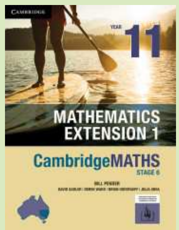
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1
YEAR 12 COURSE

Topic summary and exercises:

With references to



X1 Combinatorics

Name: ....

Initial version by H. Lam, July 2012 (Binomial Theorem)/August 2014 (Combinatorics).
Updated by A. Sun with various HSC questions. Last updated February 12, 2024.
Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

Beware! Heed warning.

Provided on NESA Reference Sheet

Facts/formulae to memorise.

Mathematics Extension 1 content.

Literacy: note new word/phrase.

Further reading/exercises to enrich your understanding and application of this topic.

Syllabus specified content

Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME11-5 uses concepts of permutations and combinations to solve problems involving counting or ordering

Syllabus subtopics

ME-A1 Working with Combinatorics

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 11 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

I Pascal's Triangle & the Binomial Theorem **5**

1 Expansion of $(a+b)^n$ **6**
- 1.1 Simple expansions of $(1+x)^n$ . . . 6
  - 1.1.1 Typical terms & coefficients . . . 9
- 1.2 Generalised expansions . . . 11
  - 1.2.1 Typical terms & coefficients . . . 12

2 Factorial notation **15**
- 2.1 Unrolling factorials . . . 15
- 2.2 Properties of binomial coefficients . . . 16
- 2.3 Greatest term/coefficient . . . 19

3 Proof of general results **23**
- 3.1 Substitution . . . 23
- 3.2 Differentiation . . . 27
- 3.3 Equating coefficients . . . 34

II Combinatorics **37**

4 Permutation **38**
- 4.1 Unrestricted cases – multiplication principle . . . 39
  - 4.1.1 With repetition . . . 39
  - 4.1.2 Without repetition . . . 40
- 4.2 Other restrictions . . . 43
- 4.3 Identical elements . . . 47
- 4.4 Circular arrangements . . . 49

5 Combinations **53**
- 5.1 Definitions . . . 53
- 5.2 Other conditions . . . 55
- 5.3 ⚠ Selection then arrangement . . . 63

6 Probability with Combinatorics **66**
- 6.1 Probability with permutations . . . 66
  - 6.1.1 Other conditions . . . 66
  - 6.1.2 Identical elements . . . 68
  - 6.1.3 Circular arrangements . . . 68

6.2 Probability with combinations 70

7 The Pigeonhole Principle **76**
7.1 Problem definition 76
7.2 Examples 77
7.2.1 Division with remainder 77
7.2.2 Additional questions 82

8 Past HSC Questions **83**
8.1 2012 Extension 1 HSC 83
8.2 2019 Extension 1 HSC 83
8.3 2017 Extension 1 HSC 83
8.4 2018 Extension 1 HSC 84
8.5 2017 Extension 1 HSC 84
8.6 1991 Extension 1 HSC 84
8.7 1993 Extension 1 HSC 85
8.8 1992 Extension 1 HSC 85
8.9 2000 Extension 1 HSC 85

III Legacy syllabus content **86**

A The Binomial Theorem **88**
A.1 Integration 88
A.2 Equate coefficients 91
A.3 Induction 95

B Combinatorics **97**
B.1 X2 Harder 3 Unit problems involving probability/combinatorics 97

References **107**

Part I

Pascal's Triangle & the Binomial Theorem

Section 1

Expansion of $(a+b)^n$

Learning Goal(s)

Knowledge
What is the Binomial Theorem

Skills
Use Pascal's Triangle and the Binomial Theorem

Understanding
The difference between terms, coefficients and flexibility on which term to increase/decrease the indices

By the end of this section am I able to:

17.1 Expand $(x+y)^n$ for small positive integers n.

17.2 Use Pascal's triangle to perform simple binomial expansions.

17.3 Determine values of unknown coefficients given information about an expansion.

17.4 Develop reasoning that the binomial coefficients are given by $\binom{n}{r}$ $({}^nC_r\ )$.

1.1 Simple expansions of $(1+x)^n$

- Expand:

* $(x+1)^2$	* $(x+1)^3$	* $(x+1)^4$
Coefficients:	Coefficients:	Coefficients: ....

- Pascal's Triangle:

- The k-th .... in the expansion of $(x+1)^n$ corresponds to values in the n-th row of Pascal's triangle.

Example 1

Expand $(1+x)^6$.

Example 2

Expand $\left(1+\frac{x}{3}\right)^4$.

Example 3

Expand $\left(1-\frac{1}{\sqrt{x}}\right)^5$.

Theorem 1

The Binomial Theorem (simplified):

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$=$$

where $k =$ ....

- The *term*(s) of a binomial expansion contains .... of x.
- The *coefficient*(s) of a binomial expansion does not.

1.1.1 Typical terms & coefficients

- Typical (generalised) term t_k
  - Write expansion in ....
  - Discard sigma to obtain typical term.

Example 4

Find the coefficient of x^3 in $(x-2)^5$. **Answer:** 40

Example 5

If $\left(1-\sqrt{2}\right)^4 \equiv a + b\sqrt{2}$ where a, $b \in \mathbb{Z}$, find the value of a and b.

Answer: $a = 17$, $b = -12$

Example 6

Calculate $(1.003)^3$ correct to five decimal places by expanding $(1+0.003)^3$.

Answer: 1.00903

Example 7

Find the coefficient of x^5 in $(x-1)(x+1)^7$.

Answer: 14

1.2 Generalised expansions

Theorem 2

Generalised Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$=$$

where $k =$ ....

- In each successive term, the powers of
  - x .... by 1 (from n)
  - y .... by 1 (from 0)

Example 8

Expand $(a-b)^5$.

Example 9

Expand $(x^2-y)^4$.

Example 10

Expand $\left(\frac{2}{x}+x\right)^4$.

1.2.1 Typical terms & coefficients

- Always write down the typical term in the form

$$t_{k+1} = \binom{n}{k} a^{n-k} b^k$$

where $k =$ ....

- Beware of counting from 0:
  - *First* term is t_1, but k commences from 0.
  - The 7th term (for example) has ordinal $k = 6$.
  - There are .... terms in total.

Example 11

Find the coefficient of x^3 in the expansion of $(3 + 2x)^4$. **Answer:** $2^5 \times 3$

Example 12

Find x^2 term in the expansion of $(2 - x)^{12}$. **Answer:** $\binom{12}{2} 2^{10} x^2$.

Example 13

Find the term independent of x in $\left(x^2 + \frac{1}{x}\right)^6$.

Answer: $\binom{6}{4}$

Example 14

Find constant term in $\left(\dfrac{2}{x^2} + x^5\right)^7$.

Answer: $\binom{7}{2}2^5$

Example 15

Without fully expanding, find the term containing x^3 in $(3-x)^4(1+x)$.

Answer: $42x^3$

Example 16

Find the coefficient of x^2 in the expansion of $\left(x+\dfrac{1}{x}\right)^7\left(x-\dfrac{1}{x}\right)^3$.

Answer: $\binom{7}{4}\binom{3}{0}-\binom{7}{3}\binom{3}{1}+\binom{7}{2}\binom{3}{2}-\binom{7}{1}\binom{3}{3}$

Further exercises

Ex 15A

- Q1-16
- Q17-19

Ex 15B

- Q1-13
- Q14-15

Ex 15C

- Q8-9

Section 2

Factorial notation

2.1 Unrolling factorials

- Useful fact for factorials:

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1 \\ &= n(n-1)! \\ &= n(n-1)(n-2)! \text{ etc etc} \end{aligned}$$

Example 17

Simplify $\dfrac{10!}{7!}$ (without a calculator). **Answer:** 720

Example 18

Simplify $\dfrac{(n+2)!}{(n-1)!}$ fully. **Answer:** $n(n+1)(n+2)$

Further exercises

Ex 14A

- Q1-12

2.2 Properties of binomial coefficients

- Definition:

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

- Property of term below:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

for $1 \leq r \leq n-1$.

Proof

- Consider $(1+x)^n$.

  $(1+x)^n = (1+x)(1+x)^{n-1}$

  $=$

- $\binom{n}{r}$ is the coefficient of the term containing x^r the expansion of $(1+x)^n$. Hence examine the term containing x^r on both sides:

- Two polynomials are equal iff all of their coefficients are equal, i.e.

Example 19

Find the value of $\binom{16}{5}$, leaving the answer factored as primes. **Answer:** $2^4 \times 3 \times 7 \times 13$

Example 20

Solve for n: $\binom{n}{2} + \binom{6}{2} = \binom{7}{2}$.

Answer: $n = 4$

Example 21

Find the values of m such that $\binom{35}{22} = \binom{34}{m} + \binom{34}{m+1}$.

Answer: 12 or 21

Example 22

In the expansion of $(1+x)^{16}$, find the ratio of the coefficients of the term in x^{13} to the term in x^{11}.

Answer: $\frac{5}{39}$

Example 23

In the expansion of $(1+x)^n$, the ratios of three consecutive coefficients are $6 : 14 : 21$. Find these coefficients.

Answer: $\binom{9}{2}$, $\binom{9}{3}$, $\binom{9}{4}$

2.3 Greatest term/coefficient

- Binomial coefficients (on the same row of Pascal's Triangle) are ....
.
- Use this property to find greatest term/coefficient: when terms are increasing, $t_{k+1} > t_k$, i.e.
- Greatest term is thus ....
- Derive all results from scratch.
- Beware of solving inequalities with unknown in the denominator.

Example 24

Find the greatest coefficient in the expansion $(1 + 2x)^6$. **Answer:** $\binom{6}{4}2^4$

Example 25

Find greatest coefficient (in terms of magnitude) in the expansion in $\left(2x^2 - \dfrac{3}{x}\right)^{11}$.

Answer: $\binom{11}{7} \times 2^4 \times 3^7$

Example 26

Find the greatest term (in terms of magnitude) in $(5-4x)^{12}$ if $x=\frac{2}{3}$. Leave your answer in its prime factorisation. **Answer:** $2^{12}\times 5^{9}\times 11\times 3^{-2}$

Example 27

[1988 3U HSC Q6] Suppose $(7+3x)^{25} = \sum_{k=0}^{25} t_k x^k$.

i. Use the Binomial Theorem to write an expression for t_k, $0 \le k \le 25$. **2**

ii. Show that $\dfrac{t_{k+1}}{t_k} = \dfrac{3(25-k)}{7(k+1)}$ **2**

iii. Hence or otherwise, find the largest coefficient t_k. **2**

You may leave your answer in the form $\dbinom{25}{k} 7^c 3^d$

Further exercises

Ex 15C

- Q1-7
- Q10-17
- Q18-19

Section 3

Proof of general results

Learning Goal(s)

Knowledge
What is the Binomial Theorem

Skills
Use Pascal's Triangle and the Binomial Theorem

Understanding
The difference between terms, coefficients and flexibility on which term to increase/decrease the indices

By the end of this section am I able to:

17.5 Derive and use simple identities associated with Pascal's triangle.

17.6 Explore the use of substitution and differentiation to develop identities using the Binomial Theorem.

17.7 Develop expressions of the general term in order to solve harder binomial problems.

3.1 Substitution

If the result to be proven involves

- a summation of $\binom{n}{r}$ with same signs, attempt a substitution into $(1+x)^n$.
- a summation of $\binom{n}{r}$ with .... signs, attempt a substitution into $(1-x)^n$.
- powers of a number (e.g. a^m), substitute $x = a$ into $(1+x)^n$ or $(1-x)^n$.

Example 28

Find the sum of the coefficients in the expansion of $(1+2x)^6$. **Answer:** 729

Example 29

Prove that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n$

Example 30

Prove that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} = 0.$

Example 31

Prove that $3^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 2^k.$

Example 32

Prove that $\displaystyle\sum_{k=0}^{n} 3^k \binom{n}{k} = 2^{2n}$.

Example 33

[1998 3U HSC Q7]

(i) Use the binomial theorem to obtain an expression for $(1+x)^{2n}+(1-x)^{2n}$, where n is a positive integer.

(ii) Hence evaluate $1+\binom{20}{2}+\binom{20}{4}+\cdots+\binom{20}{20}$.

3.2 Differentiation

- If the result to be proven involves binomial coefficients $\binom{n}{k}$ multiplied by their k value, differentiate both sides of the expansion of $(1+x)^n$.
- If the non-binomial coefficient does not correlate entirely with binomial coefficient ordinal, there may have been a multiplication by x.

Example 34

Prove that $\displaystyle\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$

Example 35

[2006 HSC Q2]

(i) By applying the binomial theorem to $(1+x)^n$ and differentiating, show that

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + r\binom{n}{r}x^{r-1} + \cdots + n\binom{n}{n}x^{n-1}$$

(ii) Hence deduce that

$$n3^{n-1} = \binom{n}{1} + \cdots + r\binom{n}{r}2^{r-1} + \cdots + n\binom{n}{n}2^{n-1}$$

Example 36

[2010 CSSA Q6]

(i) Differentiate both sides of the expansion $(1+x)^{2n} = \displaystyle\sum_{k=0}^{2n} \binom{2n}{k} x^k$.

(ii) Hence show that $\displaystyle\sum_{k=1}^{2n} k\binom{2n}{k} = n \times 4^n$.

Example 37

[2011 Ext 1 HSC Q7] ⚠ The binomial theorem states that

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

(i) Show that $\displaystyle\sum_{r=1}^{n} \binom{n}{r} r x^r = nx(1+x)^{n-1}$ **2**

(ii) By differentiating the result from part (i), or otherwise, show that **2**

$$\sum_{r=1}^{n} \binom{n}{r} r^2 = n(n+1)2^{n-2}$$

(iii) Assume now that n is even. Show that for $n \geq 4$, **3**

$$\binom{n}{2}2^2 + \binom{n}{4}4^2 + \binom{n}{6}6^2 + \cdots + \binom{n}{n}n^2 = n(n+1)2^{n-3}$$

Example 38

[2002 Ext 1 HSC Q7] ⚠ The coefficient of x^k in the expansion of $(1+x)^n$, $n \in \mathbb{N}$ is denoted by c_k, i.e. $c_k = \binom{n}{k}$.

(i) Show that $c_0 + 2c_1 + 3c_2 + \cdots + (n+1)c_n = (n+2)2^{n-1}$. **3**

(ii) See Example 106 on page 90 as it may fall out of scope with the current syllabus.

Example 39

[2016 Ext 1 HSC Q14] Consider the expansion of $(1+x)^n$, where n is a positive integer.

(i) Show that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$. **1**

(ii) Show that $n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$. **1**

(iii) Hence or otherwise, show that $\displaystyle\sum_{r=1}^{n} \binom{n}{r}(2r-n) = n$. **2**

3.3 Equating coefficients

- Look for a particular n in the expansion of $(1+x)^n$ or $(1-x)^n$ and expand.
- On some occasions, expansions of both $(1+x)^n$ and $(1-x)^n$ are required to add/subtract to obtain required result.

Example 40

[1996 3U HSC Q7] Using the fact that $(1+x)^4(1+x)^9 = (1+x)^{13}$ show that

$$\binom{4}{0}\binom{9}{4} + \binom{4}{1}\binom{9}{3} + \binom{4}{2}\binom{9}{2} + \binom{4}{3}\binom{9}{1} + \binom{4}{4}\binom{9}{0} = \binom{13}{4}$$

Example 41

[1999 3U HSC Q7] By considering $(1-x)^n\left(1+\dfrac{1}{x}\right)^n$ or otherwise, express

$$\binom{n}{2}\binom{n}{0} - \binom{n}{3}\binom{n}{1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{n-2}$$

in simplest form.

Example 42

[1986 3U HSC Q4] Factorise a^2+3a+2 and hence or otherwise find the coefficient of a^4 in

$$\left(a^2+3a+2\right)^6$$

Answer: $68\binom{6}{0}\binom{6}{4}+40\binom{6}{1}\binom{6}{3}+16\left[\binom{6}{2}\right]^2$

Example 43

Using the fact that $\left(x+\dfrac{1}{x}\right)^n\left(x+\dfrac{1}{x}\right)^n=\left(x+\dfrac{1}{x}\right)^{2n}$, prove that

$$\left[\binom{n}{0}\right]^2+\left[\binom{n}{1}\right]^2+\left[\binom{n}{2}\right]^2+\left[\binom{n}{3}\right]^2+\cdots+\left[\binom{n}{n}\right]^2=\binom{2n}{n}$$

Example 44

[2008 Ext 1 HSC Q7] Let p and q be positive integers with $p \le q$.

(i) Use the binomial theorem to expand $(1+x)^{p+q}$, and hence write down the term of $\dfrac{(1+x)^{p+q}}{x^q}$ which is independent of x. **2**

(ii) Given that $\dfrac{(1+x)^{p+q}}{x^q} = (1+x)^p\left(1+\dfrac{1}{x}\right)^q$, apply the binomial theorem and the result of part (i) to find a simpler expression for **3**

$$1 + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \binom{p}{3}\binom{q}{3} + \cdots + \binom{p}{p}\binom{q}{p}$$

Further exercises

Ex 15D

- Q1-5
- Q7 onwards

Part II

Combinatorics

Section 4

Permutation

Learning Goal(s)

Knowledge
Counting principles

Skills
Use counting principles to calculate permutations and combinations

Understanding
The difference between permutations and combinations

By the end of this section am I able to:

17.8 List and count the number of ways an event can occur.

17.9 Use factorial notation to describe and determine the number of ways n different items can be arranged in a line or a circle, including problems involving cases where some items are not distinct.

17.10 Understand and use permutations to solve problems

17.11 Solve problems involving permutations and restrictions with or without repeated objects

17.12 Solve problems involving arrangements in a circle (with no repetition).

17.13 Understand and use combinations to solve problems

17.14 Solve practical problems involving permutations and combinations, including those involving simple probability situations.

Definition 1

A *permutation* is an *arrangement* of objects into a certain order.

Keywords: **arrangement**, **order**

4.1 Unrestricted cases – multiplication principle

- Draw dashes to show positions available.
- Write numbers on dashes to indicate number of possibilities.

Example 45

(Jones & Couchman, 1983, Ex 30.1) A restaurant offers these choices:

Entree	**Main**	**Dessert**
Garlic prawns	Fillet steak	Strawberries
Soup of the day	Chicken	Apple pie and cream
Oysters	Fish	

How many different 3 course dinners can be chosen?

4.1.1 With repetition

Example 46

How many five-letter words can be formed in which the second and fourth letters are vowels and the other three letters are consonants, if letters can be repeated?

Answer: $5^2 21^3$

Steps

1. Draw dashes

2. Fill in possibilities from left

$$\_\_\ \_\_\ \_\_\ \_\_\ \_\_$$

3. Multiply

4.1.2 Without repetition

Example 47

How many ways are there to line up five students?

Steps

1. Draw dashes
2. Fill in possibilities from left

$$\_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_$$

3. Multiply

Example 48

In how many ways can a class of 16 select a committee consisting of a president, a vice-president, a treasurer and a secretary?

Example 49

In how many ways can four letters from the word `BRIDGE` be arranged if no letter is repeated?

Example 50

In a class of thirty children one prize is awarded for English, another for Science and a third for Mathematics. In how many ways can the recipients be chosen if

(a) no child can receive more than one prize?

(b) a child can receive more than one prize?

Permutation

Definition 2

The number of arrangements when permuting r objects from n objects is ${}^nP_r$. (Mnemonic: "n pick r")

- If there are 10 people in total and 4 need to be chosen for President, Vice President, Treasurer and Secretary,

  ___ ___ ___ ___

- "Complete the factorial":

- Generalise:

  ....

- A special case: 0!. Calculate from ${}^nP_n$:

Example 51

Four flags are used to send messages from a vertical flagpole. Find the total number of possible messages that can be sent:

(a) If all 4 flags must be used.

(b) If at least 1 flag must be used.

Example 52

[2005 Ext 1 HSC] Sophie has five coloured blocks: one red, one blue, one green, one yellow and one white. She stacks two, three, four or five blocks on top of one another to form a vertical tower.

(a) How many different towers are there that she could form that are three blocks high? **1**

(b) How many different towers can she form in total? **2**

Further exercises

Ex 14B

- Q9-24

4.2 Other restrictions

Important note

- Deal with restrictions first, then arrange the remaining elements.

Be aware of grouped restrictions which "travel".

Example 53

In how many ways can 5 boys and 4 girls be arranged in a line if

(a) boys and girls alternate?

(b) three boys must be at the start of the line?

(c) line begins and ends with a boy?

(d) girls are together?

(e) boys and girls are in separate groups?

(f) two girls G_1 and G_2 are together?

(g) a particular boy, B insists on being between two girls?

(a) ___ ___ ___ ___ ___ ___ ___ ___ ___

(b) ___ ___ ___ ___ ___ ___ ___ ___ ___

(c) ___ ___ ___ ___ ___ ___ ___ ___ ___

(d) ___ ___ ___ ___ ___ ___ ___ ___ ___

(e) ___ ___ ___ ___ ___ ___ ___ ___ ___

(f) ___ ___ ___ ___ ___ ___ ___ ___ ___

(g) ___ ___ ___ ___ ___ ___ ___ ___ ___

Example 54

How many ways are there of arranging the letters of the word `MEALS` if

(a) if the consonants are to be grouped together?

(b) if the vowels will be at the front and back?

Example 55

[Ex 14C Q8] Find how many arrangements of the letters of the word `UNIFORM` are possible:

(a) if the vowels must occupy the first, middle and last positions,

(b) if the word must start with `U` and end with `M`,

(c) if all the consonants must be in a group at the end of the word,

(d) if the `M` is somewhere to the right of the `U`.

Example 56

[Ex 14C Q16(a)] In how many ways can ten people be arranged in a line:

(i) without restriction?

(ii) if one particular person must sit at either end,

(iii) if two particular people must sit next to one another,

(iv) if neither of two particular people can sit on either end of the row?

Example 57

[2007 Ext 1 HSC] (2 marks) Mr and Mrs Roberts and their four children go to the theatre. They are randomly allocated six adjacent seats in a single row.

What is the probability that the four children are allocated seats next to each other?

Example 58

[2008 Ext 1 HSC] Barbara and John and six other people go through a doorway one at a time.

(i) In how many ways can the eight people go through the doorway if John goes through the doorway after Barbara with no-one in between? **1**

(ii) Find the number of ways in which the eight people can go through the doorway if John goes through the doorway after Barbara. **1**

Further exercises

Ex 14C

- Q1-21

4.3 Identical elements

Steps

1. Find permutation for the same number of elements.

2. .... by number of ways to arrange the number of identical objects. For example, ... by .... if there are 3 objects that are identical.

Example 59

[2000 3U] (2 marks) How many arrangements of the letters of the word `HOCKEYROO` are possible?

Example 60

[2012 Hurlstone Agricultural Ext 1] (2 marks) How many distinct eight letter arrangements can be made using the letters of the word `PARALLEL`?

Example 61

[2001 Ext 1 HSC] The letters `A`, `E`, `I`, `O`, and `U` are vowels.

(a) How many arrangements of the letters in the word `ALGEBRAIC` are possible? **1**

(b) How many arrangements of the letters in the word `ALGEBRAIC` are possible if the vowels must occupy the 2nd, 3rd, 5th and 8th positions? **2**

Example 62

[Ex 14D Q13] ⚠ How many five-letter words can be formed by using the letters of the word `BANANA`?

⚠ Take different cases of repeated letters.

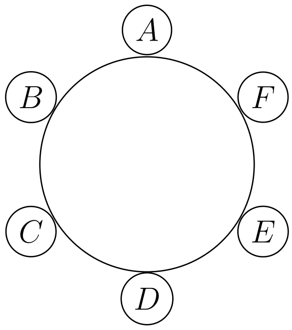
Further exercises

Ex 14D

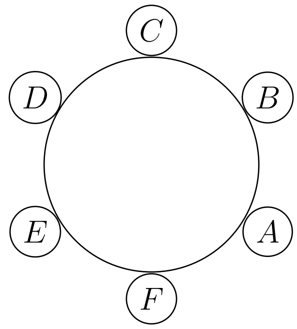
- Q4-15

4.4 Circular arrangements

- The number of way of arranging 6 people in a line: ....
- Arrange 6 people into a circle:



is the same arrangement as



(“wrap around” allowed due to circular arrangement).

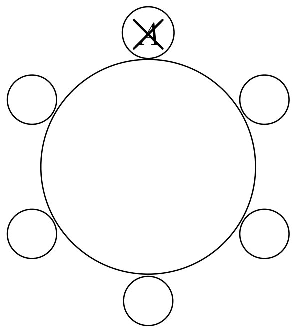
- Solution: write out all linear permutations:
  - ___ ___ ___ ___ ___ ___
  - ___ ___ ___ ___ ___ ___
  - ___ ___ ___ ___ ___ ___
  - ___ ___ ___ ___ ___ ___
  - ___ ___ ___ ___ ___ ___
  - ___ ___ ___ ___ ___ ___

  - Tedious!
  - Only one of the six arrangements required, i.e.

    .... $= 5!$
  - For n different objects,

    .... $= (n-1)!$

- Alternative solution (used more commonly): **fix** the location of one person first, then permute.



- Same rules for dealing restrictions (and glued people arising from restrictions): deal with restrictions first, then fix the “group”.

Important note

Draw picture!

Example 63

[**Ex 14G Q4**] Four boys and four girls are arranged in a circle. In how many ways can this be done:

(a) if there are no restrictions.

(b) if the boys and the girls alternate,

(c) if the boys and girls are in distinct groups,

(d) if a particular boy and girl wish to sit next to one another,

(e) if two particular boys do not wish to sit next to one another,

(f) if one particular boy wants to sit between two particular girls?

Example 64

[2013 NEAP Ext 1/2002 Ext 1 HSC] Seven people are sitting around a table.

(i) How many seating arrangements are possible? **1**

(ii) Two people, Kevin and Julia[a], do not sit next to each other. **2**

How many seating arrangements are now possible?

[a] "Jill" in 2002!

Answer: i. 6! ii. 480

Example 65

In how many ways can 10 people be seated across two tables, each seating five people?

Answer: 72 576

Example 66

Four married couples are to be seated around a circular table for dinner.

(a) In how many ways can the people be seated around the table?

(b) If each married couple is to be seated together, in how many ways can this be done?

Answer: (a) 7! (b) 96

Further exercises

Ex 14G

- Q3-11, 13

Section 5

Combinations

5.1 Definitions

Definition 3

A *combination* is an *grouping* of objects where order is irrelevant.

Keywords: **group**, **committee**, **team**

- Similar to a permutation, but divide by the number of ways to order the selection.

Laws/Results

Commence with ${}^nP_r$, i.e. choosing r objects from n objects. The number of *combinations* is given by ${}^nC_r$ and removing ....

$${}^nC_r = \frac{{}^nP_r}{r!} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Example 67

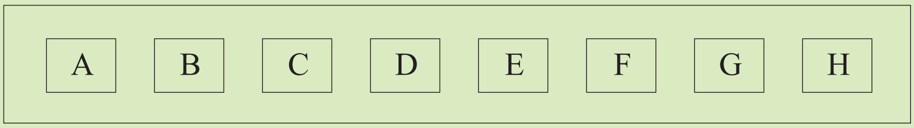
[Ex 14E Q3] Find how many possible combinations there are if, from a group of ten people:

(a) two people are chosen, (b) eight people are chosen.

Why are the answers identical?

Example 68

[1995 3U HSC] A security lock has 8 buttons labelled as shown. Each person using the lock is given a 3-letter code.



(i) How many different codes are possible if letters can be repeated and their order is important? **1**

(ii) How many different codes are possible if letters cannot be repeated and their order is important? **1**

(iii) Now suppose that the lock operates by holding 3 buttons down together, so that order is NOT important. How many different codes are possible? **1**

5.2 Other conditions

Example 69

[Ex 14E Q7] A committee of five is to be chosen from six men and eight women. Find how many committees are possible if:

(a) there are no restrictions,

(b) all members are to be female,

(c) all members are to be male,

(d) there are exactly two men,

(e) there are four women and one man,

(f) there is a majority of women,

(g) a particular man must be included,

(h) a particular man must not be included.

Example 70

Ten people meet to play doubles tennis.

(a) In how many ways can four people be selected from this group to play the first game? (Ignore the subsequent organisation into pairs.)

(b) How many of these ways will include Maria and exclude Alex?

(c) If there are four women and six men, in how many ways can two men and two women be chosen for this game?

(d) Again with four women and six men, in how many ways will women be in the majority?

Answer: (a) ${}^{10}C_4$ (b) ${}^{8}C_3$ (c) ${}^{4}C_2 \times {}^{6}C_2 = 90$ (d) ${}^{6}C_1 \times {}^{4}C_3 + 1 = 25$

Example 71

[2012 CSSA Ext 1 Q5] From six girls and four boys, a committee of 3 girls and 2 boys is to be chosen. How many different committees can be formed?

(A) 26 (B) 120 (C) 252 (D) 1 440

Example 72

[2014 CSSA Ext 1 Q8] A Mathematics department consists of 5 female and 5 male teachers. How many committees of 3 teachers can be chosen which contain at least one female and at least one male?

(A) 100 (B) 120 (C) 200 (D) 2 500

Example 73

[Ex 14E Q8]

(a) What is the number of combinations of the letters of the word `EQUATION` taken four at a time (without repetition)?

(b) How many contain four vowels?

(c) How many contain the letter `Q`?

Example 74

[Ex 14E Q13] Ten points P_1, P_2, ..., P_{10} are chosen in a plane, no three of the points being collinear.

(a) How many lines can be drawn through pairs of the points?

(b) How many triangles can be drawn using the given points as vertices?

(c) How many of these triangles have P_1 as one of their vertices?

(d) How many of these triangles have P_1 and P_2 as vertices?

Example 75

[2001 Ext 2 HSC] X2 A class of 22 students is to be divided into four groups consisting of 4, 5, 6 and 7 students.

(i) In how many ways can this be done? Leave your answer in unsimplified form. **2**

(ii) Suppose that the four groups have been chosen. **2**

In how many ways can the 22 students be arranged around a circular table if the students in each group are to be seated together? Leave your answer in unsimplified form.

Example 76

[2010 Ext 2 HSC] X2 A group of 12 people is to be divided into discussion groups.

(i) In how many ways can the discussion groups be formed if there are 8 people in one group, and 4 people in another? **1**

(ii) In how many ways can the discussion groups be formed if there are 3 groups containing 4 people each? **2**

Example 77

A hand of 5 cards is dealt from a regular pack of 52 cards. What is the number of possible hands if

(a) there are no restrictions

(b) there are 4 aces.

(c) 3 diamonds and 2 hearts

(d) all clubs

(e) all picture cards

(f) cards all the same colour

(g) at least 2 kings

(h) three cards marked '7' and two marked '8'

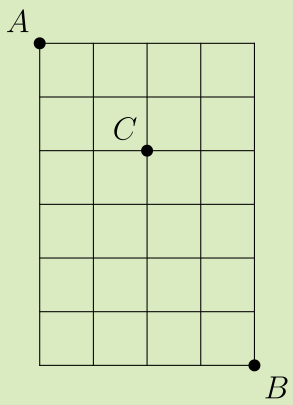
(i) 4 cards marked with the same integer?

Answer: (a) 2 598 960 (b) 48 (c) 22 308 (d) 1 287 (e) 792 (f) 131 560 (g) 108 336 (h) 24 (i) 432

Example 78

[Ex 14E Q25] The diagram shows a 6×4 grid. The aim is to walk from the point A in the top left-hand corner to the point B in the bottom right-hand corner by walking along the black lines either downwards or to the right. A single move is defined as walking along one side of a single small square, thus it takes you ten moves to get from A to B.

Find how many different routes are possible:



(i) without restriction,

(ii) if you must pass through C,

(iii) if you cannot move along the top line of the grid,

(iv) if you cannot move along the second row from the top of the grid.

Answer: (i) 210 (ii) 90 (iii) 126 (iv) 126

Example 79

[2020 Ext 1 HSC Q14]

i. Use the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$ to show that **2**

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

where n is a positive integer.

ii. A club has $2n$ members, with n women and n men. **2**

A group consisting of an even number $(0, 2, 4, \ldots, 2n)$ of members is chosen, with the number of men equal to the number of women.

Show, giving reasons, that the number of ways to do this is $\binom{2n}{n}$

iii. From the group chosen in part (ii), one of the men and one of the women are selected as leaders. **2**

Show, giving reasons, that the number of ways to choose the even number of people and then the leaders is

$$1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \cdots + n^2\binom{n}{n}^2$$

iv. The process is now reversed so that the leaders, one man and one woman, are chosen first. The rest of the group is then selected, still made up of an equal number of women and men. **2**

By considering this reversed process and using part (ii), find a simple expression for the sum in part (iii).

Answer: $n^2\binom{2n-2}{n-1}$

5.3 Selection then arrangement

Example 80

Five letter words are formed from the letters of the word PARABOLA.

(a) How many selections of 5 letters can be made?

(b) How many different five letter words are possible if

i there were no restrictions?

ii the word contained no As?

iii the word contained at least one A?

Example 81

[2020 Ext 1 HSC Q8] Out of 10 contestants, six are to be selected for the final round of a competition. Four of those six will be placed 1st, 2nd, 3rd and 4th.

In how many ways can this process be carried out?

(A) $\dfrac{10!}{6!4!}$ (B) $\dfrac{10!}{6!}$ (C) $\dfrac{10!}{4!2!}$ (D) $\dfrac{10!}{4!4!}$

Example 82

How many four letter words can be formed from the letters of `ABBOTT`?

Answer: 102

Example 83

(Lee, 2006, p.186) How many words are possible using four letters which are selected from the letters of the word SYLLABUS? **Answer:** 606

Steps

1. 0 repeated letters: ___ ___ ___ ___
   - .... unique letters that are unique to choose from, of which four are required.
     - Exclude duplicate letters
   - Permutations: ....
2. 1 repeated letter: ___ ___ ___ ___
   - Number of combinations from Y, A, B, U plus .... other repeatable, but non repeated letters: ....
   - From repeatable letters S, L: .... ways of choosing the repeated letter.
   - Permute: ....
   - Total # of words: ....
3. 2 repeated letters: ___ ___ ___ ___
   - From repeatable letters S, L, permute: ....
   - Total # of words: ....
4. Total number of words: ....

Further exercises

Ex 14E

- Q2-25

Section 6

Probability with Combinatorics

6.1 Probability with permutations

Steps

1. Find .... (................................ case) *first*.
2. Find permutations involving restrictions.

Example 84

[2013 Independent Ext 1 Trial] (2 marks) The letters of the word `NUMBER` are arranged at random in a row. Find the probability that consonants occupy both end positions.

6.1.1 Other conditions

Example 85

Eight people of whom A and B are two, arrange themselves at random in a straight line. What is the probability that

(a) A and B are next to each other,

(b) A and B occupy the end positions,

(c) there are at least 3 people between A and B?

Answer: (a) $\frac{1}{4}$ (b) $\frac{1}{28}$ (c) $\frac{5}{14}$

Example 86

The letters of the word `TUESDAY` are arranged at random in a row. What is the probability that

(a) the vowels and consonants occupy alternate positions

(b) the vowels are together

(c) the vowels are together and the letter `T` occupies the first place?

6.1.2 Identical elements

Example 87

[2013 CSSA Ext 1 Q4] The letters of the word `TWITTER` are arranged randomly. What is the probability that the three `T`s are grouped together?

(A) $\dfrac{1}{42}$ (B) $\dfrac{1}{35}$ (C) $\dfrac{1}{7}$ (D) $\dfrac{6}{7}$

6.1.3 Circular arrangements

Example 88

[2013 Ext 1 HSC Q7] A family of eight is seated randomly around a circular table. What is the probability that the two youngest members of the family sit together?

(A) $\dfrac{6!2!}{7!}$ (B) $\dfrac{6!}{7!2!}$ (C) $\dfrac{6!2!}{8!}$ (D) $\dfrac{6!}{8!2!}$

Example 89

Four Chinese, four Japanese, three Korean and two Vietnamese were invited to a conference.

(a) The delegates are lining up for a photograph. Assuming that they arrive separately, find the probability that

i. they are seated according to their nationality.

ii. the two Vietnamese are separated.

(b) At the conference dinner, the guests were seated at a round table. With the seating organised at random, find the probability that

i. they are seated in according to their nationality.

ii. the Chinese are seated together.

Answer: (a) i. $\frac{2}{75\,075}$ ii. $\frac{11}{13}$ (b) i. $\frac{1}{11\,550}$ ii. $\frac{1}{55}$

6.2 Probability with combinations

Example 90

An jar contains 9 distinguishable cubes of which 3 are white and 6 black. Two cubes are drawn at random without replacement. Calculate the probability that both cubes are black.

Answer: $\frac{5}{12}$

Example 91

Three cards are dealt from a pack of 52. Find the probability that one club and two hearts are dealt,

(a) in that order.

(b) in any order.

Example 92

From a group of 12 people of whom 8 are males and 4 are females, a sample of 4 is selected at random. What is the probability that the sample contains at least 2 females?

Example 93

[2015 Ext 1 HSC Q14] ⚠ Two players A and B play a series of games against each other to get a prize. In any game, either of the players is equally likely to win.

To begin with, the first player who wins a total of 5 games gets the prize.

(i) Explain why the probability of player A getting the prize in exactly 7 games is **1**

$$\binom{6}{4}\left(\frac{1}{2}\right)^7$$

(ii) Write an expression for the probability of player A getting the prize in at most 7 games. **1**

(iii) Suppose now that the prize is given to the first player to win a total of $(n+1)$ games, where n is a positive integer. **2**

By considering the probability that A gets the prize, prove that

$$\binom{n}{n}2^n + \binom{n+1}{n}2^{n-1} + \binom{n+2}{n}2^{n-2} + \cdots + \binom{2n}{n} = 2^{2n}$$

Important note

⚠ What type of question is this?

Further exercises

Ex 14F

- Q1-25

Section 7

The Pigeonhole Principle

Learning Goal(s)

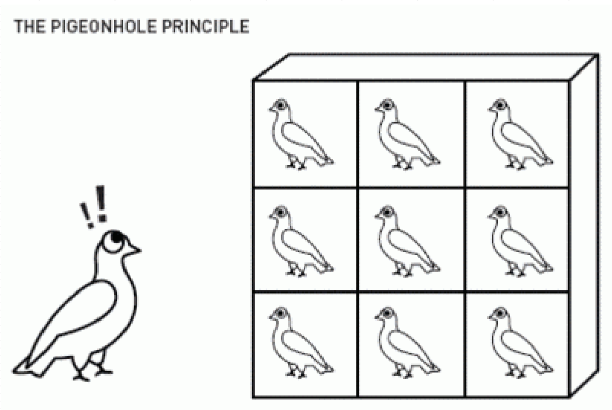
Knowledge
What is the Pigeonhole Principle (PHP)

Skills
Use the PHP to solve problems

Understanding
Clear communication is vital in proofs involving the PHP

By the end of this section am I able to:
17.16 Solve simple problems and prove results using the *pigeonhole principle*

7.1 Problem definition



- There are .... pigeonholes and ..... pigeons.
- There will be at least ... pigeonhole with at least ... pigeons.

Laws/Results

Pigeonhole Principle (PHP) If n items are distributed amongst m pigeonholes and $n > m$, then at least one pigeonhole will contain more than one item.

Important note

- This section of work mostly contains proofs.
- Questions can be asked in reverse.
- Express yourself carefully and concisely with .... !

Watch multimedia

Up and Atom - Pigeonhole Principle and applications

7.2 Examples

Example 94

- If 11 pigeons fly into 10 pigeonholes, there will be at least .... pigeonhole with at least .... pigeons.
- If .... pigeons fly into 10 pigeonholes, there will be at least .... pigeonhole with at least ... pigeons.
- If there are .... pigeonholes, then the least number of pigeons that will guarantee that there is at least one pigeonhole with at least .... pigeons is 71.

Example 95

(Pender et al., 2019)

1. Seventy guests sit at a restaurant with 23 tables. Prove that there must be at least one table with at least four guests.

2. Explain how many guests there must be to guarantee at least one of the 23 tables has at least 11 guests.

7.2.1 Division with remainder

Important note

As the number of pigeonholes increases, it's useful to find the .... from division.

Example 96

(Pender et al., 2019) A suburb has 15 large apartment blocks.

(a) If a shopkeeper knows 200 people from these blocks, explain why he must know at least 14 people from at least one block.

(b) How many people from these blocks must he know in order to know at least 25 people from at least one block?

Laws/Results

Suppose n pigeons fly into d pigeonholes. Let

$$n = dq + r$$

where the remainder is $r = 0, 1, 2, \cdots, (d-1)$.

- If $r = 0$, there must be at least one pigeonhole with ... pigeons.
- If $r > 0$, there must be at least one pigeonhole with .... pigeons.
- The least number of pigeons that will guarantee at least one pigeonhole with at least $q + 1$ pigeons is ....

Example 97

Every person has fewer than 500 000 hairs on their head. Prove that in Sydney, with a population of more than 5 000 000, there are at least eleven people with exactly the same number of hairs on their heads.

Example 98

Vikram has 30 distinct ties. Every day, including weekends, he selects a tie at random to wear. How many successive dates are needed to guarantee that there is at least one day of the week on which he has worn the same tie on at least 6 occasions?

Example 99

Fifteen people come into a room, and there are many handshakes as they meet — no pair shakes hands twice. Prove that there will be at least two people who have made the same number of handshakes.

Example 100

How many odd numbers less than 20 must be chosen to guarantee that two of the chosen numbers add to 20?

Example 101

[2023 CSSA Ext 1 Trial Q1] A bag contains a large number of red, green and blue marbles. Arlo will take out some marbles, without looking, and needs to be certain he will have at least four marbles of the same colour. What is the smallest number of marbles that he must take out to ensure this? **Answer:** (C)

(A) 4 (B) 5 (C) 10 (D) 13

Example 102

[2020 Ext 1 HSC Q12] (2 marks) To complete a course, a student must choose and pass exactly three topics.

There are eight topics from which to choose.

Last year 400 students completed the course.

Explain, using the pigeonhole principle, why at least eight students passed exactly the same three topics.

Example 103

[2021 CSSA Ext 1 Trial Q10] There are 26 cards in a bag, each has a different letter of the alphabet written on them.

A game consists of drawing cards one at a time, without replacement, until two consecutive letters of the alphabet have been drawn. A and Z are not consecutive letters.

For example, if B is drawn first and M is drawn second, if the third card is either A, C, L or N the game would stop there as A and B, or B and C, or L and M, or M and N form a consecutive pair of letters. There would be 23 letters left in the bag.

What is the least number of cards that can be left in the bag at the end of the game?

Answer: (B)

(A) 11 (B) 12 (C) 13 (D) 14

Further exercises

Ex 14H

- Q1-20

7.2.2 Additional questions

Source: Haese, Haese, and Humphries (2015, Ex 1G)

1. Show that in any group of 13 people there will be two or more people who were born in the same month.

2. Seven darts are thrown onto a circular dart board of radius 10 cm. Assuming that all the darts land on the dartboard, so there are two darts which are at most 10 cm apart.

3. 17 points are randomly placed in an equilateral triangle with side length 10 cm. Show that at least two of the points are at most 2.5 cm away from each other.

4. 10 children attended a party and each child received at least one of 50 party prizes. Show that there were at least two children who received the same number of prizes.

5. What is the minimum number of people needed to ensure that at least two of them have the same birthday, not including the year of birth?

6. There are 8 black socks and 14 white socks in a drawer. Calculate the minimum number of socks needed to be selected from the drawer without looking to ensure that

(a) a spare of the same colour is drawn

(b) two different coloured socks are drawn.

7. Prove that for every 27 word sequence in the Australian Constitution, at least two words will start with the same letter.

8. Prove that if 6 distinct numbers from the integers 1 to 10 are chosen, then there will be two of them which sum to 11.

9. Prove that if 11 integers are chosen at random, then at least two of them will have the same units digit.

10. Prove that any cocktail party with two or more people, there must be at least 2 people who have the same number of acquaintances at the party.

Hint: consider the separate cases where

(a) everyone has at least one acquaintance at the party

(b) where someone has no acquaintance at the party.

Section 8

Past HSC Questions

8.1 2012 Extension 1 HSC

Question 11

(f) i. Use the binomial theorem to find an expression for the constant term in the expansion of $\left(2x^3 - \dfrac{1}{x}\right)^{12}$. **2**

ii. For what values of n does $\left(2x^3 - \dfrac{1}{x}\right)^n$ have a non-zero constant term? **1**

8.2 2019 Extension 1 HSC

Question 13

(b) In the expansion of $(5x + 2)^{20}$, the coefficients of x^k and x^{k+1} are equal. **3**

What is the value of k?

8.3 2017 Extension 1 HSC

9. When expanded, which expression has a non-zero constant term? **1**

(A) $\left(x + \dfrac{1}{x^2}\right)^7$

(B) $\left(x^2 + \dfrac{1}{x^3}\right)^7$

(C) $\left(x^3 + \dfrac{1}{x^4}\right)^7$

(D) $\left(x^4 + \dfrac{1}{x^5}\right)^7$

8.4 2018 Extension 1 HSC

Question 14

(b) By considering the expansions of $(1+(1+x))^n$ and $(2+x)^n$, show that **3**

$$\binom{n}{r}\binom{r}{r}+\binom{n}{r+1}\binom{r+1}{r}+\binom{n}{r+2}\binom{r+2}{r}+\cdots+\binom{n}{n}\binom{n}{r}=\binom{n}{r}2^{n-r}.$$

(c) There are 23 people who have applied to be selected for a committee of 4 people. **2**

The selection process starts with Selector A choosing a group of at least 4 people from the 23 people who applied.

Selector B then chooses the 4 people to be on the committee from the group Selector A has chosen.

In how many ways could this selection process be carried out?

8.5 2017 Extension 1 HSC

Question 13

(b) Let n be a positive EVEN integer.

i. Show that $(1+x)^n+(1-x)^n=2\left[\binom{n}{0}+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n\right]$. **2**

ii. Hence show that **1**

$$n\left[(1+x)^{n-1}-(1-x)^{n-1}\right]=2\left[2\binom{n}{2}x+4\binom{n}{4}x^3+\cdots+n\binom{n}{n}x^{n-1}\right].$$

iii. Hence show that $\binom{n}{2}+2\binom{n}{4}+3\binom{n}{6}+\cdots+\frac{n}{2}\binom{n}{n}=n2^{n-3}$. **2**

8.6 1991 Extension 1 HSC

Question 4

(c) Containers are coded by different arrangements of coloured dots in a row. The colours used are red, white, and blue.

In an arrangement, at most three of the dots are red, at most two of the dots are white, and at most one is blue.

i. Find the number of different codes possible if six dots are used.

ii. On some containers only five dots are used. Find the number of different codes possible in this case. Justify your answer.

8.7 1993 Extension 1 HSC

Question 4

(b) Five travellers arrive in a town where there are five hotels.

i. How many different accommodation arrangements are there if there are no restrictions on where the travellers stay?

ii. How many different accommodation arrangements are there if each traveller stays at a different hotel?

iii. Suppose two of the travellers are husband and wife and must go to the same hotel. How many different accommodation arrangements are there if the other three can go to any of the *other* hotels?

8.8 1992 Extension 1 HSC

Question 6

(b) A total of five players is selected at random from four sporting teams. Each of the teams consists of ten players numbered from **1** to **10**.

i. What is the probability that of the five selected players, three are numbered '**6**' and two are numbered '**8**'?

ii. What is the probability that the five selected players contain at least four players from the same team?

8.9 2000 Extension 1 HSC

Question 6

(b) A standard pack of 52 cards consists of 13 cards of each of the four suits: spades, hearts, clubs and diamonds. **4**

i. In how many ways can six cards be selected without replacement so that exactly two are spades and four are clubs? (Assume that the order of selection of the six cards is not important.)

ii. In how many ways can six cards be selected without replacement if at least five must be of the same suit? (Assume that the order of selection of the six cards is not important.)

Part III

Legacy syllabus content

Important note

Previously in the legacy syllabuses, and not explicitly stated. This section would be classified as the Enrichment exercise in Pender et al. (2019).
Content here may either be:

- Vaguely resembling questions which *may* be within scope of the new syllabuses.
- Within the scope of the new syllabuses but deemed too difficult.
- Requiring additional content to be covered before questions can be attempted, such as the *Integration* or *Induction* topics.

Legacy textbook references from Pender, Sadler, Shea, and Ward (2000) remain for reference.

Section A

The Binomial Theorem

A.1 Integration

Important note

Requires a future topic (*Integration*) to be completed.

- If the result to be proven involves binomial coefficients $\binom{n}{k}$ divided by their $k+1$ value, integrate both sides of the expansion of $(1+x)^n$.
- Beware of the invisible, arbitrary .... of integration!
- Find the value of C by substituting $x = 0$.
- Prove the required identity by substituting another value of x.

Example 104

Prove that $\displaystyle\sum_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{2^{n+1}-1}{n+1}$.

Example 105

[1992 3U HSC Q6] Consider the binomial expansion

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n = (1+x)^n$$

(i) Show that $1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

(ii) Show that $1 - \dfrac{1}{2}\binom{n}{1} + \dfrac{1}{3}\binom{n}{2} - \cdots + (-1)^n\dfrac{1}{n+1}\binom{n}{n} = \dfrac{1}{n+1}$.

Example 106

[2002 Ext 1 Q7] ⚠ The coefficient of x^k in the expansion of $(1+x)^n$, $n \in \mathbb{N}$ is denoted by c_k, i.e. $c_k = \binom{n}{k}$.

(i) Show that $c_0 + 2c_1 + 3c_2 + \cdots + (n+1)c_n = (n+2)2^{n-1}$. **3**

(ii) Find the sum **3**

$$\frac{c_0}{1 \times 2} - \frac{c_1}{2 \times 3} + \frac{c_2}{3 \times 4} - \cdots + (-1)^n \frac{c_n}{(n+1)(n+2)}$$

Write your answer as a simple expression in terms of n.

A.2 Equate coefficients

Important note

May be too difficult for the scope of the new courses

Example 107

[2013 Ext 1 HSC Q14]

(i) Write down the coefficient of x^{2n} in the expansion of $(1+x)^{4n}$. **1**

(ii) Show that $\left(1+x^2+2x\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k}(x+2)^{2n-k}$. **2**

(iii) It is known that **3**

$$x^{2n-k}\,(x+2)^{2n-k} = \binom{2n-k}{0} 2^{2n-k} x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1} + \cdots + \binom{2n-k}{2n-k} 2^0 x^{4n-2k}$$

(Do NOT prove this)

Show that

$$\binom{4n}{2n} = \sum_{k=0}^{n} 2^{2n-2k} \binom{2n}{k} \binom{2n-k}{k}$$

Example 108

[2004 Ext 1 HSC Q7(b)] ⚠⚠⚠

(i) Show that for all positive integers n, **1**

$$x\left[(1+x)^{n-1}+(1+x)^{n-2}+\cdots+(1+x)^2+(1+x)+1\right]=(1+x)^n-1$$

(ii) Hence show that for $1\le k\le n$, **1**

$$\binom{n-1}{k-1}+\binom{n-2}{k-1}+\binom{n-3}{k-1}+\cdots+\binom{k-1}{k-1}=\binom{n}{k}$$

(iii) Show that **1**

$$n\binom{n-1}{k}=(k+1)\binom{n}{k+1}$$

(iv) By differentiating both sides of the identity in (i), show that for $1\le k\le n$ **3**

$$(n-1)\binom{n-2}{k-1}+(n-2)\binom{n-3}{k-1}+\cdots+k\binom{k-1}{k-1}=k\binom{n}{k+1}$$

A.3 Induction

Important note

Requires a future topic (*Induction*) to be completed.

Generally of no great difficulty as long as the principle of induction is followed – i.e. treat mathematical expression as series of propositions/statements $P(n)$ that need to be proven true over $\mathbb{N}$.

- Base case $P(1)$
- Inductive step: assume $P(k)$ is true, and then examine $P(k+1)$ by using $P(k)$ to result in the new expression.

Example 109

[2011 Ext 1 CSSA, Q4] Use mathematical induction to prove that

$$\sum_{r=1}^{n}(r^2+1)r! = n(n+1)!$$

Example 110

[2008 Ext 1 CSSA, Q4] Prove by mathematical induction that

$$\sum_{r=1}^{n} r \times r! = (n+1)! - 1$$

Example 111

[**2008 Ext 1 HSC Q5**] Use mathematical induction to prove that, for $n \geq 3$

$$\sum_{j=3}^{n} \binom{j-1}{2} = \binom{n}{3}$$

Further exercises

Ex 5F

- Q1-13 odd #

Section B

Combinatorics

B.1 X2 Harder 3 Unit problems involving probability/combinatorics

Important note

May be too difficult for the scope of the new courses

Example 112

Seven letter words are formed using the letters of `UNUSUAL`. One of these words is selected at random. Find the probability that the word: **Answer:** (a) $\frac{1}{7}$ (b) $\frac{1}{7}$ (c) $\frac{2}{35}$ (d) $\frac{2}{7}$

(a) began and ended with a `U`

(b) had three `U`s together

(c) had three `U`s at beginning or at the end

(d) had none of the `U`s together

Steps

(d) had none of the `U`s together

Question 1 Enumerating possibilities too tedious. Try:

Question 2 Draw positions where `U` can go:

___ N ___ S ___ A ___ L ___

Question 3 At any one time, the number of spaces required to place all three `U`s: .... (This also forms all of the possibilities of placing all three `U`s)

Question 4 The number of ways to permute the unique letters `N`, `S`, `A` and `L`: ....

Question 5 Use multiplication principle: total number of ways is ....

Question 6 Find probability: ....

Example 113

(Pender et al., 2000, Ex 10G Q14) The letters of `ENTERTAINMENT` are arranged in a row. Find the probability that:

Answer: (a) $\frac{1}{26}$ (b) $\frac{5}{13}$ (c) $\frac{15}{26}$ (d) $\frac{1}{26}$

(a) the letters `E` are together

(b) two `E`s are together and one is apart

(c) all the letters `E` are apart

(d) the word starts and ends with `E`

Use combination theory to show possible location of the `E`s.

Example 114

X2 (Lee, 2006, Ex 8.4 Q5) **Answer:** (a) 6 570 (b) i. $\frac{28}{73}$ ii. $\frac{8}{219}$[a] iii. $\frac{50}{219}$ iv. $\frac{20}{219}$ v. $\frac{2}{73}$

(a) How many words are possible using five letters which are selected from the letters of the word `CALCULATOR`?

(b) If a word is selected at random from these words, find the probability that the word contains:

i no repeated letters

ii all four vowels

iii two `C`s

iv two `C`s that remain together

v no vowels

[a] Error in text shows $\frac{2}{73}$

Steps

(a) Consider cases with repeated letters:

Question 1 0 repeated letters: ___ ___ ___ ___ ___

- .... unique letters that are unique to choose from, of which five are required.
  –Exclude duplicate letters
- Permutations: ....

Question 2 1 repeated letter: ___ ___ ___ ___ ___

- Number of combinations from `U`, `T`, `O`, `R` plus .... other repeatable, but non repeated letters: ....
- From repeatable letters `C`, `A`, `L`: .... ways of choosing the repeated letter.
- Permute: ....
- Total # of words: ....

Question 3 2 repeated letters: ___ ___ ___ ___ ___

- Number of combinations from `U`, `T`, `O`, `R` plus .... other repeatable, but non repeated letters: ....
- From repeatable letters `C`, `A`, `L`: .... ways of choosing the repeated letter.
- Permute: ....
- Total # of words: ....

Question 4 Total number of words: ....

Example 115

[1997 4U] In a game, two players take turns at drawing, and immediately replacing, a marble from a bag containing two green and three red marbles. The game is won by player A drawing a green marble, or player B drawing a red marble. Player A draws first.

Find the probability that:

(i) A wins on her first draw; **1**

(ii) B wins on her first draw; **1**

(iii) A wins in less than four of her turns; **1**

(iv) A wins eventually. **2**

Example 116

[2015 Ext 2 Q16]

(i) A table has 3 rows and 5 columns, creating 15 cells as shown. **2**

<table>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Counters are to be placed randomly on the table so that there is one counter in each cell. There are 5 identical black counters and 10 identical white counters.

Show that the probability that there is exactly one black counter in each column is $\dfrac{81}{1001}$.

(ii) The table is extended to have n rows and q columns. There are nq counters, where q are identical black counters and the remainder are identical white counters. The counters are placed randomly on the table with one counter in each cell. **2**

Let P_n be the probability that each column contains exactly one black counter.

Show that $P_n = \dfrac{n^q}{\binom{nq}{q}}$.

(iii) Find $\lim\limits_{n\to\infty} P_n$. **2**

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

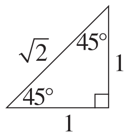
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

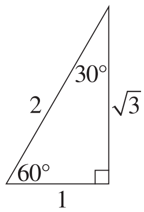
$A = \dfrac{1}{2}ab\sin C$



$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

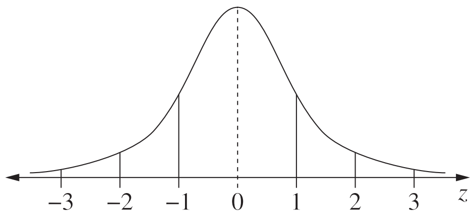
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

References

Haese, M., Haese, S., & Humphries, M. (2015). *Mathematics for Australia 11 Specialist Mathematics* (2nd ed.). Haese Mathematics.

Jones, S. B., & Couchman, K. E. (1983). *3 Unit Mathematics* (Vol. 2). Addison Wesley Longman Australia.

Lee, T. (2006). *Advanced Mathematics: A complete HSC Mathematics Extension 2 Course* (2nd ed.). Terry Lee Enterprise.

Pender, W., Sadler, D., Shea, J., & Ward, D. (2000). *Cambridge Mathematics 3 Unit Year 12* (1st ed.). Cambridge University Press.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 11* (1st ed.). Cambridge Education.