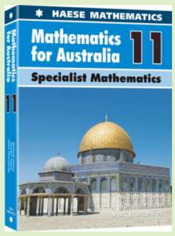
NORMANHURST BOYS HIGH SCHOOL

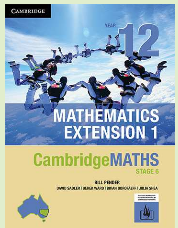
MATHEMATICS EXTENSION 1
YEAR 12 COURSE

Topic summary and exercises:

X1 X2 **Introduction to Vectors**

With references to





Name: ....

Initial version by H. Lam, September 2019 with assistance from I. Ham.
Last updated April 22, 2024
Various corrections by students & members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Parts of this handout are derived from Haese, Haese, and Humphries (2015), *MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes* (2018), So and Wong (1987) and Pender, Sadler, Ward, Dorofaeff, and Shea (2019). Most GeoGebra applets listed here are courtesy of the NSW Department of Education's Mathematics Curriculum Support team/mEsh project's efforts.

Symbols used

Beware! Heed warning.

Provided on NESA Reference Sheet

Facts/formulae to memorise.

Mathematics Extension 1 content.

Literacy: note new word/phrase.

Further reading/exercises to enrich your understanding and application of this topic.

Syllabus specified content

Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME12-2 applies concepts and techniques involving vectors and projectiles to solve problems

Syllabus subtopics

ME-V1 (1.1, 1.2) Introduction to Vectors (Introduction to vectors, Further operations with vectors)

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *Haese Mathematics for Australia - Specialist Mathematics 11* (Haese et al., 2015) *CambridgeMATHS Year 12 Extension 1* (Pender et al., 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Introduction to Vectors

1.1 Definition

Definition 1

Quantities which only have ....magnitude........ are known as *scalars*.

Example: ....speed.... of an aeroplane.

Definition 2

Quantities which have both ....magnitude........ and ....direction........ are known as *vectors*.

Example: ....velocity.... of an aeroplane.

Fill in the spaces

Other examples of vector forces:

- ....acceleration........
- ....force....
- ....displacement........
- ....momentum........

1.2 Representation

1.2.1 Geometric

Fill in the spaces

Vector quantities can be represented using a *directed line segment* or *arrow*.

- The ....length........ of the arrow represents the ....magnitude........
- The ....arrowhead........ shows its direction.

Definition 3

Two vectors are *equal* if they have the same ....magnitude........ and ....direction........ .

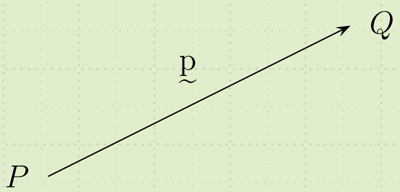
Corollary 1

The position of the starting point (tail) and ending point (arrowhead) does ....not.... matter.

Example 1

Given $\overrightarrow{PQ}$ as shown, draw two other vectors, $\underset{\sim}{a} = \overrightarrow{AB}$ and $\underset{\sim}{e} = \overrightarrow{EF}$ such that

$$\overrightarrow{PQ} = \overrightarrow{AB} = \overrightarrow{EF}$$



1.2.2 Algebraic

From Example 1, there are three ways of writing the vector drawn:

- $\overrightarrow{PQ}$: the vector ....commencing........ from the point P and ....ending........ at point Q.
- $\vec{p}$ or $\underset{\sim}{p}$: NESA syllabus documentation, handwritten.

1.3 Vector arithmetic

1.3.1 Addition

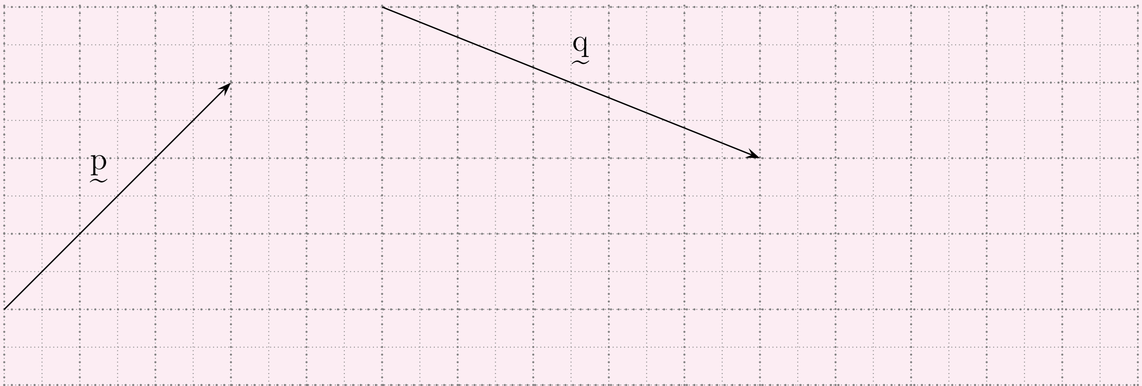
Important note

Use a ruler!

Steps

For vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$, to perform $\underset{\sim}{p} + \underset{\sim}{q}$:

1. Draw $\underset{\sim}{p}$.
2. Draw $\underset{\sim}{q}$, starting from the arrowhead of $\underset{\sim}{p}$.
3. Draw ......resultant...... vector from the ...tail... of $\underset{\sim}{p}$ to the ...head... of $\underset{\sim}{q}$.



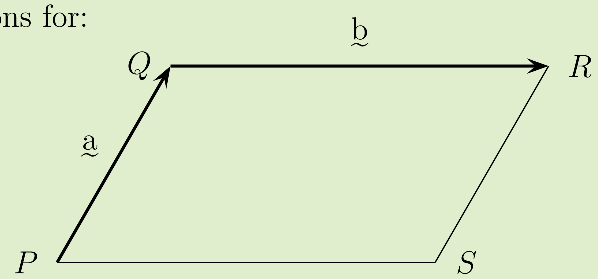
1.3.2 Subtraction/negative vectors

Definition 4

If $\underset{\sim}{p}$ is a vector, then $-\underset{\sim}{p}$ is equal in ......magnitude...... but ......opposite...... in ......direction......

Example 2

(Haese et al., 2015) $PQRS$ is a parallelogram where $\overrightarrow{PQ} = \underset{\sim}{a}$ and $\overrightarrow{QR} = \underset{\sim}{b}$. Find vector expressions for:



(a) $\overrightarrow{QP}$ (b) $\overrightarrow{RQ}$ (c) $\overrightarrow{SR}$ (d) $\overrightarrow{SP}$

1.3.3 Magnitude and zero vector

Definition 5

The *magnitude* of a vector $\underset{\sim}{p}$ is given by $\left|\underset{\sim}{p}\right|$.

Definition 6

The *zero vector* has magnitude zero, i.e. $\left|\underset{\sim}{p}\right| = 0$. Direction is not ......defined......

Usually written as $\underset{\sim}{0}$ (handwritten: $\underset{\sim}{0}$).

1.3.4 Scalar multiplication

Definition 7

If $\underset{\sim}{p}$ is a vector, and $\lambda \in \mathbb{R}$,

- If $\lambda > 1$, then $\lambda\underset{\sim}{p}$ extends the ......magnitude...... to $\lambda\left|\underset{\sim}{p}\right|$.
  (......Extends...... the vector $\underset{\sim}{p}$ by a factor of λ).
- If $0 < \lambda < 1$, then $\lambda\underset{\sim}{p}$ ......shrinks...... the magnitude.
- If $\lambda = 0$, then $\lambda\underset{\sim}{p} = \underset{\sim}{0}$.
- If $\lambda < 0$, then $\lambda\underset{\sim}{p}$ sees the magnitude scaled by $|\lambda|$ and direction ......reverses...... .

Laws/Results

- $\underset{\sim}{p}$ and $\lambda\underset{\sim}{p}$, $\lambda > 0$ are ......parallel...... .
- $\underset{\sim}{p}$ and $\lambda\underset{\sim}{p}$, $\lambda < 0$ are ......antiparallel...... .

GeoGebra

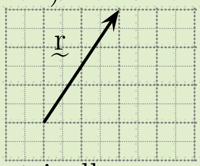
Scaling vector investigation

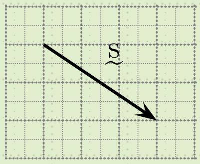
GeoGebra

Vector addition - what shape is *always* formed when adding vectors?

Example 3

(Haese et al., 2015)

Given vectors 

and 

(diagrams drawn to scale), construct geometrically:

(a) $-\underset{\sim}{r}$

(b) $2\underset{\sim}{s}$

(c) $\frac{1}{2}\underset{\sim}{r}$

(d) $-\frac{3}{2}\underset{\sim}{s}$

(e) $2\underset{\sim}{r} - \underset{\sim}{s}$

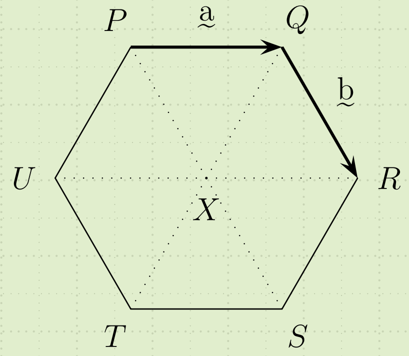
(f) $2\underset{\sim}{r} + 3\underset{\sim}{s}$

(g) $\frac{1}{2}\underset{\sim}{r} + 2\underset{\sim}{s}$

(h) $\frac{1}{2}\left(\underset{\sim}{r} + 3\underset{\sim}{s}\right)$

Example 4

(Haese et al., 2015) $PQRSTU$ is a regular hexagon. If $\overrightarrow{PQ} = \underset{\sim}{a}$ and $\overrightarrow{QR} = \underset{\sim}{b}$, find in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$:



(a) $\overrightarrow{PX}$

(b) $\overrightarrow{PS}$

(c) $\overrightarrow{QX}$

(d) $\overrightarrow{RS}$

1.4 Vector equations

Laws/Results

Vectors can be

- Added
- Have scalar multiplication applied

in the same manner as usual scalar numbers, with the same commutative/associative/distributive laws.

Example 5

(*MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes*, 2018) Fully simplify: $3(2\underset{\sim}{a} - \underset{\sim}{b}) + (\underset{\sim}{a} - 2\underset{\sim}{b})$.

Example 6

(*MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes*, 2018) Fully simplify: $2\overrightarrow{AC} - \overrightarrow{OC} + \overrightarrow{OA}$.

Important note

- Draw picture if it's slightly uncertain what is $\overrightarrow{AC}$.
- Convert the vector arrow notation $\overrightarrow{OC}$ to the single variable notation $\underset{\sim}{c}$.

Example 7

(*MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes*, 2018) In $\triangle OAB$, D and E are points such that $OD : DA = OE : EB = 1 : 2$. Prove that

(a) DE is parallel to AB

(b) the length of DE is $\dfrac{1}{3}$ times that of AB.

Further exercises

Ex 3A-3C (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8A (Pender et al., 2019)

- Q1-19
- ⚠ Q20-21

Section 2

Simple Vector Geometry

2.1 Vector components

2.1.1 Unit vector

Definition 8

The *unit vector* has ....magnitude.......... 1.

Laws/Results

For every vector $\underset{\sim}{p}$ there are two *normalised* (corresponding unit) vectors:

- $\widehat{\underset{\sim}{p}} = \dfrac{\underset{\sim}{p}}{\left|\underset{\sim}{p}\right|}$. ....Parallel.......... to $\underset{\sim}{p}$.
- $-\widehat{\underset{\sim}{p}} = -\dfrac{\underset{\sim}{p}}{\left|\underset{\sim}{p}\right|}$. ....Antiparallel.......... to $\underset{\sim}{p}$.

GeoGebra

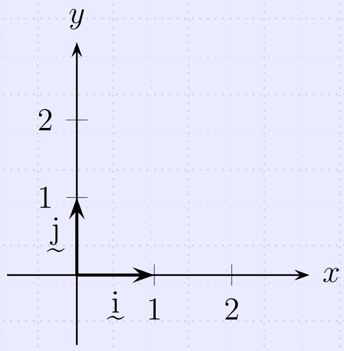
Unit vector

2.1.2 Cartesian basis vectors/components

Definition 9

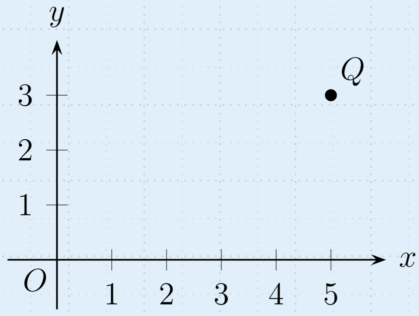
In the Cartesian plane, define the *basis vectors* to be unit vectors in the

- x direction to be $\underset{\sim}{i}$
- y direction to be $\underset{\sim}{j}$



Fill in the spaces

- The position $(5, 3)$ can now be represented via a *translation vector*.



- Component notation:

$$\overrightarrow{OQ} = 5\underset{\sim}{i} + 3\underset{\sim}{j}$$

- Column vector notation:

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Definition 10

The point $Q(x, y)$ has *position vector*

$$\overrightarrow{OQ} = \begin{pmatrix} x \\ y \end{pmatrix} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

where

- The *component form* is $x\underset{\sim}{i} + y\underset{\sim}{j}$
- Represented in *column vector* notation: $\begin{pmatrix} x \\ y \end{pmatrix}$.
- Year 7-10 *ordered pair* notation: (x, y).

Important note

Pender et al. (2019) and *Mathematics Extension 1 Stage 6 Syllabus* (2017, Revised 18/11/2019) both use brackets instead of parentheses for the column vector delimiters, e.g.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Tertiary education providers however, tend to use parentheses.

Example 8

A ship S leaves port O and sails north-east at 20 km/h. Describe its position after 6 hours by giving its position vector as a sum of multiples of the basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

2.1.3 Position vs displacement vectors

Definition 11

A *position vector* of a point P is the vector $\overrightarrow{OP}$, where the vector commences at the origin and finishes at P.

A position vector gives the absolute location of a point in the plane.

- Each point P in the Cartesian plane corresponds to the position vector $\overrightarrow{OP}$

Definition 12

A *displacement vector* is the vector $\overrightarrow{AB}$, where the vector commences at the point A and finishes at B.

A displacement vector gives the relative position of a point from another.

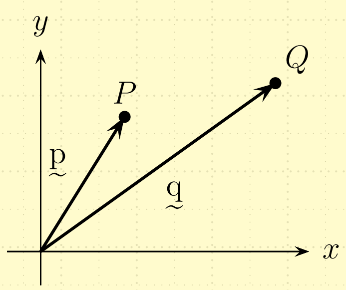
- Vector of point A relative to point B: $\overrightarrow{BA}$

GeoGebra

Position Vector Investigation - see how a position vector is different to a displacement vector.

Laws/Results

For two points P and Q with position vectors $\underset{\sim}{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ respectively,



$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$

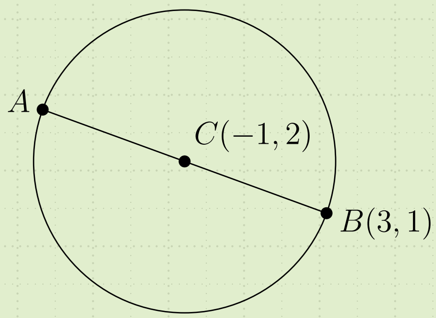
$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \underset{\sim}{q} - \underset{\sim}{p} \\ &= \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix} \end{aligned}$$

$$\overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OP}$$

$$\begin{aligned} \overrightarrow{QP} &= \overrightarrow{OP} - \overrightarrow{OQ} \\ &= \underset{\sim}{p} - \underset{\sim}{q} \\ &= \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix} \end{aligned}$$

Example 9

(Haese et al., 2015) AB is a diameter of a circle with centre $C(-1,2)$. If B is $(3,1)$, find:

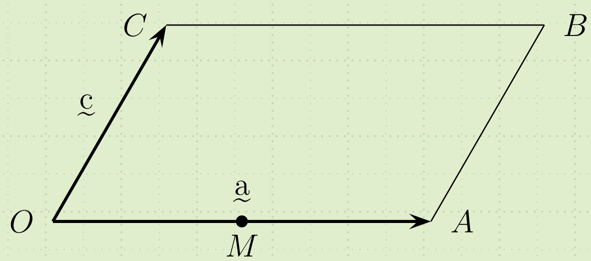


(a) $\overrightarrow{BC}$

(b) the coordinates of A.

Example 10

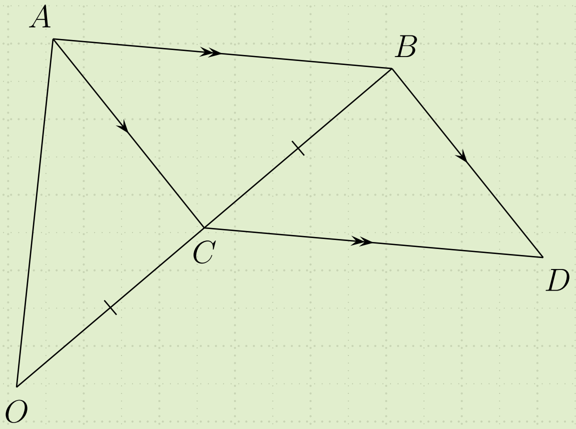
[2020 Independent Ext 1 Trial Q3] In the diagram, $OABC$ is a paralleogram. The vector $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. M is the midpoint of OA. Which of the following expressions is represented by the vector $\overrightarrow{MB}$?



(A) $\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{c}$ (B) $\frac{1}{2}\underset{\sim}{a} + \underset{\sim}{c}$ (C) $\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}$ (D) $\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$

Example 11

[2020 NEAP Ext 1 Trial Q7] The position vectors of points A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Point C is the midpoint of OB and point D is such that $ABCD$ is a parallelogram.



Which of the following is the position vector of D?

(A) $\frac{3}{2}\underset{\sim}{b} + \underset{\sim}{a}$ (B) $\frac{3}{2}\underset{\sim}{b} - \underset{\sim}{a}$ (C) $\frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$ (D) $\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$

2.2 Vector arithmetic involving column vectors

2.2.1 Addition

Laws/Results

If $\underset{\sim}{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} u \\ v \end{pmatrix}$, then

$$\underset{\sim}{p} \pm \underset{\sim}{q} = \begin{pmatrix} x \pm u \\ y \pm v \end{pmatrix}$$

$$= (x \pm u)\underset{\sim}{i} + (y \pm v)\underset{\sim}{j}$$

Important note

- Add horizontal components (x direction, $\underset{\sim}{i}$) with horizontal components
- Add vertical components (y direction, $\underset{\sim}{j}$) with vertical components

GeoGebra

Vector component addition

Example 12

(Pender et al., 2019)

1. (a) Given $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$, find:

i. $2\underset{\sim}{u}$ ii. $3\underset{\sim}{v}$ iii. $2\underset{\sim}{u} + 3\underset{\sim}{v}$

(b) Draw all five vectors as position vectors. What figure do the origin and heads of $2\underset{\sim}{u}$, $3\underset{\sim}{v}$ and $2\underset{\sim}{u} + 3\underset{\sim}{v}$ form?

2. (a) Given $\underset{\sim}{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find $\underset{\sim}{v} - \underset{\sim}{u}$.

(b) Draw $\underset{\sim}{u} = \overrightarrow{OU}$ and $\underset{\sim}{v} = \overrightarrow{OV}$ as position vectors, and mark $\underset{\sim}{v} - \underset{\sim}{u}$.

2.2.2 Scalar multiplication

Laws/Results

(Haese et al., 2015) If $\underset{\sim}{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $k \in \mathbb{R}$ then

$$k\underset{\sim}{p} = \begin{pmatrix} kx \\ ky \end{pmatrix}$$

Example 13

Given $\underset{\sim}{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find:

(a) $3\underset{\sim}{q}$ (b) $\underset{\sim}{p} + 2\underset{\sim}{q}$ (c) $\frac{1}{2}\underset{\sim}{p} - 3\underset{\sim}{q}$

2.2.3 Magnitude

Laws/Results

If $\underset{\sim}{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$\left|\underset{\sim}{p}\right| = \sqrt{x^2 + y^2}$$

Example 14

If $\underset{\sim}{p} = 3\underset{\sim}{i} - 5\underset{\sim}{j}$ and $\underset{\sim}{q} = -\underset{\sim}{i} - 2\underset{\sim}{j}$, find $\left|\underset{\sim}{p} - 2\underset{\sim}{q}\right|$. **Answer:** $\sqrt{26}$

Example 15

(Haese et al., 2015, Ex 3D) Find k such that the vector is a unit vector.

(a) $\begin{pmatrix} 0 \\ k \end{pmatrix}$ (b) $\begin{pmatrix} k \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} k \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} k \\ k \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

Answer: (a) ± 1 (b) ± 1 (c) 0 (d) $\pm\frac{1}{\sqrt{2}}$ (e) $\pm\frac{\sqrt{3}}{2}$

Example 16

(Haese et al., 2015, Ex 3D) Given $\underset{\sim}{v} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $\left|\underset{\sim}{v}\right| = \sqrt{73}$, find the possible values of p.

Answer: ± 3

Example 17

(Haese et al., 2015) If $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$, find:

(a) a unit vector in the direction of $\underset{\sim}{a}$

(b) a vector of length 4 units in the direction of $\underset{\sim}{a}$

(c) vectors of length 4 units which are parallel to $\underset{\sim}{a}$.

Answer: (a) $\frac{3}{\sqrt{10}}\underset{\sim}{i} - \frac{1}{\sqrt{10}}\underset{\sim}{j}$ (b) $\frac{12}{\sqrt{10}}\underset{\sim}{i} - \frac{4}{\sqrt{10}}\underset{\sim}{j}$ (c) $\pm\left(\frac{12}{\sqrt{10}}\underset{\sim}{i} - \frac{4}{\sqrt{10}}\underset{\sim}{j}\right)$

Example 18

[2020 NSBHS Ext 1 Trial] $\overrightarrow{OM} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ and $\overrightarrow{OT} = -3\underset{\sim}{i} + 7\underset{\sim}{j}$. What is the value of $\left|\overrightarrow{MT}\right|$?

(A) 14 (B) $\sqrt{98}$ (C) $\sqrt{29}$ (D) $\sqrt{145}$

2.2.4 Direction

Laws/Results

If $\underset{\sim}{p} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the direction of the vector is measured by the angle formed by the vector with the positive direction of the horizontal axis, i.e.

$$\tan\theta = \frac{y}{x}$$

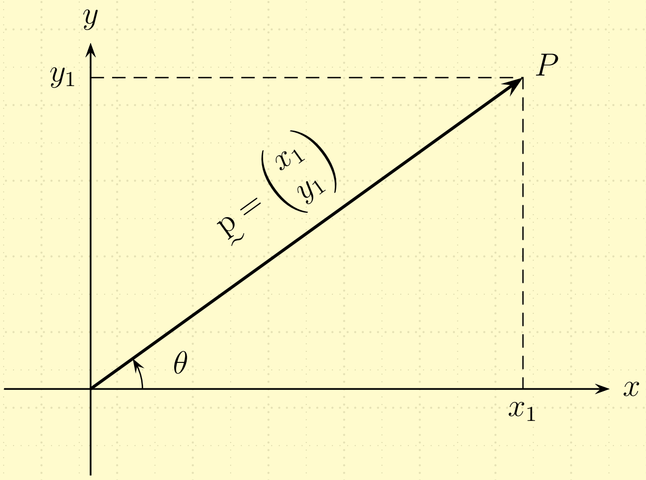
where $\theta \in [0°, 360°)$ (and radians where applicable)

Important note

X2 Later in Complex Numbers, $\theta \in (-\pi, \pi]$. The quadrants used may vary between contexts - read the context carefully.

Laws/Results

Components of a vector in terms of the magnitude and direction: if $\underset{\sim}{p} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$,



$x_1 = |p|\cos\theta$

$y_1 = |p|\sin\theta$

Example 19

(Pender et al., 2019)

(a) Find the vector $\underset{\sim}{u}$ of length 10, with angle 150° to the positive direction of the horizontal axis.

(b) Find the length and angle (exact length, angle to the nearest degree) of $\underset{\sim}{v} = -\underset{\sim}{i} - 2\underset{\sim}{j}$.

Answer: (a) $\underset{\sim}{u} = -5\sqrt{3}\underset{\sim}{i} + 5\underset{\sim}{j}$(b) $|\underset{\sim}{u}| = \sqrt{5}$, $\theta \approx 243°$.

Example 20

[2020 JRAHS Ext 1 Trial Q7] How many of the following statements are true?

- $|\underset{\sim}{a}| + |\underset{\sim}{b}| = |\underset{\sim}{a} + \underset{\sim}{b}|$ means that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same direction.
- $|\underset{\sim}{a}| + |\underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$ means that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the opposite directions.
- $|\underset{\sim}{a}| + |\underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$ means that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same magnitude.
- $|\underset{\sim}{a}| - |\underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$ means that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same direction.

(A) 1 (B) 2 (C) 3 (D) 4

Further exercises

Ex 3D-3G (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8B (Pender et al., 2019)

- Q11-18

Section 3

Vector Geometry with the Dot Product

3.1 Definitions

Definition 13

The *dot product* between two vectors $\underset{\sim}{p} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{q} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ (also known as *scalar product*):

$$\underset{\sim}{p} \cdot \underset{\sim}{q} = \dotfill x_1x_2 + y_1y_2$$

GeoGebra

Dot Product Insight

Example 21

Use $\underset{\sim}{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ etc to prove the following:

(a) $\underset{\sim}{p} \cdot \underset{\sim}{q} = \underset{\sim}{q} \cdot \underset{\sim}{p}$ (Commutativity)

(b) $\underset{\sim}{p} \cdot \underset{\sim}{p} = \left|\underset{\sim}{p}\right|^2$

(c) $\underset{\sim}{p} \cdot \left(\underset{\sim}{q} + \underset{\sim}{r}\right) = \underset{\sim}{p} \cdot \underset{\sim}{q} + \underset{\sim}{p} \cdot \underset{\sim}{r}$ (Associativity)

(d) $\left(\underset{\sim}{p} + \underset{\sim}{q}\right) \cdot \left(\underset{\sim}{r} + \underset{\sim}{s}\right) = \underset{\sim}{p} \cdot \underset{\sim}{r} + \underset{\sim}{p} \cdot \underset{\sim}{s} + \underset{\sim}{q} \cdot \underset{\sim}{r} + \underset{\sim}{q} \cdot \underset{\sim}{s}$

Example 22

(Haese et al., 2015, Ex 3I) Explain why $\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c}$ is meaningless.

3.2 Angle between two vectors

The dot product enables the ....angle.... θ between two non-zero vectors to be found.

Definition 14

The angle θ between two non-zero vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ is related by:

$$\therefore \underset{\sim}{p} \cdot \underset{\sim}{q} = \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos\theta$$

Steps

Proof

1. Draw situation representing the angle θ between two vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$, and vector $\underset{\sim}{q} - \underset{\sim}{p}$.

2. Apply the cosine rule to the triangle:

$$\left|\underset{\sim}{q} - \underset{\sim}{p}\right|^2 = \left|\underset{\sim}{p}\right|^2 + \left|\underset{\sim}{q}\right|^2 - 2\left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos\theta$$

3. Let $\underset{\sim}{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and replace magnitudes with components of $\underset{\sim}{p}$ and $\underset{\sim}{q}$:

$$\therefore \underset{\sim}{p} \cdot \underset{\sim}{q} = \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos\theta$$

4. Consequently,

$$\cos\theta = \frac{\underset{\sim}{p} \cdot \underset{\sim}{q}}{\left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right|}$$

Laws/Results

Geometric results of the dot product

- If $\underset{\sim}{p}$ is *perpendicular* to $\underset{\sim}{q}$, then $\theta = 90°$:

$$\underset{\sim}{p} \cdot \underset{\sim}{q} = \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos\theta$$

$$= \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos 90°$$

$$= 0$$

- If $\underset{\sim}{p}$ is *parallel* to $\underset{\sim}{q}$, then $\theta = 0°$ or $180°$:

$$\underset{\sim}{p} \cdot \underset{\sim}{q} = \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos\theta$$

$$= \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos 0° \quad \text{or} \quad \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right| \cos 180°$$

$$= \pm \left|\underset{\sim}{p}\right| \left|\underset{\sim}{q}\right|$$

GeoGebra

Parallel Vectors investigation

Example 23

[2022 Ext HSC Q11] (2 marks) The vectors $\underset{\sim}{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$ are perpendicular.

What are the possible values of a? **Answer:** 2 or $-\frac{7}{4}$

Example 24

(Haese et al., 2015) Consider the points $A(2, 1)$ and $B(6, -1)$ and $C(5, -3)$. Use the dot product to determine whether $\triangle ABC$ is right angled or not. If so, locate the right angle. **Answer:** At B

Example 25

(Haese et al., 2015) Find the form of *all* vectors which are perpendicular to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Answer: $k\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Example 26

(Haese et al., 2015) Find a vector of length 3 units which is perpendicular to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Answer: $\frac{3}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Example 27

(Haese et al., 2015) Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have lengths 5 and $\sqrt{7}$ respectively. The angle between them is $110°$.

Find the value of $\underset{\sim}{a} \cdot \underset{\sim}{b}$

Answer: -4.525

Example 28

(Haese et al., 2015) Find the angle between $\underset{\sim}{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ **Answer:** $\approx 116°$

Example 29

(Haese et al., 2015) Given $A(2, -1)$, $B(3, 4)$ and $C(-1, 3)$, find the size of $\angle ABC$.

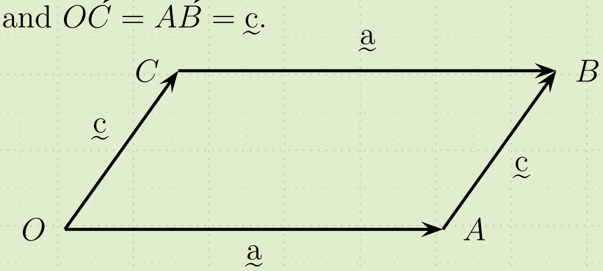
Answer: $\approx 64.7°$

Important note

Draw a simple diagram first!

Example 30

[2020 NEAP Ext 1 Trial Q10] The diagram shows $OABC$, a rhombus in which $\overrightarrow{OA} = \overrightarrow{CB} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underset{\sim}{c}$.



To prove that the diagonals of $OABC$ are perpendicular, it is required to show that

(A) $(\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$

(B) $(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = 0$

(C) $(\underset{\sim}{a} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} + \underset{\sim}{c}) = 0$

(D) $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

Example 31

[2020 JRAHS Ext 1 Trial Q10] It is given that $\underset{\sim}{a} = \begin{pmatrix} -3 \\ m \end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. If the angle between vector $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is obtuse angle, what is true for the value of m?

(A) $m < 4$

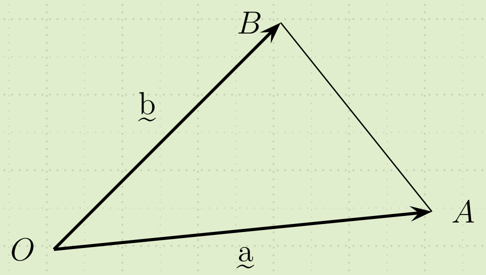
(B) $m < 4$ and $m \neq -\frac{9}{4}$

(C) $m > 4$

(D) $m \neq 4$ and $m > -\frac{9}{4}$

Example 32

[2020 Independent Ext 1 Trial Q11] In the diagram, OAB is an acute angled triangle in which $OA = OB$. The vector $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



(i) Show that $\cos\angle OAB = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{a} - \underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\ |\underset{\sim}{a}-\underset{\sim}{b}|}$ and $\cos\angle OBA = \dfrac{\underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\ |\underset{\sim}{a}-\underset{\sim}{b}|}$ **4**

(ii) Hence show that $\angle OAB = \angle OBA$ by using vector methods. **2**

Example 33

[2020 JRAHS Ext 1 Trial Q12] In the isosceles triangle ABC, $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right|$. D is the midpoint of side AB and E is the midpoint of side AC. $\overrightarrow{CD}$ is perpendicular to $\overrightarrow{BE}$.

(i) Draw the diagram and label $\angle BAC = \theta$ and $\left|\overrightarrow{AD}\right| = r$. **1**

(ii) Hence noting that $\overrightarrow{CD}$ may be written as $\overrightarrow{AD} - \overrightarrow{AC}$, or otherwise, use vector methods to find the value of $\angle BAC$. **3**

Further exercises

Ex 3I-3J (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8C (Pender et al., 2019)

- All questions

3.3 Other geometric proofs

Important note

Geometric proofs involving vectors are vastly different to the Euclidean Geometry proofs!

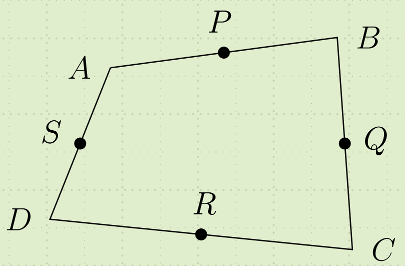
Steps

Use the following as a guide to these proofs:

- Draw picture!
- Introduce vectors, by choosing one of the vertices or a point outside the figure as a reference point/origin.
- Use the sum and scalar multiple of vectors to complete a triangle.
- Parallel vectors are scalar multiples of each other.
- The dot product is zero when two non - zero vectors are at right angles .
- The dot product may also allow access to the angle between some of the vectors.

Example 34

[Ex 8D Q2] P, Q, R and S are midpoints of AB, BC, CD and DA respectively. Let $\overrightarrow{AB} = \underset{\sim}{a}$, $\overrightarrow{BC} = \underset{\sim}{b}$ and $\overrightarrow{AD} = \underset{\sim}{d}$ and $\overrightarrow{DC} = \underset{\sim}{c}$.



(a) Explain why $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{d} + \underset{\sim}{c}$.

(b) Express $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

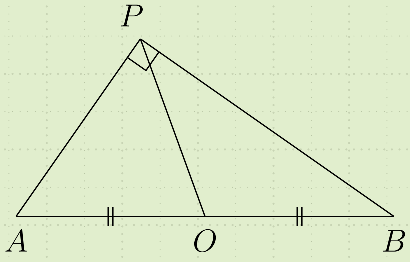
(c) Express $\overrightarrow{SR}$ in terms of $\underset{\sim}{d}$ and $\underset{\sim}{c}$.

(d) Hence show that $\overrightarrow{PQ} = \overrightarrow{SR}$.

(e) Deduce that the quadrilateral $PQRS$ is a parallelogram.

Example 35

(Pender et al., 2019) An interval AB with midpoint O subtends a right angle at point P.



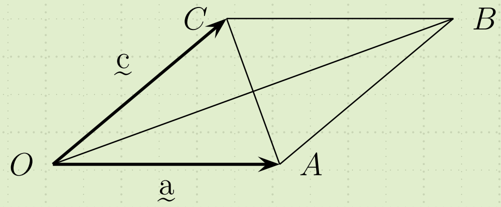
Prove that a circle with diameter AB passes through P.

Important note

Commence by letting $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{p} = \overrightarrow{OP}$.

Example 36

[Ex 8D Q6] In the diagram $OABC$ is a parallelogram whose diagonals OB and AC are equal. The points A and C have respective position vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ relative to O.



(a) Explain why $\overrightarrow{CB} = \underset{\sim}{a}$.

(b) Write $\overrightarrow{AC}$ in terms of $\underset{\sim}{c}$ and $\underset{\sim}{a}$.

(c) Explain why $\left|\underset{\sim}{c} + \underset{\sim}{a}\right| = \left|\underset{\sim}{c} - \underset{\sim}{a}\right|$.

(d) Use the result in the previous part, and the fact that $\left|\underset{\sim}{v}\right|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$, to show that $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$.

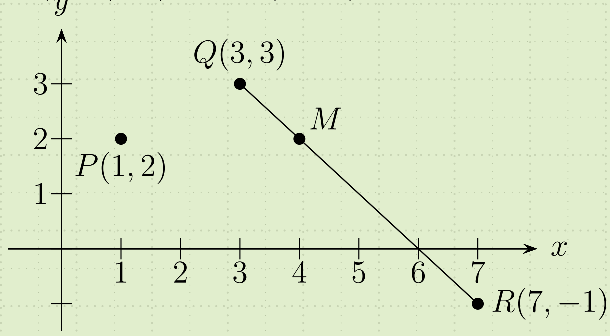
(e) What conclusions can be made about a parallelogram whose diagonals are equal?

Example 37

[**Ex 8D Q10**] Use vectors to prove that the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides.

Example 38

Given three points $P(1,2)$, $Q(3,3)$ and $R(7,-1)$,



(a) Express $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

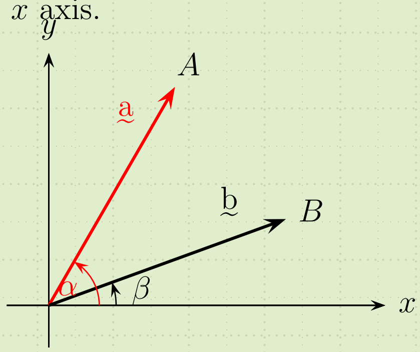
(b) If $\overrightarrow{QM} = \dfrac{1}{4}\overrightarrow{QR}$, express $\overrightarrow{PQ}$ and $\overrightarrow{OM}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

(c) Deduce that $PQ \parallel OM$.

Answer: (a) $\overrightarrow{OP} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{OQ} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$, $\overrightarrow{OR} = 7\underset{\sim}{i} - \underset{\sim}{j}$. (b) $\overrightarrow{PQ} = 2\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OM} = 4\underset{\sim}{i} + 2\underset{\sim}{j}$.

Example 39

In the following figure, the position vectors of the points A and B with respect to the origin are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. They are unit vectors making angles α and β with the positive direction of the x axis.



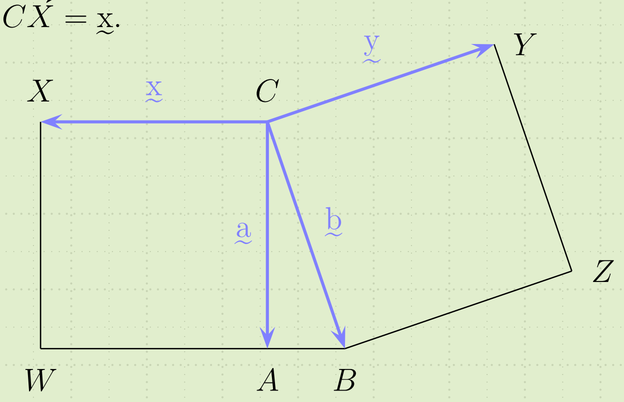
Prove that:

(a) $\underset{\sim}{a} = \cos\alpha\,\underset{\sim}{i} + \sin\alpha\,\underset{\sim}{j}$ and $\underset{\sim}{b} = \cos\beta\,\underset{\sim}{i} + \sin\beta\,\underset{\sim}{j}$

(b) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Example 40

(So & Wong, 1987, Ex 12-5) In the figure, $ACXW$ and $BCYZ$ are squares. $\overrightarrow{CA} = \underset{\sim}{a}$, $\overrightarrow{CB} = \underset{\sim}{b}$, $\overrightarrow{CY} = \underset{\sim}{y}$ and $\overrightarrow{CX} = \underset{\sim}{x}$.

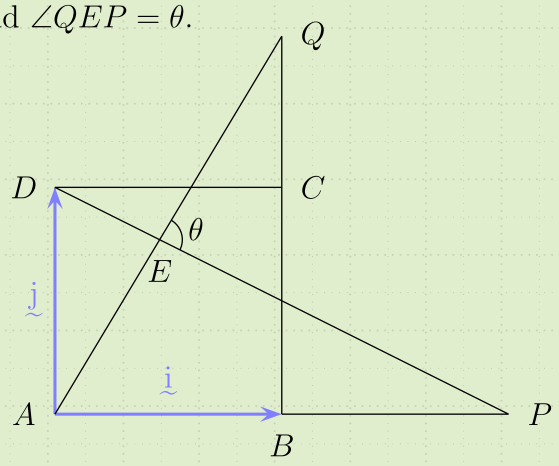


(a) Prove that $\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{y} \cdot \underset{\sim}{x} = 0$.

(b) Deduce that $AY \perp BX$.

Example 41

In the diagram, $ABCD$ is a square with $\overrightarrow{AB} = \underset{\sim}{i}$ and $\overrightarrow{AD} = \underset{\sim}{j}$. P and Q are respectively points on AB and BC produced with $BP = k$ and $CQ = m$. AQ and DP intersect at E and $\angle QEP = \theta$.



(a) By calculating $\overrightarrow{AQ} \cdot \overrightarrow{DP}$, find $\cos\theta$ in terms of m and k.

(b) Given that $\dfrac{DE}{EP} = \dfrac{1}{4}$,

i. Express $\overrightarrow{AE}$ in terms of k.

ii. Let $\dfrac{AE}{AQ} = r$. Express $\overrightarrow{AE}$ in terms of r and m.

iii. If $\theta = 90°$, use the results above to find the values of k, m and r.

Answer: (a) $\frac{k-m}{\sqrt{(m^2+2m+2)(k^2+2k+2)}}$ (b) i. $\frac{1+k}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$ ii. $r\underset{\sim}{i} + r(1+m)\underset{\sim}{j}$ iii. $r = \frac{2}{5}$, $m = k = 1$

Further exercises

Ex 3L (Haese et al., 2015)
(More introductory type questions)

- Every second question

Ex 8D (Pender et al., 2019)

- All questions

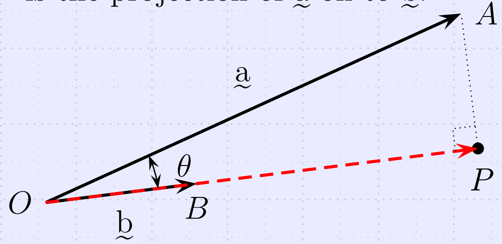
3.4 Projection of a vector on to another vector

Definition 15

The *projection* of a vector $\underset{\sim}{a}$ on another vector $\underset{\sim}{b}$ is a vector parallel to $\underset{\sim}{b}$ with the following notation:

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}$$

In the diagram given, $\overrightarrow{OP}$ is the projection of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$.



Assumption: $\underset{\sim}{b}$ is a ......non...... ......zero...... ......vector...... .

Important note

Plain English: the projection of vector $\underset{\sim}{a}$ on to vector $\underset{\sim}{b}$ is the ......'shadow cast'...... from ..$\underset{\sim}{a}$.. on to ..$\underset{\sim}{b}$..

Laws/Results

The projection of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$, $\underset{\sim}{b} \neq \underset{\sim}{0}$:

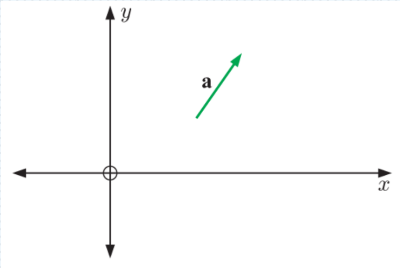
$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}\right) \underset{\sim}{b}$$

GeoGebra

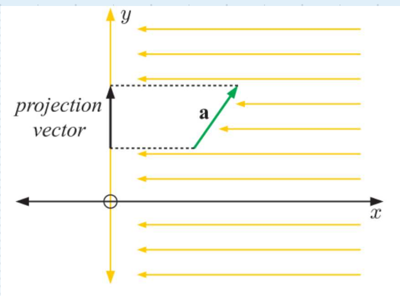
Projection of vector $\underset{\sim}{a}$ on to $\underset{\sim}{b}$

Fill in the spaces

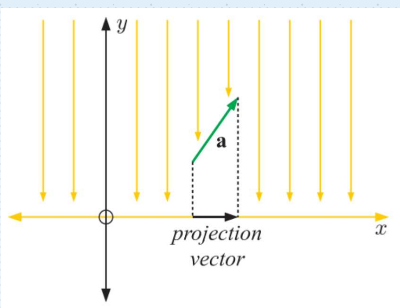
The following diagrams are taken from Haese et al. (2015), explanations paraphrased:



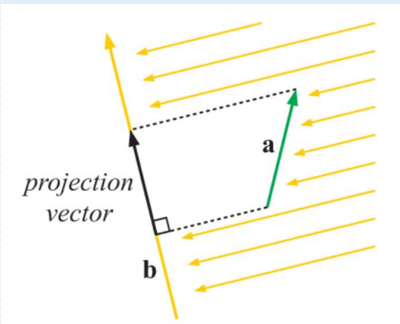
- Consider a vector $\underset{\sim}{a}$ in the ....Cartesian........ plane.



- If a light is shone on to the vector from the ....right...., it will cast a ....shadow.... on the y axis.
- This ....shadow.... is the ....projection.... ....vector.... of $\underset{\sim}{a}$ on the y axis
- Also known as the y component of the vector $\underset{\sim}{a}$.
- If $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then $\text{proj}_{\underset{\sim}{j}}\,\underset{\sim}{a} =$ ...$3\underset{\sim}{j}$...



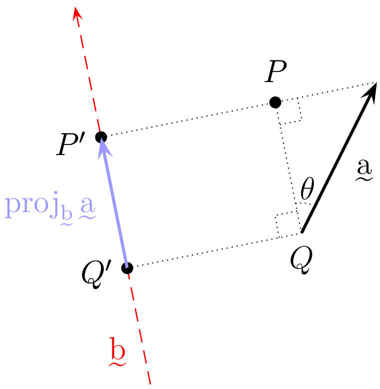
- Similarly, if a light is shone from the ....top...., the ....shadow.... is the ....projection.... ....vector.... of $\underset{\sim}{a}$ on the x axis.
- If $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then $\text{proj}_{\underset{\sim}{i}}\,\underset{\sim}{a} =$ ...$2\underset{\sim}{i}$...



- If the light is shone through vector $\underset{\sim}{a}$ on to another vector $\underset{\sim}{b}$ so that the light is ....perpendicular.... to $\underset{\sim}{b}$, the ....shadow.... cast on $\underset{\sim}{b}$ is the ....projection.... ....vector.... of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$.

3.4.1 Derivation

Consider the diagram:



- Suppose θ is the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- Give the relationship between PQ, $|a|$ and θ:

$$\frac{PQ}{|\underset{\sim}{a}|} = \cos\theta$$

$$\therefore PQ = |\underset{\sim}{a}|\cos\theta$$

- As $\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}$ is a vector in the same direction to $\underset{\sim}{b}$, it is also the unit vector in $\underset{\sim}{b}$'s direction, multiplied by a scalar multiple:*

$$\begin{aligned}
\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} &= PQ\ \widehat{\underset{\sim}{b}} \\
&= \underbrace{\left|\underset{\sim}{a}\cos\theta\right|}_{\text{scalar multiple, `magnitude'}}\ \widehat{\underset{\sim}{b}} \qquad (\ddagger) \\
&= |\underset{\sim}{a}|\,|\cos\theta|\ \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}
\end{aligned}$$

- Recall also, that $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos\theta$, rearranging:

$$\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$$

Substitute in the result from ($\ddagger$):

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \cancel{|\underset{\sim}{a}|}\,\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\cancel{|\underset{\sim}{a}|}\,|\underset{\sim}{b}|}\,\frac{\underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$$

Laws/Results

The projection of vector $\underset{\sim}{a}$ on to vector $\underset{\sim}{b}$:

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|^2}\underset{\sim}{b} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b}$$

*See Section 2.1.1 on page 13

3.4.2 Examples and proofs

Example 42

Find the projection of $\underset{\sim}{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ on to $\underset{\sim}{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, and state the length of the vector.

Answer: $\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a} = \frac{9}{34}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, length $\frac{9}{\sqrt{34}}$

Example 43

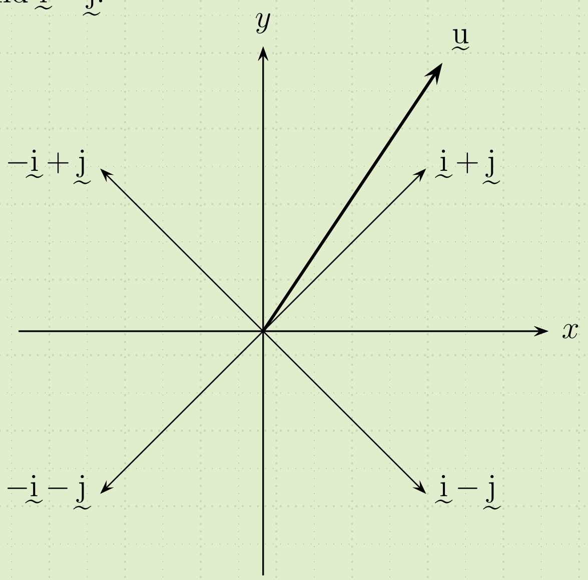
[2020 JRAHS Ext 1 Trial Q11] It is given that $A(-2, 3)$, $B(4, -5)$, $C(-7, -6)$ and $D(-5, -2)$.

i. Find the vector projection of $\overrightarrow{AB}$ on to $\overrightarrow{CD}$. **2**

ii. What is the vector component of $\overrightarrow{AB}$ perpendicular to $\overrightarrow{CD}$? **1**

Example 44

[2022 Ext 1 HSC Q6] The following diagram shows the vector $\underset{\sim}{u}$ and the vectors $\underset{\sim}{i}+\underset{\sim}{j}$, $-\underset{\sim}{i}+\underset{\sim}{j}$, $-\underset{\sim}{i}-\underset{\sim}{j}$ and $\underset{\sim}{i}-\underset{\sim}{j}$.



(Drawn to scale)

Which statement regarding this diagram could be true?

(A) The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i}+\underset{\sim}{j}$ is the vector $1.1\underset{\sim}{i}+1.8\underset{\sim}{j}$.

(B) The projection of $\underset{\sim}{u}$ onto $-\underset{\sim}{i}+\underset{\sim}{j}$ is the vector $-0.4\underset{\sim}{i}+0.4\underset{\sim}{j}$.

(C) The projection of $\underset{\sim}{u}$ onto $-\underset{\sim}{i}-\underset{\sim}{j}$ is the vector $3.2\underset{\sim}{i}+3.2\underset{\sim}{j}$.

(D) The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{i}-\underset{\sim}{j}$ is the vector $0.5\underset{\sim}{i}-0.5\underset{\sim}{j}$.

Example 45

[2022 Ext 1 HSC Q14] (3 marks) The vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are not parallel. The vector $\underset{\sim}{p}$ is the projection of $\underset{\sim}{u}$ onto the vector $\underset{\sim}{v}$.

The vector $\underset{\sim}{p}$ is parallel to $\underset{\sim}{v}$ so it can be written as $\lambda_0\underset{\sim}{v}$ for some real number λ_0. (Do NOT prove this).

Prove that $\left|\underset{\sim}{u} - \lambda\underset{\sim}{v}\right|$ is smallest when $\lambda = \lambda_0$ by showing that, for all real numbers λ, $\left|\underset{\sim}{u} - \lambda_0\underset{\sim}{v}\right| \leq \left|\underset{\sim}{u} - \lambda\underset{\sim}{v}\right|$.

Further exercises

Ex 3K (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8E (Pender et al., 2019)

- All questions

Section A

Past examination questions

- Most questions in this section originate from various VCEs.
- Two additional terms which are not used in the NSW Syllabuses but have equivalents:

Definition 16

Vector resolute is synonymous with the the vector projection.

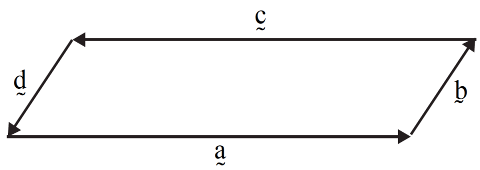
Definition 17

Scalar projection is the length of the vector projection, with a negative sign if the projection has an opposite direction with respect to $\underset{\sim}{b}$

A.1 2006 VCE Specialist Mathematics

A.1.1 Paper 2 Section 1

15. In the parallelogram shown, $\left|\underset{\sim}{a}\right| = 2\left|\underset{\sim}{b}\right|$. **1**



Which one of the following statements is true?

(A) $\underset{\sim}{a} = 2\underset{\sim}{b}$

(B) $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$

(C) $\underset{\sim}{b} - \underset{\sim}{d} = \underset{\sim}{0}$

(D) $\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{0}$

(E) $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

A.1.2 Paper 2 Section 2

Question 8

Point A has position vector $\underset{\sim}{a} = -\underset{\sim}{i} - 4\underset{\sim}{j}$, point B has position vector $\underset{\sim}{b} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$, point C has position vector $\underset{\sim}{c} = 5\underset{\sim}{i} - 4\underset{\sim}{j}$, and point D has position vector $\underset{\sim}{d} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ relative to the origin O.

(a) Show that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular. **2**

(b) Use a vector method to find the cosine of $\angle ADC$, the angle between $\overrightarrow{DA}$ and $\overrightarrow{DC}$. **2**

(c) Find the cosine of $\angle ABC$, and hence show that $\angle ADC$ and $\angle ABC$ are supplementary. **3**

Point P has position vector $\underset{\sim}{p} = 2\underset{\sim}{i}$.

(d) Use the cosine of $\angle APC$ and an appropriate trigonometric formula to prove that $\angle APC = 2\angle ADC$. **3**

A.2 2008 VCE Specialist Mathematics

A.2.1 Paper 2 Section 1

17. If P, Q and R are three collinear points with position vectors $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where Q lies between P and R. If $\left|\overrightarrow{QR}\right| = \frac{1}{2}\left|\overrightarrow{PQ}\right|$, then $\underset{\sim}{r}$ is equal to **1**

(A) $\frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

(B) $\frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

(C) $\frac{3}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{p}$

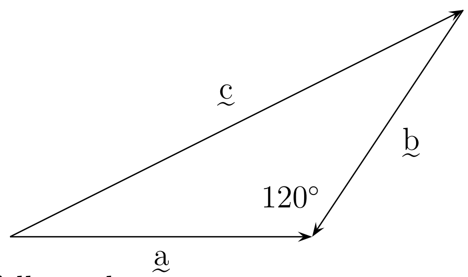
(D) $\frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

(E) $\frac{3}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

A.3 2009 VCE Specialist Mathematics

A.3.1 Paper 2 Section 1

17. Vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are shown below. **1**



From the diagram it follows that

(A) $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2$

(B) $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 - \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$

(C) $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a} \cdot \underset{\sim}{b}\right|$

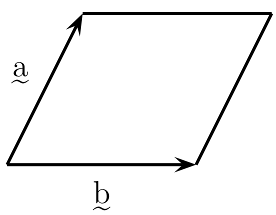
(D) $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|$

(E) $\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 - \left|\underset{\sim}{a} \cdot \underset{\sim}{b}\right|$

A.4 2011 VCE Specialist Mathematics

A.4.1 Paper 2 Section 1

10. The diagram below shows a rhombus, spanned by the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**



It follows that

(A) $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

(B) $\underset{\sim}{a} = \underset{\sim}{b}$

(C) $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

(D) $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \left|\underset{\sim}{a} - \underset{\sim}{b}\right|$

(E) $2\underset{\sim}{a} + 2\underset{\sim}{b} = \underset{\sim}{0}$

A.5 2018 VCE Specialist Mathematics

A.5.1 Paper 2 Section 1

11. Consider the vectors given by $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$, where $m \in \mathbb{R}$. If the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 30°, then m equals **1**

(A) $\sqrt{2} \pm 1$

(B) $2 \pm \sqrt{3}$

(C) $\sqrt{3}, \dfrac{1}{\sqrt{3}}$

(D) $\dfrac{\sqrt{3}}{4 - \sqrt{3}}$

(E) $\dfrac{\sqrt{39}}{13}$

12. If $\left|\underset{\sim}{a}+\underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right| + \left|\underset{\sim}{b}\right|$ and $\underset{\sim}{a}$, $\underset{\sim}{b} \neq 0$, which one of the following is **necessarily true**? **1**

(A) $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

(B) $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right|$

(C) $\underset{\sim}{a} = \underset{\sim}{b}$

(D) $\underset{\sim}{a} = -\underset{\sim}{b}$

(E) $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$

A.6 2020 Ext 1 HSC

4. Maria starts at the origin and walks along all of the vector $2\underset{\sim}{i}+3\underset{\sim}{j}$, then walks along all of the vector $3\underset{\sim}{i}-2\underset{\sim}{j}$ and finally along all of the vector $4\underset{\sim}{i}-3\underset{\sim}{j}$. **1**

How far from the origin is she?

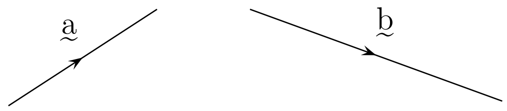
(A) $\sqrt{77}$

(B) $\sqrt{85}$

(C) $2\sqrt{13}+\sqrt{5}$

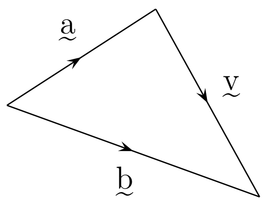
(D) $\sqrt{5}+\sqrt{7}+\sqrt{13}$

6. The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are shown. **1**

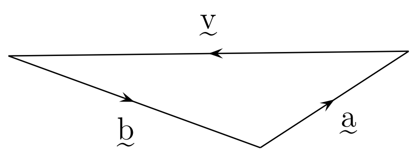


Which diagram below shows the vector $\underset{\sim}{v} = \underset{\sim}{a} - \underset{\sim}{b}$?

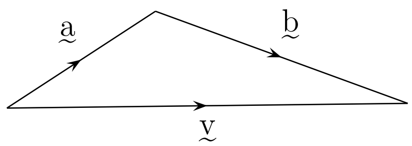
(A)



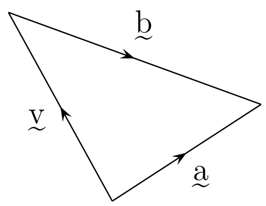
(C)



(B)



(D)



9. The projection of the vector $\begin{pmatrix}6\\7\end{pmatrix}$ on to the line $y=2x$ is $\begin{pmatrix}4\\8\end{pmatrix}$. **1**

The point $(6,7)$ is reflected in the line $y=2x$ to a point A.

What is the position vector of the point A?

(A) $\begin{pmatrix}6\\12\end{pmatrix}$ (B) $\begin{pmatrix}2\\9\end{pmatrix}$ (C) $\begin{pmatrix}-6\\7\end{pmatrix}$ (D) $\begin{pmatrix}-2\\1\end{pmatrix}$

Question 11

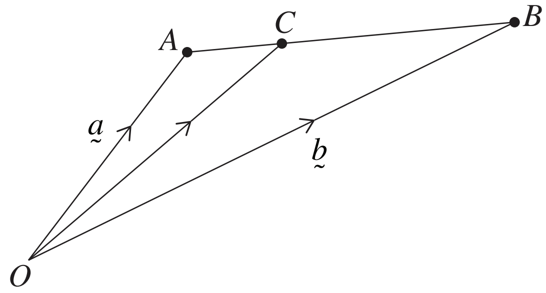
(b) For what value(s) of a are the vectors $\begin{pmatrix} a \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2a-3 \\ 2 \end{pmatrix}$ perpendicular? **3**

A.7 2020 Ext 2 HSC

This question does not contain any Extension 2 specific content, and so is placed in this summary instead.

Question 15

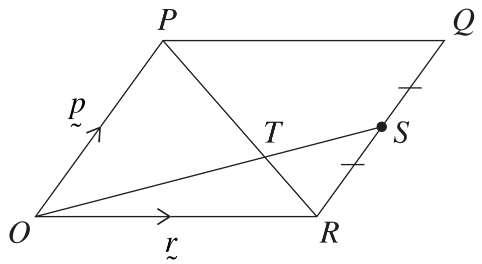
(b) The point C divides the interval AB so that $\dfrac{CB}{AC} = \dfrac{m}{n}$. The position vectors of A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, as shown in the diagram.



i. Show that $\overrightarrow{AC} = \dfrac{n}{m+n}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$. **2**

ii. Prove that $\overrightarrow{OC} = \dfrac{m}{m+n}\underset{\sim}{a} + \dfrac{n}{m+n}\underset{\sim}{b}$. **1**

Let $OPQR$ be a parallelogram with $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OR} = \underset{\sim}{r}$. The point S is the midpoint of QR and T is the intersection of PR And OS, as shown in the diagram.



iii. Show that $\overrightarrow{OT} = \dfrac{2}{3}\underset{\sim}{r} + \dfrac{1}{3}\underset{\sim}{p}$. **3**

iv. Using parts (ii) and (iii), or otherwise, prove that T is the point that divides the interval PR in the ratio $2:1$. **1**

A.8 2021 Ext 1 HSC

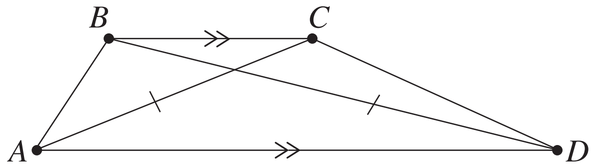
Question 14

(a) A plane needs to travel to a destination that is on a bearing of 063°. The engine is set to fly at a constant 175 km/h. However, there is a wind from the south with a constant speed of 42 km/h. **3**

On what constant bearing, to the nearest degree, should the direction of the plane be set in order to reach the destination? **Answer:** 075°

(c) i. For vector $\underset{\sim}{v}$, show that $\underset{\sim}{v} \cdot \underset{\sim}{v} = \left|\underset{\sim}{v}\right|^2$. **1**

ii. In the trapezium $ABCD$, BC is parallel to AD and $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BD}\right|$. **3**



NOT TO SCALE

Let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\overrightarrow{AD} = k\overrightarrow{BC}$, where $k > 0$.

Using part (i) or otherwise, show $2\underset{\sim}{a} \cdot \underset{\sim}{b} + (1-k)\left|\underset{\sim}{b}\right|^2 = 0$.

A.9 2022 Ext 1 HSC

8. The angle between two unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is θ and $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| < 1$. **1**

Which of the following best describes the possible range of values of θ?

(A) $0 \le \theta < \dfrac{\pi}{3}$

(B) $0 \le \theta < \dfrac{2\pi}{3}$

(C) $\dfrac{\pi}{3} < \theta \le \pi$

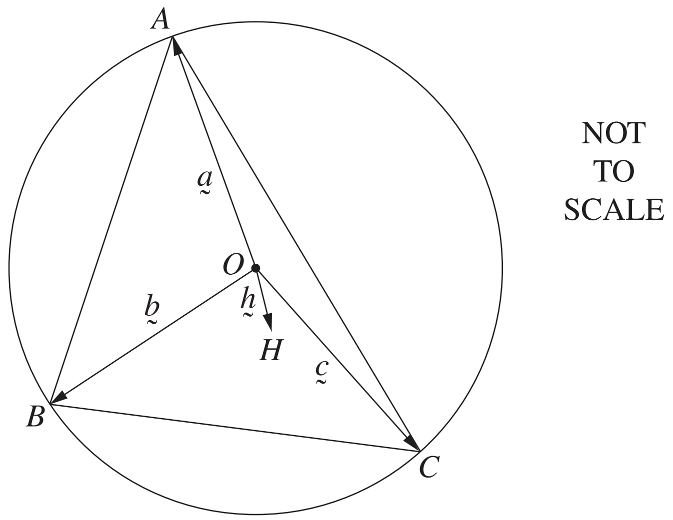
(D) $\dfrac{2\pi}{3} < \theta \le \pi$

Note: attempt this after Topic 9 (Radians) has been completed.

Question 13

(a) Three different points A, B and C are chosen on a circle centred at O. **3**

Let $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$ and $\underset{\sim}{c} = \overrightarrow{OC}$. Let $\underset{\sim}{h} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ and let H be the point such that $\overrightarrow{OH} = \underset{\sim}{h}$, as shown in the diagram.



Show that $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

A.10 2023 Ext 1 HSC

6. Given two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, let $\underset{\sim}{c}$ be the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. **1**

What is the projection of $10\underset{\sim}{a}$ onto $2\underset{\sim}{b}$?

(A) $2\underset{\sim}{c}$ (B) $5\underset{\sim}{c}$ (C) $10\underset{\sim}{c}$ (D) $20\underset{\sim}{c}$

Question 14

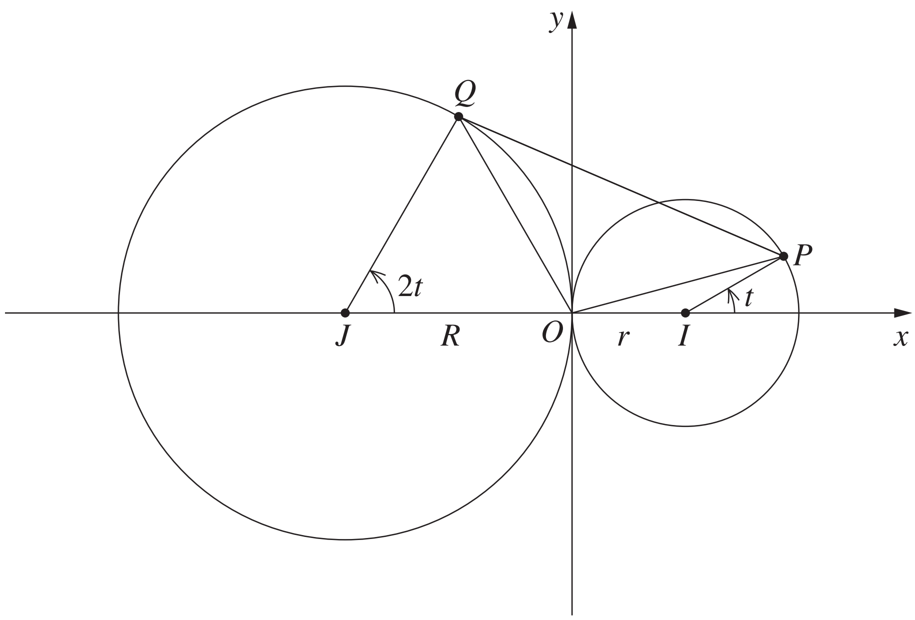
(c) i. Given a non-zero vector $\begin{pmatrix} p \\ q \end{pmatrix}$, it is known that the vector $\begin{pmatrix} q \\ -p \end{pmatrix}$ is perpendicular to $\begin{pmatrix} p \\ q \end{pmatrix}$ and has the same magnitude. (Do NOT prove this). **3**

Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, respectively.

Using the given information, or otherwise, show that the area of $\triangle OAB$ is $\dfrac{1}{2}\left|a_1b_2 - a_2b_1\right|$.

ii. The point P lies on the circle centred at $I(r, 0)$ with radius $r > 0$, such that $\overrightarrow{IP}$ makes an angle of t to the horizontal. **4**

The point Q lies on the circle centred at $J(-R, 0)$ with radius $2t$ to the horizontal.



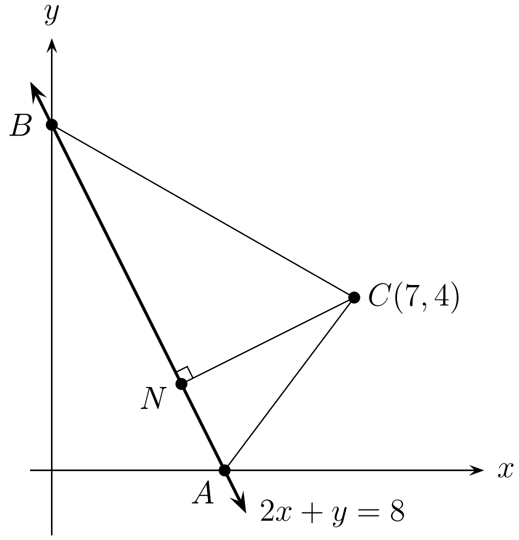
Note that $\overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IP}$ and $\overrightarrow{OQ} = \overrightarrow{OJ} + \overrightarrow{JQ}$.

Using part (i), or otherwise, find the values of t, where $-\pi \leq t \leq \pi$, that maximise the area of $\triangle OPQ$.

A.11 2024 NBHS Assessment Task 1

Question 4

(a) The diagram shows $\triangle ABC$, and the line $2x + y = 8$ with x and y intercepts at A and B respectively. The point C has coordinates $(7, 4)$ and N is a point on AB such that $CN \perp AB$.



i. Show that $\overrightarrow{BA} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$. **2**

ii. By using vector methods, show that **2**

$$\cos \angle ABC = \frac{3}{\sqrt{13}}$$

iii. Find a vector $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ such that $\underset{\sim}{u}$ is perpendicular to $\overrightarrow{BA}$. **1**

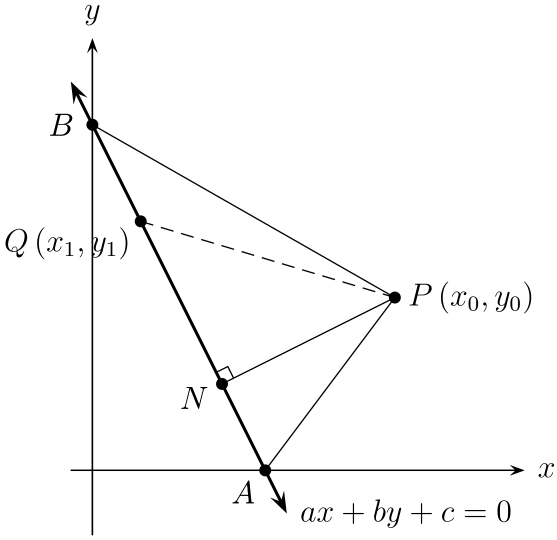
iv. Hence show that **2**

$$\overrightarrow{NC} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

v. Use vector methods to find the coordinates of N. **1**

Question continues overleaf...

A line ℓ with equation $ax + by + c = 0$ is drawn in the Cartesian plane, and $P(x_0, y_0)$ is located off the line ℓ, and point Q has coordinates $Q(x_1, y_1)$ which lies on the line ℓ. N is a point that is located on the line ℓ such that $PN \perp AB$.



You are given that $\underset{\sim}{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is a vector that is perpendicular to $\overrightarrow{BA}$ (Do NOT prove this).

vi. By finding $\overrightarrow{QP}$ using the coordinates given, show that **3**

$$\overrightarrow{NP} = \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \hat{\underset{\sim}{v}}$$

vii. Briefly explain why the shortest distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is given by **1**

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Answers

iii. Any scalar multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

iv. Show

v. $(3, 2)$

vi. Hint: use $ax_1 + by_1 + c = 0$

vii. Show. Insufficient to state the perpendicular distance formula.

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

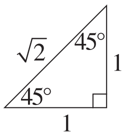
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

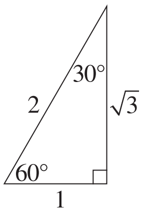
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

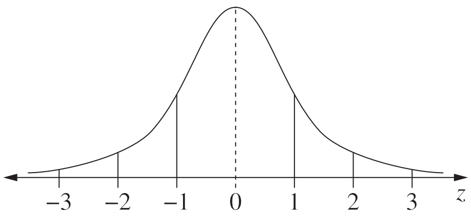
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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