



NORMANHURST BOYS HIGH SCHOOL

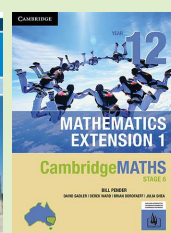
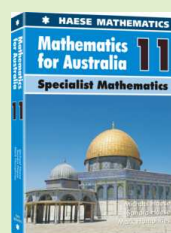
MATHEMATICS EXTENSION 1 YEAR 12 COURSE



Topic summary and exercises:

(X1) (X2) **Introduction to Vectors**

With references to



Name:

Initial version by H. Lam, September 2019 with assistance from I. Ham.

Last updated May 7, 2024

Various corrections by students & members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at <http://www.flaticon.com>, used under  CC BY 2.0.

Parts of this handout are derived from Haese, Haese, and Humphries (2015), *MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes* (2018), So and Wong (1987) and Pender, Sadler, Ward, Dorofaeff, and Shea (2019). Most GeoGebra applets listed here are courtesy of the NSW Department of Education's Mathematics Curriculum Support team/mEsh project's efforts.

Symbols used

-  Beware! Heed warning.
-  Provided on NESA Reference Sheet
-  Facts/formulae to memorise.
-  Mathematics Extension 1 content.
-  Literacy: note new word/phrase.
-  Further reading/exercises to enrich your understanding and application of this topic.
-  Syllabus specified content
-  Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME12-2 applies concepts and techniques involving vectors and projectiles to solve problems

Syllabus subtopics

ME-V1 (1.1, 1.2) Introduction to Vectors (Introduction to vectors, Further operations with vectors)

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *Haese Mathematics for Australia - Specialist Mathematics 11* (Haese et al., 2015) *CambridgeMATHS Year 12 Extension 1* (Pender et al., 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

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Section 1

Introduction to Vectors

1.1 Definition

Definition 1

Quantities which only have are known as *scalars*.

Example: of an aeroplane.

Definition 2

Quantities which have both and are known as *vectors*.

Example: of an aeroplane.

Fill in the spaces

Other examples of vector forces:

•

•

•

•

1.2 Representation

1.2.1 Geometric

Fill in the spaces

Vector quantities can be represented using a *directed line segment* or *arrow*.

- The of the arrow represents the
- The shows its direction.

Definition 3

Two vectors are *equal* if they have the same and

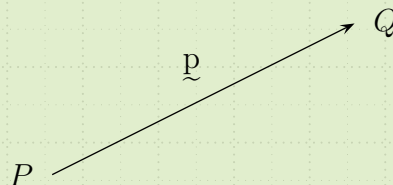
Corollary 1

The position of the starting point (tail) and ending point (arrowhead) does matter.

Example 1

Given $\overrightarrow{PQ}$ as shown, draw two other vectors, $\underline{a} = \overrightarrow{AB}$ and $\underline{e} = \overrightarrow{EF}$ such that

$$\overrightarrow{PQ} = \overrightarrow{AB} = \overrightarrow{EF}$$



1.2.2 Algebraic

From Example 1, there are three ways of writing the vector drawn:

- $\overrightarrow{PQ}$: the vector from the point P and at point Q .
- $\vec{p}$ or $\underline{p}$: NESA syllabus documentation, handwritten.

1.3 Vector arithmetic

1.3.1 Addition

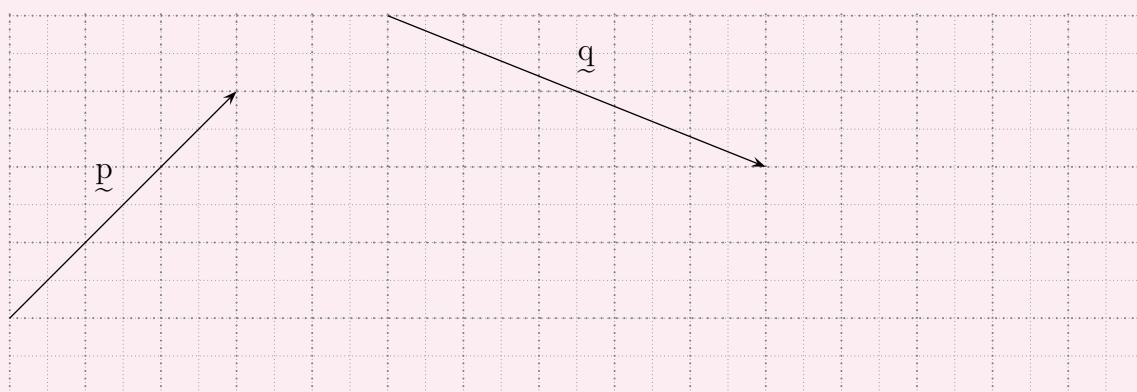
! Important note

⚠ Use a ruler!

📋 Steps

For vectors $\underline{p}$ and $\underline{q}$, to perform $\underline{p} + \underline{q}$:

1. Draw $\underline{p}$.
2. Draw $\underline{q}$, starting from the arrowhead of $\underline{p}$.
3. Draw vector from the of $\underline{p}$ to the of $\underline{q}$.



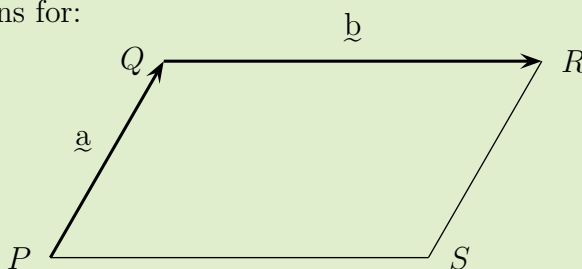
1.3.2 Subtraction/negative vectors

📖 Definition 4

If $\underline{p}$ is a vector, then $-\underline{p}$ is equal in but in

🎓 Example 2

(Haese et al., 2015) $PQRS$ is a parallelogram where $\overrightarrow{PQ} = \underline{a}$ and $\overrightarrow{QR} = \underline{b}$. Find vector expressions for:



(a) $\overrightarrow{QP}$

(b) $\overrightarrow{RQ}$

(c) $\overrightarrow{SR}$

(d) $\overrightarrow{SP}$

1.3.3 Magnitude and zero vector

Definition 5

The *magnitude* of a vector $\underline{p}$ is given by $|\underline{p}|$.

Definition 6

The *zero vector* has magnitude zero, i.e. $|\underline{0}| = 0$. Direction is not

Usually written as $\underline{0}$ (handwritten: $\underline{0}$).

1.3.4 Scalar multiplication

Definition 7

If $\underline{p}$ is a vector, and $\lambda \in \mathbb{R}$,

- If $\lambda > 1$, then $\lambda \underline{p}$ extends the to $\lambda |\underline{p}|$.
(..... the vector $\underline{p}$ by a factor of λ).
- If $0 < \lambda < 1$, then $\lambda \underline{p}$ the magnitude.
- If $\lambda = 0$, then $\lambda \underline{p} = \underline{0}$.
- If $\lambda < 0$, then $\lambda \underline{p}$ sees the magnitude scaled by $|\lambda|$ and direction

Laws/Results

- $\underline{p}$ and $\lambda \underline{p}$, $\lambda > 0$ are
- $\underline{p}$ and $\lambda \underline{p}$, $\lambda < 0$ are

GeoGebra

Scaling vector investigation

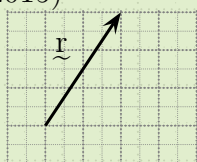


Vector addition - what shape is *always* formed when adding vectors?

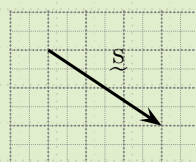
Example 3

(Haese et al., 2015)

Given vectors



and



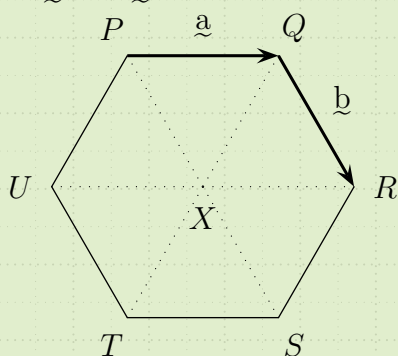
(diagrams drawn to scale),

construct geometrically:

- | | | | |
|----------------------|---------------------------------|---------------------------------------|---|
| (a) $-\underline{r}$ | (c) $\frac{1}{2}\underline{r}$ | (e) $2\underline{r} - \underline{s}$ | (g) $\frac{1}{2}\underline{r} + 2\underline{s}$ |
| (b) $2\underline{s}$ | (d) $-\frac{3}{2}\underline{s}$ | (f) $2\underline{r} + 3\underline{s}$ | (h) $\frac{1}{2}(\underline{r} + 3\underline{s})$ |


Example 4

(Haese et al., 2015) $PQRSTU$ is a regular hexagon. If $\overrightarrow{PQ} = \underline{a}$ and $\overrightarrow{QR} = \underline{b}$, find in terms of $\underline{a}$ and $\underline{b}$:



(a) $\overrightarrow{PX}$

(b) $\overrightarrow{PS}$

(c) $\overrightarrow{QX}$

(d) $\overrightarrow{RS}$

1.4 Vector equations

Laws/Results

Vectors can be

- Added
- Have scalar multiplication applied in the same manner as usual scalar numbers, with the same commutative/associative/distributive laws.



Example 5

(MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes, 2018) Fully simplify: $3(2\underline{a} - \underline{b}) + (\underline{a} - 2\underline{b})$.



Example 6

(MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes, 2018) Fully simplify: $2\overrightarrow{AC} - \overrightarrow{OC} + \overrightarrow{OA}$.



Important note

- Draw picture if it's slightly uncertain what is $\overrightarrow{AC}$.
- Convert the vector arrow notation $\overrightarrow{OC}$ to the single variable notation $\underline{c}$.

**Example 7**

(*MATH1131 Mathematics 1A and MATH1141 Higher Mathematics 1A Algebra Notes*, 2018) In $\triangle OAB$, D and E are points such that $OD : DA = OE : EB = 1 : 2$. Prove that

- (a) DE is parallel to AB
- (b) the length of DE is $\frac{1}{3}$ times that of AB .

**Further exercises**

Ex 3A-3C (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8A (Pender et al., 2019)

- Q1-19
-  Q20-21

Section 2

Simple Vector Geometry

2.1 Vector components

2.1.1 Unit vector

Definition 8

The *unit vector* has 1.

Laws/Results

For every vector $\underline{p}$ there are two *normalised* (corresponding unit) vectors:

- $\hat{\underline{p}} = \frac{\underline{p}}{|\underline{p}|}$ to $\underline{p}$.
- $-\hat{\underline{p}} = -\frac{\underline{p}}{|\underline{p}|}$ to $\underline{p}$.



GeoGebra

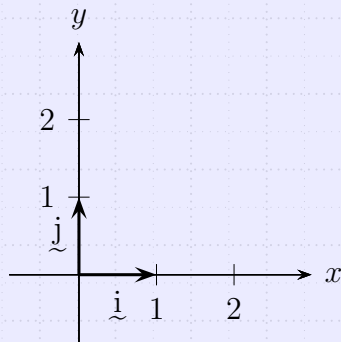
Unit vector

2.1.2 Cartesian basis vectors/components

Definition 9

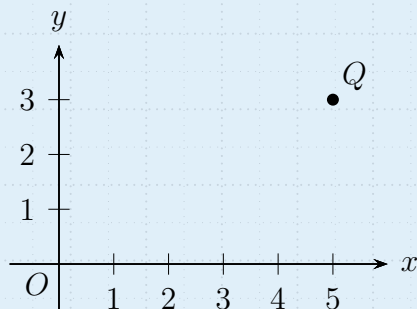
In the Cartesian plane, define the *basis vectors* to be unit vectors in the

- x direction to be $\underline{\hat{i}}$
- y direction to be $\underline{\hat{j}}$



Fill in the spaces

- The position $(5, 3)$ can now be represented via a *translation vector*.



..... notation:

$$\overrightarrow{OQ} = \dots\dots\dots$$

..... notation:

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Definition 10

The point $Q(x, y)$ has *position vector*

$$\overrightarrow{OQ} = \begin{pmatrix} x \\ y \end{pmatrix} = x\underline{\underline{i}} + y\underline{\underline{j}}$$

where

- The *component form* is $x\underline{\underline{i}} + y\underline{\underline{j}}$
- Represented in *column vector* notation: $\begin{pmatrix} x \\ y \end{pmatrix}$.
- Year 7-10 *ordered pair* notation: (x, y) .

Important note

Pender et al. (2019) and *Mathematics Extension 1 Stage 6 Syllabus* (2017, Revised 18/11/2019) both use brackets instead of parentheses for the column vector delimiters, e.g.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Tertiary education providers however, tend to use parentheses.

Example 8

A ship S leaves port O and sails north-east at 20 km/h. Describe its position after 6 hours by giving its position vector as a sum of multiples of the basis vectors $\underline{\underline{i}}$ and $\underline{\underline{j}}$.

2.1.3 Position vs displacement vectors

Definition 11

A *position vector* of a point P is the vector $\overrightarrow{OP}$, where the vector commences at the and finishes at P .

A **position vector** gives the of a point in the plane.

- Each point P in the Cartesian plane corresponds to the position vector

Definition 12

A *displacement vector* is the vector $\overrightarrow{AB}$, where the vector commences at the point A and finishes at B .

A **displacement vector** gives the of a point from another.

- Vector of point A relative to point B :

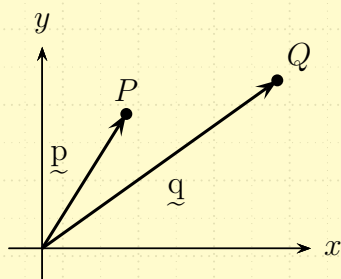


GeoGebra

 Position Vector Investigation - see how a position vector is different to a displacement vector.

Laws/Results

For two points P and Q with position vectors $\underline{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ respectively,



$$\overrightarrow{OP} + \overrightarrow{PQ} = \dots\dots\dots$$

$$\overrightarrow{OQ} + \overrightarrow{QP} = \dots\dots\dots$$

$$\overrightarrow{PQ} = \dots\dots\dots$$

$$\overrightarrow{QP} = \dots\dots\dots$$

$$= \dots\dots\dots$$

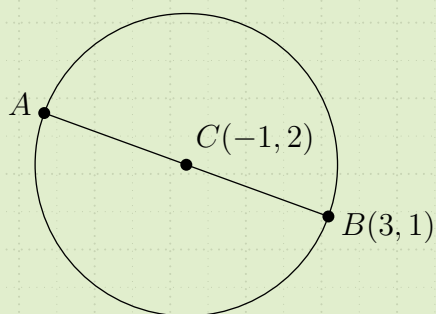
$$= \dots\dots\dots$$

$$= \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$$

$$= \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \end{pmatrix}$$

**Example 9**

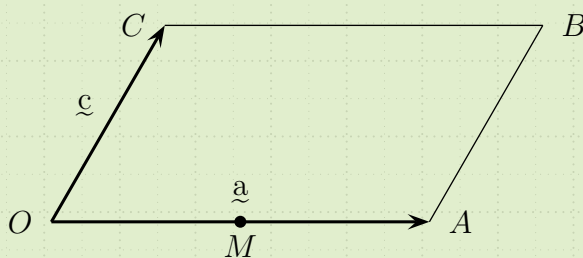
(Haese et al., 2015) AB is a diameter of a circle with centre $C(-1, 2)$. If B is $(3, 1)$, find:



- (a) $\overrightarrow{BC}$
- (b) the coordinates of A .

**Example 10**

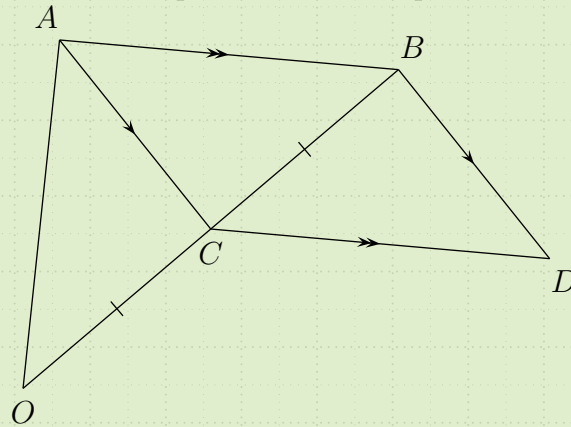
[2020 Independent Ext 1 Trial Q3] In the diagram, $OABC$ is a parallelogram. The vector $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OC} = \underline{c}$. M is the midpoint of OA . Which of the following expressions is represented by the vector $\overrightarrow{MB}$?



- (A) $\frac{1}{2}\underline{a} - \underline{c}$ (B) $\frac{1}{2}\underline{a} + \underline{c}$ (C) $\underline{a} - \frac{1}{2}\underline{c}$ (D) $\underline{a} + \frac{1}{2}\underline{c}$

**Example 11**

[2020 NEAP Ext 1 Trial Q7] The position vectors of points A and B are $\underline{a}$ and $\underline{b}$ respectively. Point C is the midpoint of OB and point D is such that $ABCD$ is a parallelogram.



Which of the following is the position vector of D ?

- (A) $\frac{3}{2}\underline{b} + \underline{a}$ (B) $\frac{3}{2}\underline{b} - \underline{a}$ (C) $\frac{1}{2}\underline{b} - \frac{1}{2}\underline{a}$ (D) $\frac{1}{2}\underline{b} - \underline{a}$

2.2 Vector arithmetic involving column vectors

2.2.1 Addition

Laws/Results

If $\underline{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} u \\ v \end{pmatrix}$, then

$$\underline{p} \pm \underline{q} = \dots\dots\dots$$

$$= \dots\dots\dots$$

Important note

- Add components (x direction, $\underline{i}$) with components
- Add components (y direction, $\underline{j}$) with

GeoGebra

Vector component addition

Example 12

(Pender et al., 2019)

1. (a) Given $\underline{u} = 2\underline{i} - \underline{j}$ and $\underline{v} = -\underline{i} + 2\underline{j}$, find:

i. $2\underline{u}$

ii. $3\underline{v}$

iii. $2\underline{u} + 3\underline{v}$

- (b) Draw all five vectors as position vectors. What figure do the origin and heads of $2\underline{u}$, $3\underline{v}$ and $2\underline{u} + 3\underline{v}$ form?

2. (a) Given $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find $\underline{v} - \underline{u}$.

- (b) Draw $\underline{u} = \overrightarrow{OU}$ and $\underline{v} = \overrightarrow{OV}$ as position vectors, and mark $\underline{v} - \underline{u}$.

2.2.2 Scalar multiplication

Laws/Results

(Haese et al., 2015) If $\underline{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $k \in \mathbb{R}$ then

$$k\underline{p} = \dots\dots\dots$$

Example 13

Given $\underline{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find:

(a) $3\underline{q}$

(b) $\underline{p} + 2\underline{q}$

(c) $\frac{1}{2}\underline{p} - 3\underline{q}$

2.2.3 Magnitude

Laws/Results

 If $\underline{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$|\underline{p}| = \dots\dots\dots$$

Example 14

If $\underline{p} = 3\underline{i} - 5\underline{j}$ and $\underline{q} = -\underline{i} - 2\underline{j}$, find $|\underline{p} - 2\underline{q}|$.

Answer: $\sqrt{26}$

**Example 15**

(Haese et al., 2015, Ex 3D) Find k such that the vector is a unit vector.

- (a) $\begin{pmatrix} 0 \\ k \end{pmatrix}$ (b) $\begin{pmatrix} k \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} k \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} k \\ k \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

Answer: (a) ± 1 (b) ± 1 (c) 0 (d) $\pm \frac{1}{\sqrt{2}}$ (e) $\pm \frac{\sqrt{3}}{2}$

**Example 16**

(Haese et al., 2015, Ex 3D) Given $\underline{y} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $|\underline{y}| = \sqrt{73}$, find the possible values of p .

Answer: ± 3

**Example 17**

(Haese et al., 2015) If $\underline{a} = 3\underline{i} - \underline{j}$, find:

- (a) a unit vector in the direction of $\underline{a}$
- (b) a vector of length 4 units in the direction of $\underline{a}$
- (c) vectors of length 4 units which are parallel to $\underline{a}$.

Answer: (a) $\frac{3}{\sqrt{10}}\underline{i} - \frac{1}{\sqrt{10}}\underline{j}$ (b) $\frac{12}{\sqrt{10}}\underline{i} - \frac{4}{\sqrt{10}}\underline{j}$ (c) $\pm \left(\frac{12}{\sqrt{10}}\underline{i} - \frac{4}{\sqrt{10}}\underline{j} \right)$

**Example 18**

[2020 NSBHS Ext 1 Trial] $\overrightarrow{OM} = 2\underline{i} + 5\underline{j}$ and $\overrightarrow{OT} = -3\underline{i} + 7\underline{j}$. What is the value

of $\left| \overrightarrow{MT} \right|$?

- (A) 14
- (B) $\sqrt{98}$
- (C) $\sqrt{29}$
- (D) $\sqrt{145}$

2.2.4 Direction

 Laws/Results

If $\underline{p} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the direction of the vector is measured by the formed by the vector with the positive direction of the horizontal axis, i.e.

$$\tan \theta = \frac{y}{x}$$

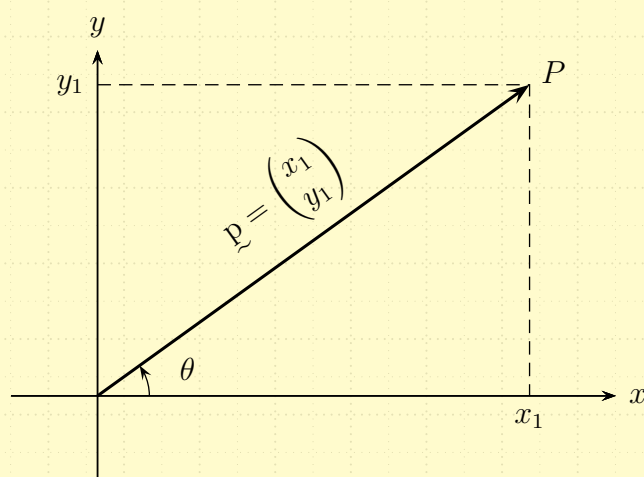
where $\theta \in [0^\circ, 360^\circ)$ (and radians where applicable)

 Important note

 (x2) Later in Complex Numbers, $\theta \in (-\pi, \pi]$. The quadrants used may vary between contexts - read the context carefully.

 Laws/Results

Components of a vector in terms of the magnitude and direction: if $\underline{p} = x_1 \underline{i} + y_1 \underline{j}$,



$$x_1 = \dots\dots\dots$$

$$y_1 = \dots\dots\dots$$

**Example 19**

(Pender et al., 2019)

- (a) Find the vector $\underline{u}$ of length 10, with angle 150° to the positive direction of the horizontal axis.
- (b) Find the length and angle (exact length, angle to the nearest degree) of $\underline{v} = -\underline{i} - 2\underline{j}$.

Answer: (a) $\underline{u} = -5\sqrt{3}\underline{i} + 5\underline{j}$ (b) $|\underline{u}| = \sqrt{5}$, $\theta \approx 243^\circ$.**Example 20****[2020 JRAHS Ext 1 Trial Q7]** How many of the following statements are true?

- $|\underline{a}| + |\underline{b}| = |\underline{a} + \underline{b}|$ means that $\underline{a}$ and $\underline{b}$ have the same direction.
- $|\underline{a}| + |\underline{b}| = |\underline{a} - \underline{b}|$ means that $\underline{a}$ and $\underline{b}$ have the opposite directions.
- $|\underline{a}| + |\underline{b}| = |\underline{a} - \underline{b}|$ means that $\underline{a}$ and $\underline{b}$ have the same magnitude.
- $|\underline{a}| - |\underline{b}| = |\underline{a} - \underline{b}|$ means that $\underline{a}$ and $\underline{b}$ have the same direction.

(A) 1

(B) 2

(C) 3

(D) 4

**Further exercises**

Ex 3D-3G (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8B (Pender et al., 2019)

- Q11-18

Section 3

Vector Geometry with the Dot Product

3.1 Definitions

Definition 13

The *dot product* between two vectors $\underline{p} = x_1\underline{i} + y_1\underline{j}$ and $\underline{q} = x_2\underline{i} + y_2\underline{j}$ (also known as *scalar product*):

$$\underline{p} \cdot \underline{q} = \dots\dots\dots$$



GeoGebra

Dot Product Insight



Example 21

Use $\underline{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ etc to prove the following:

(a) $\underline{p} \cdot \underline{q} = \underline{q} \cdot \underline{p}$ (Commutativity)

(b) $\underline{p} \cdot \underline{p} = |\underline{p}|^2$

(c) $\underline{p} \cdot (\underline{q} + \underline{r}) = \underline{p} \cdot \underline{q} + \underline{p} \cdot \underline{r}$ (Associativity)

(d) $(\underline{p} + \underline{q}) \cdot (\underline{r} + \underline{s}) = \underline{p} \cdot \underline{r} + \underline{p} \cdot \underline{s} + \underline{q} \cdot \underline{r} + \underline{q} \cdot \underline{s}$

**Example 22**

(Haese et al., 2015, Ex 3I) Explain why $\underline{a} \cdot \underline{b} \cdot \underline{c}$ is meaningless.

3.2 Angle between two vectors

The dot product enables the θ between two non-zero vectors to be found.

Definition 14

 The angle θ between two non-zero vectors $\underline{p}$ and $\underline{q}$ is related by:

$$\therefore \underline{p} \cdot \underline{q} = \dots\dots\dots$$

Steps

Proof

1. Draw situation representing the angle θ between two vectors $\underline{p}$ and $\underline{q}$, and vector $\underline{q} - \underline{p}$.

2. Apply the cosine rule to the triangle:

$$|\underline{q} - \underline{p}|^2 = \dots\dots\dots$$

3. Let $\underline{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and replace magnitudes with components of $\underline{p}$ and $\underline{q}$:

$$\therefore \underline{p} \cdot \underline{q} = \dots\dots\dots$$

4. Consequently,

$$\cos \theta = \dots\dots\dots$$

Laws/Results

Geometric results of the dot product

- If $\underline{p}$ is *perpendicular* to $\underline{q}$, then $\theta = 90^\circ$:

$$\begin{aligned}\underline{p} \cdot \underline{q} &= |\underline{p}| |\underline{q}| \cos \theta \\ &= \dots\dots\dots \\ &= \dots\dots\dots\end{aligned}$$

- If $\underline{p}$ is *parallel* to $\underline{q}$, then $\theta = 0^\circ$ or 180° :

$$\begin{aligned}\underline{p} \cdot \underline{q} &= |\underline{p}| |\underline{q}| \cos \theta \\ &= \dots\dots\dots \text{ or } \dots\dots\dots \\ &= \dots\dots\dots\end{aligned}$$

GeoGebra

Parallel Vectors investigation

Example 23

[2022 Ext HSC Q11] (2 marks) The vectors $\underline{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$ are perpendicular.

What are the possible values of a ?

Answer: 2 or $-\frac{7}{4}$

Example 24

(Haese et al., 2015) Consider the points $A(2, 1)$ and $B(6, -1)$ and $C(5, -3)$. Use the dot product to determine whether $\triangle ABC$ is right angled or not. If so, locate the right angle.

Answer: At B

**Example 25**

(Haese et al., 2015) Find the form of *all* vectors which are perpendicular to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Answer: $k \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

**Example 26**

(Haese et al., 2015) Find a vector of length 3 units which is perpendicular to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Answer: $\frac{3}{\sqrt{17}} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

**Example 27**

(Haese et al., 2015) Vectors $\underline{a}$ and $\underline{b}$ have lengths 5 and $\sqrt{7}$ respectively. The angle between them is 110° .

Find the value of $\underline{a} \cdot \underline{b}$

Answer: -4.525

**Example 28**

(Haese et al., 2015) Find the angle between $\underline{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ **Answer:** $\approx 116^\circ$

**Example 29**

(Haese et al., 2015) Given $A(2, -1)$, $B(3, 4)$ and $C(-1, 3)$, find the size of $\angle ABC$.

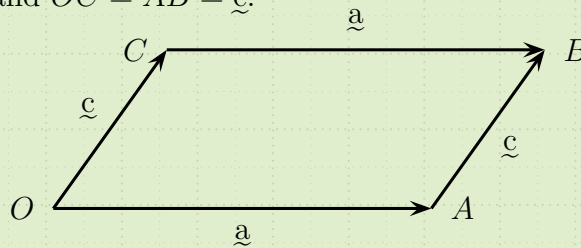
Answer: $\approx 64.7^\circ$

**Important note**

Draw a simple diagram first!

**Example 30**

[2020 NEAP Ext 1 Trial Q10] The diagram shows $OABC$, a rhombus in which $\overrightarrow{OA} = \overrightarrow{CB} = \underline{a}$ and $\overrightarrow{OC} = \overrightarrow{AB} = \underline{c}$.



To prove that the diagonals of $OABC$ are perpendicular, it is required to show that

- | | |
|---|---|
| (A) $(\underline{a} + \underline{c}) \cdot (\underline{a} + \underline{c}) = 0$ | (C) $(\underline{a} - \underline{c}) \cdot (\underline{a} + \underline{c}) = 0$ |
| (B) $(\underline{a} - \underline{c}) \cdot (\underline{a} - \underline{c}) = 0$ | (D) $\underline{a} \cdot \underline{c} = 0$ |

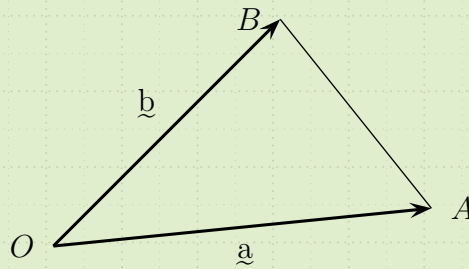
**Example 31**

[2020 JRAHS Ext 1 Trial Q10] It is given that $\underline{a} = \begin{pmatrix} -3 \\ m \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. If the angle between vector $\underline{a}$ and $\underline{b}$ is obtuse angle, what is true for the value of m ?

- | | |
|---------------------------------------|---------------------------------------|
| (A) $m < 4$ | (C) $m > 4$ |
| (B) $m < 4$ and $m \neq -\frac{9}{4}$ | (D) $m \neq 4$ and $m > -\frac{9}{4}$ |


Example 32

[2020 Independent Ext 1 Trial Q11] In the diagram, OAB is an acute angled triangle in which $OA = OB$. The vector $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OB} = \underline{b}$.



- (i) Show that $\cos \angle OAB = \frac{\underline{a} \cdot \underline{a} - \underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{a} - \underline{b}|}$ and $\cos \angle OBA = \frac{\underline{b} \cdot \underline{b} - \underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{a} - \underline{b}|}$ 4
- (ii) Hence show that $\angle OAB = \angle OBA$ by using vector methods. 2

**Example 33**

[2020 JRAHS Ext 1 Trial Q12] In the isosceles triangle ABC , $|\vec{AB}| = |\vec{AC}|$. D is the midpoint of side AB and E is the midpoint of side AC . $\vec{CD}$ is perpendicular to $\vec{BE}$.

- (i) Draw the diagram and label $\angle BAC = \theta$ and $|\vec{AD}| = r$. 1
- (ii) Hence noting that $\vec{CD}$ may be written as $\vec{AD} - \vec{AC}$, or otherwise, use vector methods to find the value of $\angle BAC$. 3

**Further exercises**

Ex 3I-3J (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8C (Pender et al., 2019)

- All questions

3.3 Other geometric proofs



Important note

Geometric proofs involving vectors are vastly different to the Euclidean Geometry proofs!



Steps

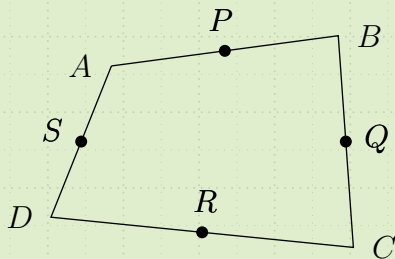
Use the following as a guide to these proofs:

- !
- Introduce vectors, by choosing one of the vertices or a point outside the figure as a reference point/origin.
- Use the and of vectors to complete a triangle.
- vectors are of each other.
- The is zero when two - vectors are at
- The may also allow access to the angle between some of the vectors.



Example 34

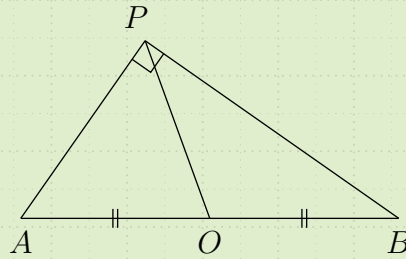
[Ex 8D Q2] **¶¶** P, Q, R and S are midpoints of AB, BC, CD and DA respectively. Let $\overrightarrow{AB} = \underline{a}$, $\overrightarrow{BC} = \underline{b}$ and $\overrightarrow{AD} = \underline{d}$ and $\overrightarrow{DC} = \underline{c}$.



- Explain why $\underline{a} + \underline{b} = \underline{d} + \underline{c}$.
- Express $\overrightarrow{PQ}$ in terms of $\underline{a}$ and $\underline{b}$.
- Express $\overrightarrow{SR}$ in terms of $\underline{d}$ and $\underline{c}$.
- Hence show that $\overrightarrow{PQ} = \overrightarrow{SR}$.
- Deduce that the quadrilateral $PQRS$ is a parallelogram.

**Example 35**

(Pender et al., 2019) An interval AB with midpoint O subtends a right angle at point P .



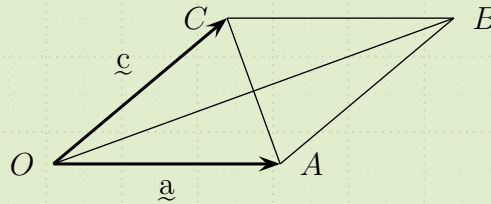
Prove that a circle with diameter AB passes through P .

**Important note**

Commence by letting $\underline{a} = \overrightarrow{OA}$ and $\underline{p} = \overrightarrow{OP}$.


Example 36

[Ex 8D Q6] In the diagram $OABC$ is a parallelogram whose diagonals OB and AC are equal. The points A and C have respective position vectors $\underline{a}$ and $\underline{c}$ relative to O .



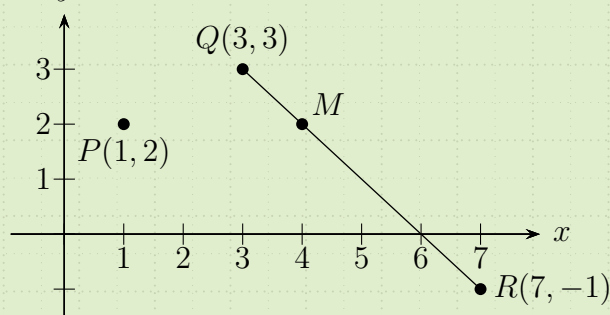
- Explain why $\overrightarrow{CB} = \underline{a}$.
- Write $\overrightarrow{AC}$ in terms of $\underline{c}$ and $\underline{a}$.
- Explain why $|\underline{c} + \underline{a}| = |\underline{c} - \underline{a}|$.
- Use the result in the previous part, and the fact that $|\underline{y}|^2 = \underline{y} \cdot \underline{y}$, to show that $\underline{a} \cdot \underline{c} = 0$.
- What conclusions can be made about a parallelogram whose diagonals are equal?

**Example 37**

[Ex 8D Q10] “ Use vectors to prove that the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides.

**Example 38**

Given three points $P(1, 2)$, $Q(3, 3)$ and $R(7, -1)$,

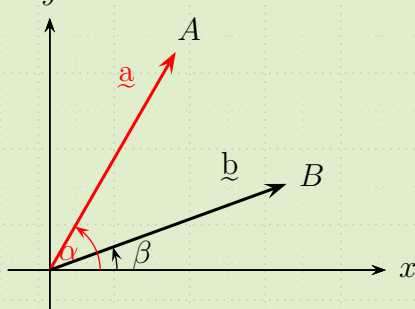


- Express $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ in terms of $\underline{i}$ and $\underline{j}$.
- If $\overrightarrow{QM} = \frac{1}{4}\overrightarrow{QR}$, express $\overrightarrow{PQ}$ and $\overrightarrow{OM}$ in terms of $\underline{i}$ and $\underline{j}$.
- Deduce that $PQ \parallel OM$.

Answer: (a) $\overrightarrow{OP} = \underline{i} + 2\underline{j}$, $\overrightarrow{OQ} = 3\underline{i} + 3\underline{j}$, $\overrightarrow{OR} = 7\underline{i} - \underline{j}$. (b) $\overrightarrow{PQ} = 2\underline{i} + \underline{j}$, $\overrightarrow{OM} = 4\underline{i} + 2\underline{j}$.

**Example 39**

In the following figure, the position vectors of the points A and B with respect to the origin are $\underline{a}$ and $\underline{b}$ respectively. They are unit vectors making angles α and β with the positive direction of the x axis.

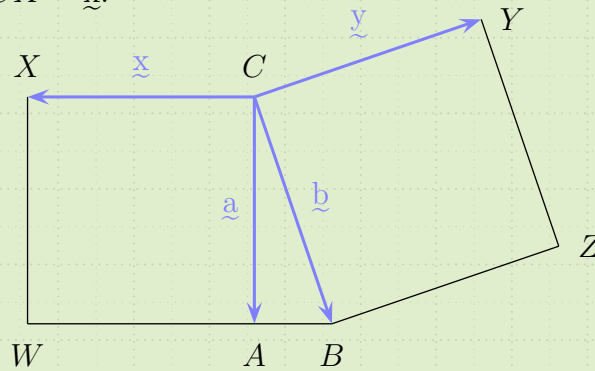


Prove that:

- (a) $\underline{a} = \cos \alpha \underline{i} + \sin \alpha \underline{j}$ and $\underline{b} = \cos \beta \underline{i} + \sin \beta \underline{j}$
- (b) $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$.


Example 40

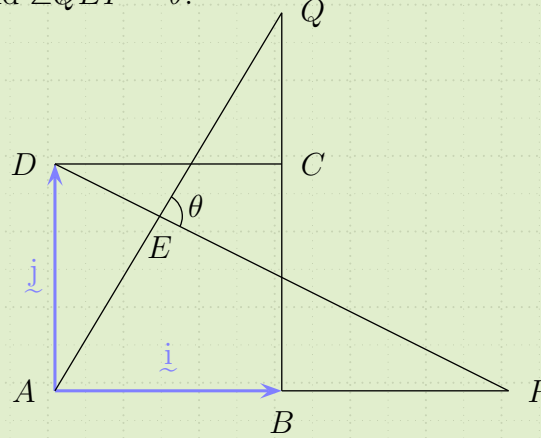
(So & Wong, 1987, Ex 12-5) In the figure, $ACXW$ and $BCYZ$ are squares. $\overrightarrow{CA} = \underline{a}$, $\overrightarrow{CB} = \underline{b}$, $\overrightarrow{CY} = \underline{y}$ and $\overrightarrow{CX} = \underline{x}$.



- (a) Prove that $\underline{a} \cdot \underline{b} + \underline{y} \cdot \underline{x} = 0$.
- (b) Deduce that $AY \perp BX$.


Example 41

A In the diagram, $ABCD$ is a square with $\overrightarrow{AB} = \underline{i}$ and $\overrightarrow{AD} = \underline{j}$. P and Q are respectively points on AB and BC produced with $BP = k$ and $CQ = m$. AQ and DP intersect at E and $\angle QEP = \theta$.



- (a) By calculating $\overrightarrow{AQ} \cdot \overrightarrow{DP}$, find $\cos \theta$ in terms of m and k .
- (b) Given that $\frac{DE}{EP} = \frac{1}{4}$,
 - i. Express $\overrightarrow{AE}$ in terms of k .
 - ii. Let $\frac{AE}{AQ} = r$. Express $\overrightarrow{AE}$ in terms of r and m .
 - iii. If $\theta = 90^\circ$, use the results above to find the values of k , m and r .

Answer: (a) $\frac{k-m}{\sqrt{(m^2+2m+2)(k^2+2k+2)}}$ (b) i. $\frac{1+k}{5}\underline{i} + \frac{4}{5}\underline{j}$ ii. $r\underline{i} + r(1+m)\underline{j}$ iii. $r = \frac{2}{5}$, $m = k = 1$

**Further exercises**

Ex 3L (Haese et al., 2015)
(More introductory type questions)

- Every second question

Ex 8D (Pender et al., 2019)

- All questions

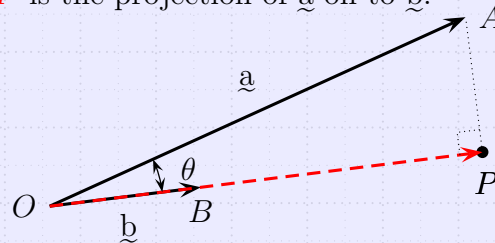
3.4 Projection of a vector on to another vector

Definition 15

The *projection* of a vector $\underline{a}$ on another vector $\underline{b}$ is a vector parallel to $\underline{b}$ with the following notation:

$$\text{proj}_{\underline{b}} \underline{a}$$

In the diagram given, $\overrightarrow{OP}$ is the projection of $\underline{a}$ on to $\underline{b}$.



Assumption: $\underline{b}$ is a

Important note

Plain English: the projection of vector $\underline{a}$ on to vector $\underline{b}$ is the
..... from ... on to ...

Laws/Results

The projection of $\underline{a}$ on to $\underline{b}$, $\underline{b} \neq \underline{0}$:

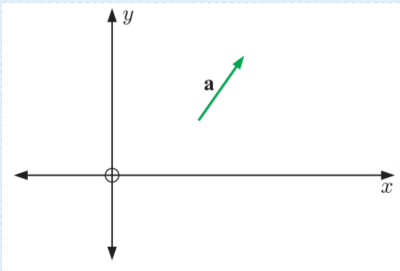
$$\text{proj}_{\underline{b}} \underline{a} = \dots\dots\dots$$

GeoGebra

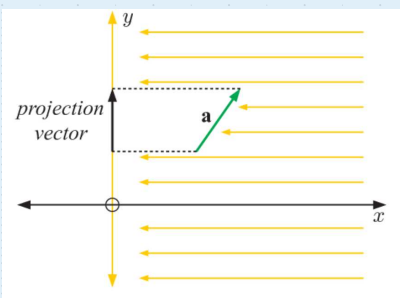
Projection of vector $\underline{a}$ on to $\underline{b}$

 Fill in the spaces

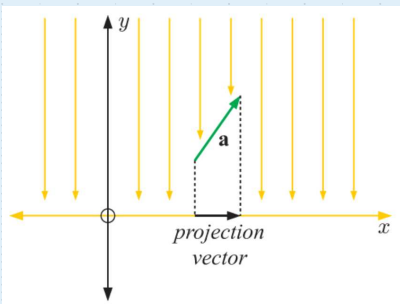
The following diagrams are taken from Haese et al. (2015), explanations paraphrased:



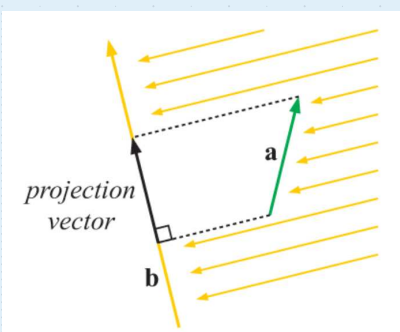
- Consider a vector $\underline{a}$ in the plane.



- If a light is shone on to the vector from the, it will cast a on the y axis.
- This is the of $\underline{a}$ on the y axis
- Also known as the y component of the vector $\underline{a}$.
- If $\underline{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then $\text{proj}_{\underline{j}} \underline{a} = \dots$



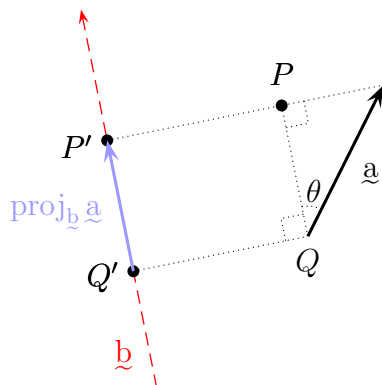
- Similarly, if a light is shone from the, the is the of $\underline{a}$ on the x axis.
- If $\underline{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then $\text{proj}_{\underline{i}} \underline{a} = \dots$



- If the light is shone through vector $\underline{a}$ on to another vector $\underline{b}$ so that the light is to $\underline{b}$, the cast on $\underline{b}$ is the of $\underline{a}$ on to $\underline{b}$.

3.4.1 Derivation

Consider the diagram:



- Suppose θ is the acute angle between $\underline{a}$ and $\underline{b}$.
- Give the relationship between PQ , $|a|$ and θ :

$$\frac{PQ}{|a|} = \dots\dots\dots$$

$$\therefore PQ = \dots\dots\dots$$

- As $\text{proj}_{\underline{b}} \underline{a}$ is a vector in the same $\dots\dots\dots$ to $\underline{b}$, it is also the unit vector in $\underline{b}$'s direction, multiplied by a scalar multiple:*

$$\begin{aligned} \text{proj}_{\underline{b}} \underline{a} &= \dots\dots\dots \hat{\underline{b}} \\ &= \underbrace{\dots\dots\dots}_{\text{scalar multiple, 'magnitude'}} \hat{\underline{b}} \\ &= \dots\dots\dots \frac{\underline{b}}{|\underline{b}|} \end{aligned} \quad (\ddagger)$$

- Recall also, that $\underline{a} \cdot \underline{b} = \dots\dots\dots$, rearranging:

$$\cos \theta = \dots\dots\dots$$

Substitute in the result from $(\ddagger)$:

$$\text{proj}_{\underline{b}} \underline{a} = \dots\dots\dots = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|^2} \underline{b}$$

Laws/Results

The projection of vector $\underline{a}$ on to vector $\underline{b}$:

$$\text{proj}_{\underline{b}} \underline{a} = \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|^2} \underline{b} = \frac{\underline{a} \cdot \underline{b}}{\underline{b} \cdot \underline{b}} \underline{b}$$

*See Section 2.1.1 on page 13

3.4.2 Examples and proofs



Example 42

Find the projection of $\underline{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ on to $\underline{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, and state the length of the vector.

Answer: $\text{proj}_{\underline{b}} \underline{a} = \frac{9}{34} \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, length $\frac{9}{\sqrt{34}}$



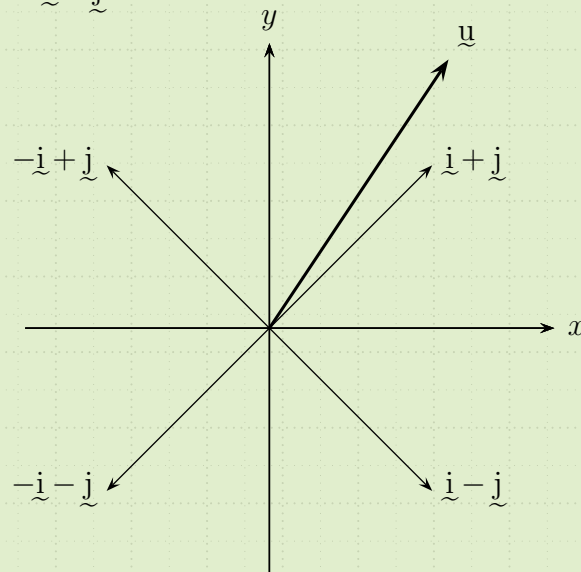
Example 43

[2020 JRAHS Ext 1 Trial Q11] It is given that $A(-2, 3)$, $B(4, -5)$, $C(-7, -6)$ and $D(-5, -2)$.

- | | | |
|-----|--|----------|
| i. | Find the vector projection of $\overrightarrow{AB}$ on to $\overrightarrow{CD}$. | 2 |
| ii. | What is the vector component of $\overrightarrow{AB}$ perpendicular to $\overrightarrow{CD}$? | 1 |

**Example 44**

[2022 Ext 1 HSC Q6] The following diagram shows the vector $\underline{u}$ and the vectors $\underline{i} + \underline{j}$, $-\underline{i} + \underline{j}$, $-\underline{i} - \underline{j}$ and $\underline{i} - \underline{j}$.



(Drawn to scale)

Which statement regarding this diagram could be true?

- (A) The projection of $\underline{u}$ onto $\underline{i} + \underline{j}$ is the vector $1.1\underline{i} + 1.8\underline{j}$.
- (B) The projection of $\underline{u}$ onto $-\underline{i} + \underline{j}$ is the vector $-0.4\underline{i} + 0.4\underline{j}$.
- (C) The projection of $\underline{u}$ onto $-\underline{i} - \underline{j}$ is the vector $3.2\underline{i} + 3.2\underline{j}$.
- (D) The projection of $\underline{u}$ onto $\underline{i} - \underline{j}$ is the vector $0.5\underline{i} - 0.5\underline{j}$.

**Example 45**

[2022 Ext 1 HSC Q14] (3 marks) The vectors $\underline{u}$ and $\underline{v}$ are not parallel. The vector $\underline{p}$ is the projection of $\underline{u}$ onto the vector $\underline{v}$.

The vector $\underline{p}$ is parallel to $\underline{v}$ so it can be written as $\lambda_0 \underline{v}$ for some real number λ_0 . (Do NOT prove this).

Prove that $|\underline{u} - \lambda \underline{v}|$ is smallest when $\lambda = \lambda_0$ by showing that, for all real numbers λ , $|\underline{u} - \lambda_0 \underline{v}| \leq |\underline{u} - \lambda \underline{v}|$.

**Further exercises**

Ex 3K (Haese et al., 2015)
(More introductory type questions)

- Every second subpart

Ex 8E (Pender et al., 2019)

- All questions

Section A

Past examination questions

- Most questions in this section originate from various VCEs.
- Two additional terms which are not used in the NSW Syllabuses but have equivalents:

Definition 16

Vector resolute is synonymous with the the vector projection.

Definition 17

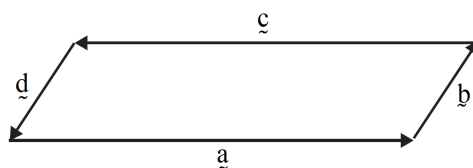
Scalar projection is the length of the vector projection, with a negative sign if the projection has an opposite direction with respect to $\underline{b}$

A.1 2006 VCE Specialist Mathematics

A.1.1 Paper 2 Section 1

15. In the parallelogram shown, $|\underline{a}| = 2|\underline{b}|$.

1



Which one of the following statements is true?

- | | | |
|---|---|---|
| (A) $\underline{a} = 2\underline{b}$ | (C) $\underline{b} - \underline{d} = \underline{0}$ | (E) $\underline{a} - \underline{b} = \underline{c} - \underline{d}$ |
| (B) $\underline{a} + \underline{b} = \underline{c} + \underline{d}$ | (D) $\underline{a} + \underline{c} = \underline{0}$ | |

A.1.2 Paper 2 Section 2

Question 8

Point A has position vector $\underline{a} = -\underline{i} - 4\underline{j}$, point B has position vector $\underline{b} = 2\underline{i} - 5\underline{j}$, point C has position vector $\underline{c} = 5\underline{i} - 4\underline{j}$, and point D has position vector $\underline{d} = 2\underline{i} + 5\underline{j}$ relative to the origin O .

- (a) Show that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular. 2
- (b) Use a vector method to find the cosine of $\angle ADC$, the angle between $\overrightarrow{DA}$ and $\overrightarrow{DC}$. 2
- (c) Find the cosine of $\angle ABC$, and hence show that $\angle ADC$ and $\angle ABC$ are supplementary. 3

Point P has position vector $\underline{p} = 2\underline{i}$.

- (d) Use the cosine of $\angle APC$ and an appropriate trigonometric formula to prove that $\angle APC = 2\angle ADC$. 3

⚠ Tip: attempt this question after **Topic 9: Radians & (x1) Further Trigonometric Identities 1**. Requires $\cos 2x = 2\cos^2 x - 1$.

A.2 2008 VCE Specialist Mathematics

A.2.1 Paper 2 Section 1

17. If P, Q and R are three collinear points with position vectors $\underline{p}, \underline{q}$ and $\underline{r}$ respectively, where Q lies between P and R . If $|\overrightarrow{QR}| = \frac{1}{2}|\overrightarrow{PQ}|$, then $\underline{r}$ is equal to 1

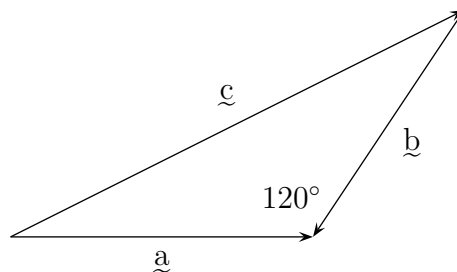
- (A) $\frac{3}{2}\underline{q} - \frac{1}{2}\underline{p}$ (C) $\frac{3}{2}\underline{q} - \frac{3}{2}\underline{p}$ (E) $\frac{3}{2}\underline{p} - \frac{3}{2}\underline{q}$
- (B) $\frac{3}{2}\underline{p} - \frac{1}{2}\underline{q}$ (D) $\frac{1}{2}\underline{p} - \frac{3}{2}\underline{q}$

A.3 2009 VCE Specialist Mathematics

A.3.1 Paper 2 Section 1

17. Vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ are shown below.

1



From the diagram it follows that

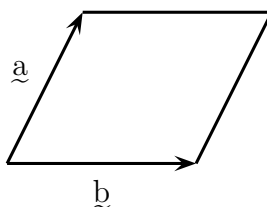
- (A) $|\underline{c}|^2 = |\underline{a}|^2 + |\underline{b}|^2$ (D) $|\underline{c}|^2 = |\underline{a}|^2 + |\underline{b}|^2 + |\underline{a}| |\underline{b}|$
 (B) $|\underline{c}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - |\underline{a}| |\underline{b}|$ (E) $|\underline{c}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - |\underline{a} \cdot \underline{b}|$
 (C) $|\underline{c}|^2 = |\underline{a}|^2 + |\underline{b}|^2 + |\underline{a} \cdot \underline{b}|$

A.4 2011 VCE Specialist Mathematics

A.4.1 Paper 2 Section 1

10. The diagram below shows a rhombus, spanned by the two vectors $\underline{a}$ and $\underline{b}$.

1



It follows that

- (A) $\underline{a} \cdot \underline{b} = 0$ (C) $(\underline{a} + \underline{b}) \cdot (\underline{a} - \underline{b}) = 0$ (D) $|\underline{a} + \underline{b}| = |\underline{a} - \underline{b}|$
 (B) $\underline{a} = \underline{b}$ (E) $2\underline{a} + 2\underline{b} = \underline{0}$

A.5 2018 VCE Specialist Mathematics

A.5.1 Paper 2 Section 1

11. Consider the vectors given by $\underline{a} = m\underline{i} + \underline{j}$ and $\underline{b} = \underline{i} + m\underline{j}$, where $m \in \mathbb{R}$. If the acute angle between $\underline{a}$ and $\underline{b}$ is 30° , then m equals

1

- (A) $\sqrt{2} \pm 1$ (C) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (E) $\frac{\sqrt{39}}{13}$
 (B) $2 \pm \sqrt{3}$ (D) $\frac{\sqrt{3}}{4 - \sqrt{3}}$

12. If $|\underline{a} + \underline{b}| = |\underline{a}| + |\underline{b}|$ and $\underline{a}, \underline{b} \neq 0$, which one of the following is **necessarily true**? 1

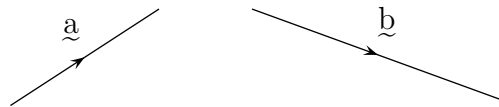
- (A) $\underline{a}$ is parallel to $\underline{b}$ (D) $\underline{a} = -\underline{b}$
 (B) $|\underline{a}| = |\underline{b}|$ (E) $\underline{a}$ is perpendicular to $\underline{b}$
 (C) $\underline{a} = \underline{b}$

A.6 2020 Ext 1 HSC

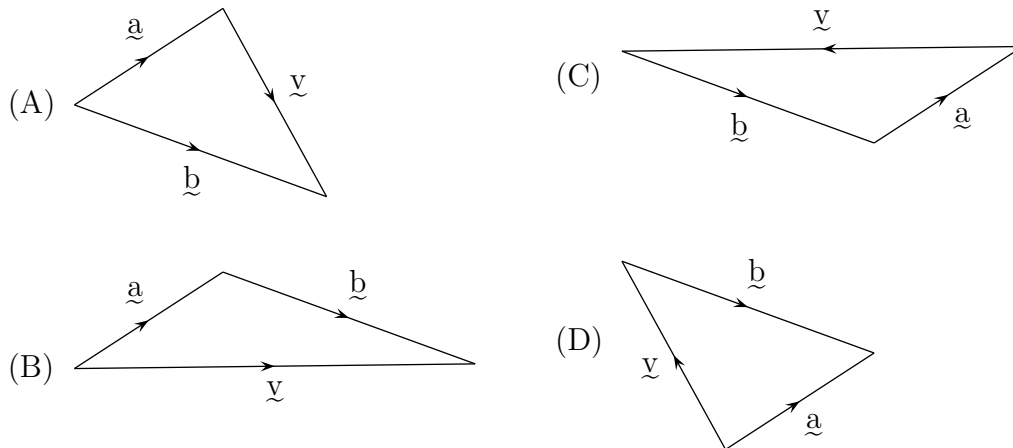
4. Maria starts at the origin and walks along all of the vector $2\underline{i} + 3\underline{j}$, then walks along all of the vector $3\underline{i} - 2\underline{j}$ and finally along all of the vector $4\underline{i} - 3\underline{j}$. 1

How far from the origin is she?

- (A) $\sqrt{77}$ (C) $2\sqrt{13} + \sqrt{5}$
 (B) $\sqrt{85}$ (D) $\sqrt{5} + \sqrt{7} + \sqrt{13}$
 6. The vectors $\underline{a}$ and $\underline{b}$ are shown. 1



Which diagram below shows the vector $\underline{v} = \underline{a} - \underline{b}$?



9. The projection of the vector $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ on to the line $y = 2x$ is $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$. 1

The point $(6, 7)$ is reflected in the line $y = 2x$ to a point A .

What is the position vector of the point A ?

- (A) $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$ (B) $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$ (C) $\begin{pmatrix} -6 \\ 7 \end{pmatrix}$ (D) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Question 11

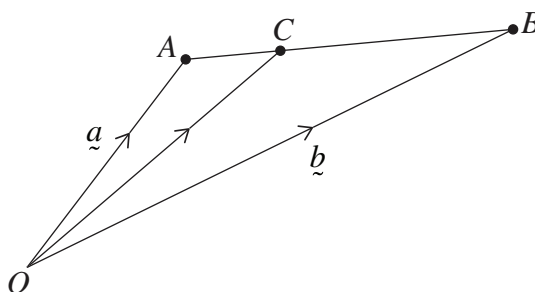
- (b) For what value(s) of a are the vectors $\begin{pmatrix} a \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2a-3 \\ 2 \end{pmatrix}$ perpendicular? **3**

A.7 2020 Ext 2 HSC

This question does not contain any Extension 2 specific content, and so is placed in this summary instead.

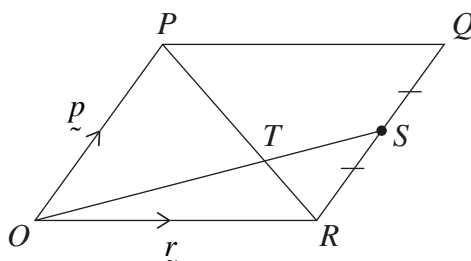
Question 15

- (b) The point C divides the interval AB so that $\frac{CB}{AC} = \frac{m}{n}$. The position vectors of A and B are $\underline{a}$ and $\underline{b}$ respectively, as shown in the diagram.



- i. Show that $\overrightarrow{AC} = \frac{n}{m+n} (\underline{b} - \underline{a})$. **2**
- ii. Prove that $\overrightarrow{OC} = \frac{m}{m+n} \underline{a} + \frac{n}{m+n} \underline{b}$. **1**

Let $OPQR$ be a parallelogram with $\overrightarrow{OP} = \underline{p}$ and $\overrightarrow{OR} = \underline{r}$. The point S is the midpoint of QR and T is the intersection of PR and OS , as shown in the diagram.



- iii. Show that $\overrightarrow{OT} = \frac{2}{3} \underline{r} + \frac{1}{3} \underline{p}$. **3**
- iv. Using parts (ii) and (iii), or otherwise, prove that T is the point that divides the interval PR in the ratio $2 : 1$. **1**

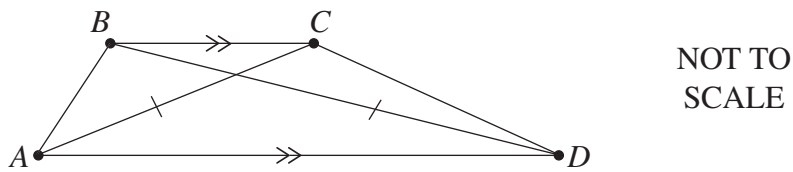
A.8 2021 Ext 1 HSC

Question 14

- (a) A plane needs to travel to a destination that is on a bearing of 063° . The engine is set to fly at a constant 175 km/h . However, there is a wind from the south with a constant speed of 42 km/h . **3**

On what constant bearing, to the nearest degree, should the direction of the plane be set in order to reach the destination? **Answer:** 075°

- (c) i. For vector $\underline{v}$, show that $\underline{v} \cdot \underline{v} = |\underline{v}|^2$. **1**
- ii. In the trapezium $ABCD$, BC is parallel to AD and $|\overrightarrow{AC}| = |\overrightarrow{BD}|$. **3**



NOT TO
SCALE

Let $\underline{a} = \overrightarrow{AB}$, $\underline{b} = \overrightarrow{BC}$ and $\overrightarrow{AD} = k\overrightarrow{BC}$, where $k > 0$.

Using part (i) or otherwise, show $2\underline{a} \cdot \underline{b} + (1 - k) |\underline{b}|^2 = 0$.

A.9 2022 Ext 1 HSC

8. The angle between two unit vectors $\underline{a}$ and $\underline{b}$ is θ and $|\underline{a} + \underline{b}| < 1$. **1**

Which of the following best describes the possible range of values of θ ?

(A) $0 \leq \theta < \frac{\pi}{3}$

(D) $\frac{2\pi}{3} < \theta \leq \pi$

(B) $0 \leq \theta < \frac{2\pi}{3}$

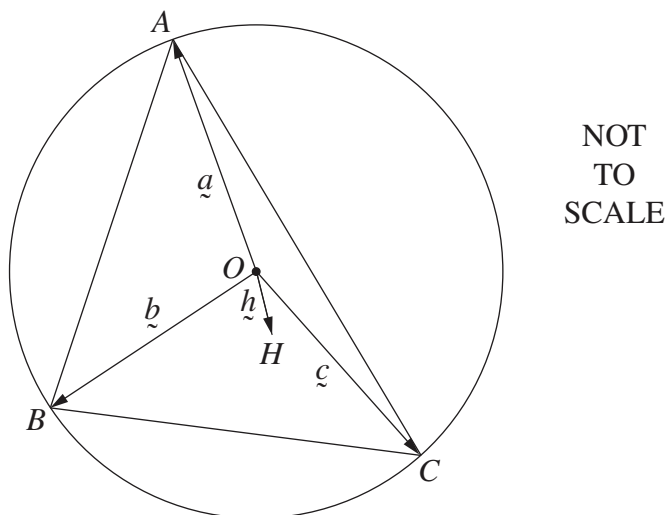
Note: attempt this after Topic 9 (Radians) has been completed.

(C) $\frac{\pi}{3} < \theta \leq \pi$

Question 13

- (a) Three different points A , B and C are chosen on a circle centred at O . **3**

Let $\underline{a} = \overrightarrow{OA}$, $\underline{b} = \overrightarrow{OB}$ and $\underline{c} = \overrightarrow{OC}$. Let $\underline{h} = \underline{a} + \underline{b} + \underline{c}$ and let H be the point such that $\overrightarrow{OH} = \underline{h}$, as shown in the diagram.



Show that $\overrightarrow{BH}$ and $\overrightarrow{CA}$ are perpendicular.

A.10 2023 Ext 1 HSC

6. Given two non-zero vectors $\underline{a}$ and $\underline{b}$, let $\underline{c}$ be the projection of $\underline{a}$ onto $\underline{b}$. **1**

What is the projection of $10\underline{a}$ onto $2\underline{b}$?

- (A) $2\underline{c}$ (B) $5\underline{c}$ (C) $10\underline{c}$ (D) $20\underline{c}$

Question 14

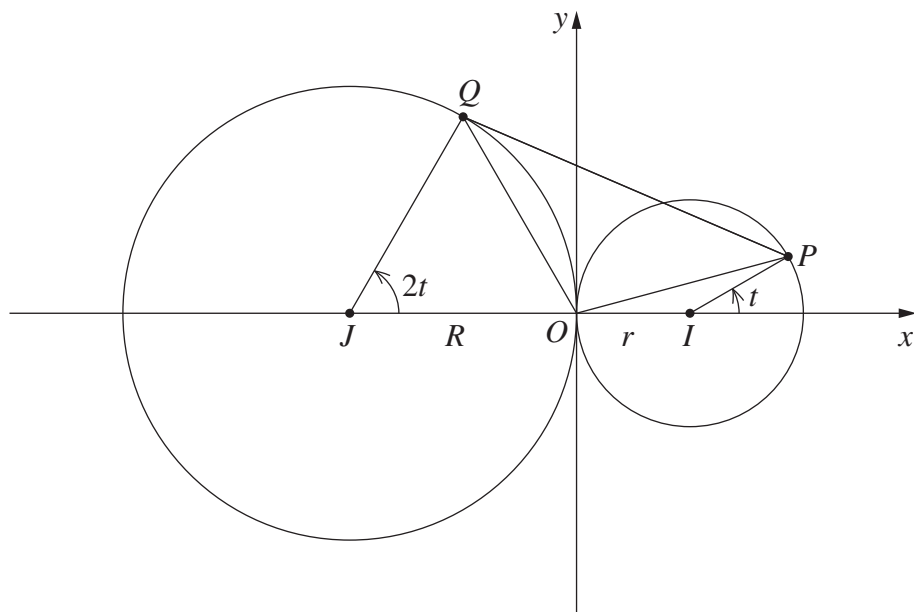
- (c) i. Given a non-zero vector $\begin{pmatrix} p \\ q \end{pmatrix}$, it is known that the vector $\begin{pmatrix} q \\ -p \end{pmatrix}$ is **3**
perpendicular to $\begin{pmatrix} p \\ q \end{pmatrix}$ and has the same magnitude. (Do NOT prove this).

Points A and B have position vectors $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, respectively.

Using the given information, or otherwise, show that the area of $\triangle OAB$ is $\frac{1}{2} |a_1 b_2 - a_2 b_1|$.

- ii. The point P lies on the circle centred at $I(r, 0)$ with radius $r > 0$, such that $\overrightarrow{IP}$ makes an angle of t to the horizontal. **4**

The point Q lies on the circle centred at $J(-R, 0)$ with radius $2t$ to the horizontal.



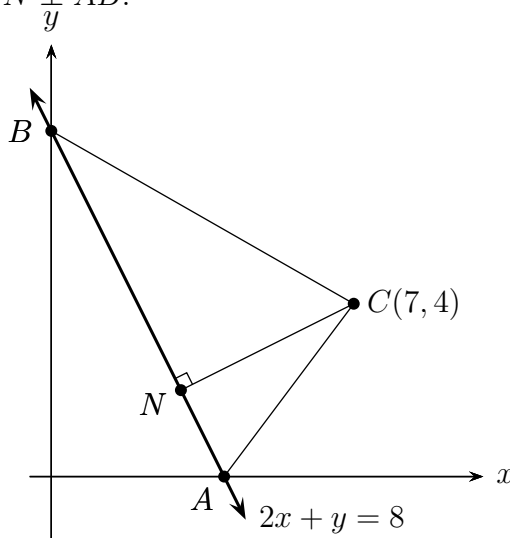
Note that $\overrightarrow{OP} = \overrightarrow{OI} + \overrightarrow{IP}$ and $\overrightarrow{OQ} = \overrightarrow{OJ} + \overrightarrow{JQ}$.

Using part (i), or otherwise, find the values of t , where $-\pi \leq t \leq \pi$, that maximise the area of $\triangle OPQ$.

A.11 2024 NBHS Assessment Task 1

Question 4

- (a) The diagram shows $\triangle ABC$, and the line $2x + y = 8$ with x and y intercepts at A and B respectively. The point C has coordinates $(7, 4)$ and N is a point on AB such that $CN \perp AB$.



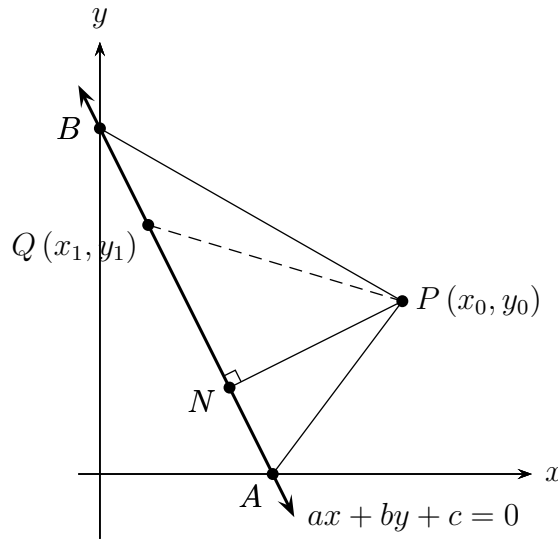
- i. Show that $\overrightarrow{BA} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$. 2
- ii. By using vector methods, show that 2

$$\cos \angle ABC = \frac{3}{\sqrt{13}}$$
- iii. Find a vector $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ such that $\underline{u}$ is perpendicular to $\overrightarrow{BA}$. 1
- iv. Hence show that 2

$$\overrightarrow{NC} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$
- v. Use vector methods to find the coordinates of N . 1

Question continues overleaf...

A line ℓ with equation $ax + by + c = 0$ is drawn in the Cartesian plane, and $P(x_0, y_0)$ is located off the line ℓ , and point $Q(x_1, y_1)$ which lies on the line ℓ . N is a point that is located on the line ℓ such that $PN \perp AB$.



You are given that $\underline{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is a vector that is perpendicular to $\overrightarrow{BA}$ (Do NOT prove this).

vi. By finding $\overrightarrow{QP}$ using the coordinates given, show that

3

$$\overrightarrow{NP} = \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}} \hat{\underline{v}}$$

vii. Briefly explain why the shortest distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is given by

1

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Answers

iii. Any scalar multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

v. $(3, 2)$

vii. Show. Insufficient to state the perpendicular distance formula.

iv. Show

vi. Hint: use $ax_1 + by_1 + c = 0$

NESA Reference Sheet – calculus based courses



NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced Mathematics Extension 1 Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\text{and } \alpha\beta\gamma = -\frac{d}{a}$$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

$$S = \frac{a}{1 - r}, |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2} ab \sin C$$

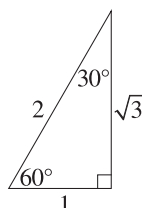
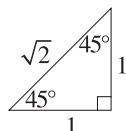
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2} r^2 \theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \quad \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \quad \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\text{If } t = \tan \frac{A}{2} \text{ then } \sin A = \frac{2t}{1+t^2}$$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)]$$

$$\sin^2 nx = \frac{1}{2} (1 - \cos 2nx)$$

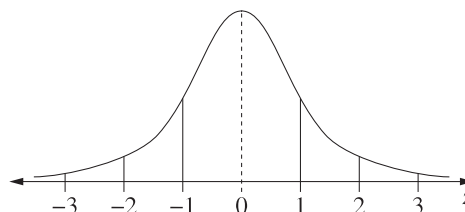
$$\cos^2 nx = \frac{1}{2} (1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z -scores between -1 and 1
- approximately 95% of scores have z -scores between -2 and 2
- approximately 99.7% of scores have z -scores between -3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E[(X - \mu)^2] = E(X^2) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x) dx$$

$$P(a < X < b) = \int_a^b f(x) dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function

Derivative

$$y = f(x)^n$$

$$\frac{dy}{dx} = n f'(x) [f(x)]^{n-1}$$

$$y = uv$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$y = g(u) \text{ where } u = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$y = \sin f(x)$$

$$\frac{dy}{dx} = f'(x) \cos f(x)$$

$$y = \cos f(x)$$

$$\frac{dy}{dx} = -f'(x) \sin f(x)$$

$$y = \tan f(x)$$

$$\frac{dy}{dx} = f'(x) \sec^2 f(x)$$

$$y = e^{f(x)}$$

$$\frac{dy}{dx} = f'(x) e^{f(x)}$$

$$y = \ln f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$y = a^{f(x)}$$

$$\frac{dy}{dx} = (\ln a) f'(x) a^{f(x)}$$

$$y = \log_a f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{(\ln a) f(x)}$$

$$y = \sin^{-1} f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \cos^{-1} f(x)$$

$$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \tan^{-1} f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{1 + [f(x)]^2}$$

Integral Calculus

$$\int f'(x) [f(x)]^n dx = \frac{1}{n+1} [f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x) \sin f(x) dx = -\cos f(x) + c$$

$$\int f'(x) \cos f(x) dx = \sin f(x) + c$$

$$\int f'(x) \sec^2 f(x) dx = \tan f(x) + c$$

$$\int f'(x) e^{f(x)} dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$$

$$\int f'(x) a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1} \frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1} \frac{f(x)}{a} + c$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int_a^b f(x) dx$$

$$\approx \frac{b-a}{2n} \{f(a) + f(b) + 2[f(x_1) + \dots + f(x_{n-1})]\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$|\underline{u}| = |x\underline{i} + y\underline{j}| = \sqrt{x^2 + y^2}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos \theta = x_1x_2 + y_1y_2,$$

$$\text{where } \underline{u} = x_1\underline{i} + y_1\underline{j}$$

$$\text{and } \underline{v} = x_2\underline{i} + y_2\underline{j}$$

$$\underline{r} = \underline{a} + \lambda \underline{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos \theta + i \sin \theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} [r(\cos \theta + i \sin \theta)]^n &= r^n(\cos n\theta + i \sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

$$x = a \cos(nt + \alpha) + c$$

$$x = a \sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

References

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