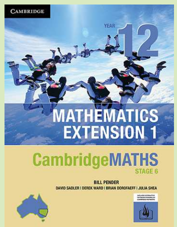
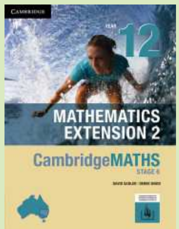
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1 (INCORPORATING EXTENSION 2) YEAR 12 COURSE

Topic summary and exercises:

With references to





X1 X2 **Mathematical Induction**

Name: ....

Initial version by H. Lam, November 2013 (former 3U content), July 2014 (Strong Induction). Last update November 30, 2021
Various corrections by students and members of the Department of Mathematics at North Sydney Boys High School and Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.
Parts of this document are also sourced from:

- Brown (2008)
- Fitzpatrick (1984, §23.4)
- (Haese, Haese, & Humphries, 2015b, Ch 6)

Symbols used

- Beware! Heed warning.
- Provided on NESA Reference Sheet
- Facts/formulae to memorise.
- Mathematics Extension 1 content.
- Mathematics Extension 2 content.
- Literacy: note new word/phrase.
- Further reading/exercises to enrich your understanding and application of this topic.
- Syllabus specified content
- Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME12-1 applies techniques involving proof or calculus to model and solve problems

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

Syllabus subtopics

ME-P1 Proof by Mathematical Induction

MEX-P2 Further Proof by Mathematical Induction

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Extension 1* and/or *CambridgeMATHS Extension 2* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Part I

X1 X2 Mathematics Extension 1 and 2 common content

Section 1

Induction into (mathematical) induction

Learning Goal(s)

Knowledge
Steps involved for a formal induction proof

Skills
Prove summation and divisibility results by induction

Understanding
What is *proof by induction* vs *proof by deduction*

By the end of this section am I able to:

18.1 Understand the nature of inductive proof, including the 'initial statement' and the inductive step

18.2 Prove results using mathematical induction

18.3 Identify errors in false 'proofs by induction', such as cases where only one of the required two steps of a proof by induction is true, and understand that this means that the statement has not been proved

1.1 Statements & propositions

- Mathematical induction is a *method* to prove a ....series.... of statements/propositions, usually ....infinitely.... many.

Example 1

State whether these propositions are true or false:

Propositon 1: ....$P(1)$:.... (T/F) $1+3=4$

Propositon 2: ....$P(2)$:.... (T/F) $2\times 7=15$

Propositon 3: ....$P(3)$:.... (T/F) The Prime Minister is handsome

- Notation for propositions:
  - (Serious) algebra required.
  - **1st** proposition: denote ....$P(1)$.... .
  - **2nd** proposition: denote ....$P(2)$.... .
  - k th proposition: denote ....$P(k)$.... .
  - n th proposition: denote ....$P(n)$.... .

1.2 General structure of induction proofs

1.2.1 Plain English

Proving a series of infinitely many propositions with finite time.

1.2.2 Logic

- Base case (normally, the first proposition, or $P(1)$):
  - Prove $P(1)$, is true
- Inductive step :
  - Assume some arbitrary statement $P(k)$ is true. (Inductive hypothesis)
  - If $P(k)$ is true, and overall all the statements are true, then the subsequent statement $P(k+1)$ should also be true.
    * Use algebraic skills to show the $k+1$-th statement is also true.

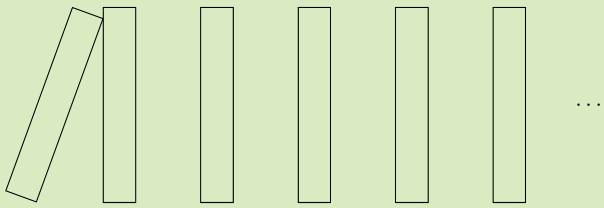
Watch multimedia

"Dominos": `https://fb.watch/9BL-OlVXWX/`

Example 2

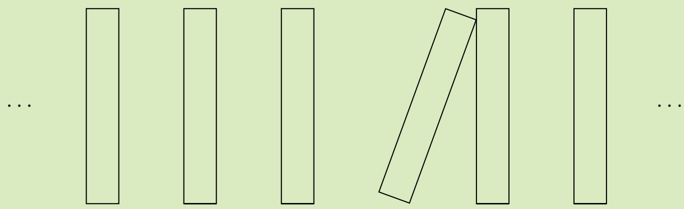
Domino analogy – well constructed set of dominos.

- Truth of k-th proposition $\to$ k-th domino knocked down.
- Base case : knock over first domino.



$P(1)$ $P(2)$ $P(3)$ $P(4)$ $P(5)$ $P(6)$

- Inductive step :
  - Find an arbitrary domino somewhere in the entire set of dominos, and assume it knocks over.
  - Show that the next domino also knocks over.



$P(k)$ $P(k+1)$?

- Hence, all dominos will eventually fall.

Section 2

Proofs involving sum/products

2.1 Introduction

Example 3

Prove by induction:

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

where $n \in \mathbb{N}$

Solution

- Let $P(n)$ be the proposition

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n+1)$$

- ....Base.... ....case.... : $P(1)$:

- ....Inductive........ ....step....
  - (Hypothesis) Assume that $P(k)$ is true for some $k \in \mathbb{N}$, $1 < k < n$, i.e.
  - Examine $P(k+1)$:

2.2 Basic proofs

Example 4

Prove by induction:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

where $n \in \mathbb{N}$

Example 5

Prove by mathematical induction that

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \cdots + n \times 2^{n-1} = 1 + (n-1)2^n$$

$\forall n \in \mathbb{Z}^+$, $n \geq 1$.

Example 6

[2016 Ext 1 HSC Q14]

i. Show that $4n^3 + 18n^2 + 23n + 9$ can be written as **1**

$$(n+1)\left(4n^2 + 14n + 9\right)$$

ii. Using the result in part (i), or otherwise, prove by mathematical induction that, for $n \geq 1$, **3**

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2n-1)(2n+1) = \frac{1}{3}n\left(4n^2 + 6n - 1\right)$$

Important note

Show part (i) adequately, otherwise it could look like it's being *fudged*.

Example 7

(a) Show that $(n+1)! = (n+1) \times n!$

(b) Use induction to show that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1$$

2.3 Harder proofs

Example 8

Prove by induction:

$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$$

where $n \in \mathbb{N}$

Example 9

Use mathematical induction to prove that, for all integers n with $n \geq 1$.

$$\frac{1}{\sec x} + \frac{1}{\sec 3x} + \frac{1}{\sec 5x} + \cdots + \frac{1}{\sec(2n-1)x} = \frac{\operatorname{cosec} x}{2\operatorname{cosec}(2nx)}$$

Further exercises

Ex 2A (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019)

- Q2-5, 8, 9, 12

2.4 When induction fails

Important note

A single ....counter........ ....example........ is sufficient to disprove a result.

Further reading

- brilliant.org - Flawed induction proofs, including when you *shouldn't* use induction.
- AMSI Proofs by Induction, p.7-8

Example 10

Pender et al. (2019, Ex 2A Q7)

(a) Attempt to prove by mathematical induction that

$$3 + 6 + 9 + \cdots + 3n = n(n+1) + 1$$

for all $n \in \mathbb{Z}^+$.

(b) Where does the proof break down?

Further exercises

Ex 2A (Pender et al., 2019)

- Q6-7

2.4.1 Further questions

1. Prove by mathematical induction that

$$1^2+3^2+\cdots(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$$

$\forall n\in\mathbb{Z}^+$, $n\geq 1$.

2. (a) By considering the sum of the terms of an arithmetic series, show that

$$(1+2+\cdots+n)^2=\frac{1}{4}n^2(n+1)^2$$

(b) Use induction to prove

$$1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$$

$\forall n\geq 1$.

3. Use mathematical induction to prove that, for all positive integers n,

$$1+2+4+\cdots+2^{n-1}=2^n-1$$

4. Use the Principle of Mathematical Induction to show that

$$2\times 1!+5\times 2!+10\times 3!+\cdots+(n^2+1)n!=n(n+1)!$$

$\forall n\in\mathbb{Z}^+$.

5. Use mathematical induction to prove that, for integers $n\geq 1$,

$$1\times 3+2\times 4+3\times 5+\cdots+n(n+2)=\frac{n}{6}(n+1)(2n+7)$$

6. Let $S(n)=\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\cdots+\dfrac{n}{(n+1)!}$, where $n\in\mathbb{Z}^+$.

(a) Use induction to show that

$$S_n=1-\frac{1}{(n+1)!}$$

$\forall n\in\mathbb{Z}^+$.

(b) Find the value of $\lim\limits_{n\to\infty}S_n$.

(c) Find the smallest positive integer n such that $|S_n-1|<1\times 10^{-6}$.

7. Use induction to show that for all positive integers $n\geq 1$,

$$\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+n}=\frac{2n}{n+1}$$

8. Use Mathematical Induction to show that for all positive integers $n\geq 1$,

$$\frac{3}{1\times 2\times 2}+\frac{4}{2\times 3\times 2^2}+\cdots+\frac{n+2}{n(n+1)2^n}=1-\frac{1}{(n+1)2^n}$$

9. Use mathematical induction to prove that

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

for all positive integers n.

10. (a) Use the fact that

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

to show that

$$1+\tan n\theta\tan(n+1)\theta=\cot\theta\,(\tan(n+1)\theta-\tan n\theta)$$

(b) ⚠ Use mathematical induction to prove that for all integers $n\geq 1$,

$$\tan\theta\tan 2\theta+\tan 2\theta\tan 3\theta+\cdots+\tan n\theta\tan(n+1)\theta=-(n+1)+\cot\theta\tan(n+1)\theta$$

11. (a) Prove that

$$\frac{\cos A-\cos(A+2B)}{2\sin B}=\sin(A+B)$$

(b) ⚠ By rewriting $\cos 2\theta$ in terms of $\sin^2\theta$ or otherwise, prove

$$\sin\theta+\sin 3\theta+\sin 5\theta+\cdots+\sin(2n-1)\theta=\frac{1-\cos 2n\theta}{2\sin\theta}$$

for all $n\in\mathbb{N}$

Section 3

Proofs involving divisibility

3.1 Divisibility revisited

Fill in the spaces

- If 20 is divisible by 5, then $20 = 5 \times 4$.
- If $4^n - 1$ is divisible by 3, then $4^n = 3M$.

Example 11

Prove that $4^n - 1$ is divisible by 3 $\forall n \in \mathbb{N}$.

Example 12

Prove that $5^n + 3$ is divisible by 4 $\forall n \in \mathbb{N}$.

Example 13

Prove that $4^n + 14$ is a multiple of 6 $\forall n \geq 1$.

Example 14

Prove by induction that $47^n + 53 \times 147^{n-1}$ is divisible by 100 $\forall n \geq 1$.

Example 15

Prove that $3^{3n} + 2^{n+2}$ is divisible by 5 $\forall n \in \mathbb{N}$.

Example 16

Prove that $3^n + 7^n$ is divisible by 10, if n is odd.

Important note

Inductive hypothesis still requires the assumption of the truth $P(k)$ where $k = 2m+1$, but need to examine $P(\ k+2\ )$.

Example 17

Prove that $n^3 + 5n$ is divisible by 3 $\forall n \in \mathbb{N}$.

- ⚠ (2013 CSSA Ext 1 Q14) Prove it's also divisible by 6.

Example 18

Prove that $x^n - 1$ is divisible by $(x-1)$ for all $n \in \mathbb{N}$.

3.2 When induction fails

Learning Goal(s)

Knowledge
Steps involved for a formal induction proof

Skills
Prove summation and divisibility results by induction

Understanding
What is *proof by induction* vs *proof by deduction*

By the end of this section am I able to:
18.4 Recognise situations where proof by mathematical induction is not appropriate

Example 19

(Haese et al., 2015b) Prove the following *incorrect* statements using induction:

1. $n^2 + 2$ is divisible by 3 $\forall n \in \mathbb{Z}^+$.

2. $3^n + 4$ is divisible by 7 for all $n \in \mathbb{Z}^+$.

Further exercises

Ex 2B (Pender et al., 2019)

- Q2-3

3.2.1 Further questions

1. Use mathematical induction to prove that, $\forall n \in \mathbb{N}$, $13 \times 6^n + 2$ is divisible by 5.

2. Prove that $4^n + 14$ is a multiple of 6 $\forall n \geq 1$.

3. Use the mathematical induction to prove that $7^{2n-1} + 5$ is divisible by 12 $\forall n \in \mathbb{N}$.

Part II

X2 Mathematics Extension 2 only content

Section 4

Proofs involving harder sums & divisibility

Learning Goal(s)

Knowledge
Steps involved for a formal induction proof

Skills
Algebraic manipulation of $P(k)$ to fit into the expression for $P(k+1)$

Understanding
Different proofs require various algebraic techniques

By the end of this section am I able to:

18.5 Prove results using mathematical induction where the initial value of n is greater than 1,and/or n does not increase strictly by 1.

18.6 Understand and use sigma notation to prove results for sums, for example:

$$\sum_{n=1}^{N} \frac{1}{(2n+1)(2n-1)} = \frac{N}{2N+1}$$

18.7 Understand and prove results using mathematical induction, including inequalities and results in algebra, calculus, probability and geometry.

18.8 Prove De Moivre's Theorem for integral powers using induction

4.1 Sum

Important note

Extension 2 sum proofs build upon Extension 1 proofs with

- sigma notation
- n commencing at values greater than 1, or not necessarily increasing by 1

Example 20

Prove by induction for $n \in \mathbb{N}$.

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

Further exercises

Ex 2E (Sadler & Ward, 2019)

- Q1, 3, 20

4.2 Divisibility

Important note

Extension 2 divisibility proofs build upon Extension 1 proofs with

- Polynomial division

Further exercises

Ex 2E (Sadler & Ward, 2019)

- 2, 4, 5, 16

4.2.1 Further questions

Some questions are sourced from Haese, Haese, and Humphries (2015a).

1. Let

$$S_n = 1 \times 2 + 2 \times 3 + \cdots + (n-1) \times n$$

Use mathematical induction to prove that, for all integers n with $n \geq 2$,

$$S_n = \frac{1}{3}(n-1)n(n+1)$$

2. Prove that $2^{3n} - 3^n$ is divisible by 5 $\forall n \geq 1$.

3. Use induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n.

4. (a) Use the method of mathematical induction to show that if x is a positive integer then $(1+x)^n - 1$ is divisible by x $\forall n \in \mathbb{N}$.

(b) Factorise $12^n - 4^n - 3^n + 1$. Without using induction again, use the previous part to show deduce that $12^n - 4n - 3^n + 1$ is divisible by 6 for all integers $n \geq 1$.

5. Prove by induction: $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$

6. Prove by induction: $2^{4n+3} + 3^{3n+1}$ is divisible by 11 $\forall n \in \mathbb{Z}$, $n \geq 0$.

7. Prove by induction: $\dfrac{2^n - (-1)^n}{3}$ is an odd number for all $n \in \mathbb{Z}^+$.

8. Prove by mathematical induction that

$$\sum_{r=1}^{n} r \times r! = (n+1)! - 1$$

9. Use mathematical induction to prove for all integers $n \geq 3$,

$$\left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{5}\right)\cdots\left(1 - \frac{2}{n}\right) = \frac{2}{n(n-1)}$$

10. Prove by induction that, for all integers $n \geq 1$,

$$(n+1)(n+2)\cdots(2n-1)2n = 2^n\left(1 \times 3 \times 5 \times \cdots \times (2n-1)\right)$$

11. Prove by induction that

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 9 $\forall n \in \mathbb{Z}^+$.

Section 5

Proofs involving inequalities, calculus, geometry and well known theorems

5.1 Inequalities

- If $x > 5$, then ....$x > -4$.....
- Look for quantity larger/smaller than what needs to be proven.

Example 21

Prove for $n \geq 2$, that

$$3^n > 1 + 2n$$

Solution

Steps

- Let $P(n)$ be the proposition $3^n > 1 + 2n$, $n \geq 2$.
- Base case $P(2)$:
- Inductive step:

Example 22

Prove by induction for $n \geq 6$,

$$n^2 - 11n + 30 \geq 0$$

Example 23

Prove by induction for $n \geq 5$,

$$2^n > n^2$$

Example 24

Prove for $x > 0$, $n \geq 1$,

$$(1+x)^n \geq 1 + nx$$

Example 25

Prove by induction, for x, $y > 0$ and $n \geq 2$,

$$(x+y)^n > x^n + y^n$$

Example 26

Prove by induction $\forall n \geq 2$,

$$12^n > 7^n + 5^n$$

Further exercises

Ex 2E (Sadler & Ward, 2019)

- 6-8, 15, 18

5.1.1 Further questions

1. Use the principle of mathematical induction to show that $4^n - 1 - 7n > 0$ for all integers $n \geq 2$.

2. Use the method of Mathematical Induction to show that $n! > 2^n$ for all positive integers $n \geq 4$.

3. Use Mathematical Induction to show that $5^n > 4^n + 3^n$ for all integers $n \geq 3$.

4. (a) Show that for $k > 0$,

$$\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$$

(b) Use mathematical induction to prove that $\forall n \geq 2, n \in \mathbb{Z}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

5.2 Functions/Calculus proofs

Example 27

[2009 Ext 1 HSC Q7]

i. Use differentiation from first principles to show that **1**

$$\frac{d}{dx}(x) = 1$$

ii. Use mathematical induction and the product rule for differentiation to prove that $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ for $n \in \mathbb{Z}^+$. **2**

Example 28

Use induction to show that

$$\cos(x + n\pi) = (-1)^n \cos x$$

$\forall n \in \mathbb{Z}^+$, $n \geq 1$.

Example 29

A function $f(x)$ is such that $f(x) > 0$ for $x \in \mathbb{R}$ and $f(a+b) = f(a) \times f(b)$ for $a, b \in \mathbb{R}$.

(a) Show that $f(0) = 1$ and deduce that $f(-x) = \dfrac{1}{f(x)}$.

(b) Use induction to show that

$$f(nx) = \Big(f(x)\Big)^n$$

for all positive integers n.

(c) Without using induction again, deduce that

$$f(-nx) = \Big(f(x)\Big)^{-n}$$

for all positive integers n.

5.3 Geometric proofs

Example 30

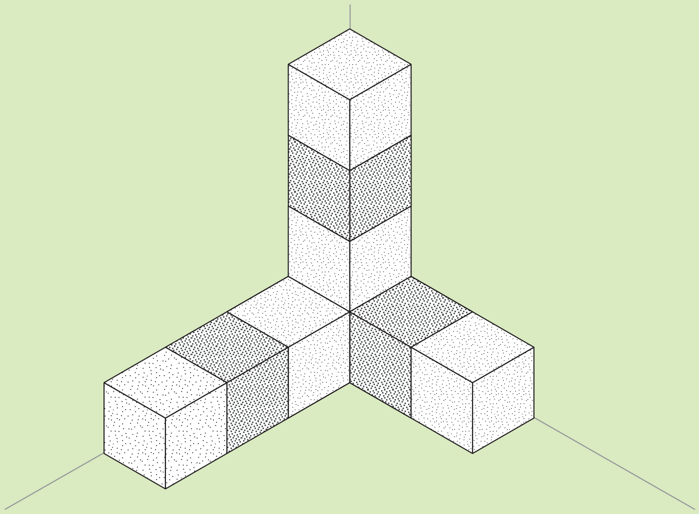
Prove by induction, that the sum of the angles of a polygon of n sides is $(2n - 4)$ right angles, where $n \geq 3$.

Example 31

[Ex 2E Q13] Sadler and Ward (2019) Suppose there are n lines in a plane, arranged such that no three of the lines are concurrent, and no two of the lines are parallel. Show that for $n \geq 1$, n such lines divide the plane into $\dfrac{1}{2}\left(n^2 + n + 2\right)$ regions.

Example 32

[2020 Ext 2 HSC Sample Q12] Two vertical walls and the floor meet at a corner of a room. One cube is placed in the corner. A solid shape is then formed by placing identical cubes to form horizontal rows on the floor against the walls or by stacking vertically against the two walls. An example is the solid shape shown in the diagram. This example is formed from nine cubes.



Let n be the number of cubes used to make a solid shape in this way.

Use mathematical induction to show that the number of exposed faces of the cubes in this shape is $2n + 1$.

Further exercises

Ex 2E (Sadler & Ward, 2019) - note some of already been done in class.

- Q12-14

5.4 Well known theorems

Example 33

The Binomial Theorem

(a) Prove *Pascal's formula*:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

(b) Hence, prove the *Binomial Theorem* by induction for $n \geq 1$:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Example 34

De Moivre's Theorem Prove *De Moivre's Theorem* by induction:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for

(a) $n \in \mathbb{Z}^+$.

(b) $n \in \mathbb{Z}^-$.

Section 6

Proofs involving recurrence relations

Example 35

[2011 NSGHS Ext 2 Trial Q8] The sequence u_1, u_2, u_3, $\ldots$ is defined by

$$u_1 = 2 \quad \text{and} \quad u_{k+1} = 2u_k + 1$$

i. Prove by induction that, for all integers $n \geq 1$, **3**

$$u_n = 3 \times 2^{n-1} - 1$$

ii. Show that **2**

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n+2)$$

Example 36

[2018 Independent Ext 2 Trial Q14] (3 marks) A sequence of numbers T_1, T_2, T_3 is given by $T_1 = 2$ and $T_n = \dfrac{5T_{n-1} - 3}{3T_{n-1} - 1}$ for $n \geq 2$.

Use mathematical induction to show that $T_n = \dfrac{3n+1}{3n-1}$ for all positive integers $n \geq 1$.

Example 37

[2008 NEAP Ext 2 Trial Q7] (4 marks) A sequence is defined by the relationship

$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{2}{u_n}\right)$$

where $u_1 = 1$ and $n \in \mathbb{Z}^+$.

Use mathematical induction to show that

$$\frac{u_n - \sqrt{2}}{u_n + \sqrt{2}} = \left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{2^{n-1}}$$

Example 38

[2016 JRAHS Ext 2 Trial Q13] A sequence is defined by the formula $U_0 = 0$ and $U_n = \sqrt{U_{n-1} + 2}$ for $n = 1, 2, 3, \cdots$.

Prove by mathematical induction that

$$U_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

for $n = 0, 1, 2, 3\cdots$

Current syllabus would start at $n = 1$.

Example 39

[1981 HSC Q8] Using induction, show that for each $n \in \mathbb{Z}^+$, there are unique positive integers p_n and q_n such that

$$\left(1+\sqrt{2}\right)^n = p_n + q_n\sqrt{2}$$

Show also that $(p_n)^2 - 2\,(q_n)^2 = (-1)^n$. *Hint:* use induction!

Further exercises

Ex 2E (Sadler & Ward, 2019)

- Q9, 17

6.0.1 Further questions

Source: Haese et al. (2015a)

1. A sequence is defined by $u_1 = 1$, $u_{n+1} = 2u_n + 1$, $\forall n \in \mathbb{Z}^+$.

Prove that $u_n = 2^n - 1$, $\forall n \in \mathbb{Z}^+$.

2. A sequence t_n is defined by $t_1 = 5$ and $t_{n+1} = t_n + 8n + 5$, $\forall n \in \mathbb{Z}^+$.

Prove that $t_n = 4n^2 + n$.

3. A sequence $u_1 = 1$, and subsequent terms are $u_{n+1} = 2 + 3u_n$, prove that

$$u_n = 2\left(3^{n-1} - 1\right)$$

4. A sequence is defined by $t_1 = 2$ and $t_{n+1} = \dfrac{t_n}{2(n+1)}$ for all $n \in \mathbb{Z}^+$.

Prove that $t_n = \dfrac{2^{2-n}}{n!}$.

5. A sequence is defined by $u_1 = 1$ and $u_{n+1} = u_n + (-1)^n(n+1)^2$ for all $n \in \mathbb{Z}^+$.

Prove that $u_n = \dfrac{(-1)^{n-1}n(n+1)}{2}$

6. A sequence is defined by $t_1 = 1$, $t_{n+1} = t_n + (2n+1)$, $\forall n \in \mathbb{Z}^+$.

(a) By finding t_n for $n = 2, 3$ and 4, conjecture a formula for t_n in terms of n only (not recursively)

(b) Prove that your conjecture is true using mathematical induction.

(c) Find the value of t_{20}.

7. Prove that if $u_1 = 9$ and $u_{n+1} = 2u_n + 3\left(5^n\right)$, then $u_n = 2^{n+1} + 5^n$, $\forall n \in \mathbb{Z}^+$.

8. Prove that if $t_1 = 5$ and $t_{n+1} = 2t_n - 3(-1)^n$, then $t_n = 3 \times 2^n + (-1)^n$, $\forall n \in \mathbb{Z}^+$.

9. A sequence is defined by $u_1 = \dfrac{1}{3}$ and $u_{n+1} = u_n + \dfrac{1}{(2n+1)(2n+3)}$, $\forall n \in \mathbb{Z}^+$.

(a) By finding t_n for $n = 2, 3$ and 4, conjecture a formula for t_n in terms of n only (not recursively)

(b) Prove that your conjecture is true using mathematical induction.

(c) Find the value of t_{50}.

10. Given $\left(2 + \sqrt{3}\right)^n = A_n + B_n\sqrt{3}$, $\forall n \in \mathbb{Z}^+$, where A_n and B_n are integers,

(a) Find A_n and B_n for $n = 1, 2, 3$ and 4.

(b) Without using induction, show that $A_{n+1} = 2A_n + 3B_n$ and $B_{n+1} = A_n + 2B_n$.

(c) Calculate $\left(A_n\right)^2 - 3\left(B_n\right)^2$ for $n = 1, 2, 3$ and 4 and hence conjecture a result.

(d) Prove that your conjecture is true.

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \; r \neq 1$$

$$S = \frac{a}{1-r}, \; |r| < 1$$

Logarithmic and Exponential Functions

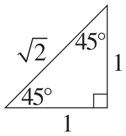
$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

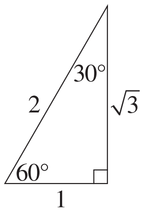
$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$



$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$



$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \ne 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \ne 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \ne 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

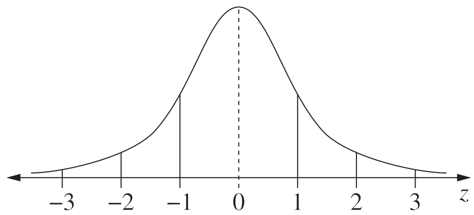
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \ne 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

References

Brown, P. G. (2008). Mathematical Induction. *Parabola*, *44*(1).

Fitzpatrick, J. B. (1984). *New Senior Mathematics – Three Unit Course for Years 11 & 12*. Harcourt Education.

Haese, M., Haese, S., & Humphries, M. (2015a). *Mathematics for Australia 11 Mathematical Methods* (2nd ed.). Haese Mathematics.

Haese, M., Haese, S., & Humphries, M. (2015b). *Mathematics for Australia 11 Specialist Mathematics* (2nd ed.). Haese Mathematics.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 12* (1st ed.). Cambridge Education.

Sadler, D., & Ward, D. (2019). *CambridgeMATHS Stage 6 Mathematics Extension 2* (1st ed.). Cambridge Education.