHM

Skills
Solve problems involving SHM in terms of time and displacement

Understanding
Special positions for a particle undergoing SHM

By the end of this section am I able to:

30.6 Derive equations for displacement, velocity and acceleration in terms of time, given that a motion is simple harmonic and describe the motion modelled by these equations

30.7 Prove that motion is simple harmonic when given an equation of motion for acceleration, velocity or displacement and describe the resulting motion

30.8 Sketch graphs of x, $\dot{x}$ and $\ddot{x}$ as functions of t and interpret and describe features of the motion.

30.9 Prove that motion is simple harmonic when given graphs of motion for acceleration, velocity or displacement and determine equations for the motion and describe the resulting motion

30.10 Use relevant formulae and graphs to solve problems involving simple harmonic motion

2.1 Introduction

Definition 1

Simple harmonic motion is modelled by a single (simple) sinusoid (harmonic). Physically, the restoring force is directly proportional to (and directed against) the displacement from the centre of motion, i.e.

$$\ddot{x} = -n^2 x \qquad \text{(centred about } x = 0)$$

$$\ddot{x} = -n^2 (x - x_0) \qquad \text{(centred about } x = x_0)$$

where

- x is the displacement of the particle
- $\ddot{x}$ is the acceleration of the particle

x_0 is the centre of motion

2.1.1 Physical phenomena modelled by SHM

Ignoring friction, air resistance etc.

- Spring
- Floating buoy
- Pendulum

GeoGebra

- Point moving along unit circle and relationship to sinusoids
- Simple Harmonic Motion - Diagrams

2.1.2 Observations:

- At the extremities $x = \pm a$:
  - Velocity is zero
  - Acceleration is maximum
- At the centre of motion $x = 0$:
  - Velocity is maximum
  - Acceleration is zero

2.2 Derivation of velocity/displacement equations

2.2.1 Velocity equation

Steps

Differential equation defining simple harmonic motion:

$$\ddot{x} = -n^2 x$$

1. Use $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \ddot{x}$ to integrate:

$$\frac{1}{2}v^2 = \int -n^2 x \, dx$$

$$= -\frac{n^2x^2}{2} + C_1$$

$$\therefore v^2 = -n^2x^2 + C_2$$

2. At the extremities, $x = a$ and $v = 0$:

$$0 = -n^2a^2 + C_2$$

$$\therefore C_2 = n^2a^2$$

3. Hence,

$$v^2 = -n^2x^2 + n^2a^2 = n^2\left(a^2 - x^2\right)$$

Proving a particle is in SHM requires demonstration of

$$\ddot{x} = -n^2 x$$

Note the shape of the v^2 against x graph: parabolic

2.2.2 Displacement equation

Steps

Differential equation for velocity in terms of displacement:

$$v^2 = n^2\left(a^2 - x^2\right)$$

1. Take appropriate square root:

$$v = \pm n\sqrt{a^2 - x^2}$$

2. Rewrite velocity as differential and separate variables:

$$\frac{dx}{dt} = \pm n\sqrt{a^2 - x^2}$$

$$\frac{dx}{\pm\sqrt{a^2 - x^2}} = n\,dt$$

3. Use positive square root (first case), and integrate:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int n\,dt$$

$$\sin^{-1}\tfrac{x}{a} = nt + C_1$$

$$x = a\sin(nt + \phi)$$

4. Use negative square root (second case), and integrate:

$$\int \cos^{-1}\tfrac{x}{a} = \int nt + C_2$$

$$\frac{x}{a} = \cos(nt + C_2)$$

$$x = a\cos(nt + \phi)$$

Remarks

- ϕ is known as the phase shift.
- Use initial conditions to determine which square root is used.
- Choose appropriate sinusoid and negative signs such that the phase shift is thus minimised and becomes an acute angle.

Extremity	Centre of motion	Extremity
$\dot{x} = 0$	$\dot{x} =$ max	$\dot{x} = 0$
$\lvert\ddot{x}\rvert =$ max	$\ddot{x} = 0$	$\lvert\ddot{x}\rvert =$ max

2.3 SHM in terms of time

- Questions of this type generally rely on x, $\dot{x}$ and $\ddot{x}$ in terms of time.
- Mask time factor with displacement factor in $\ddot{x}$ where asked.

⚠ Use graphs of trigonometric functions!

Example 16

[Ex 3D Q2] (Pender et al., 2000) A particle is moving in simple harmonic motion with period 4 seconds and centre at the origin, and starts from rest 12 cm on the positive side of the origin.

(a) Find x as a function of t

(b) Differentiate to find v and $\ddot{x}$ as functions of t, and show that $\ddot{x} = -n^2x$.

(c) How long is it between visits to the origin?

Example 17

A particle is moving in simple harmonic motion according to the equation

$$x = 2 + 4\cos\left(2t + \frac{\pi}{3}\right)$$

(a) Find the centre, period, amplitude and extremes of the motion.

(b) What is the initial phase, and where is the particle at $t = 0$?

(c) Find the first time when the particle is at:

i. the centre of motion,

ii. the origin,

iii. the maximum displacement

iv. the minimum displacement

Answer: (a) $c = 2$, extremities $x = 6$ or -2, $a = 4$. (b) Initial phase $\frac{\pi}{3}$, $x = 4$ (c) i. $\frac{\pi}{12}$ ii. $\frac{\pi}{6}$ iii. $\frac{5\pi}{6}$ iv. $\frac{\pi}{3}$

Laws/Results

Differential equation defining simple harmonic motion with centre of motion $x = c$:

$$\ddot{x} = -n^2(x - c)$$

Example 18

[2020 Ext 2 HSC Sample Q15] A particle is moving in a straight line according to the equation $x = 5 + 6\cos 2t + 8\sin 2t$, where x is the displacement in metres and t is the time in seconds.

i. Prove that the particle is moving in simple harmonic motion by showing that x satisfies an equation of the form **2**

$$\ddot{x} = -n^2(x - c)$$

ii. When is the displacement of the particle zero for the first time? **3**

Answer: $t = \frac{1}{2}\left(\frac{2\pi}{3} + \tan^{-1}\frac{4}{3}\right) \approx 1.511$

Important note

Free revision for **Topic 26 - (X1) Further Trigonometric Identities 2**

Example 19

[1998 3U HSC Q3]

(b) Express $\sin 4t + \sqrt{3}\cos 4t$ in the form $R\sin(4t+\alpha)$, where α is in radians.

(c) A particle moves in a straight line and its position at time t is given by

$$x = 1 + \sin 4t + \sqrt{3}\cos 4t$$

i. Prove that the particle is undergoing simple harmonic motion about $x = 1$. **2**

ii. Find the amplitude of the motion. **1**

iii. When does the particle first reach maximum speed after time $t = 0$? **2**

Answer: i. Proof ii. $a = 2$ iii. $t = \frac{\pi}{6}$

Differentiate w.r.t. t to obtain defining equation for SHM.

Example 20

[1999 3U HSC Q6] A particle moves in a straight line and its displacement x metres from the origin after t seconds is given by

$$x = \cos^2 3t \quad t > 0$$

i. When is the particle first at $x = \frac{3}{4}$? **1**

ii. In what direction is the particle travelling when it is first at $x = \frac{3}{4}$? **1**

iii. Express the acceleration of the particle in terms of x. **2**

iv. Hence, or otherwise, show that the particle is undergoing simple harmonic motion. **1**

v. State the period of the motion. **1**

Answer: (i) $t = \frac{\pi}{18}$ (ii) Negative (iii) $\ddot{x} = -36\left(x - \frac{1}{2}\right)$ (iv) Proof (v) $T = \frac{\pi}{3}$

Example 21

A weight on a spring is moving in simple harmonic motion with a period of $\frac{2\pi}{5}$ seconds. A laser observation at a certain instant shows it to be 15 cm below the origin, moving upwards at 60 cm/s.

Answer: (a) $x = 3\sqrt{41}\sin(5t - \alpha)$ (b) $t \approx 0.179$ s (c) $t \approx 0.987$ s

(a) Find the displacement x of the weight above the origin as a function of the time t after the laser observation. Use the form $x = a\sin(nt - \alpha)$.

(b) Find how long the weight takes to reach the origin (three significant figures).

(c) ⚠ Find when the weight returns to where it was first observed. Use the $\sin A = \sin B$ approach to solve the trigonometric equation.

Example 22

Two particles A and B move in simple harmonic motion in a straight line with the same amplitude and period about the origin. Initially, particle A is at the origin and moving towards B with a speed of $13\,\text{ms}^{-1}$, while particle B is $2\,\text{m}$ from O, moving with a speed of $5\,\text{ms}^{-1}$ away from O.

(a) By choosing a suitable mathematical model for the motion of particle A, show that the amplitude is $\dfrac{13}{6}$ metres, and the period of the motion is $\dfrac{\pi}{3}$ seconds.

(b) Show that particle A is $\dfrac{5}{6}$ metres from O when particle B is at rest for the first time.

Further exercises

Ex 6B (Sadler & Ward, 2019)

- All questions

Further exercises (Legacy Textbooks)

Ex 3D (Pender et al., 2000)

- Q4-7
- Q9-13
- Q21

2.4 SHM in terms of displacement

- Questions generally rely on
  - velocity squared

    $$v^2 = n^2\left(a^2 - (x - x_0)^2\right)$$

  - or, acceleration in terms of displacement

    $$\ddot{x} = -n^2\left(x - x_0\right)$$

  equations as opposed to time equations.
- Use these positions to assist with answering questions:

Extremity	Centre of motion	Extremity
$\dot{x} = 0$	$\dot{x} =$ max	$\dot{x} = 0$
$\lvert\ddot{x}\rvert =$ max	$\ddot{x} = 0$	$\lvert\ddot{x}\rvert =$ max

Example 23

The speed of a particle moving along the x axis is given by $v^2 = 24 + 8x - 2x^2$.

(a) Find the acceleration of the particle.

(b) Explain why the motion of the particle is simple harmonic and find its amplitude.

(c) Find the displacement of the particle in terms of time t given that initially it started to move from the centre of its motion towards the right.

Answer: (a) $\ddot{x} = -2(x-2)$ (b) Explain (c) $x = 4\sin\left(\sqrt{2}t\right) + 2$

Example 24

[2009 Ext 1 HSC Q5] The equation of motion for a particle moving in simple harmonic motion is given by

$$\frac{d^2x}{dt^2} = -n^2x$$

where n is a positive constant, x is the displacement of the particle and t is time.

i. Show that the square of the velocity of the particle is given by **3**

$$v^2 = n^2\left(a^2 - x^2\right)$$

where $v = \dfrac{dx}{dt}$ and a is the amplitude of the motion.

ii. Find the maximum speed of the particle. **1**

iii. Find the maximum acceleration of the particle. **1**

iv. The particle is initially at the origin. Write down a formula for x as a function of t, and hence find the first time that the particle's speed is half its maximum speed. **2**

Answer: (i) Proof (ii) $|v|_{\max} = an$ (iii) $|\ddot{x}|_{\max} = an^2$ (iv) $t = \frac{\pi}{3n}$

Example 25

[2000 3U HSC Q4] (2 marks) A particle is moving in simple harmonic motion about a fixed point O. Its amplitude is 3 cm and its period is 4π seconds.

Find its speed at the point O. **Answer:** $\frac{3}{2}$ cm s^{-1}

Example 26

[2008 Ext 1 HSC Q5] (3 marks) A particle is moving in simple harmonic motion in a straight line. Its maximum speed is $2\,\text{ms}^{-1}$ and its maximum acceleration is $6\,\text{ms}^{-2}$.

Find the amplitude and the period of the motion. **Answer:** $a = \frac{2}{3}$, $T = \frac{2\pi}{3}$

Example 27

[**Ex 6C Q17**] A particle is moving according to

$$x = 4\cos 3t - 6\sin 3t$$

(a) Prove that the acceleration is proportional to the displacement but oppositely directed, and hence that the motion is simple harmonic.

(b) Find the period, amplitude and maximum speed of the particle, and find the acceleration when the particle is halfway between its mean position and one of its extreme positions.

⚠ Rewrite sum of sinusoids into single sinusoid.

Example 28

[2001 Ext 1 HSC Q4] (5 marks) A particle, whose displacement is x, moves in simple harmonic motion.

Find x as a function of t if $\ddot{x} = -4x$ and if $x = 3$, $\dot{x} = -6\sqrt{3}$ when $t = 0$.

Answer: $x = 6\cos\left(2t + \frac{\pi}{3}\right)$

Example 29

[1999 NEAP 3U Trial Q4] A particle moves in a straight line and at time t seconds, its velocity, v metres per second, is related to its displacement, x metres, by

$$v^2 = 108 + 36x - 9x^2$$

i. By deriving an expression for acceleration, show that the motion is simple harmonic. **1**

ii. Find the period of the motion. **1**

iii. Find the amplitude of the motion. **1**

iv. For what value of x does the particle have maximum speed? **2**

Example 30

[2011 CSSA Ext 1 Trial Q5] The velocity, v metres per second, of a particle moving in simple harmonic motion along the x axis is given by $v^2 = 8 - 2x - x^2$, where x is in metres.

i. Between which two points is the particle oscillating? **2**

ii. Find the centre of the motion. **1**

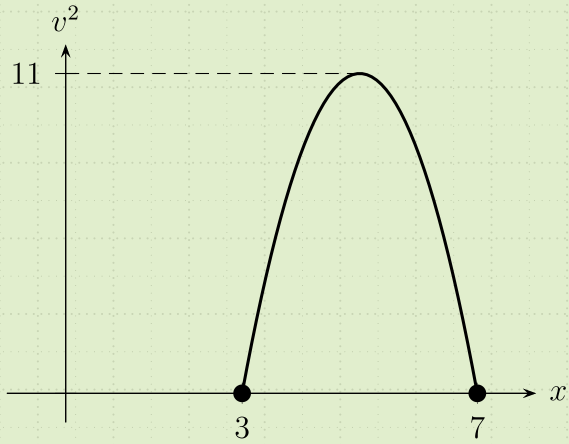
iii. Find the maximum speed. **1**

iv. Find an expression for the acceleration of the particle in terms of x. **1**

Answer: i. $x = -4, 2$ ii. $x = -1$ iii. $v = 3$ iv. $\ddot{x} = -1 - x$

Example 31

[2015 Ext 1 HSC Q13] A particle is moving along the x axis in simple harmonic motion. The displacement of the particle is x metres and its velocity is v ms^{-1}. The parabola below shows v^2 as a function of x.



i. For what value(s) of x is the particle at rest? **1**

ii. What is the maximum speed of the particle? **1**

iii. The velocity v of the particle is given by the equation **3**

$$v^2 = n^2\left(a^2 - (x-c)^2\right)$$

where a, c and n are positive constants.

What are the values of a, c and n?

Example 32

The velocity v metres per second, of a particle moving along the x axis is given by

$$v^2 = 28 + 24x - 4x^2$$

(a) Show that the particle is moving in simple harmonic motion, and write down the centre and period of motion.

(b) Between what two points it the particle oscillating? Hence state the amplitude of the motion.

(c) Initially, the particle is at its furthest point to the right. Draw a graph of the displacement of the particle against time over the two complete oscillations.

(d) Write down an expression for the displacement of the particle, x metres, in terms of time t.

(e) When is the particle at the origin for the first time? Give your answer to the nearest 0.1 second.

Further exercises

Ex 6B (Sadler & Ward, 2019)

- Q1-19

Further exercises (Legacy Textbooks)

Ex 3F (Pender et al., 2000)

- Q1,3
- Q5-19

Part II

X1 X2 Vectors and mechanics in two dimensions

Section 3

Displacement, Velocity, Forces

Learning Goal(s)

Knowledge
Velocity and displacement represented by forces

Skills
Solve problems by calculus and vector component techniques

Understanding
Speed of a particle is given by the sum of squares of the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components

By the end of this section am I able to:
30.11 Solve problems involving displacement, force and velocity involving vector concepts in two dimensions

Steps

General steps for solving problems

- Draw vector diagram of velocity/forces.
- Relate the problem to **Topic 15 - X1 Introduction to Vectors** work.
  - Vector addition/subtraction
  - Magnitude/unit vector
  - Dot product/projection

3.1 Review of vectors

3.1.1 Components of a vector

Example 33

[2007 VCE Specialist Mathematics Paper 2 Q16] A force of magnitude 18 newtons acts on a body at an angle of 150° in the anticlockwise direction to the vector $\underset{\sim}{i}$.

A vector representation of this force could be:

(A) $9\sqrt{3}\underset{\sim}{i} + 9\underset{\sim}{j}$

(B) $-9\underset{\sim}{i} + 9\sqrt{3}\underset{\sim}{j}$

(C) $-9\sqrt{3}\underset{\sim}{i} + 9\underset{\sim}{j}$

(D) $-9\underset{\sim}{i} - 9\sqrt{3}\underset{\sim}{j}$

(E) $9\sqrt{3}\underset{\sim}{i} - 9\underset{\sim}{j}$

3.2 Displacement and velocity as 2D vectors

Laws/Results

Comparison of one and two dimensional displacement/velocity/acceleration

- **Displacement:**

  One dimensional (from earlier work) **Two dimensional** Vector based

  $$\underset{\cdot\cdot}{x} = f(t)$$

  $$\underset{\sim}{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

- **Velocity:**

  One dimensional **Two dimensional**

  $$v = \underset{\cdot\cdot}{\dot{x}} = f'(t)$$

  $$\underset{\sim}{v}(t) = \underset{\sim}{\dot{r}} = \dot{x}(t)\,\underset{\sim}{i} + \dot{y}(t)\,\underset{\sim}{j} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

- **Acceleration:**

  One dimensional **Two dimensional**

  $$a = \underset{\cdot\cdot}{\ddot{x}} = f''(t)$$

  $$\underset{\sim}{a}(t) = \underset{\sim}{\ddot{r}} = \ddot{x}(t)\,\underset{\sim}{i} + \ddot{y}(t)\,\underset{\sim}{j} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}$$

Important note

Cartesian path of a particle $\underset{\sim}{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$:

- Reworded from **Topic 3 - X1 Further work with Functions** as a pair of ......parametric...... ......equations...... .
- Express y in terms of x only.

Important note

Speed of a particle

- One dimensional: the magnitude of the ......velocity...... of the particle.
- Two dimensional: the magnitude of the ......velocity...... ......vector...... (found via Pythagoras' Theorem):

  $$\text{Speed} = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2}$$

Example 34

[2010 VCE Specialist Mathematics Paper 1 Q8] The path of a particle is given by $\underset{\sim}{r}(t) = t\sin(t)\,\underset{\sim}{i} - t\cos(t)\,\underset{\sim}{j}$, $t \geq 0$. The particle leaves the origin at $t = 0$ and then spirals outwards.

(a) Show that the **second time** that the particle crosses the x axis after leaving the origin occurs when $t = \dfrac{3\pi}{2}$. **1**

(b) Find the speed of the particle when $t = \dfrac{3\pi}{2}$. **3**

(c) Let θ be the acute angle at which the path of the particle crosses the x axis. Find $\tan\theta$ when $t = \dfrac{3\pi}{2}$. **1**

Answer: (a) Show (b) $\sqrt{1+\frac{9\pi^2}{4}}$ (c) $\frac{3\pi}{2}$

Example 35

[2012 VCE Specialist Mathematics Paper 1 Q9] The position of a particle at time t is given by

$$\underset{\sim}{r}(t) = \left(2\sqrt{t^2+2} - t^2\right)\underset{\sim}{i} + \left(2\sqrt{t^2+2} + 2t\right)\underset{\sim}{j} \qquad t \geq 0$$

(a) Find the velocity of the particle at time t. **1**

(b) Find the speed of the particle at time $t = 1$ in the form $\dfrac{a\sqrt{b}}{c}$, where a, b and c are positive integers. **2**

(c) Show that at time $t = 1$, $\dfrac{dy}{dx} = \dfrac{1+\sqrt{3}}{1-\sqrt{3}}$. **2**

(d) Find the angle in terms of π, between the vector $-\sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}$ and the vector $\underset{\sim}{r}(t)$ at time $t = 0$. **2**

Answer: (a) $\underset{\sim}{v} = \left(\frac{2t}{\sqrt{t^2+2}} - 2t\right)\underset{\sim}{i} + \left(\frac{2t}{\sqrt{t^2+2}} + 2\right)\underset{\sim}{j}$ (b) $\frac{4\sqrt{6}}{3}$ (c) Show (d) $\frac{7\pi}{12}$

Laws/Results

Two particles **collide** when their x and y position coordinates are respectively equal.

Example 36

[**2015 VCE Specialist Mathematics Paper 2 Q18**] The position vectors of two moving particles are given by

$$\underset{\sim}{r}_1 = \left(2 + 4t^2\right) \underset{\sim}{i} + (3t + 2) \underset{\sim}{j} \quad \text{and} \quad \underset{\sim}{r}_2(t) = 6t \underset{\sim}{i} + (4 + t) \underset{\sim}{j}$$

where $t \geq 0$.

The particles will collide at

(A) $3\underset{\sim}{i} + 3.5\underset{\sim}{j}$

(B) $6\underset{\sim}{i} + 5\underset{\sim}{j}$

(C) $3\underset{\sim}{i} + 4.5\underset{\sim}{j}$

(D) $0.5\underset{\sim}{i} + \underset{\sim}{j}$

(E) $5\underset{\sim}{i} + 6\underset{\sim}{j}$

Further exercises

Ex 13.1 (Fitzpatrick & Aus, 2019)

- All questions

3.3 Sum of forces

Theorem 3


[12]

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Projectilia perseverant in motibus suis nisi quatenus a resistentia aeris retardantur & vi gravitatis impelluntur deorsum. Trochus, cujus partes cohaerendo perpetuo retrahunt sese a motibus rectilineis, non cessat rotari nisi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus suos & progressivos & circulares in spatiis minus resistentibus factos conservant diutius.

Lex. II.

Mutationem motus proportionalem esse vi motrici impressae, & fieri secundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, sive simul & semel, sive gradatim & successive impressa fuerit. Et hic motus quoniam in eandem semper plagam cum vi generatrice determinatur, si corpus antea movebatur, motui ejus vel conspiranti additur, vel contrario subducitur, vel obliquo oblique adjicitur, & cum eo secundum utriusq; determinationem componitur.

Lex. III.

Newton's Laws in Latin

Newton's Laws of Motion:

1. When viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity, unless acted upon by an external force.

2. The vector sum of forces $\sum \underset{\sim}{F}$ on an object is equal to the mass m of that object, multiplied by the acceleration vector $\underset{\sim}{a}$ of the object.

$$\underset{\sim}{F} = m\underset{\sim}{a} \tag{3.1}$$

(units: $\text{kg}\,\text{ms}^{-2}$, Newtons)

3. When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

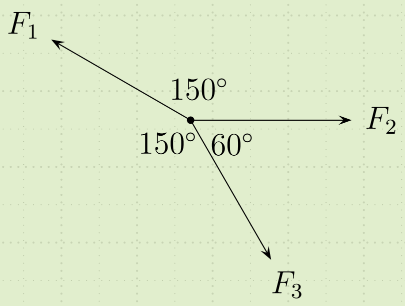
3.3.1 Equilibrium

Definition 2

Forces in **equilibrium** $\Leftrightarrow$ object has ....constant.... ....velocity.... .

Example 37

[2006 VCE Specialist Mathematics Paper 2 Q20] If three co-planar forces, F_1, F_2 and F_3, act on a particle which is in equilibrium[a] as shown in the above diagram, then



(A) $F_1 = F_2 = \sqrt{3}F_3$

(B) $F_2 = F_3 = \sqrt{3}F_1$

(C) $F_1 = F_3 = \sqrt{3}F_2$

(D) $F_2 = F_3 = \dfrac{\sqrt{3}}{3}F_1$

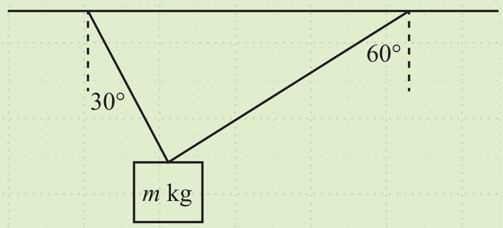
(E) $F_1 = F_3 = \dfrac{\sqrt{3}}{3}F_2$

[a] sum of forces is zero

Example 38

[2011 VCE Specialist Mathematics Paper 1 Q7] A flowerpot of mass m kg is held in equilibrium by two light ropes, both of which are connected to a ceiling. The first rope makes an angle of $30°$ to the vertical and has tension T_1 newtons. The second makes an angle of $60°$ to the vertical and has tension T_2 newtons.

(Use $g = 9.8\,\mathrm{ms}^{-2}$)



(a) Show that $T_2 = \dfrac{T_1}{\sqrt{3}}$. **1**

(b) The first rope is strong, but the second rope will break if the tension in it exceeds 98 newtons. Find the maximum value of m for which the flowerpot will remain in equilibrium. **3**

Answer: 20 kg

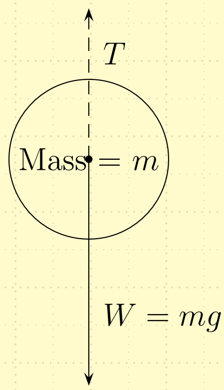
Laws/Results

Weight of the mass due to gravitational force

$$W = mg$$

where

- W is the weight.
- m is the mass.
- g is the downwards acceleration experienced, typically $g = 9.8\,\mathrm{ms}^{-2}$.



Resolution of forces on a mass

3.3.2 Projection

Example 39

[2015 VCE Specialist Mathematics Paper 2 Q15] The component of force $\underset{\sim}{F} = a\,\underset{\sim}{i} + b\,\underset{\sim}{j}$, where a and b are non-zero real constants, in the direction of the vector $\underset{\sim}{w} = \underset{\sim}{i} + \underset{\sim}{j}$ is

(A) $\left(\dfrac{a+b}{2}\right)\underset{\sim}{w}$

(B) $\dfrac{\underset{\sim}{F}}{a+b}$

(C) $\left(\dfrac{a+b}{a^2+b^2}\right)\underset{\sim}{F}$

(D) $(a+b)\underset{\sim}{w}$

(E) $\left(\dfrac{a+b}{\sqrt{2}}\right)\underset{\sim}{w}$

Example 40

[2008 VCE Specialist Mathematics Paper 2 Q18] A force $\underset{\sim}{F}$ is applied to a body causing it to accelerate in the direction of vector $\underset{\sim}{d}$.

The magnitude of the force which causes the body to accelerate in this direction is given by:

(A) $\underset{\sim}{F} \cdot \underset{\sim}{d}$ (B) $\dfrac{\underset{\sim}{d}}{\underset{\sim}{F}}$ (C) $\dfrac{\underset{\sim}{F} \cdot \underset{\sim}{d}}{|\underset{\sim}{F}|}$ (D) $\dfrac{\underset{\sim}{F}}{\underset{\sim}{d}}$ (E) $\dfrac{\underset{\sim}{F} \cdot \underset{\sim}{d}}{|\underset{\sim}{d}|}$

Example 41

[2020 Ext 1 HSC Sample Q12] A force described by the vector $\underset{\sim}{F} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ newtons is applied to an object lying on a line ℓ which is parallel to the vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

i. Find the component of the force $\underset{\sim}{F}$ in the direction of the line ℓ. **2**

ii. What is the component of the force $\underset{\sim}{F}$ in the direction perpendicular to the line? **1**

Answer: i. $\begin{pmatrix} 6/5 \\ 8/5 \end{pmatrix}$ ii. $\underset{\sim}{F} - \begin{pmatrix} 6/5 \\ 8/5 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$

Further exercises

Ex 8F (Pender et al., 2019)

- Q1-19

Ex 13.2 (Fitzpatrick & Aus, 2019)

- All questions

Section 4

Projectile motion

Learning Goal(s)

Knowledge
Horizontal and vertical components of displacement/velocity/acceleration applied in projectile motion

Skills
Apply calculus to solve problems involving projectiles

Understanding
Concept of projectile motion

By the end of this section am I able to:

30.12 Understand the concept of projectile motion, and model and analyse a projectile's path

30.13 Model the motion of a projectile as a particle moving with constant acceleration due to gravity and derive the equations of motion of a projectile

30.14 Use equations for horizontal and vertical components of velocity and displacement to solve problems on projectiles

30.15 Apply calculus to the equations of motion to solve problems involving projectiles

4.1 Rationale

- Motion involving two dimensions, simultaneously:
  - up/down
  - left/right
- Notation: identical to usual notation for xy (Cartesian) plane. See Section 3.2 on page 50.

Laws/Results

Single governing equation for projectile motion (ignoring air resistance):

$$\underset{\sim}{a} = \ddot{\underset{\sim}{r}} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix} \tag{4.1}$$

where g is the acceleration due to gravity.

Important note

⚠ **Derive equations** for $\underset{\sim}{v}$ and $\underset{\sim}{r}$ unless they are provided.

4.2 Vertical projection

- Occurs when particle is projected vertically, at 90° to ground.
- Characteristic equations:

➔ $\dot{x} = 0$. ⬆ $\ddot{y} = -g$.

Laws/Results

Maximum height is reached when $\dot{y} = 0$.

Example 42

A steel ball is projected vertically and reaches its maximum height in 6 seconds. What height was attained and what was the velocity of projection? (Use $g = 10$).

4.3 Horizontal projection

- Occurs when particle is projected horizontally, parallel to ground.
- Characteristic equations:

$\dot{x} = V$ $\ddot{y} = -g$ $t = 0,\ y = h$

Example 43

A body is projected horizontally from the top of a building 125 metres high with a speed of 10 metres/second. Find:

(a) the time taken to reach the ground,

(b) the horizontal distance travelled.

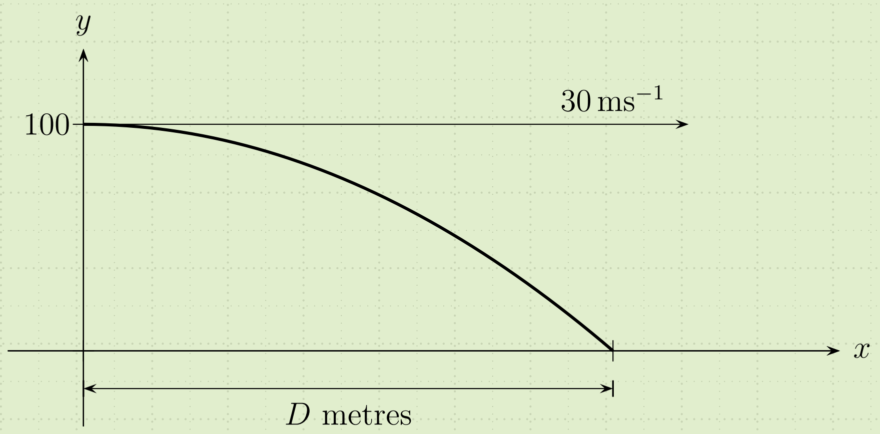
Explain why the time taken to reach the ground is independent of the velocity of projection. Use $g = 10\,\text{ms}^{-2}$.

Important note

- Draw diagram depicting projection.
- Show direction of initial velocity.

Example 44

[1999 NEAP 3U Trial] A plane designed to drop waterbombs flies in a horizontal line towards a spot fire. It maintains a steady speed of $30\,\mathrm{ms}^{-1}$, and an altitude of 100 m. The plane releases its water when at the horizontal distance D metres from the fire.



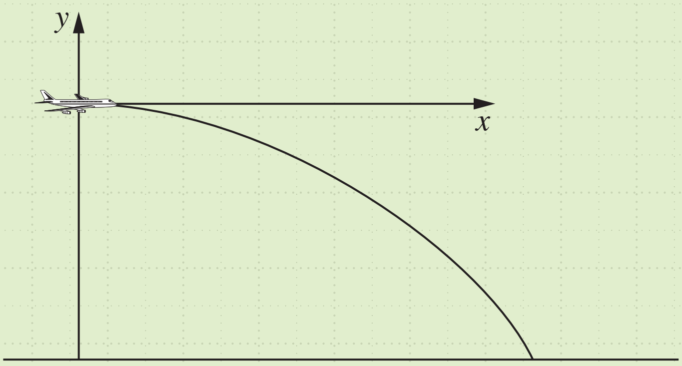
Let the origin O be as shown in the diagram, and let x and y metres be the horizontal and vertical displacements of the centre of the water bomb at time t seconds after its release. Then $\dot{x}$ and $\dot{y}$ are the horizontal and vertical components of its velocity in metres per second. Let $g = 10$ metres per second per second, and assume that air resistance has negligible effect.

i. Use calculus to derive expressions for $\dot{x}$, $\dot{y}$, x and y in terms of t. **2**

ii. Hence determine the value of D correct to the nearest metre. **2**

Example 45

[2001 Ext 1 HSC Q4] An aircraft flying horizontally at V ms^{-1} releases a bomb that hits the ground 4 000 m away, measured horizontally. The bomb hits the ground at an angle of 45° to the vertical.



Assume that, t seconds after release, the position of the bomb is given by

$$x = Vt, y = -5t^2$$

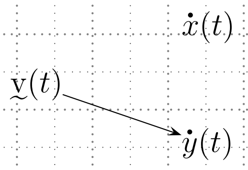
Find the speed V of the aircraft.

4.4 Angled projection

4.4.1 Time equations

Derivation of time equations

- Resolve the *tangent* to the motion (velocity) into two components (vectors):
  - up/down (y axis)
  - left/right (x axis)



Vector components:

$$\underset{\sim}{v}(t) = \dot{x}(t)\,\underset{\sim}{i} + \dot{y}(t)\,\underset{\sim}{j}$$

- Similar approach with displacement & acceleration.

Definition 3

Initial velocity $\underset{\sim}{v}(0)$, the velocity of projectile at $t = 0$ in the direction of arrow.

Definition 4

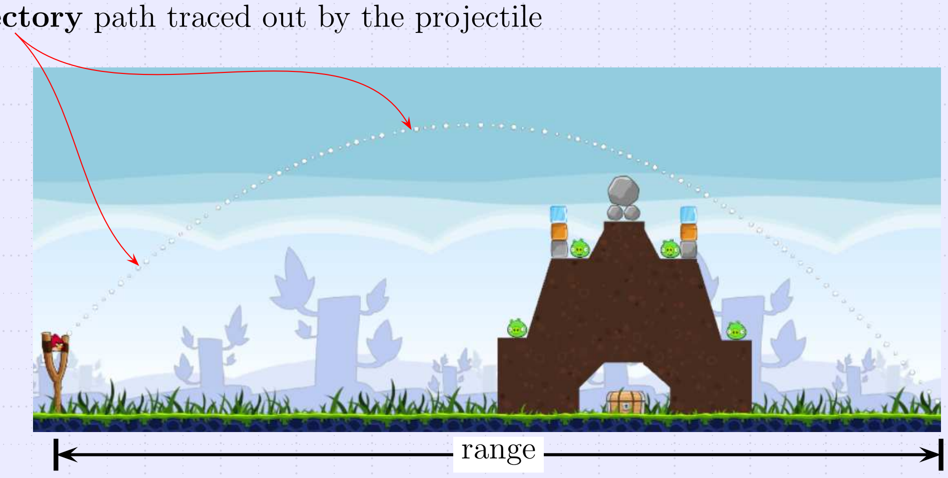
Initial position (x, y) coordinates of object being fired at $t = 0$.

Definition 5

Angle of inclination α acute angle between the tangent to the trajectory and the horizontal.

Definition 6

Range Total horizontal distance travelled by the projectile
Trajectory path traced out by the projectile



- By convention, the direction of travel is
  - positive: up / right
  - negative: down / left
- All projectile motion is governed by Equation (4.1), or when separated into its components:

$$\ddot{x} = 0 \tag{4.2}$$
$$\ddot{y} = -g \tag{4.3}$$

  where g is the acceleration due to gravity, generally given as $g = 10\,\text{ms}^{-2}$

- Commence with these two equations for *every* question, work with the components individually.

Example 46

A ball is thrown with initial velocity 40 m/s and angle of inclination 30° from the top of a stand 25 metres above the ground. The base of the stand is at ground level.

(a) Using the stand as the origin and $g = 10\,\text{ms}^{-2}$, find the six equations of motion.

(b) Find how high the ball rises, how long it takes to get there, what its speed is then, and how far it is horizontally from the stand.

(c) Find the flight time, the horizontal range, and the impact speed and angle.

Example 47

A gun at O fires shells with an initial speed of $200\,\text{ms}^{-1}$ but a variable angle of inclination α. Take $g = 10\,\text{ms}^{-2}$.

(a) Find the two possible angles at which the gun can be set so that it will hit a fortress F, 2 km away on top of a mountain 1 000 metres high.

(b) Show that the two angles are equally inclined to OF and to the vertical.

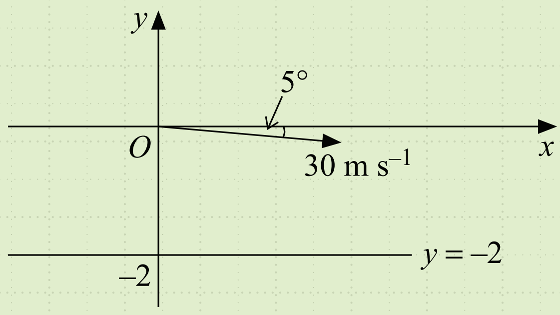
(c) Find the corresponding flight times and the impact speeds and angles.

Example 48

[1999 3U HSC Q7] A cricket ball leaves the bowler's hand 2 metres above the ground with a velocity of $30\,\text{ms}^{-1}$ at an angle of $5°$ below the horizontal. The equations of motion for the ball are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -10$$

Take the origin to be the point where the ball leaves the bowler's hand.



i. Using calculus, prove that the coordinates of the ball at time t are given by **3**

$$x = 30t\cos 5° \qquad \text{and}$$
$$y = -30t\sin 5° - 5t^2$$

ii. Find the time at which the ball strikes the ground. **2**

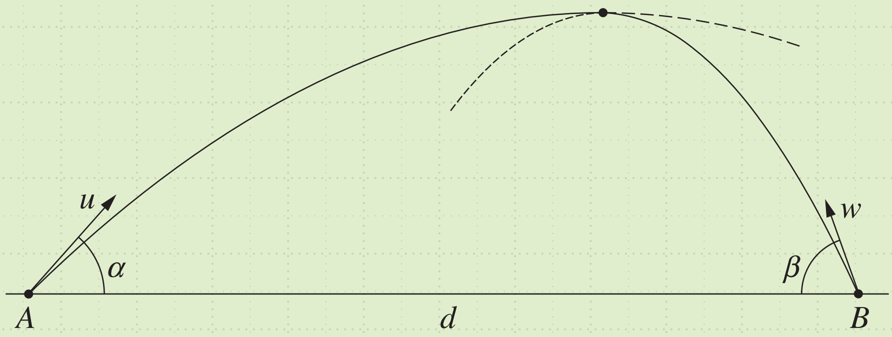
iii. Calculate the angle at which the ball strikes the ground. **3**

Example 49

[2013 Ext 1 HSC Q13] Points A and B are located d metres apart on a horizontal plane. A projectile is fired from A towards B with initial velocity u ms^{-1} at angle α to the horizontal.

At the same time, another projectile is fired from B towards A with initial velocity w ms^{-1} at angle β to the horizontal, as shown on the diagram.

The projectiles collide when they both reach their maximum height.



The equations of motion of a projectile fired from the origin with initial velocity V ms^{-1} at angle θ to the horizontal are

$$x = Vt\cos\theta \text{ and } y = Vt\sin\theta - \frac{g}{2}t^2$$

(Do NOT prove this).

i. How long does the projectile fired from A take to reach its maximum height? **2**

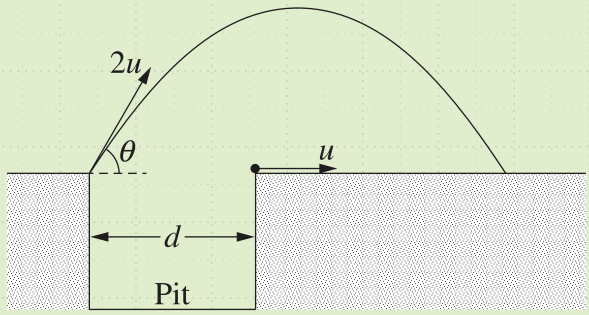
ii. Show that $u\sin\alpha = w\sin\beta$. **1**

iii. Show that $d = \dfrac{uw}{g}\sin(\alpha+\beta)$. **2**

Example 50

[2022 Ext 1 HSC Q14] (4 marks) A video game designer wants to include an obstacle in the game they are developing. The player will reach one side of a pit and must shoot a projectile to hit a target on the other side of the pit in order to be able to cross. However, the instant the player shoots, the target begins to move away from the player at a constant speed that is half the initial speed of the projectile shot by the player, as shown in the diagram below.

The initial distance between the player and the target is d, the initial speed of the projectile is $2u$ and it is launched at an angle of θ to the horizontal. The acceleration due to gravity is g. The launch angle is the ONLY parameter that the player can change.



Taking the position of the player when the projectile is launched as the origin, the positions of the projectile and target at time t after the projectile is launched are as follows.

$$\underset{\sim}{r}_P = \begin{pmatrix} 2ut\cos\theta \\ 2ut\sin\theta - \dfrac{g}{2}t^2 \end{pmatrix} \quad \text{Projectile}$$

$$\underset{\sim}{r}_T = \begin{pmatrix} d + ut \\ 0 \end{pmatrix} \quad \text{Target}$$

(Do NOT prove these)

Show that, for the player to have a chance of hitting the target, d must be less than 37% of the maximum possible range of the projectile (to 2 significant figures).

Further exercises

Ex 10A (Pender et al., 2019)

- Q1-16

Further exercises (Legacy Textbooks)

Ex 3G X1 (Pender et al., 2000)

- Q7-13
- Q15-18

Additional questions

1. [**2008 VCE Specialist Mathematics Paper 2 Q13**] A cricket ball is hit from an origin at ground level so that its position vector at time t is given by $\underset{\sim}{r}(t) = 15t\,\underset{\sim}{i} + (20t - 5t^2)\,\underset{\sim}{j}$ for $t \geq 0$ where $\underset{\sim}{i}$ is a unit vector in the forward direction and $\underset{\sim}{j}$ is a unit vector vertically up.

When the cricket ball reaches its maximum height, its position vector is

(A) $\underset{\sim}{r} = 20\,\underset{\sim}{i} + 30\,\underset{\sim}{j}$ (C) $\underset{\sim}{r} = 60\,\underset{\sim}{i}$ (E) $\underset{\sim}{r} = 30\,\underset{\sim}{i} + 20\,\underset{\sim}{j}$

(B) $\underset{\sim}{r} = 15\,\underset{\sim}{i} + 20\,\underset{\sim}{j}$ (D) $\underset{\sim}{r} = 30\,\underset{\sim}{i} + 10\,\underset{\sim}{j}$

2. [**2017 VCE Specialist Mathematics NHT Paper 2 Q4**] A cricketer hits a ball at time $t = 0$ seconds from an origin O at ground level across a level playing field.

The position vector $\underset{\sim}{r}(t)$, from O, of the ball after t seconds is given by

$$\underset{\sim}{r}(t) = 15t\,\underset{\sim}{i} + \left(15\sqrt{3}t - 4.9t^2\right)\underset{\sim}{j}$$

where $\underset{\sim}{i}$ is a unit vector in the forward direction, $\underset{\sim}{j}$ is a unit vector vertically up and displacement components are measured in metres.

(a) Find the initial velocity of the ball and the initial angle, in degrees, of its trajectory to the horizontal. **2**

(b) Find the maximum height reached by the ball, giving your answer in metres, correct to two decimal places. **2**

(c) Find the time of flight of the ball. Give your answer in seconds, correct to three decimal places. **1**

(d) Find the range of the ball in metres, correct to one decimal place. **1**

(e) A fielder, more than 40 m from O, catches the ball at a height of 2 m above the ground. **2**

How far horizontally from O is the fielder when the ball is caught? Give your answer in metres, correct to one decimal place.

Answers

1. (E) 2. (a) $15\underset{\sim}{i} + 15\sqrt{3}\underset{\sim}{j}$, 60° (b) 34.44 (c) 5.302 (d) 79.5 (e) 78.4

4.4.2 Cartesian equation

Derivation

- Once the time equations have been derived, the *path* of the particle can be found.
- Parameter to eliminate: t.
- Look for other quadratic equations that appear after the Cartesian equation has been derived.

Steps

1. Time equations:

$$x = Vt\cos\alpha \tag{4.4}$$

$$y = Vt\sin\alpha - \frac{1}{2}gt^2 \tag{4.5}$$

2. Change subject to t in (4.4)

$$t = \frac{x}{V\cos\alpha} \tag{4.6}$$

3. Substitute (4.6) into (4.5):

$$\begin{aligned} y &= V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 \\ &= x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \\ &= x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha \\ &= x\tan\alpha - \frac{gx^2}{2V^2}\left(1+\tan^2\alpha\right) \end{aligned}$$

- Observe equation now contains term in $\tan\alpha$: up to two possible angles of projection for each point.

Example 51

[2020 Ext 1 HSC Sample Q11] A particle is fired from the origin O with initial velocity $18\,\mathrm{ms}^{-1}$ at an angle $60°$ to the horizontal.

The equations of motion are $\dfrac{d^2x}{dt^2} = 0$ and $\dfrac{d^2y}{dt^2} = -10$.

i. Show that $x = 9t$. **1**

ii. Show that $y = 9\sqrt{3}t - 5t^2$. **2**

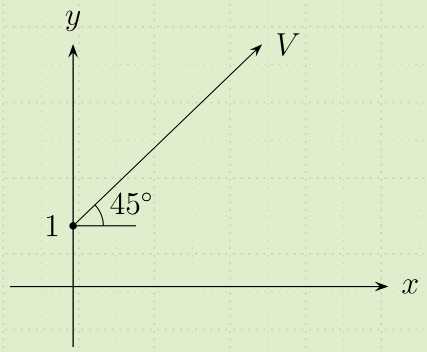
iii. Hence find the Cartesian equation for the trajectory of the particle. **1**

Answer: $y = \sqrt{3}x - \frac{5}{81}x^2$

Example 52

[1998 3U HSC] A particle is projected from the point $(0, 1)$ at an angle of 45° with a velocity of V metres per second. The equations of motion of the particle are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$



i. Using calculus, derive the expressions for the position of the particle at time t. Hence show that the path of the particle is given by **3**

$$y = 1 + x - g\frac{x^2}{V^2}$$

A volleyball player serves a ball with initial speed V metres per second and angle of projection 45°. At that moment the bottom of the ball is 1 metre above the ground and its horizontal distance from the net is 9.3 metres. The ball just clears the net, which is 2.3 metres high.

ii. Show that the initial speed of the ball is approximately $10.3\,\mathrm{ms}^{-1}$ (Take $g = 9.8\,\mathrm{ms}^{-2}$) **2**

iii. What is the horizontal distance from the net to the point where the ball lands? **3**

Example 53

[2021 Ext 1 HSC Q13] (4 marks) When an object is projected from a point h metres above the origin with initial speed V m/s at an angle of $\theta°$ to the horizontal, its displacement vector, t seconds after projection, is

$$\underset{\sim}{r}(t) = \left(Vt\cos\theta\right)\underset{\sim}{i} + \left(-5t^2 + Vt\sin\theta + h\right)\underset{\sim}{j}$$

(Do NOT prove this.)

A person, standing in an empty room which is 3 m high, throws a ball at the far wall of the room. The ball leaves their hand 1 m above the floor and 10 m from the far wall. The initial velocity of the ball is 12 m/s at an angle of 30° to the horizontal.

Show that the ball will NOT hit the ceiling of the room but that it will hit the far wall without hitting the floor. **Answer:** $t = \frac{6+\sqrt{56}}{10}$, $x \approx 14$ when it hits the floor

Steps

1. Is there a picture? If not, draw one!

2. Are there constants that can replace variables?

3. See if the $y(t)$ value will hit the ceiling.

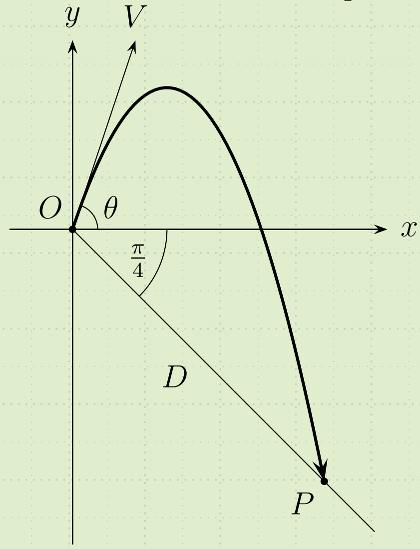
Steps

4. Check the time for when $y(t) = 0$ and find the corresponding $x(t)$ value.

4.4.3 Landing on non-ground level

Example 54

[2014 Ext 1 HSC Q14] The take-off point on a ski jump is located at the top of a downslope. The angle between the downslope and the horizontal is $\dfrac{\pi}{4}$. A skier takes off from O with velocity V ms^{-1} at an angle θ to the horizontal, where $0 \le \theta \le \dfrac{\pi}{2}$. The skier lands on the downslope at some point P, a distance D metres from O.



The flight path of the skier is given by

$$\begin{cases} x = Vt\cos\theta \\ y = -\dfrac{1}{2}gt^2 + Vt\sin\theta \end{cases}$$

(Do NOT prove this)

where t is the time in seconds after take-off.

i. Show that the Cartesian equation of the flight path of the skier is given by **2**

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

ii. Show that $D = 2\sqrt{2}\dfrac{V^2}{g}\cos\theta\,(\cos\theta + \sin\theta)$ **3**

iii. Show that $\dfrac{dD}{d\theta} = 2\sqrt{2}\dfrac{V^2}{g}\,(\cos 2\theta - \sin 2\theta)$ **2**

iv. Show that D has a maximum value and find the value of θ for which this occurs. **3**

Further exercises

Ex 10B (Pender et al., 2019)

- Q1-10

Further exercises (Legacy Textbooks)

Ex 3H (Pender et al., 2000)

- Q4-10

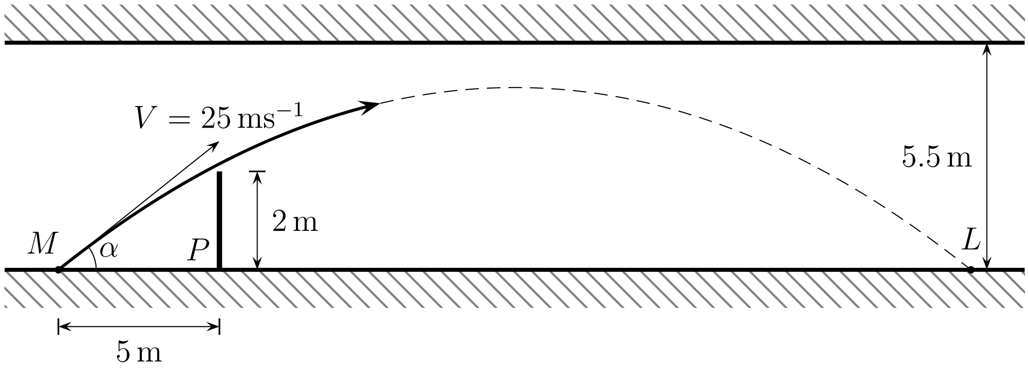
4.4.4 Additional questions

1. **[2013 NSBHS Assessment Task 4]** A movie scene is to be filmed inside a tunnel of height 5.5 m. In this scene, a stunt man (of negligible height) will ride a motorcycle off an inclined plane (in the shape of a wedge, of negligible height) positioned at point M at α degrees, with velocity $V = 25\,\text{ms}^{-1}$.

The stunt man must just clear a policeman of height 2 m, standing 5 m away from the inclined plane at point P.

The stunt man will then land safely at another point L in the tunnel.

Assume that there is no air resistance in the tunnel, and that the acceleration due to gravity is $g = 10\,\text{ms}^{-2}$.



You may also assume the equations of motion are $\begin{cases} x = 25t\cos\alpha \\ y = 25t\sin\alpha - 5t^2 \end{cases}$

and the equations for the velocity are $\begin{cases} \dot{x} = 25\cos\alpha \\ \dot{y} = 25\sin\alpha - 10t \end{cases}$.

(a) Show that the trajectory can be expressed as **2**

$$y = -\frac{x^2}{125}\sec^2\alpha + x\tan\alpha$$

(b) Hence show that $\tan^2\alpha - 25\tan\alpha + 11 = 0$. **2**

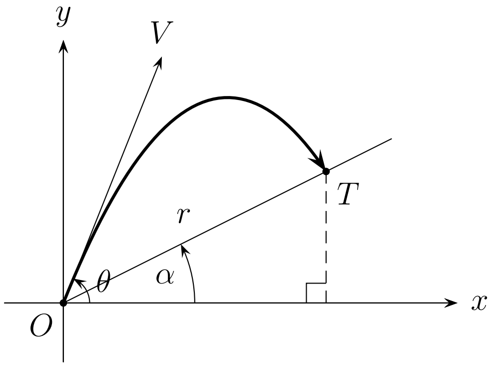
(c) Hence or otherwise, show that the value(s) of α (correct to the nearest degree) are **2**

$$\alpha = 24^\circ \text{ or } 88^\circ$$

(d) Explain (with mathematical reasoning) which of these values of α is valid. **3**

For simplicity, the rounded off values from part (c) may be used in your explanation

2. **[2000 3U HSC Q7]** (8 marks) The diagram shows an inclined plane that makes an angle of α radians with the horizontal. A projectile is fired from O, at the bottom of the incline, with a speed of V ms^{-1} at an angle of elevation θ to the horizontal, as shown.



With the above axes, you may assume that the position of the projectile is given by

$$\begin{cases} x = Vt\cos\theta \\ y = Vt\sin\theta - \frac{1}{2}gt^2 \end{cases}$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity.

For simplicity we assume that the unit of length has been chosen so that

$$\frac{2V^2}{g} = 1$$

i. Show that the path of the trajectory of the projectile is

$$y = x\tan\theta - x^2\sec^2\theta$$

ii. Show that the range of the projectile, $r = OT$ metres, up the inclined plane is given by

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha}$$

iii. Hence or otherwise, deduce that the maximum range, R metres up the incline is

$$R = \frac{1}{2(1 + \sin\alpha)}$$

(You may assume that $2\sin A\cos B = \sin(A + B) + \sin(A - B)$)

iv. Consider the trajectory of the projectile for which the maximum range R is achieved.

Show that, for this trajectory, the initial direction is perpendicular to the direction at which the projectile hits the inclined plane.

Part III

X2 Resisted Motion

Section 5

Horizontal

Learning Goal(s)

Knowledge
When to use particular parts of the motion in terms of displacement differential equation, $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Skills
Solve problems using differential equations

Understanding
Newton's laws of motion and relationship to integration

By the end of this section am I able to:

30.16 Examine force, acceleration, action and reaction under constant and non-constant force

30.17 Use Newton's laws to obtain equations of motion in situations involving motion other than projectile motion or simple harmonic motion

30.18 Examine motion of a body under concurrent forces

30.19 Solve problems involving resisted motion of a particle moving along a horizontal line

5.1 Review of formulae

- Crucial formula: derivative of velocity w.r.t. displacement:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} \\ &= v\frac{dv}{dx} \\ &= \frac{dx}{dt}\times\frac{dv}{dx} \\ &= \ddot{x}\end{aligned} \tag{5.1}$$

- Questions in Section 7 and 6 will build upon
  - All of Equation (5.1) and Section 1 on page 7.
  - Newton's Second Law: Section 3.3 on page 54.
  - Manipulation of differentials: **Topic 29 - X1 Differential Equations**
  - Integration techniques: **Topic 27 - X1 X2 Further Integration**

Example 55

Initially a particle is observed to be travelling in a straight line with speed V ms^{-1}. Throughout the subsequent motion, the particle is slowing down at a rate proportional to its speed.

(a) Find an expression for the velocity v of the particle when it is x metres from its initial position

(b) Find an expression for the velocity of the particle and its displacement from its initial position after t seconds.

(c) Show that this displacement has a limiting value as $t \to \infty$.

Answer: (a) $v = V - kx$ (b) $v = Ve^{-kt}$ (c) $x \to \left(\frac{V}{k}\right)^-$

Example 56

[Ex 7.1 Q4] (Lee, 2006) The acceleration of a particle moving in a straight line is given by

$$a = k(1 - v^2) \qquad (k > 0)$$

where v is the velocity at time t.

(a) Find an expression for the velocity in terms of t, hence, its terminal velocity.

(b) Find an expression for the position in terms of the velocity, hence, its limiting position, if possible.

Answer: (a) $v = \frac{Ae^{2kt}-1}{Ae^{2kt}+1}$, $t \to \infty$, $v \to 1$ (b) $x = -\frac{1}{2k}\ln B(1-v^2)$. No limiting position.

5.2 Resistance

Important note

- Write down the sum of forces via equation (3.1).
- ...Draw... ...picture...!

Example 57

A particle of mass 2 kg moves in a straight line such that at time t seconds, its displacement from a fixed origin is x metres and speed v metres per second. If the resultant force in Newtons is

(a) $6 - 4x$, find v in terms of x, given $v = 2$ when $x = 1$

(b) $8 - 2v^2$, find t in terms of v, given $t = 0$ when $v = 0$.

Answer: (a) $v = \sqrt{6x - 2x^2}$, $x \in [0, 3]$ (b) $t = \frac{1}{4}\ln\left(\frac{2+v}{2-v}\right)$, $v \in (-2, 2)$

Example 58

A particle of mass m moves in a straight line subject to a resistance force of magnitude $mk(v+v^3)$, where v is the velocity of the particle. Initially the particle has velocity V. If x is the displacement of the particle from its initial position at time t,

(a) Find the expression of x in terms of v,

(b) Find the expression of v^2 in terms of t.

(c) Deduce the limiting values of v and x as $t \to \infty$.

Answer: (a) $x = \frac{1}{k}\left(\tan^{-1} V - \tan^{-1} v\right)$ (b) $v^2 = \frac{V^2 e^{-2kt}}{1+V^2(1-e^{-2kt})}$ (c) $x \to \frac{1}{k}\tan^{-1} V$

Example 59

[Ex 6D Q1] (Sadler & Ward, 2019) A certain drag-racing car of mass M kg is capable of a top speed of 288 km/h. After it reaches this top speed, two different retarding forces combine to bring it to rest. First there is a constant braking force of magnitude $\frac{2}{3}M$ Newtons. Secondly there is a resistive force of magnitude $\frac{Mv^2}{180}$ Newtons, where v metres per second is the speed of the car, acting against a parachute released from the rear-end of the vehicle. Let x metres be the distance of the car from the point at which the two retarding forces are activated.

(a) Show that $x = 90\ln\left(\dfrac{120+80^2}{120+v^2}\right)$.

(b) Hence calculate, to the nearest metre, the distance that the drag-racing car travels as it is brought from its top speed to rest. **Answer:** 360 m

Example 60

[2013 ACE Ext 2 Trial Q13] A speed boat of mass m is travelling at maximum power. At maximum power, its engine delivers a force F on the speed boat. The water exerts a resistive force proportional to the square of the speed boat's speed v.

(a) Show that $\ddot{x} = \dfrac{1}{m}(F - kv^2)$, where k is a positive constant. **2**

(b) The speed boat increases its speed from v_1 to v_2. Show that the distance travelled during this period is **3**

$$x = \frac{m}{2k} \ln \left(\frac{F - k{v_1}^2}{F - k{v_2}^2} \right)$$

Example 61

[2014 CSSA Ext 2 Trial Q15] An object of mass 70 kg, initially at rest, is pulled along a horizontal surface by a constant force of 140 N. It experiences a resistance proportional to its speed.

When the speed is $10\,\mathrm{ms}^{-1}$, the acceleration is $1\,\mathrm{ms}^{-2}$. Let x be represent the displacement in metres from the initial position of the object.

i. Show that the equation of motion is $\ddot{x} = 2 - \dfrac{1}{10}v$. **2**

ii. Find an expression for x as a function of v. **3**

iii. Show that the object's speed cannot exceed $20\,\mathrm{ms}^{-1}$. **1**

Answer: $x = 200\ln\left(\frac{20}{20-v}\right) - 10v$

Section 6

Vertical

Learning Goal(s)

Knowledge
When to use particular parts of the motion in terms of displacement differential equation, $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Skills
Solve problems using differential equations

Understanding
Newton's laws of motion and relationship to integration

By the end of this section am I able to:

30.20 Solve problems involving the motion of a particle moving vertically (upwards or downwards) in a resisting medium and under the influence of gravity

Definition 7

Terminal velocity Limiting value of v as $\ddot{x} \to 0$.

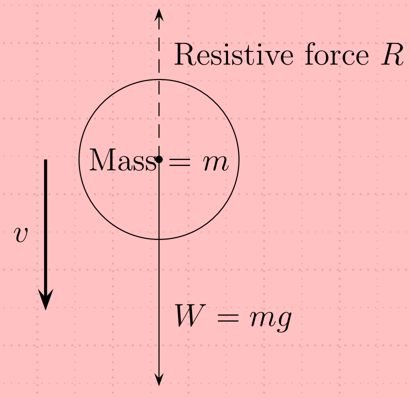
Steps

General steps to solving vertical resisted motion questions

- Draw ....vector........ ....diagram........ showing forces acting on the body of mass, use $\sum \underset{\sim}{F} = m\underset{\sim}{a}$.
- Use value of acceleration due to gravity given.
- ⚠ Be careful of direction.

Important note

- Object in travelling downwards, subjected to gravitational force and a resistive force upwards.
- Be consistent!
  - If *up* is the positive direction, stick with this for all of your solutions.
  - Otherwise, break into two cases: travelling up (take up to be positive) and travelling down (take down to be positive)
- Sum of forces:
$$\sum \underset{\sim}{F} = \ldots mg - R \ldots$$
- Terminal velocity when $\sum \underset{\sim}{F} = 0$, i.e. $\ldots mg = R \ldots$



Resolution of forces on a mass experiencing resisted motion

6.1 Downward only motion

Example 62

A particle falls from rest through a medium for which the resistance is proportional to the velocity and opposite in direction. Let y be the distance in metres below its initial position after t seconds. Assume that

$$\ddot{y} = g - kv$$

(a) Find v as a function of t.

(b) Hence show that the terminal velocity is $\lim\limits_{t\to\infty} v = \dfrac{g}{k}$

(c) Find y as a function of t.

(d) Show that if h is large enough then the approximate time to reach the ground at $y = h$ is given by

$$t \approx \frac{kh}{g} + \frac{1}{k}$$

Answer: (a) $v = \frac{g}{k}\left(1 - e^{-kt}\right)$ (b) Show (c) $y = \frac{g}{k}\left(t + \frac{1}{k}\left(e^{-kt} - 1\right)\right)$ (d) Show

Example 63

A small body is projected vertically downwards with velocity u in a medium which offers a resistance kv per unit mass when the speed is v.

(a) Prove that the speed approaches a limit which is independent of u.

t seconds later, an exactly similar body is projected vertically downwards with velocity u' .

(b) Show that the distance between the two bodies is ultimately

$$\frac{u - u' + gt}{k}$$

Example 64

[2020 Ext 2 HSC Sample Q13] A particle of mass 0.5 kilograms is dropped from the top of a tower of height h metres above the ground. The particle experiences a force due to air resistance of magnitude $\dfrac{v^2}{1\,000}$ newtons, where v is the speed of the particle in metres per second. Let the displacement, x metres, at time t seconds, be measured in a downwards direction.

i. Show that the equation of motion of the particle is **1**

$$\ddot{x} = g - \frac{v^2}{500}$$

where g is the acceleration due to gravity.

Use $g = 9.8\,\text{ms}^{-2}$ in the following parts.

ii. Show, by integrating using partial fractions, that $v = 70\left(\dfrac{e^{0.28t}-1}{e^{0.28t}+1}\right)$. **5**

iii. The particle takes 3 seconds to reach the ground. By finding a formula for x as a function of v, or otherwise, find h, the height of the tower. **3**

Answer: $h \approx 42.86$

6.2 Upwards, then downwards motion

Example 65

⚠ Point P, Q are h metres apart in a vertical line. A particle is projected with speed C from P towards Q and at the same instant, another is projected with speed C from Q towards P.

If the resistance of the medium is proportional to the speed, i.e. $R = mkv$ and the terminal speed is C, show that the particles meet after time

$$\frac{C}{g}\log_e\left(\frac{2C^2}{2C^2 - gh}\right)$$

provided $2C^2 > gh$.

Important note

- Draw picture!
- Carefully think through which equations (x-v, x-t or v-t) are required.
- For the particle to be projected
  - *up*, take the upwards direction as positive.
  - *down*, take the *downwards* direction as positive.

  Why would this 'inconsistency' be required to solve the problem?

Example 66

[2013 Ext 2 HSC Q15] A ball of mass m is projected vertically into the air from the ground with initial velocity u. After reaching the maximum height H it falls back to the ground. While in the air, the ball experiences a resistive force kv^2, where v is the velocity of the ball and k is a constant.

The equation of motion when the ball falls can be written as

$$m\dot{v} = mg - kv^2$$

(Do NOT prove this)

i. Show that the terminal velocity v_T of the ball when it falls is $\sqrt{\dfrac{mg}{k}}$. **1**

ii. Show that when the ball goes up, the maximum height is **3**

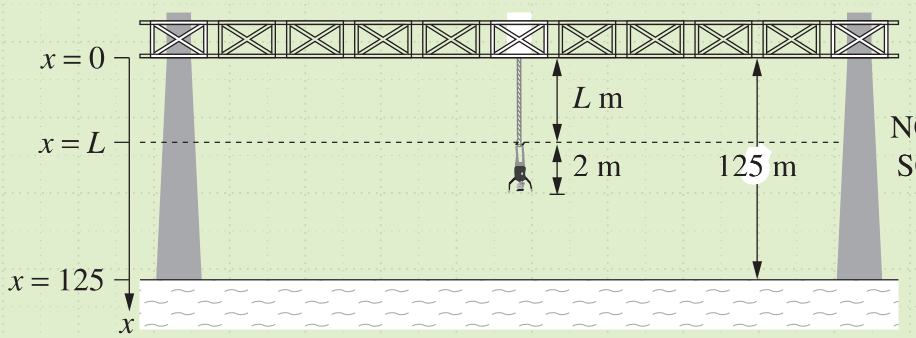
$$H = \frac{{v_T}^2}{2g} \ln\left(1 + \frac{u^2}{{v_T}^2}\right)$$

iii. When the ball falls from height H it hits the ground with velocity w. **2**

Show that $\dfrac{1}{w^2} = \dfrac{1}{u^2} + \dfrac{1}{{v_T}^2}$.

Example 67

[**2009 Ext 2 HSC Q7**] A bungee jumper of height 2 m falls from a bridge which is 125 m above the surface of the water, as shown in the diagram. The jumper's feet are tied to an elastic cord of length L m. The displacement of the jumper's feet, measured downwards from the bridge, is x m.



The jumper's fall can be examined in two stages. In the first stage of the fall, where $0 \leq x \leq L$, the jumper falls a distance of L m subject to air resistance, and the cord does not provide resistance to the motion. In the second stage of the fall, where $x > L$, the cord stretches and provides additional resistance to the downward motion.

a. The equation of motion for the jumper in the first stage of the fall is

$$\ddot{x} = g - rv$$

where g is the acceleration due to gravity, r is a positive constant, and v is the velocity of the jumper.

i. Given that $x = 0$ and $v = 0$ initially, show that **3**

$$x = \frac{g}{r^2} \ln\left(\frac{g}{g - rv}\right) - \frac{v}{r}$$

ii. Given that $g = 9.8\,\text{ms}^{-2}$ and $r = 0.2\,\text{s}^{-1}$, find the length L of the cord such that the jumper's velocity is $30\,\text{ms}^{-1}$ when $x = L$. Give your answer to two significant figures. **1**

b. In the second stage of the fall, where $x > L$, the displacement x is given by **4**

$$x = e^{-\frac{t}{10}}(29\sin t - 10\cos t) + 92$$

where t is the time in seconds after the jumpers feet pass $x = L$.

Determine whether or not the jumper's head stays out of the water.

Further exercises

Ex 6E (Sadler & Ward, 2019)

- Q1-11

Further exercises (Legacy Textbooks)

Ex 7.2 (Arnold & Arnold, 2000)
Ex 8C (Patel, 2004)

Section 7

Forces encountered

Learning Goal(s)

Knowledge **Skills** **Understanding**

By the end of this section am I able to:

30.16 Examine force, acceleration, action and reaction under constant and non-constant force

7.1 Resolution of forces

Definition 8

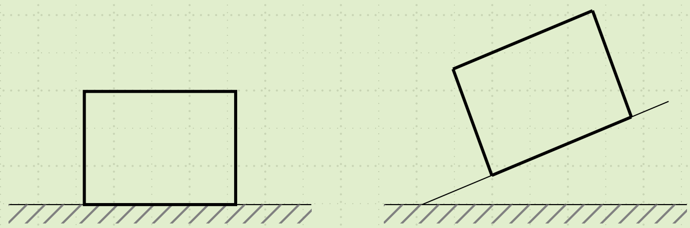
Normal reaction force: (from Newton's Third Law) a body in contact with a surface will exert a force on the surface, and the surface will exert an equal but oppositely-directed force on the body. Always perpendicular to the surface.

Important note

Resolve forces via similar triangles/Euclidean Geometry/elementary trigonometry

Example 68

Draw the normal reaction force N and gravitational weight W acting on these boxes.



Further exercises (Legacy Textbooks)

Patel (2004, Ex 7A)

- Most objects are regarded as particles, "ideal" objects of no dimension but with mass m.

1. Gravitational weight (gravity)

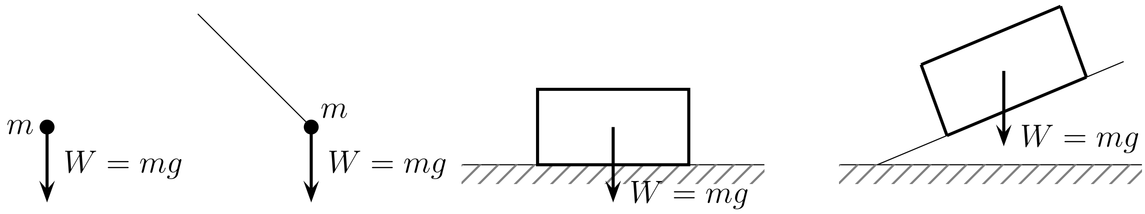
Definition 9

Gravitational weight W is a force acting on an object towards centre of earth. Always vertically down.

$$W = mg$$

with $g = 9.8\,\text{ms}^{-2}$

- Already substantially dealt with in Extension 1 Projectile Motion.
- Usual Extension 1 treatment: $g = 10\,\text{ms}^{-2}$ (unless otherwise specified)
- ⚠ Usual Extension 2 treatment: $g = 9.8\,\text{ms}^{-2}$



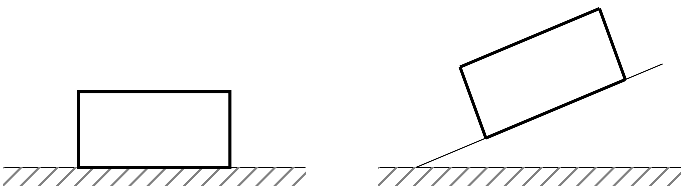
2. Friction

Definition 10

Friction is a force that acts along the surface between two objects, and is always directed against the motion.

(Already substantially dealt with in Section 3.3 on page 54 and within this section)

Draw examples of frictional forces:



3. Tension

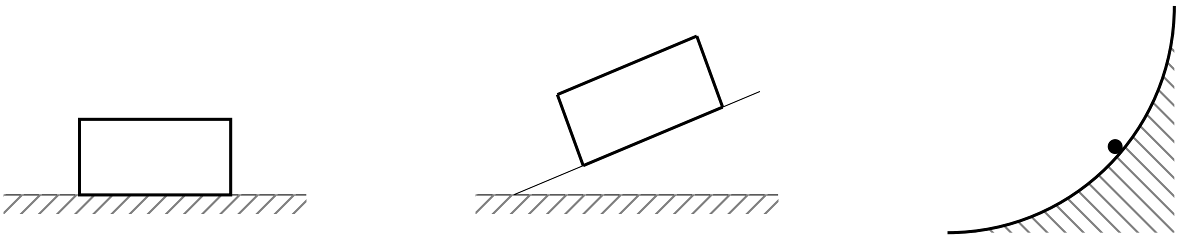
Definition 11

Tension is a force acting in a rod or string which joins two objects.

Draw tension on the following strings/rods

4. Normal reaction (See Definition 8 on page 107)

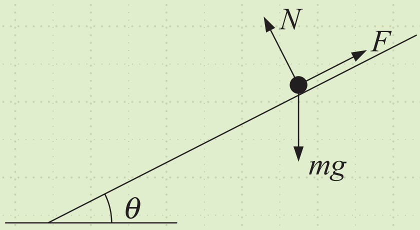
Draw examples of normal forces, weight and friction:



7.2 Inclined planes

Example 69

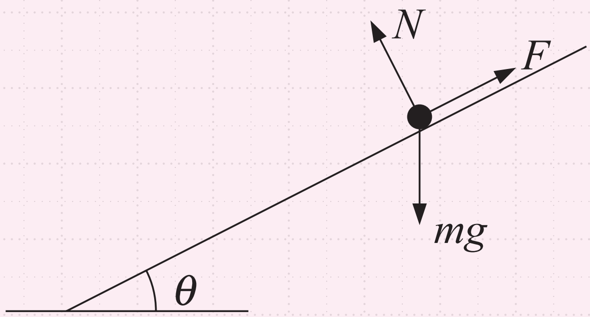
[1997 HSC] (3 marks) A particle of mass m is lying on an inclined plane and does not move. The plane is at an angle θ to the horizontal. The particle is subject to a gravitational force mg, a normal reaction force N, and a frictional force F parallel to the plane, as shown in the diagram.



Resolve the forces acting on the particle, and hence find an expression for $\dfrac{F}{N}$ in terms of θ.

Steps

1. Reproduce diagram, draw vectors.



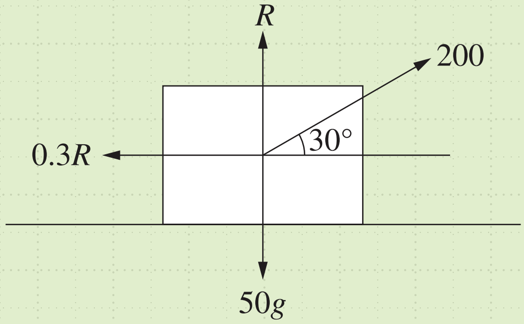
2. Key phrase: **does not move**. Net force acting on particle is ...zero...:

3. Find $\dfrac{F}{N}$:

Example 70

[2020 Ext 2 HSC Q12] A 50-kilogram box is initially at rest. The box is pulled along the ground with a force of 200 newtons at an angle of 30° to the horizontal. The box experiences a resistive force of $0.3R$ newtons, where R is the normal force, as shown in the diagram.

Take the acceleration g due to gravity to be $10\,\mathrm{ms}^{-2}$. **Answer:** $3.192\,\mathrm{ms}^{-1}$



i. By resolving the forces vertically, show that $R = 400$. **2**

ii. Show that the net force horizontally is approximately 53.2 newtons. **2**

iii. Find the velocity of the box after the first three seconds. **2**

Further exercises

Ex 6D (Sadler & Ward, 2019)

- Q2-11

Further exercises (Legacy Textbooks)

Ex 7.1 (Arnold & Arnold, 2000)

7.3 Pulleys

Two objects are connected by a string and placed over a smooth pulley:

- string is *inextensible* and *light* – tension is ......constant...... through.
- pulley is *smooth* – no ......friction...... .

Example 71

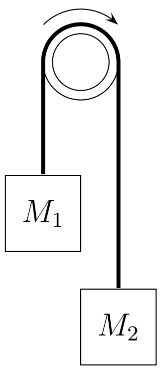
Two particles have mass M_1 and M_2 kg, and are connected by a light, inextensible string. The string passes over a smooth pulley. Let the tension in the string be T Newtons, and acceleration due to gravity be g ms^{-2}.

Assume that the table is smooth, and $M_2 > M_1$. In each of the cases below:

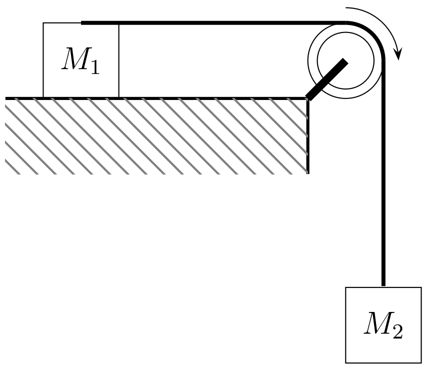
(a) Draw the forces acting on each particle.

(b) Find the acceleration of the system in terms of M_1, M_2, g and/or x.

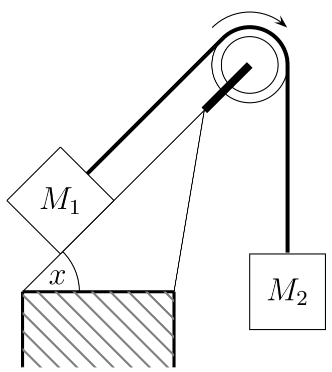
Case 1:



Case 2:



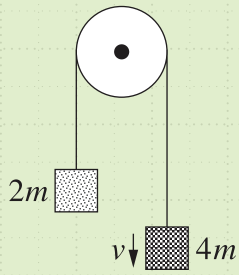
Case 3:



Example 72

[2020 Ext 2 HSC Q16] Two masses, $2m$ kg and $4m$ kg, are attached by a light string. The string is placed over a smooth pulley as shown.

The two masses are at rest before being released and v is the velocity of the larger mass at time t seconds after they are released.



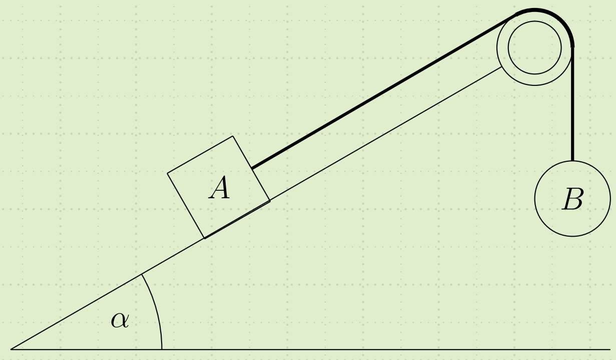
The force due to air resistance on each mass has magnitude kv, where k is a positive constant.

(i) Show that $\dfrac{dv}{dt} = \dfrac{gm - kv}{3m}$. **2**

(ii) Given that $v < \dfrac{gm}{k}$, show that when $t = \dfrac{3m}{k}\ln 2$, the velocity of the large mass is $\dfrac{gm}{2k}$. **3**

Example 73

(Madas, 2021) Two particles A and B, of mass 3 kg and 4 kg respectively, are attached to each of the ends of a light inextensible string. The string passes over a smooth pulley P, at the top of a fixed rough plane, inclined at α to the horizontal, where $\tan\alpha = 0.75$.



Particle A is placed at rest on the incline plane while B is hanging freely at the end of the incline plane vertically below P, as shown in the figure above.

The two particles, the pulley and the string lie in a vertical plane parallel to the line of greatest slope of the incline plane.

The particles are released from rest with the string taut.

Particle A begins to move up the incline plane, where the constant ground friction between A and the plane has magnitude 10.5 N.

Ignoring air resistance, calculate:

(a) the acceleration of the system immediately after the particles are released.

(b) the tension of the string.

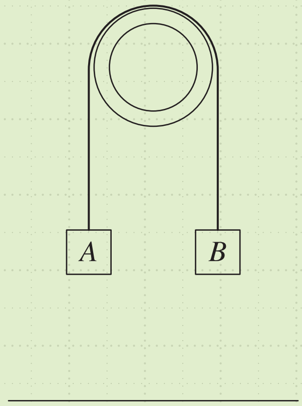
Two seconds after release, while both particles are moving, the string breaks.

(c) Calculate the total distance A moves up the plane from the instant since the particles were released, assuming that A does not reach the pulley.

Answer: (a) $a = 1.58$ ms^{-2} (b) 32.88 N (c) 3.69 m

Example 74

[2021 NEAP Ext 2 Trial] Two particles, A and B, have masses $5m$ kg and km kg respectively, where $k < 5$. The particles are connected by a light, inextensible string that passes over a smooth light fixed pulley. The system is held at rest with the string taut, the hanging parts of the string are vertical and A and B are at the same height above a horizontal floor as shown.



The system is released from rest and particle A descends with acceleration $\dfrac{g}{4}$ m/s^2.

(i) Show that the tension in the string as particle A descends is $\dfrac{15mg}{4}$ newtons. **2**

(ii) Find the value of k. **2**

(iii) After descending for 1 second, particle A impacts the floor and is immediately brought to rest. In its subsequent motion, particle B does not reach the pulley. **3**

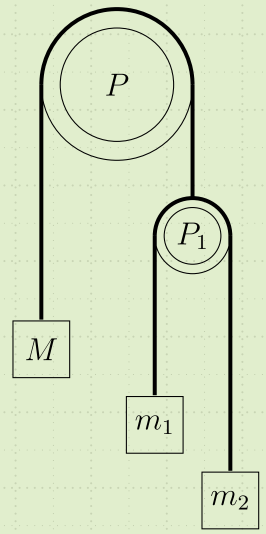
Show that the greatest height reached by particle B above the floor is $\dfrac{9g}{32}$ metres.

Answer: (i) Show (ii) $k = 3$ (iii) Show

7.3.1 System of pulleys

Example 75

The diagram below show a system of particles, strings and pulleys. In the system, the pulleys are smooth and light, the strings are light and inextensible, the particles move vertically and the pulley labelled with P is fixed. The masses of the particles are as indicated on the diagram.



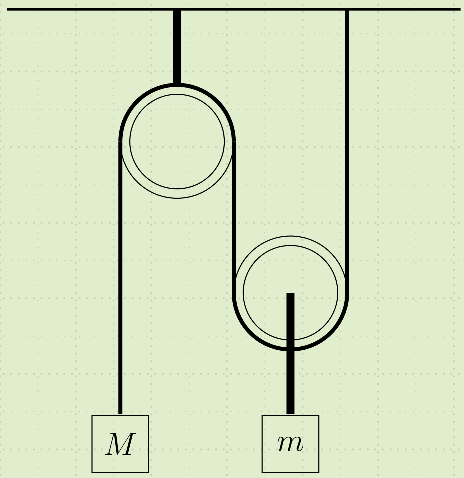
For the pulley system shown above, show that the acceleration, a, of the particle of mass M, measured in the downwards direction, is given by **5**

$$a = \frac{M - 4\mu}{M + 4\mu} g$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$.

Example 76

In the following diagram, the two pulleys are massless and frictionless. The first pulley is suspended from the ceiling by a rigid rod, and the mass m is suspended from the pulley by another rigid rod. The two masses are initially held at the same height to prevent them from moving.



(i) What is the condition on M and m for the system not to move when released?

(ii) If mass m falls 10 cm after being released, how far will M rise in the same time?

What does this imply about the acceleration of M versus the accceleration of m?

(iii) Find the acceleration of each of the masses in terms of M, m and g.

(iv) Let $M = 1$ kg, $m = 6$ kg and $g = 10$ ms^{-2}. After being released, m falls 8 metres before hitting the floor.

Assuming there is enough rope for M not to hit the pulley, how far above the floor will M ultimately rise?

Answer: (i) $m = 2M$ (ii) 20 cm (iii) $\frac{2M-m}{4M+m}g$ and $\frac{2(2M-m)}{4M+m}g$ (iv) 36.8 metres

Section 8

Further projectile motion

Learning Goal(s)

Knowledge
Horizontal and vertical components of displacement/velocity/acceleration applied in projectile motion

Skills
Apply calculus to solve problems involving projectiles

Understanding
Concept of projectile motion with resistance

By the end of this section am I able to:

30.21 Solve problems involving projectiles in a variety of contexts.

30.22 Solve problems involving projectile motion in a resisting medium and under the influence of gravity which include consideration of the complete motion of a particle projected vertically upwards or at an angle to the horizontal.

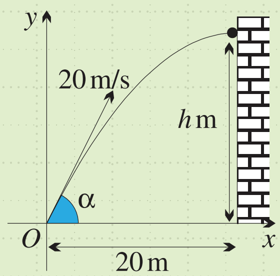
Important note

- ⚠ Hard, difficult algebra

8.1 Cartesian equation/final speed/angle

Example 77

[Ex 6F Q13] (Sadler & Ward, 2019) A cricketer hits the ball from ground level with a speed of $20\,\text{ms}^{-1}$ and an angle of elevation α. It flies towards a high wall 20 metres away on level ground. Take the origin at the point where the ball was hit, and take $g = 10\,\text{ms}^{-2}$.



(a) Show that the ball hits the wall when $h = 20\tan\alpha - 5\sec^2\alpha$.

(b) Show that $\dfrac{d}{d\alpha}(\sec\alpha) = \sec\alpha\tan\alpha$.

(c) Show that the maximum value of h occurs when $\tan\alpha = 2$.

(d) Find the maximum height.

(e) Find the speed and angle (to the nearest minute) at which the ball hits the wall.

⚠ Velocity contains a ....magnitude.......... and ....direction.......... .

8.2 Questions disguised as projectile motion

Example 78

[Ex 6F Q18/1993 3U HSC] A projectile is fired from the origin with velocity V and angle of elevation α, where α is acute. Assume the usual equations of motion.

(a) Let $k = \dfrac{V^2}{2g}$. Show that the Cartesian equation of the parabolic path of the projectile can be written as

$$x^2 \tan^2 \alpha - 4kx \tan \alpha + \left(4ky + x^2\right) = 0$$

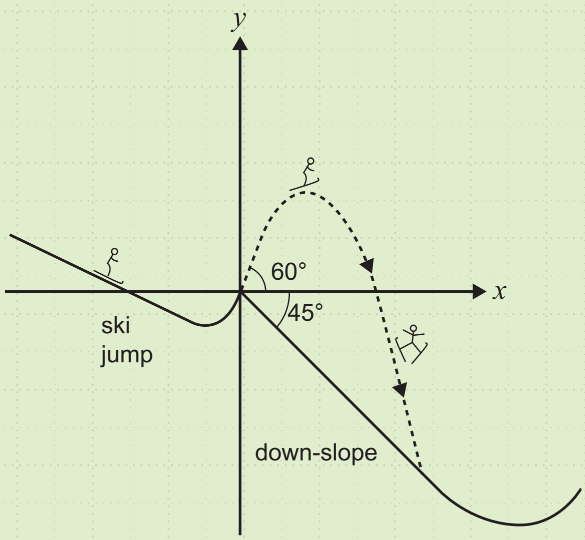
(b) Show that the projectile can pass through point (X, Y) in the first quadrant by firing at two different initial angles α_1 and α_2, only if $X^2 < 4k^2 - 4kY$.

(c) Suppose that $\tan \alpha_1$ and $\tan \alpha_2$ are the two real roots of the quadratic equation in $\tan \alpha$ in part (a). Show that $\tan \alpha_1 \tan \alpha_2 > 1$, and hence explain why it is impossible for α_1 and α_2 both to be less than $45°$.

8.3 Projectile motion experiencing resistance

Example 79

[**2005 VCE Specialist Mathematics Paper 2 Q4**] A skier accelerates down a slope and then skis up a short ski jump. The skier leaves the jump at a speed of $12\,\mathrm{ms}^{-1}$ and at an angle of $60°$ to the horizontal. The skier performs various gymnastic twists and lands on a straight line section of the $45°$ down-slope T seconds after leaving the jump.



Let the origin O of a cartesian coordinate system be at the point where the skier leaves the jump, with $\underset{\sim}{i}$ a unit vector in the positive x direction, and $\underset{\sim}{j}$ a unit vector in the positive y direction. Displacements are measured in metres, and time in seconds.

(a) Show that the initial velocity of the skier when leaving the jump is $6\underset{\sim}{i} + 6\sqrt{3}\underset{\sim}{j}$ **1**

(b) The acceleration of the skier, t seconds after leaving the ski jump, is given by **3**

$$\ddot{\underset{\sim}{r}}(t) = -0.1t\,\underset{\sim}{i} - (g - 0.1t)\underset{\sim}{j} \quad 0 \le t \le T$$

Show that the position vector of the skier, t seconds after leaving the jump, is given by

$$\underset{\sim}{r}(t) = \left(6t - \frac{1}{60}t^3\right)\underset{\sim}{i} + \left(6t\sqrt{3} - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right)\underset{\sim}{j} \quad 0 \le t \le T$$

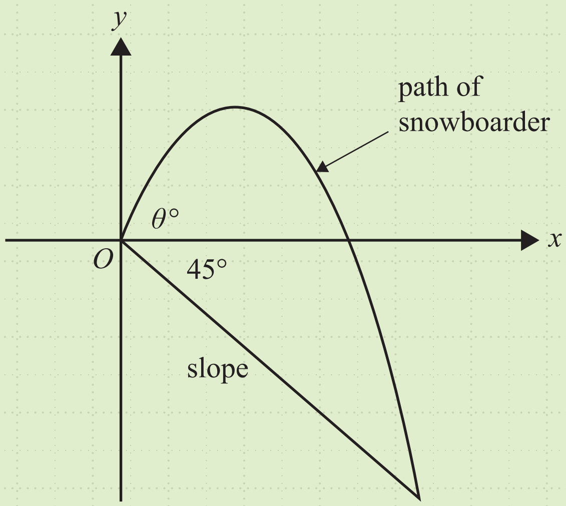
(c) Show that $T = \dfrac{12}{g}\left(\sqrt{3}+1\right)$. **3**

(d) At what speed, in metres per second, does the skier land on the down-slope? Give your answer correct to one decimal place. **3**

Answer: $22.5\,\mathrm{ms}^{-1}$

Example 80

[2019 VCE Specialist Mathematics NHT Paper 2 Q4] A snowboarder at the Winter Olympics leaves a ski jump at an angle θ degrees to the horizontal, rises up in the air, performs various tricks and then lands at a distance down a straight slope that makes an angle of 45° to the horizontal, as shown below.



Let the origin O of a cartesian coordinate system be at the point where the snowboarder leaves the jump, with a unit vector in the positive x direction being represented by $\underset{\sim}{i}$ and a unit vector in the positive y direction being represented by $\underset{\sim}{j}$. Distances are measured in metres and time is measured in seconds.

The position vector of the snowboarder t seconds after leaving the jump is given by

$$\underset{\sim}{r}(t) = \left(6t - 0.01t^3\right)\underset{\sim}{i} + \left(6\sqrt{3}t - 4.9t^2 + 0.01t^3\right)\underset{\sim}{j} \qquad t \geq 0$$

(a) Find the angle $\theta°$. **2**

(b) Find the speed, in metres per second, of the snowboarder when she leaves the jump at O. **1**

(c) Find the maximum height above O reached by the snowboarder. Give your answer in metres, correct to one decimal place. **2**

(d) Show that the time spent in the air by the snowboarder is $\dfrac{60\left(\sqrt{3}+1\right)}{49}$ seconds. **3**

Answer: (a) 60° (b) 12 (c) 5.5 (d) Show

Example 81

A particle is projected upward over horizontal ground with velocity U inclined at an angle θ to the horizontal. A second particle is projected simultaneously from the same point in the same direction with velocity $V > U$.

(a) Show that throughout the motion, the line joining the positions of the particles makes a constant angle with the horizontal.

(b) Given that the range and time of flight of the first particle are respectively $\frac{1}{g}U^2 \sin 2\theta$ and $\frac{2}{g}U \sin\theta$,

i. Find the position of the faster particle and the direction of its velocity vector when the slower particle hits the ground.

ii. If this velocity vector is directed upward at an angle $\beta = \frac{1}{2}\theta$ to the horizontal, find $\frac{U}{V}$ in terms of θ.

iii. Deduce that $\frac{1}{4} < \frac{U}{V} < \frac{1}{2}$.

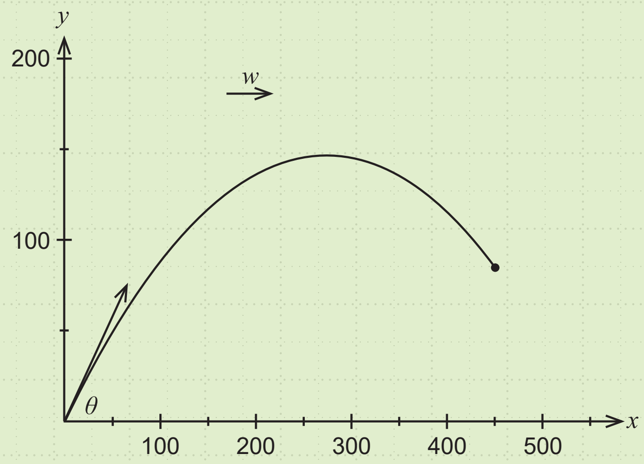
Example 82

[2018 WACE Mathematics Specialist Section 2 Q19] A small rocket is fired from the ground at an angle of $\theta°$ to the horizontal with a speed of 70 metres per second. The rocket has the assistance of a steady wind that is blowing horizontally at w metres per second.

A coordinate system is set up to track the path of the rocket as shown below.

Let

$$\begin{aligned} t &= \text{the number of seconds elapsed after the rocket is fired} \\ \underset{\sim}{r}(t) &= \text{the position vector (metres)} \\ \underset{\sim}{v}(t) &= \text{the velocity vector (ms)}^{-1} \\ \underset{\sim}{a}(t) &= \text{the acceleration vector (due to gravity)} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \text{ ms}^{-2} \end{aligned}$$



(a) Given $\underset{\sim}{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$, show that $\underset{\sim}{r}(t) = \begin{pmatrix} (70\cos\theta + w)\,t \\ (70\sin\theta)t - 4.9t^2 \end{pmatrix}$. **3**

(b) Obtain the Cartesian equation for the path of the rocket, in terms of θ and w. **2**

(c) The range of the rocket is defined as the horizontal distance travelled from its launch to the point at which it strikes the ground. **4**

Assuming that the wind speed $w = 2$ metres per second, determine the optimum angle θ so that the range of the rocket is maximised, correct to the nearest 0.1 degree.

Answer: (a) Show (b) $y = \left(\frac{70\sin\theta}{70\cos\theta + w}\right)x - \frac{4.9x^2}{(70\cos\theta + w)^2}$ (c) $\approx 45.6°$

Further exercises

Ex 6F (Sadler & Ward, 2019)

- Q1-17

Ex 6H (Sadler & Ward, 2019)

- Q1-4, 6-12

Further exercises (Legacy Textbooks)

Ex 7.3 (Arnold & Arnold, 2000)
Ex 8A (Patel, 2004)

8.4 ⚠ Other harder problems

1. [**2003 Ext 1 HSC Q7**] A particle is projected from the origin with velocity v ms^{-1} at an angle α to the horizontal. The position of the particle at time t seconds is given by the parametric equations

$$x = vt\cos\alpha \qquad y = vt\sin\alpha - \frac{1}{2}gt^2$$

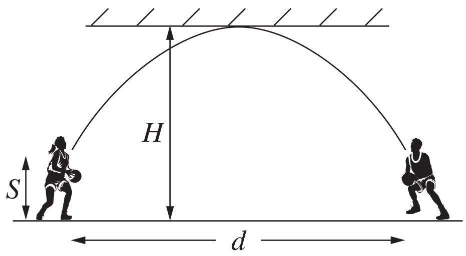
where g ms^{-2} is the acceleration due to gravity. (You are NOT required to derive these).

i. Show that the maximum height reached, h metres is given by **2**

$$h = \frac{v^2\sin^2\alpha}{2g}$$

ii. Show that it returns to the initial height at $x = \dfrac{v^2}{g}\sin 2\alpha$. **2**

iii. Chris and Sandy are tossing a ball to each other in a long hallway. The ceiling height is H metres and the ball is thrown and caught at shoulder height, which is S metres for both Chris and Sandy. **4**

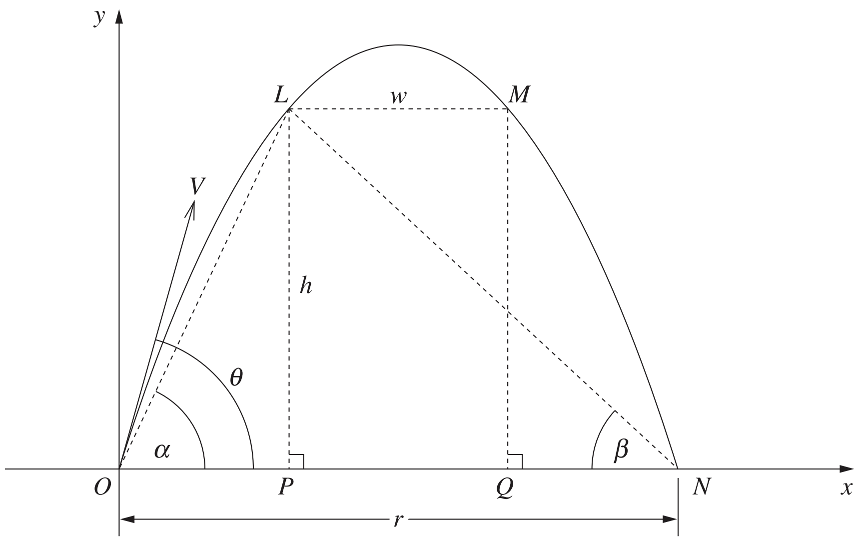


The ball is thrown with a velocity $v$ms^{-1}. Show that the maximum separation, d metres, that Chris and Sandy can have and still catch the ball is given by

$$d = 4\times\sqrt{(H-S)\left(\frac{v^2}{2g}\right) - (H-S)^2} \quad \text{if } v^2 \geq 4g(H-S) \quad \text{and}$$

$$d = \frac{v^2}{g} \qquad \text{if } v^2 \leq 4g(H-S)$$

2. **[2008 Ext 1 HSC Q7]** A projectile is fired from O with velocity V at an angle of inclination θ across level ground. The projectile passes through the points L and M, which are both h metres above the ground, at times t_1 and t_2 respectively. The projectile returns to the ground at N.



The equations of motion of the projectile are

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{1}{2}gt^2$$

(Do NOT prove this).

(a) Show that $t_1 + t_2 = \dfrac{2V}{g}\sin\theta$ AND $t_1t_2 = \dfrac{2h}{g}$. **2**

Let $\angle LON = \alpha$ and $\angle LNO = \beta$. it can be shown that

$$\tan\alpha = \frac{h}{Vt_1\cos\theta} \text{ and } \tan\beta = \frac{h}{Vt_2\cos\theta}$$

(Do NOT prove this).

(b) Show that $\tan\alpha + \tan\beta = \tan\theta$. **2**

(c) Show that $\tan\alpha\tan\beta = \dfrac{gh}{2V^2\cos^2\theta}$ **1**

Let $ON = r$ and $LM = w$.

(d) Show that $r = h(\cot\alpha + \cot\beta)$ and $w = h(\cot\beta - \cot\alpha)$. **2**

Let the gradient of the parabola at L be $\tan\phi$.

(e) Show that $\tan\phi = \tan\alpha - \tan\beta$. **3**

(f) Show that $\dfrac{w}{\tan\phi} = \dfrac{r}{\tan\theta}$. **2**

Part IV

Past examination questions

Section A

X1 X2 Vectors and mechanics

- Questions in this section originate from various VCEs and are largely to cater for the relatively small number of exercises for work in Part II. Answers can be found in the respective Specialist Mathematics examination reports.
- Some Projectile Motion questions are also amongst this group of vector based questions.
- Other Projectile Motion questions can be found in the respective HSC Exam Packs (2001-2019) or the legacy BOSTES site (1995-2000).

A.1 2006 VCE Specialist Mathematics

A.1.1 Paper 1

Question 7

The position vector of a moving particle is given by $\underset{\sim}{r}(t) = \sqrt{t-2}\,\underset{\sim}{i} + 2t\,\underset{\sim}{j}$ for $2 \leq t \leq 6$.

(a) Find the cartesian equation of the path followed by the particle. **2**

(b) Sketch the path of the particle. **2**

A.1.2 Paper 2 Section 1

13. Two particles, R and S, have position vectors $\underset{\sim}{r} = (2t - 10)\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{s} = 2\underset{\sim}{i} + (t - 1)\underset{\sim}{j}$ respectively at time t seconds, $t \geq 0$. **1**

Then

(A) R and S have the same position when $t = 1$

(B) R and S have the same position when $t = 4$

(C) R and S have the same position when $t = 5$

(D) R and S have the same position when $t = 6$

(E) R and S are never in the same position

A.1.3 Paper 2 Section 2

Question 4

A ball rolling along a horizontal plane has position vector $\underset{\sim}{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$, $t \geq 0$, $y > 0$ and velocity vector $\dot{\underset{\sim}{r}}(t) = \dfrac{1}{y(t)}\underset{\sim}{i} + (1 - y(t))\,\underset{\sim}{j}$.

(a) The component of velocity in the $\underset{\sim}{j}$ direction gives the differential equation **2**

$$\frac{dy}{dt} = 1 - y$$

Use calculus to show that the solution to this differential equation is

$$\log_e |1 - y| = c - t$$

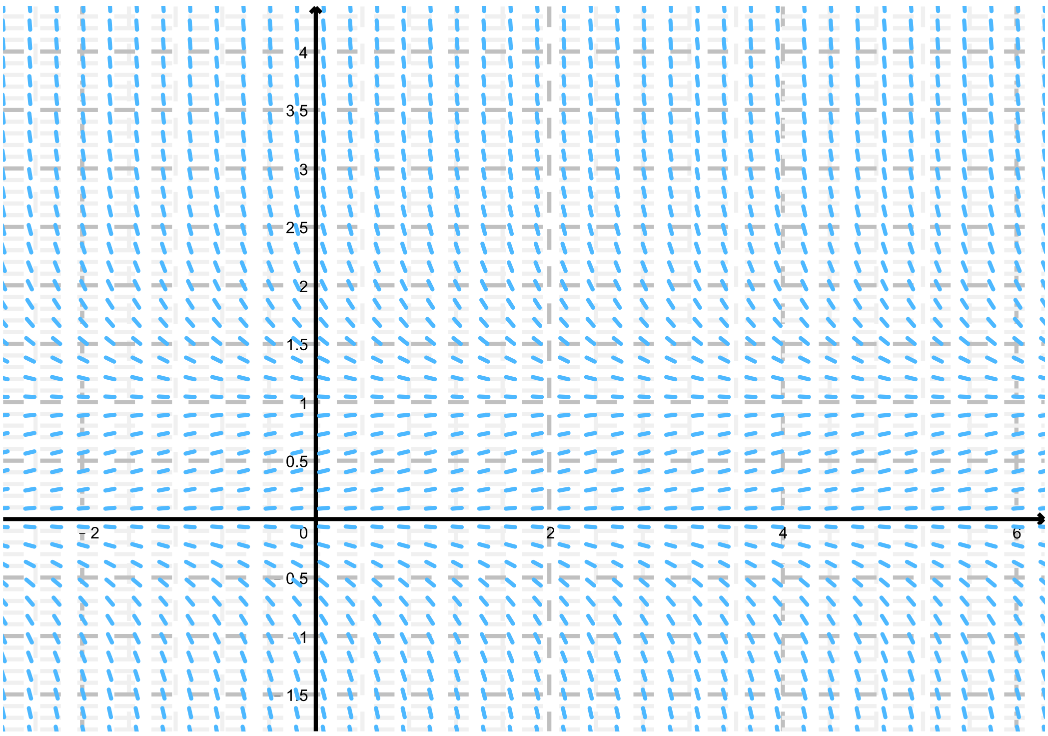
where c is the constant of integration.

(b) Write down an equation for $\dfrac{dx}{dt}$ and **hence** use calculus to show that the gradient of the curve along which the ball rolls is given by $\dfrac{dy}{dx} = y(1 - y)$. **2**

(c) i. Show that $\dfrac{d^2y}{dx^2} = (1 - 2y)y(1 - y)$. **1**

ii. **Hence**, for $0 < y < 1$, find the y coordinate of any points of inflection on the curve along which the ball rolls and verify that they are points of inflection. **2**

(d) If the position of the ball at a particular time is given by $\underset{\sim}{r} = 0.5\,\underset{\sim}{j}$, sketch the path of the ball on the direction field below. **1**



A.2 2007 VCE Specialist Mathematics

A.2.1 Paper 1

Question 9

A particle moves in the Cartesian plane with position vector $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$, where x and y are functions of t. **3**

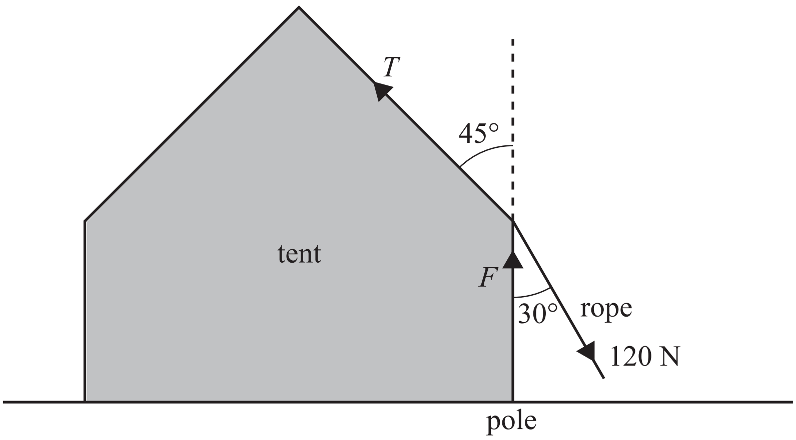
If its velocity vector is $\underset{\sim}{v} = -y\underset{\sim}{i} + x\underset{\sim}{j}$, find the acceleration vector of the particle in terms of the position vector $\underset{\sim}{r}$.

A.3 2008 VCE Specialist Mathematics

A.3.1 Paper 1

Question 7

The side of a tent is supported by a vertical pole supplying a force of F newtons and a rope with a tension of 120 newtons. The tension in the tent fabric is T newtons as shown in the accompanying diagram. **3**



Find the exact values of T and F.

A.3.2 Paper 2 Section 1

15. Two forces $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ act on a body. $\underset{\sim}{P}$ acts in the direction of $\underset{\sim}{i}$ with magnitude one newton and $\underset{\sim}{Q}$ acts in the direction of $\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$ with magnitude 4 newtons. **1**

The magnitude of total force acting on the body, in newtons, is

(A) 3 (B) $\sqrt{15}$ (C) $\sqrt{17}$ (D) $\sqrt{21}$ (E) 5

A.4 2009 VCE Specialist Mathematics

A.4.1 Paper 2 Section 1

15. A force of magnitude 4 newtons acts in the northeasterly direction and another force of 3 newtons acts in the easterly direction. **1**

The magnitude, in newtons, of the resultant of these two forces is

(A) 7

(B) $25 + 12\sqrt{2}$

(C) $\sqrt{25 + 12\sqrt{2}}$

(D) $\sqrt{25 - 12\sqrt{2}}$

(E) $25 - 12\sqrt{2}$

A.5 2010 VCE Specialist Mathematics

A.5.1 Paper 2 Section 1

4. The position vector of a particle at time $t \geq 0$ is given by $\underset{\sim}{r} = (\sin t)\,\underset{\sim}{i} + (\cos 2t)\,\underset{\sim}{j}$. The path of the particle has cartesian equation **1**

(A) $y = 2x^2 - 1$

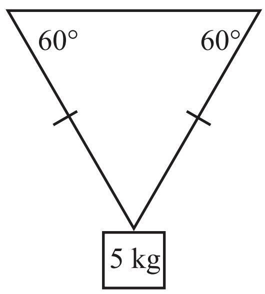
(B) $y = 1 - 2x^2$

(C) $y = \sqrt{1 - x^2}$

(D) $y = \sqrt{x^2 - 1}$

(E) $y = 2x\sqrt{1 - x^2}$

18. A 5 kg mass is suspended from a horizontal ceiling by two strings of equal length. Each string makes an angle of 60° to the ceiling, as shown in the diagram. **1**



Correct to one decimal place, the tension in each string in newtons is

(A) 24.5 (B) 28.3 (C) 34.6 (D) 49.0 (E) 84.9

A.6 2011 VCE Specialist Mathematics

A.6.1 Paper 2 Section 1

11. Consider the three forces **1**

$$\underset{\sim}{F}_1 = -\sqrt{3}\underset{\sim}{j} \qquad \underset{\sim}{F}_2 = \underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} \qquad \underset{\sim}{F}_3 = -\frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}$$

The magnitude of the sum of these three forces is equal to the magnitude of

(A) $\underset{\sim}{F}_3 - \underset{\sim}{F}_1$ (B) $\underset{\sim}{F}_2 - \underset{\sim}{F}_1$ (C) $\underset{\sim}{F}_1$ (D) $\underset{\sim}{F}_2$ (E) $\underset{\sim}{F}_3$

13. The position of a particle at time t is given by $\underset{\sim}{r}(t) = \left(\sqrt{t-2}\right)\underset{\sim}{i} + (2t)\underset{\sim}{j}$ for $t \geq 2$. The cartesian equation of the path of the particle is **1**

(A) $y = 2x^2 + 4$, $x \geq 2$

(B) $y = 2x^2 + 2$, $x \geq 2$

(C) $y = 2x^2 + 4$, $x \geq 0$

(D) $y = \sqrt{\dfrac{x-4}{2}}$, $x \geq 4$

(E) $y = 2x^2 + 2$, $x \geq 0$

A.7 2012 VCE Specialist Mathematics

A.7.1 Paper 2 Section 1

14. A particle is acted on by two forces, one of 6 newtons acting due south, the other of 4 newtons acting in the direction N60°W **1**

The magnitude of the resultant force, acting on the particle is

(A) 10

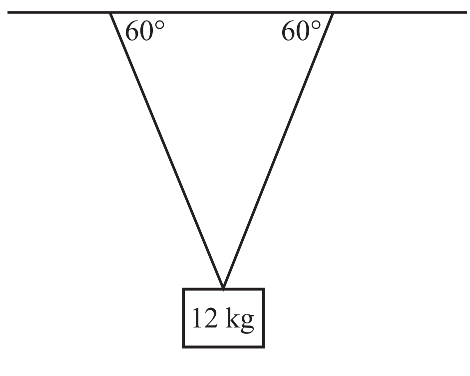
(B) $2\sqrt{7}$

(C) $2\sqrt{19}$

(D) $\sqrt{52 - 24\sqrt{3}}$

(E) $\sqrt{52 + 24\sqrt{3}}$

22. A 12 kg mass is suspended in equilibrium from a horizontal ceiling by two identical light strings. Each string makes an angle of 60° with the ceiling, as shown. **1**



The magnitude, in newtons, of the tension in each string is equal to

(A) $6g$

(B) $12g$

(C) $24g$

(D) $4\sqrt{3}\,g$

(E) $8\sqrt{3}\,g$

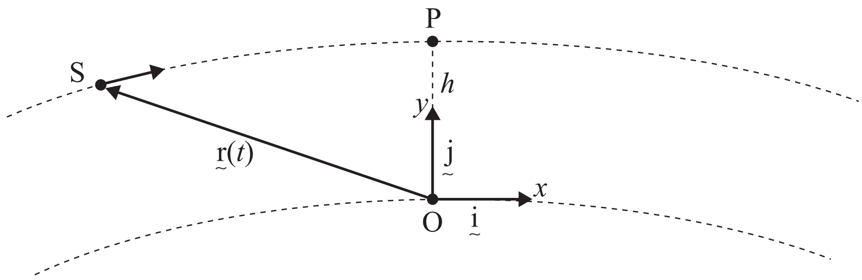
A.7.2 Paper 2 Section 2

Question 4

The position vector of the *International Space Station* (S), when visible above the horizon from a radar tracking location (O) on the surface of Earth, is modelled by

$$\underset{\sim}{r}(t) = 6\,800\sin\big(\pi(1.3t - 0.1)\big)\,\underset{\sim}{i} + \Big(6\,800\cos\big(\pi(1.3t - 0.1)\big) - 6\,400\Big)\,\underset{\sim}{j} \quad \text{for } t \in [0, 0.154]$$

where $\underset{\sim}{i}$ is a unit vector relative to O as shown and $\underset{\sim}{j}$ is a unit vector vertically up from point O. Time t is measured in hours and displacement components are measured in kilometres.



(a) Find the height, h km, of the space station above the surface of Earth when it is at point P, directly above point O. **1**

Give your answer correct to the nearest km.

(b) Find the acceleration of the space station, and show that its acceleration is perpendicular to its velocity. **3**

(c) Find the speed of the space station in km/h. Give your answer correct to the nearest integer. **2**

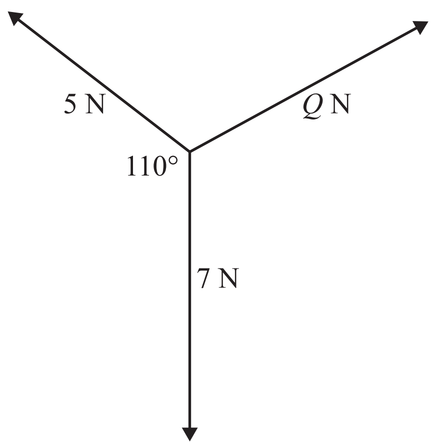
(d) Find the equation of the path followed by the space station in cartesian form. **2**

(e) Find the times when the space station is at a distance of 1 000 km from the radar tracking location O. Give your answers in hours, correct to two decimal places. **3**

A.8 2013 VCE Specialist Mathematics

A.8.1 Paper 2 Section 1

16. Forces of magnitude 5 N, 7 N and Q N act on a particle that is in equilibrium, as shown in the diagram below. **1**



The magnitude of Q, in newtons, can be found by evaluating

(A) $\sqrt{5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 70°}$

(B) $5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 110°$

(C) $\sqrt{5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 110°}$

(D) $5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 70°$

(E) $\sqrt{5^2 + 7^2 - 2 \times 5 \times 7 \times \cos 20°}$

A.8.2 Paper 2 Section 2

Question 5

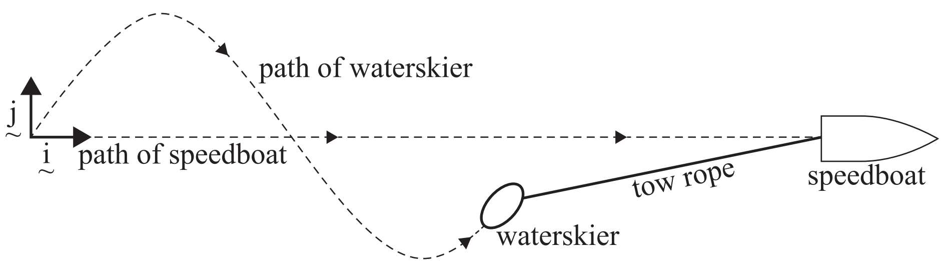
A waterskier moves from side to side behind a speedboat that is travelling in a straight line along a still lake.

The position vector of the waterskier relative to a fixed observation point on the shore of the lake is given by

$$\underset{\sim}{r}(t) = 7.5t\,\underset{\sim}{i} + \left(50 - 10\sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{j}$$

t seconds after passing a marker buoy.

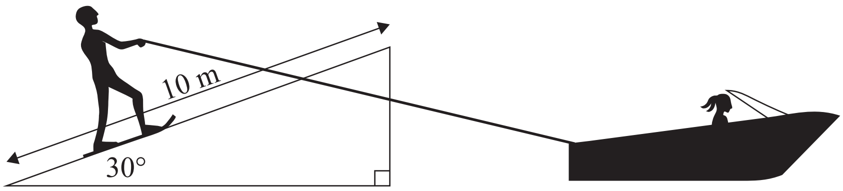
The components of the position vector are measured in metres, where $\underset{\sim}{i}$ is a unit vector in the direction of motion of the speedboat, and $\underset{\sim}{j}$ is a unit vector perpendicular to $\underset{\sim}{i}$ to the left of the direction of motion of the speedboat. The diagram below represents the view from above.



(a) Find $\dot{\underset{\sim}{r}}(t)$ and **hence** determine the minimum and maximum speeds of the waterskier. Give your answers in metres per second, correct to one decimal place. **3**

(b) Find the times when the acceleration of the waterskier is zero. **2**

Later, the waterskier performs a jump from the top of a 10 m long ramp, which is inclined at 30° to the horizontal. To do this, the speedboat travels beside the ramp and keeps the waterskier's speed at 15 ms^{-1} directly up the ramp, until the waterskier reaches the top. Then the speedboat slows, so that the tow rope slackens and remains slack after the waterskier leaves the top of the ramp. Assume negligible air resistance on the waterskier, who is then subject only to the force of gravity.



(c) Find the time it takes for the waterskier to hit the water after leaving the top of the ramp. Give your answer in seconds, correct to the nearest one hundredth of a second. **2**

(d) After leaving the top of the ramp, how far in the horizontal direction does the waterskier travel before hitting the water? **1**

Give your answer in metres, correct to the nearest metre.

A.9 2014 VCE Specialist Mathematics

A.9.1 Paper 1

Question 2

The position vector of a particle at time $t \geq 0$ is given by

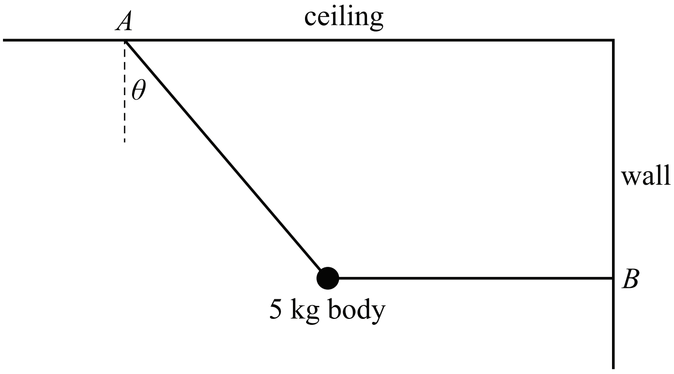
$$\underset{\sim}{r}(t) = (t-2)\underset{\sim}{i} + \left(t^2 - 4t + 1\right)\underset{\sim}{j}$$

(a) Show that the Cartesian equation of the path followed by the particle is $y = x^2 - 3$. **1**

(b) Sketch the path followed by the particle, labelling all important features. **2**

Question 8

A body of mass 5 kg is held in equilibrium by two light inextensible strings. One string is attached to a ceiling at A and the other to a wall at B. The string attached to the ceiling is at an angle θ to the vertical and has tension T_1 newtons, and the other string is horizontal and has tension T_2 newtons. Both strings are made of the same material.



(a) i. Resolve the forces on the body vertically and horizontally, and express T_1 in terms of θ. **2**

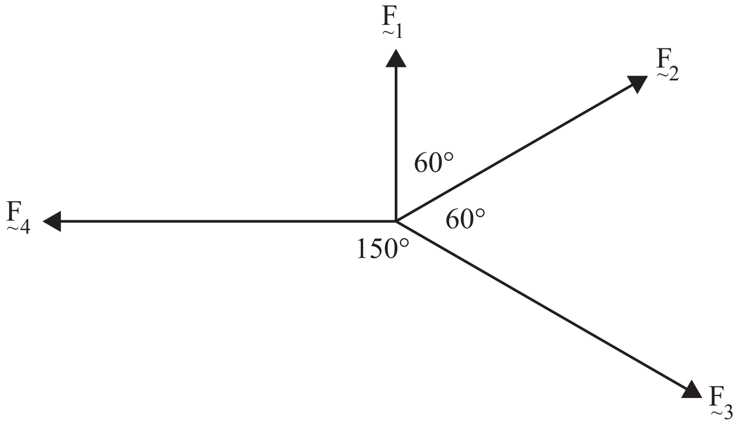
ii. Express T_2 in terms of θ. **1**

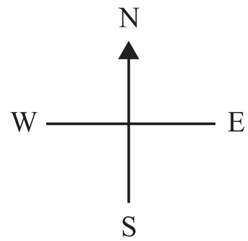
(b) Show that $\tan\theta < \sec\theta$ for $0 < \theta < \frac{\pi}{2}$. **1**

(c) The type of string used will break if it is subjected to a tension of more than 98 N. Find the maximum allowable value of θ so that **neither** string will break. **3**

A.9.2 Paper 2 Section 1

18. A body on a horizontal smooth plane is acted upon by four forces $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$, $\underset{\sim}{F}_3$ and $\underset{\sim}{F}_4$, as shown. The force $\underset{\sim}{F}_1$ acts in a northerly direction and force $\underset{\sim}{F}_4$ acts in a westerly direction. **1**





Given that $\left|\underset{\sim}{F}_1\right| = 1$, $\left|\underset{\sim}{F}_2\right| = 2$, $\left|\underset{\sim}{F}_3\right| = 4$ and $\left|\underset{\sim}{F}_4\right| = 5$, the motion of the body is such that

(A) is in equilibrium

(B) moves to the west

(C) moves to the north

(D) moves in the direction 30° south of west

(E) moves to the east

19. The velocity vector of a 5 kg mass moving in the cartesian plane is given by **1**

$$\underset{\sim}{v}(t) = 3\sin(2t)\,\underset{\sim}{i} + 4\cos(2t)\,\underset{\sim}{j}$$

where velocity components are measured in m/s.

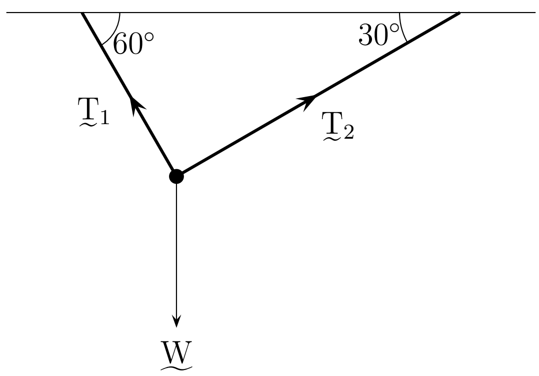
During its motion, the maximum magnitude of the net force, in newtons, acting on the mass is

(A) 8 (B) 30 (C) 40 (D) 50 (E) 70

A.10 2015 VCE Specialist Mathematics

A.10.1 Paper 2 Section 1

16. The diagram above shows a mass suspended in equilibrium by two light strings that make angles of 60° and 30° with a ceiling. The tensions in the strings are T_1 and T_2, and the weight force acting on the mass is $\underset{\sim}{W}$. **1**



The correct statement relating the given forces is

(A) $\underset{\sim}{T}_1 + \underset{\sim}{T}_2 + \underset{\sim}{W} = \underset{\sim}{0}$

(B) $\underset{\sim}{T}_1 + \underset{\sim}{T}_2 - \underset{\sim}{W} = \underset{\sim}{0}$

(C) $\underset{\sim}{T}_1 \times \dfrac{1}{2} + \underset{\sim}{T}_2 \times \dfrac{\sqrt{3}}{2} = \underset{\sim}{0}$

(D) $\underset{\sim}{T}_1 \times \dfrac{\sqrt{3}}{2} + \underset{\sim}{T}_2 \times \dfrac{1}{2} = \underset{\sim}{W}$

(E) $\underset{\sim}{T}_1 \times \dfrac{1}{2} + \underset{\sim}{T}_2 \times \dfrac{\sqrt{3}}{2} = \underset{\sim}{W}$

A.11 2016 VCE Specialist Mathematics

A.11.1 Paper 1

Question 8

The position of a body with mass 3 kg from a fixed origin at time t seconds, is given by $\underset{\sim}{r}(t) = \Big(3\sin(2t) - 2\Big)\underset{\sim}{i} + \Big(3 - 2\cos(2t)\Big)\underset{\sim}{j}$ where components are in metres.

(a) Find an expression for the speed, in metres per second, of the body at time t. **2**

(b) Find the speed of the body, in metres per second, when $t = \dfrac{\pi}{12}$. **1**

(c) Find the maximum magnitude of the net force acting on the body in newtons. **3**

A.11.2 Paper 2 Section 1

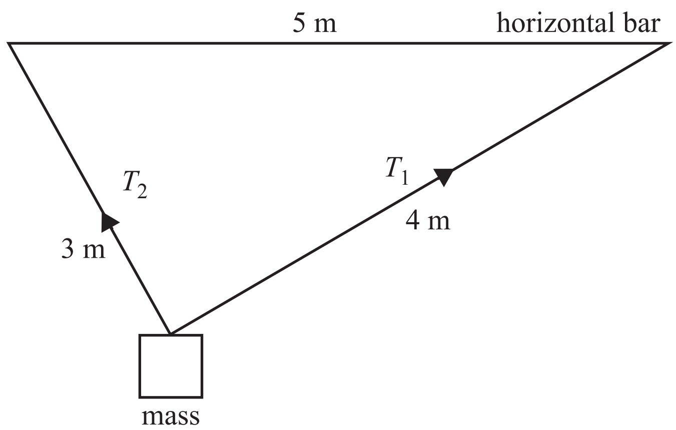
13. A particle of mass 5 kg is subject to forces $12\underset{\sim}{i}$ newtons and $9\underset{\sim}{j}$ newtons. **1**

If no other forces act on the particle, the magnitude of the particle's acceleration, in ms^{-2} is

(A) 3 (C) 4.2 (E) $60\underset{\sim}{i} + 45\underset{\sim}{j}$

(B) $2.4\underset{\sim}{i} + 1.8\underset{\sim}{j}$ (D) 9

14. Two light strings of length 4 m and 3 m connect a mass to a horizontal bar, as shown below. The strings are attached to the horizontal bar 5 m apart. **1**



Given the tension in the longer string is T_1 and the tension in the shorter string is T_2, the ratio of the tensions $\dfrac{T_1}{T_2}$ is

(A) $\dfrac{3}{5}$ (B) $\dfrac{3}{4}$ (C) $\dfrac{4}{5}$ (D) $\dfrac{5}{4}$ (E) $\dfrac{4}{3}$

16. A cricket ball is hit from the ground at an angle of 30° to the horizontal with a velocity of 20 ms^{-1}. The ball is subject only to gravity and air resistance is negligible. **1**

Given that the field is level, the horizontal distance travelled by the ball, in metres, to the point of impact is

(A) $\dfrac{10\sqrt{3}}{g}$ (B) $\dfrac{20}{g}$ (C) $\dfrac{100\sqrt{3}}{g}$ (D) $\dfrac{200\sqrt{3}}{g}$ (E) $\dfrac{400}{g}$

A.11.3 Paper 2 Section 2

Question 4

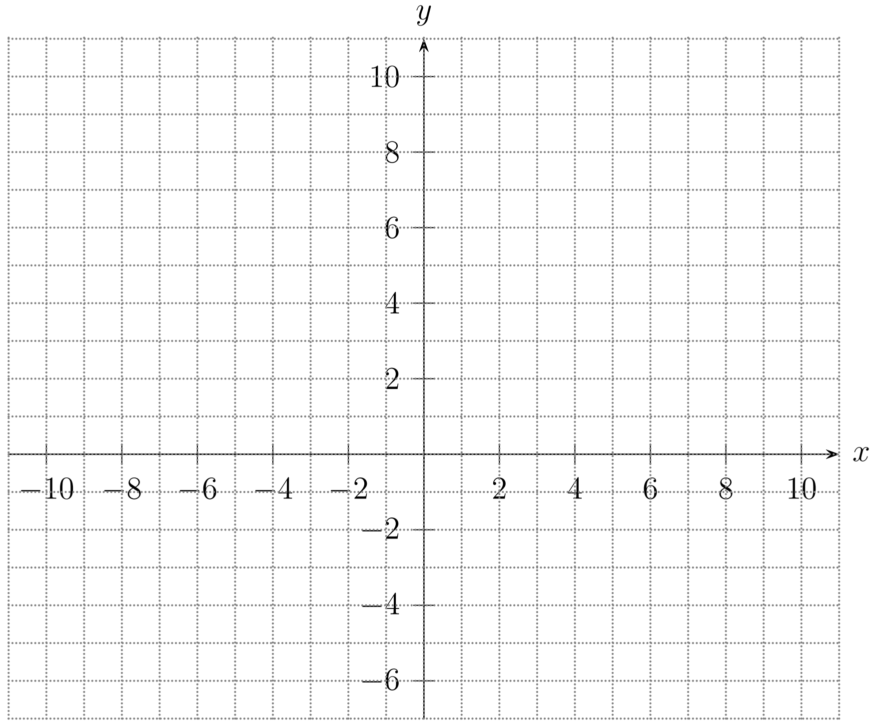
Two ships, A and B, are observed from a lighthouse at origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\underset{\sim}{r}_A = 5(1-t)\,\underset{\sim}{i} + 3(1+t)\,\underset{\sim}{j}$$
$$\underset{\sim}{r}_B = 4(t-2)\,\underset{\sim}{i} + (5t-2)\,\underset{\sim}{j}$$

where displacements are measured in kilometres.

(a) Show that the two ships will not collide, clearly stating your reason. **2**

(b) Sketch and label the path of each ship on the axes below. Show the direction of motion of each ship with an arrow. **3**



(c) Find the obtuse angle between the paths of the two ships. Give your answer in degrees, correct to one decimal place. **2**

(d) i. Find the value of t, correct to three decimal places, when the ships are closest. **2**

ii. Find the minimum distance between the two ships, in kilometres, correct to two decimal places. **1**

A.12 2017 VCE Specialist Mathematics

A.12.1 Paper 1

Question 9

A particle of mass 2 kg with initial velocity $3\underset{\sim}{i} + 2\underset{\sim}{j}$ ms^{-1} experiences a constant force for 10 seconds.

The particle's velocity at the end of the 10 second period is $43\underset{\sim}{i} - 18\underset{\sim}{j}$ ms^{-1}

(a) Find the magnitude of the constant force in newtons. **2**

(b) Find the displacement of the particle from its initial position after 10 seconds. **3**

A.12.2 Paper 2 Section 1

12. Let $\underset{\sim}{r}(t) = (1 - \sqrt{a}\sin t)\,\underset{\sim}{i} + \left(1 - \frac{1}{b}\cos t\right)\underset{\sim}{j}$ for $t \geq 0$ and $a,\ b \in \mathbb{R}^+$ be the path of the particle moving in the cartesian plane. **1**

The path of the particle will always be a circle if

(A) $ab^2 = 1$ (C) $ab^2 \neq 1$ (E) $a^2b \neq 1$

(B) $a^2b = 1$ (D) $ab = 1$

15. A body has displacement of $3\underset{\sim}{i} + \underset{\sim}{j}$ metres at a particular time. The body moves with constant velocity and two seconds later its displacement is $-\underset{\sim}{i} + 5\underset{\sim}{j}$ metres. **1**

The velocity, in ms^{-1}, of the body is

(A) $2\underset{\sim}{i} + 6\underset{\sim}{j}$ (C) $-4\underset{\sim}{i} + 4\underset{\sim}{j}$ (E) $\underset{\sim}{i} + 3\underset{\sim}{j}$

(B) $-2\underset{\sim}{i} + 2\underset{\sim}{j}$ (D) $4\underset{\sim}{i} - 4\underset{\sim}{j}$

A.13 2018 VCE Specialist Mathematics

A.13.1 Paper 1

Question 9

A curve is specified parametrically by $\underset{\sim}{\mathrm{r}}(t) = (\sec t)\underset{\sim}{\mathrm{i}} + \left(\dfrac{\sqrt{2}}{2}\tan t\right)\underset{\sim}{\mathrm{j}}$, $t \in \mathbb{R}$.

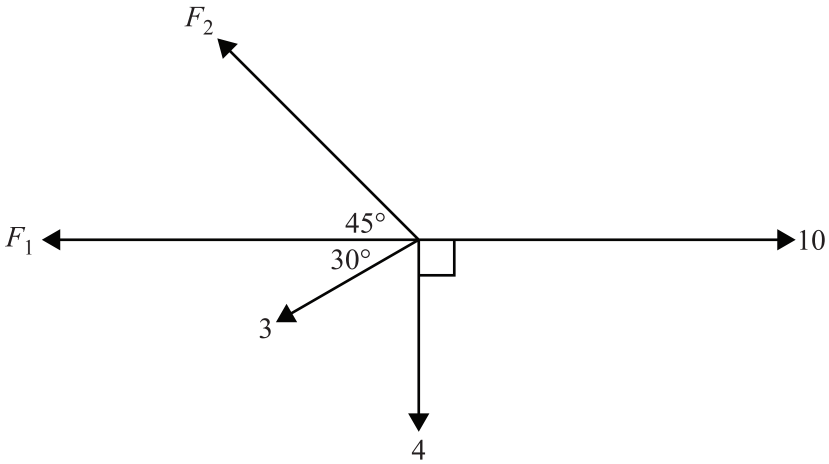
(a) Show that the Cartesian equation of the curve is $x^2 - 2y^2 = 1$. **2**

(b) Find the x-coordinates of the points of intersection of the curve $x^2 - 2y^2 = 1$ and the line $y = x - 1$. **1**

(c) Find the volume of the solid of revolution formed when the region bounded by the curve and the line is rotated about the x-axis. **2**

A.13.2 Paper 2 Section 1

16. The diagram below shows a mass being acted on by a number of forces whose magnitudes are labelled. All forces are measured in newtons and the system is in equilibrium. **1**



The value of F_2 is:

(A) $\dfrac{\sqrt{2}}{2}\left(8 + 3\sqrt{3}\right)$

(B) $\dfrac{11\sqrt{2}}{2}$

(C) $\dfrac{3\sqrt{2}}{2}$

(D) 7.78

(E) 7.0

A.13.3 Paper 2 Section 2

Question 4

The yachts A and B, are competing in a race and their position vectors on a certain section of the race after time t hours are given by

$$\underset{\sim}{r}_A(t) = (t+1)\,\underset{\sim}{i} + (t^2+2t)\,\underset{\sim}{j} \qquad \underset{\sim}{r}_B(t) = t^2\,\underset{\sim}{i} + (t^2+3)\,\underset{\sim}{j} \qquad t \geq 0$$

where displacement components are measured in kilometres from a given reference buoy at origin O.

(a) Find the Cartesian equation of the path for each yacht. **2**

(b) Show that the two yachts will not collide if they follow these paths. **2**

(c) Find the coordinates of the point where the paths of the two yachts cross. Give your coordinates correct to three decimal places. **2**

(d) For what values of t is yacht A travelling faster than yacht B? **2**

A.14 2019 VCE Specialist Mathematics

A.14.1 Paper 1

Question 4

The position vectors of two particles A and B at time t seconds after they have started moving are given by **3**

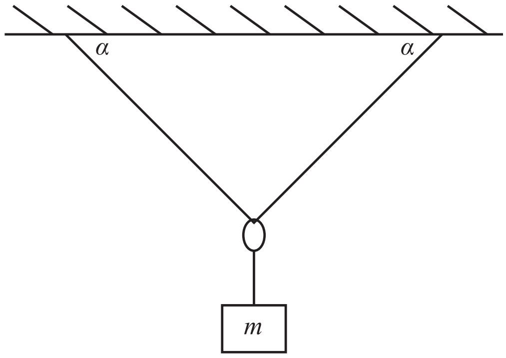
$$\underset{\sim}{r}_A(t) = \left(t^2+1\right)\,\underset{\sim}{i} + \left(a + \frac{t}{3}\right)\,\underset{\sim}{j} \quad \text{and} \quad \underset{\sim}{r}_B(t) = \left(t^3 - t\right)\,\underset{\sim}{i} + \cos^{-1}\left(\frac{t}{2}\right)\,\underset{\sim}{j}$$

respectively, where a is a real constant and $0 \leq t \leq 2$.

Find the value of a if the particles collide after they have started moving.

Question 9

A light inextensible string is connected at each end to a horizontal ceiling. A mass of m kilograms hangs in equilibrium from a smooth ring on the string, as shown in the diagram. The string makes an angle α with the ceiling. **1**



Express the tension, T newtons, in the string in terms of m, g and α.

A.14.2 Paper 2 Section 1

13. Two forces, $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$, both measured in newtons, act on a mass of 3 kg, producing an acceleration of $\sqrt{3}\,\underset{\sim}{i} + \underset{\sim}{j}$ ms^{-2}. **1**

Given $\underset{\sim}{F}_1 = 2\,\underset{\sim}{j}$, the acute angle between $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ is

(A) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

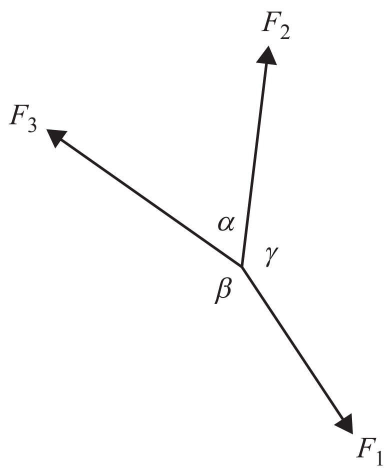
(B) $\pi - \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

(C) $\dfrac{\pi}{6}$

(D) $\cos^{-1}\left(\dfrac{1}{2\sqrt{7}}\right)$

(E) $\pi - \cos^{-1}\left(\dfrac{1}{2\sqrt{7}}\right)$

17. A particle is held in equilibrium by three coplanar forces of magnitudes F_1, F_2 and F_3. **1**
The angles between these forces are α, β and γ, as shown in the diagram.



If $\beta = 2\alpha$, then $\dfrac{F_1}{F_2}$ is equal to

(A) $\dfrac{1}{2}\sin\alpha$

(B) $2\sin\alpha$

(C) $\dfrac{1}{2}\operatorname{cosec}\alpha$

(D) $\dfrac{1}{2}\cos\alpha$

(E) $\dfrac{1}{2}\sec\alpha$

A.15 2020 VCE Specialist Mathematics

15. Two forces, $\underset{\sim}{F}_A = 4\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{F}_B = 2\underset{\sim}{i} + 5\underset{\sim}{j}$, act on a particle of mass 3 kg. The particle is initially at rest at position $\underset{\sim}{i} + \underset{\sim}{j}$. All force components are measured in newtons and displacements are measured in metres. **1**

The Cartesian equation of the path of the particle is:

(A) $y = \dfrac{x}{2}$

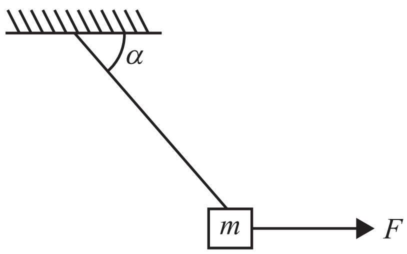
(B) $y = \dfrac{x}{2} - \dfrac{1}{2}$

(C) $y = \dfrac{(x+1)^2}{2} + 1$

(D) $y = \dfrac{(x-1)^2}{2} + 1$

(E) $y = \dfrac{x}{2} + \dfrac{1}{2}$

18. A particle of mass m kilograms hangs from a string that is attached to a fixed point. The particle is acted on by a horizontal force of magnitude F newtons. The system is in equilibrium when the string makes an angle α to the horizontal, as shown in the diagram below. The tension in the string has magnitude T newtons. **1**



The value of $\tan\alpha$ is

(A) $\dfrac{mg}{T}$ (B) $\dfrac{T}{mg}$ (C) $\dfrac{T}{F}$ (D) $\dfrac{F}{mg}$ (E) $\dfrac{mg}{F}$

Section B

X2 Resisted Motion

Questions here are for Part III.

B.1 1997 4U HSC

Question 6

(a) A ball of mass 2 kilograms is thrown vertically upward from the origin with an initial speed of 8 metres per second. The ball is subject to a downward gravitational force of 20 newtons and air resistance of $\frac{v^2}{5}$ newtons in the opposite direction to the velocity, v metres per second. Hence, until the ball reaches its highest point, the equation of motion is **8**

$$\ddot{y} = -\frac{v^2}{10} - 10$$

where y metres is its height.

i. Using the fact that $\ddot{y} = v\frac{dv}{dy}$, show that, while the ball is rising,

$$v^2 = 164e^{-\frac{y}{5}} - 100$$

ii. Hence find the maximum height reached.

iii. Using the fact that $\ddot{y} = \frac{dv}{dt}$, find how long the ball takes to reach this maximum height.

iv. How fast is the ball travelling when it returns to the origin?

B.2 1998 4U HSC

Question 7

(b) i. Differentiate $\sin^{-1}(u) - \sqrt{1-u^2}$. **1**

ii. Hence show that **2**

$$\int_0^\alpha \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} du = \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2}$$

for $0 < \alpha < 1$.

(c) A bead of mass m slides along a wire in the shape of the curve **6**

$$y = \frac{3}{2}x^{\frac{2}{3}} \quad \text{where } 0 \leq x \leq 1$$

At time t, the bead is at $(x(t), y(t))$ and its velocity is $(\dot{x}(t), \dot{y}(t))$. The motion of the bead is governed by the equation

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy = E$$

where E and g are constants, and

$$\dot{x} = x^{\frac{1}{3}}\dot{y}$$

When $t = 0$, the bead is released from rest at the point $\left(1, \frac{3}{2}\right)$. It accelerates along the wire towards the origin, where it arrives at time t_1.

i. Find E, and show that $\dot{y}^2 = \dfrac{3g(3-2y)}{3+2y}$.

ii. Find $\dot{x}\,(t_1)$ and $\dot{y}\,(t_1)$

iii. Using the result of part (b), or otherwise, find the time it takes for the bead to travel from $\left(\frac{1}{8}, \frac{3}{8}\right)$ to the origin.

B.3 1999 4U HSC

Question 6

(b) A ball of unit mass is projected vertically upwards from ground level with initial speed U. Assume that air resistance is kv, where v is the ball's speed and k is a positive constant. **11**

We wish to consider the ball's motion as it falls back to ground level. Let y be the displacement of the ball measured *vertically downwards* from the point of maximum height, t be the time elapsed after the ball has reached maximum height, and g be the acceleration due to gravity.

i. Explain why $v(0) = 0$, and $\dfrac{dv}{dt} = g - kv$ while the ball is in motion.

ii. Deduce that $v = \dfrac{g}{k}\left(1 - e^{-kt}\right)$ for $t \geq 0$.

iii. By writing $\dfrac{dv}{dt} = v\dfrac{dv}{dy}$, deduce from part (i) that

$$\frac{g}{k}\log_e\left(\frac{g - kv}{g}\right) + v = -ky$$

iv. Using parts (ii) and (iii), deduce that

$$t = \frac{v + ky}{g}$$

v. You are given that the ball reaches maximum height

$$h = \frac{1}{k}\left(U - \frac{g}{k}\log_e\left(\frac{g + kU}{g}\right)\right)$$

in time $t_h = \dfrac{1}{k}\log_e\left(\dfrac{g + kU}{g}\right)$. (Do NOT prove these results).

Deduce that the total time T that the ball is in the air is $T = \dfrac{U + V}{g}$, where V is the final speed that the ball reaches when returning to ground level.

vi. If air resistance is ignored, the total time T_0 that the ball is in the air is

$$T_0 = \frac{U + V_0}{g}$$

where V_0 is the final speed of the ball then reaches when returning to ground level. By considering V_0 and V, determine which is larger: T or T_0.

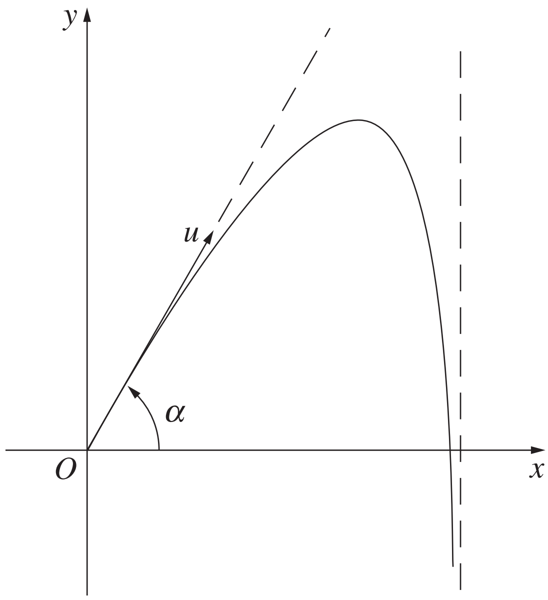
B.4 2003 Ext 2 HSC

Question 5

(b) A particle of mass m is thrown from the top, O, of a very tall building with an initial velocity u at an angle α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \qquad \text{and} \qquad \ddot{y} = -k\dot{y} - g$$

where k is a constant and the acceleration due to gravity is g. (You are NOT required to show these.)



i. Derive the result $\dot{x} = ue^{-kt}\cos\alpha$ from the relevant equation of motion. **2**

ii. Verify that $\dot{y} = \dfrac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$ satisfies the appropriate equation of motion and initial condition. **2**

iii. Find the value of t when the particle reaches its maximum height. **2**

iv. What is the limiting value of the horizontal displacement of the particle? **2**

B.5 2004 Ext 2 HSC

Question 6

(b) A particle is released from the origin O with an initial velocity of A ms^{-1} directed vertically downward. The particle is subject to a constant gravitational force and a resistance which is proportional to the velocity, v ms^{-1}, of the particle.

Let x be the displacement in metres of the particle below O at time t seconds after the release of the particle, so that the equation of motion is

$$\ddot{x} = g - kv$$

where g ms^{-2} is the acceleration due to gravity.

i. The terminal velocity of the particle is B ms^{-1}. Show that $k = \dfrac{g}{B}$. **1**

ii. Verify that v satisfies the equation $\dfrac{d}{dt}ve^{kt} = ge^{kt}$. **2**

iii. Hence show that the velocity of the particle is given by **2**

$$v = B - (B - A)e^{-\frac{gt}{B}}$$

iv. Deduce that $x = Bt - \dfrac{B}{g}(B - A)\left(1 - e^{-\frac{gt}{B}}\right)$. **2**

At the same time as the particle is released from O, an identical particle is released from the point P which is h metres below O. The second particle has an initial velocity of A ms^{-1} directed vertically upward.

Its displacement below O is given by $x = h + Bt - \dfrac{B}{g}\,(B + A)\left(1 - e^{-\frac{gt}{B}}\right)$. (Do NOT prove this).

v. Suppose that the two particles meet after T seconds. Show that **2**

$$T = \frac{B}{g}\log_e\left(\frac{2AB}{2AB - gh}\right)$$

vi. The value of A can be varied. What condition must A satisfy so that the two particles can meet? **1**

B.6 2005 Ext 2 HSC

Question 7

(a) The acceleration of any body towards the centre of a star due to the force of gravity is proportional to x^{-2}, where x is the distance of the body from the centre of the star. That is,

$$\ddot{x} = -\frac{k}{x^2}$$

where k is a positive constant.

i. No longer in new syllabus. However, the following result may be helpful for the next two parts: **2**

$$k = \frac{4\pi^2 R^3}{T^2}$$

ii. The satellite is stopped suddenly. It then falls in a straight line towards the centre of the star under the influence of gravity. **3**

Show that the satellite's velocity, v, satisfies the equation

$$v^2 = \frac{8\pi^2 R^2}{T^2}\left(\frac{R-x}{x}\right)$$

where x is the distance of the satellite from the centre of the star.

iii. Show that, if the mass of the star were concentrated at a single point, the satellite would reach this point after time $\dfrac{T}{4\sqrt{2}}$. **3**

You may assume that

$$\int \sqrt{\frac{x}{R-x}}\,dx = R\sin^{-1}\left(\sqrt{\frac{x}{R}}\right) - \sqrt{x(R-x)}$$

B.7 2006 Ext 2 HSC

Question 6

(a) In an alien universe, the gravitational attraction between two bodies is proportional to x^{-3}, where x is the distance between their centres.

A particle is projected upward from the surface of a planet with velocity u at time $t = 0$. Its distance x from the centre of the planet satisfies the equation

$$\ddot{x} = -\frac{k}{x^3}$$

i. Show that $k = gR^3$, where g is the magnitude of the acceleration due to gravity at the surface of the planet and R is the radius of the planet. **1**

ii. Show that v, the velocity of the particle, is given by **3**

$$v^2 = \frac{gR^3}{x^2} - \left(gR - u^2\right)$$

iii. It can be shown that **2**

$$x = \sqrt{R^2 + 2uRt - \left(gR - u^2\right)t^2}$$

(Do NOT prove this).

Show that if $u \geq \sqrt{gR}$ the particle will not return to the planet.

iv. If $u < \sqrt{gR}$ the particle reaches a point whose distance from the centre of the planet is D, and then falls back.

(1) Use the formula in part (ii) to find D in terms of u, R and g. **1**

(2) Use the formula in part (iii) to find the time taken for the particle to return to the surface of the planet in terms of u, R and g. **1**

B.8 2007 Ext 2 HSC

Question 6

(b) A raindrop falls vertically from a high cloud. The distance it has fallen is given by

$$x = 5\log_e\left(\frac{e^{1.4t}+e^{-1.4t}}{2}\right)$$

where x is in metres and t is the time elapsed in seconds.

i. Show that the velocity of the raindrop, v metres per second, is given by **2**

$$v = 7\left(\frac{e^{1.4t}-e^{-1.4t}}{e^{1.4t}+e^{-1.4t}}\right)$$

ii. Hence show that **2**

$$v^2 = 49\left(1-e^{-\frac{2x}{5}}\right)$$

iii. Hence or otherwise, show that **2**

$$\ddot{x} = 9.8 - 0.2v^2$$

iv. The physical significance of the 9.8 in part (iii) is that it represents the acceleration due to gravity. **1**

What is the physical significance of the term $-0.2v^2$?

v. Estimate the velocity at which the raindrop hits the ground. **1**

B.9 2008 Ext 2 HSC

Question 7

(c) A fishing boat drifts with a current in a straight line across a fishing ground.

The boat's velocity v, at time t after the start of this drift is given by

$$v = b - (b - v_0)\,e^{-\alpha t}$$

where v_0, b and α are positive constants, and $v_0 < b$.

i. Show that $\dfrac{dv}{dt} = \alpha(b-v)$. **1**

ii. The physical significance of v_0 is that it represents the initial velocity of the boat. **1**

What is the physical significance of b?

iii. Let x be the distance travelled by the boat from the start of the drift. **3**

Find x as a function of t. Hence show that

$$x = \frac{b}{\alpha}\log_e\left(\frac{b-v_0}{b-v}\right) + \frac{v_0 - v}{\alpha}$$

B.10 2011 Ext 2 HSC

Question 6

(a) Jac jumps out of an aeroplane and falls vertically. His velocity at time t after his parachute is opened is given by $v(t)$, where $v(0) = v_0$ and $v(t)$ is positive in the downwards direction. The magnitude of the resistive force provided by the parachute is kv^2, where k is a positive constant. Let m be Jac's mass and g the acceleration due to gravity. Jac's terminal velocity with the parachute open is v_T.

Jac's equation of motion with the parachute open is

$$m\frac{dv}{dt} = mg - kv^2$$

(Do NOT prove this)

i. Explain why Jac's terminal velocity v_T is given by $\sqrt{\dfrac{mg}{k}}$. **1**

ii. By integrating the equation of motion, show that t and v are related by the equation **3**

$$t = \frac{v_T}{2g}\ln\left[\frac{(v_T + v)(v_T - v_0)}{(v_T - v)(v_T + v_0)}\right]$$

iii. Jac's friend Gil also jumps out of the aeroplane and falls vertically. Jac and Gil have the same mass and identical parachutes. **3**

Jac opens his parachute when his speed is $\dfrac{1}{3}v_T$. Gil opens her parachute when her speed is $3v_T$. Jac's speed increases and Gil's speed decreases, both towards v_T.

Show that in the time taken for Jac's speed to double, Gil's speed has halved.

B.11 2012 Ext 2 HSC

Question 13

(a) An object on the surface of a liquid is released at time $t = 0$ and immediately sinks. Let x be its displacement in metres in a downward direction from the surface at time t seconds.

The equation of motion is given by $\dfrac{dv}{dt} = 10 - \dfrac{v^2}{40}$ where v is the velocity of the object.

i. Show that $v = \dfrac{20\left(e^t - 1\right)}{e^t + 1}$. **4**

ii. Use $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$ to show that $x = 20\log_e\left(\dfrac{400}{400 - v^2}\right)$. **2**

iii. How far does the object sink in the first 4 seconds? **2**

B.12 2014 Ext 2 HSC

9. A particle is moving along a straight line so that initially its displacement is $x = 1$, its velocity is $v = 2$, and its acceleration is $a = 4$. **1**

Which is a possible equation describing the motion of the particle?

(A) $v = 2\sin(x - 1) + 2$

(B) $v = 2 + 4\log_e x$

(C) $v^2 = 4\left(x^2 - 2\right)$

(D) $v = x^2 + 2x + 4$

Question 14

(c) A high speed train of mass m starts from rest and moves along a straight track. At time t hours, the distance travelled by the train from its starting point is x km, and its velocity is v km/h. The train is driven by a constant force F in the forward direction. The resistive force in the opposite direction is Kv^2, where K is a positive constant. The terminal velocity of the train is 300 km/h.

i. Show that the equation of motion for the train is **2**

$$m\ddot{x} = F\left[1 - \left(\frac{v}{300}\right)^2\right]$$

ii. Find, in terms of F and m, the time it takes the train to reach a velocity of 200 km/h. **4**

B.13 2015 Ext 2 HSC

Question 15

(a) A particle A of unit mass travels horizontally through a viscous medium. When $t = 0$, the particle is at point O with initial speed u. The resistance on particle A due to the medium is kv^2, where v is the velocity of the particle at time t and k is a positive constant.

When $t = 0$, a second particle B of equal mass is projected vertically upwards from O with the same initial speed u through the same medium. It experiences both a gravitational force and a resistance due to the medium. The resistance on particle B is kw^2, where w is the velocity of the particle B at time t. The acceleration due to gravity is g.

i. Show that the velocity v of the particle A is given by $\dfrac{1}{v} = kt + \dfrac{1}{u}$. **2**

ii. By considering the velocity w of particle B, show that **3**

$$t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(\mu\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right)$$

iii. Show that the velocity V of particle A when particle B is at rest is given by **1**

$$\frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

iv. Hence, if u is very large, explain why $V \approx \dfrac{2}{\pi}\sqrt{\dfrac{g}{k}}$. **1**

B.14 2016 Ext 2 HSC

Question 15

(c) A particle is initially at rest at the point B which is b metres to the right of O. The particle then moves in a straight line towards O.

For $x \neq 0$, the acceleration of the particle is given by $\dfrac{-\mu^2}{x^2}$, where x is the distance from O and μ is a positive constant.

i. Prove that $\dfrac{dx}{dt} = -\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$. **2**

ii. Using the substitution $x = b\cos^2\theta$, show that the time taken to reach a distance d metres to the right of O is given by **3**

$$t = \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta \, d\theta$$

It can be shown that $t = \dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\left(\sqrt{bd-d^2} + b\cos^{-1}\sqrt{\dfrac{d}{b}}\right)$. (Do NOT prove this).

iii. What is the limiting time taken for the particle to reach O? **1**

B.15 2017 Ext 2 HSC

Question 13

(c) A particle is projected upwards from ground level with initial velocity $\dfrac{1}{2}\sqrt{\dfrac{g}{k}}$ ms^{-1}, where g is the acceleration due to gravity and k is a positive constant. The particle moves through the air with speed v ms^{-1} and experiences a resistive force. **4**

The acceleration of the particle is given by

$$\ddot{x} = -g - kv^2 \text{ ms}^{-2}$$

(Do NOT prove this).

The particle reaches a maximum height, H, before returning to the ground.

Using $\ddot{x} = v\dfrac{dv}{dx}$, or otherwise, show that

$$H = \frac{1}{2k}\log_e\left(\frac{5}{4}\right) \text{ metres}$$

B.16 2018 Ext 2 HSC

Question 14

(b) A falling particle experiences forces due to gravity and air resistance. The acceleration of the particle is $g - kv^2$, where g and k are positive constants and v is the speed of the particle. (Do NOT prove this.) **3**

Prove that, after falling from rest through a distance, h, the speed of the particle will be

$$\sqrt{\frac{g}{k}\left(1 - e^{-2kh}\right)}$$

B.17 2019 Ext 2 HSC

Question 13

(c) Two objects are projected from the same point on a horizontal surface. Object 1 is projected with an initial velocity of $20\,\mathrm{ms}^{-1}$ directed at an angle of $\dfrac{\pi}{3}$ to the horizontal. Object 2 is projected 2 seconds later.

The equations of motion of an object projected from the origin with initial velocity v at an angle θ to the x axis are

$$x = vt\cos\theta$$
$$y = -4.9t^2 + vt\sin\theta$$

where t is the time after the projection of the object. Do NOT prove these equations.

i. Show that Object 1 will land at a distance of $\dfrac{100\sqrt{3}}{4.9}$ m from the point of projection. **2**

ii. The two objects hit the horizontal plane at the same place and time. **3**

Find the initial speed and the angle of projection of Object 2, giving your answer correct to 1 decimal place.

Question 14

(b) A parachutist jumps from a plane, falls freely for a short time and then opens the parachute. Let t be the time in seconds after the parachute opens, $x(t)$ be the distance in metres travelled after the parachute opens, and $v(t)$ be the velocity of the parachutist in ms^{-1}.

The acceleration of the parachutist after the parachute opens is given by

$$\ddot{x} = g - kv$$

where g ms^{-2} is the acceleration due to gravity and k is a positive constant.

i. With an open parachute the parachutist has a terminal velocity of w ms^{-1}. **1**

Show that $w = \dfrac{g}{k}$.

At the time the parachute opens, the speed of descent is $1.6w$ ms^{-1}.

ii. Show that it takes $\dfrac{1}{k}\log_e 6$ seconds to slow down to a speed of $1.1w$ ms^{-1}. **4**

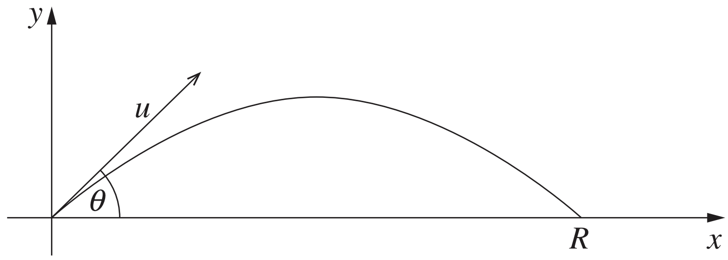
iii. Let D be the distance the parachutist travels between opening the parachute and reaching the speed $1.6w$ ms^{-1}. **3**

Show that $D = \dfrac{g}{k^2}\left(\dfrac{1}{2} + \log_e 6\right)$.

B.18 2020 Ext 2 HSC

Question 12

(b) A particle is projected from the origin with initial velocity of u m/s at an angle θ to the horizontal. The particle lands at $x = R$ on the x axis. The acceleration vector is given by $\underset{\sim}{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$, where g is the acceleration due to gravity. (Do NOT prove this.)



i. Show that the position vector $\underset{\sim}{r}(t)$ of the particle is given by **3**

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ ut\sin\theta - \frac{1}{2}gt^2 \end{pmatrix}$$

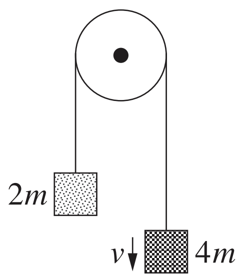
ii. Show that the Cartesian equation of the path of flight is given by **3**

$$y = \frac{-gx^2}{2u^2}\left(\tan^2\theta - \frac{2u^2}{gx}\tan\theta + 1\right)$$

iii. Given that $u^2 > gR$, prove that there are 2 distinct values of θ for which the particle with land at $x = R$. **2**

Question 16

(a) Two masses, $2m$ kg and $4m$ kg, are attached by a light string. The string is placed over a smooth pulley as shown. The two masses are at rest before being released and v is the velocity of the larger mass at time t seconds after they are released.



The force due to air resistance on each mass has magnitude kv, where k is a positive constant.

i. Show that $\dfrac{dv}{dt} = \dfrac{gm - kv}{3m}$. **2**

ii. Given that $v < \dfrac{gm}{k}$, show that when $t = \dfrac{3m}{k}\ln 2$, the velocity of the larger mass is $\dfrac{gm}{2k}$. **3**

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

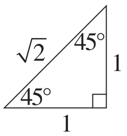
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

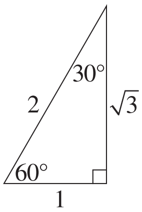
$A = \dfrac{1}{2}ab\sin C$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

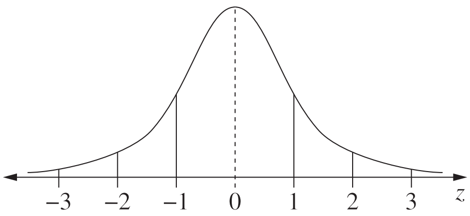
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a)+f(b)+2\left[f(x_1)+\cdots+f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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