



NORMANHURST BOYS HIGH SCHOOL

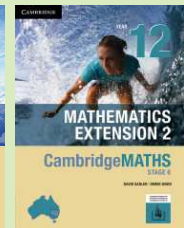
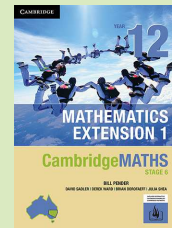
MATHEMATICS EXTENSION 1 (INCORPORATING EXTENSION 2) YEAR 12 COURSE



Topic summary and exercises:

(X1) (X2) **Mathematical Induction**

With references to



Name:

Initial version by H. Lam, November 2013 (former 3U content), July 2014 (Strong Induction). Last update February 3, 2025
Various corrections by students and members of the Department of Mathematics at North Sydney Boys High School and Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at <http://www.flaticon.com>, used under  CC BY 2.0.
Parts of this document are also sourced from:

- Brown (2008)
- Fitzpatrick (1984, §23.4)
- (Haese, Haese, & Humphries, 2015b, Ch 6)

Symbols used

-  Beware! Heed warning.
-  Provided on NESA Reference Sheet
-  Facts/formulae to memorise.
-  Mathematics Extension 1 content.
-  Mathematics Extension 2 content.
-  Literacy: note new word/phrase.
-  Further reading/exercises to enrich your understanding and application of this topic.
-  Syllabus specified content
-  Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME12-1 applies techniques involving proof or calculus to model and solve problems

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

Syllabus subtopics

ME-P1 Proof by Mathematical Induction

MEX-P2 Further Proof by Mathematical Induction

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Extension 1* and/or *CambridgeMATHS Extension 2* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

ⓧ1 ⓧ2 Simple proofs

✓ Content/learning intentions

- ☐ 19.1 Understand the nature of inductive proof, including the ‘initial statement’ and the inductive step  (ACMSM064)

➤ Use $P(n)$ notation to explicitly help students understand the series of propositions being proven.

-
- ☐ 19.2 Prove results using mathematical induction

➤ Prove results for sums, for example

$$1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for any positive integer n  (ACMSM065)

➤ Prove divisibility results, for example  (ACMSM066)

$3^{2n+4} - 2^{2n}$ is divisible by 5 for any positive integer n

-
- ☐ 19.3 Identify errors in false ‘proofs by induction’, such as cases where only one of the required two steps of a proof by induction is true, and understand that this means that the statement has not been proved

-
- ☐ 19.4 Recognise situations where proof by mathematical induction is not appropriate
-

ⓧ Further proofs

✓ Content/learning intentions

- ☐ 19.5 Prove results using mathematical induction where the initial value of n is greater than 1, and/or n does not increase strictly by 1, for example

Prove that $n^2 + 2n$ is a multiple of 8 if n is an even positive integer

-
- ☐ 19.6 Understand and use sigma notation to prove results for sums, for example:

$$\sum_{n=1}^N \frac{1}{(2n+1)(2n-1)} = \frac{N}{2N+1}$$

-
- ☐ 19.7 Understand and prove results using mathematical induction, including inequalities and results in algebra, calculus, probability and geometry. For example:

- Prove inequality results, e.g. $2^n > n^2$ for $n \in \mathbb{Z}^+$, $n \geq 5$
- Prove results in calculus, e.g. $\frac{d}{dx}(x^n) = nx^{n-1}$ for $n \in \mathbb{Z}^+$
- Prove results related to probability, e.g. the Binomial Theorem $(x+a)^n = \sum_{r=0}^n \binom{n}{r} x^{n-r} a^r$
- Prove geometric results, e.g. The sum of the exterior angles of an n -sided plane convex polygon is 360°

-
- ☐ 19.8 Prove De Moivre's Theorem for integral powers using induction
 (ACMSM083)
-

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Part I

ⓧ1 ⓧ2 Mathematics Extension 1 and 2 common content

Section 1

Induction into (mathematical) induction



Learning Goal(s)

Knowledge

Steps involved for a formal induction proof

Skills

Prove summation and divisibility results by induction

Understanding

What is *proof by induction* vs *proof by deduction*

By the end of this section am I able to:

- 19.1 Understand the nature of inductive proof, including the ‘initial statement’ and the inductive step
- 19.2 Prove results using mathematical induction
- 19.3 Identify errors in false ‘proofs by induction’, such as cases where only one of the required two steps of a proof by induction is true, and understand that this means that the statement has not been proved

1.1 Statements & propositions

- Mathematical induction is a *method* to prove a of statements/propositions, usually many.



Example 1

State whether these propositions are true or false:

Proposition 1: (T/F) $1 + 3 = 4$

Proposition 2: (T/F) $2 \times 7 = 15$

Proposition 3: (T/F) Mr Ho is handsome

- Notation for propositions:
 - (Serious) algebra required.
 - 1st proposition: denote ➤ k th proposition: denote
 - 2nd proposition: denote ➤ n th proposition: denote

1.2 General structure of induction proofs

1.2.1 Plain English

Proving a of many propositions with time.

1.2.2 Logic

- (normally, the first proposition, or):
 - Prove $P(1)$, is true
- :
 - Assume some arbitrary statement $P(k)$ is true.
(.....)
 - If $P(k)$ is true, and overall all the statements are true, then the subsequent statement $P(k + 1)$ should also be true.
 - * Use algebraic skills to show the $k + 1$ -th statement is also true.

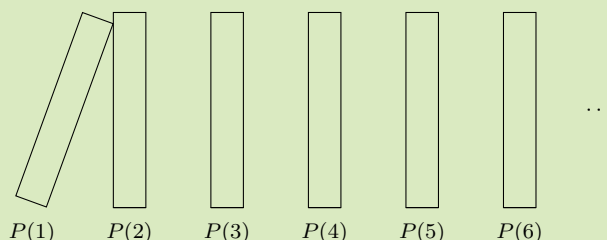
Watch multimedia

“Dominoes”: <https://fb.watch/9BL-01VXWX/>

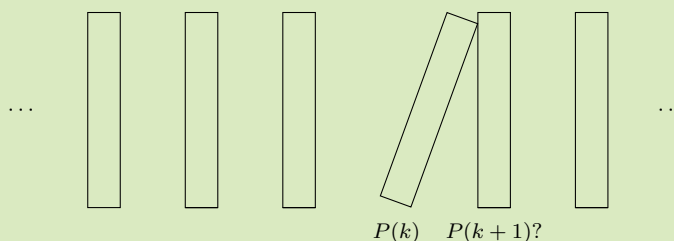
Example 2

Domino analogy – well constructed set of dominos.

- Truth of k -th proposition $\rightarrow k$ -th domino knocked down.
- : knock over first domino.



- :
 - Find an arbitrary domino somewhere in the entire set of dominos, and assume it knocks over.
 - Show that the next domino also knocks over.



- Hence, all dominos will eventually fall.

Section 2

Proofs involving sum/products

2.1 Introduction



Example 3

Prove by induction:

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

where $n \in \mathbb{N}$

Solution

- Let $P(n)$ be the proposition

$$1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$$

- : $P(1)$:

-

➤ (Hypothesis) Assume that $P(k)$ is true for some $k \in \mathbb{N}$, $1 < k < n$, i.e.

➤ Examine $P(k + 1)$:

2.2 Basic proofs



Example 4

Prove by induction:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

where $n \in \mathbb{N}$



Example 5

Prove by mathematical induction that

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \cdots + n \times 2^{n-1} = 1 + (n-1)2^n$$

$\forall n \in \mathbb{Z}^+, n \geq 1.$



Example 6

[2016 Ext 1 HSC Q14]

- i. Show that $4n^3 + 18n^2 + 23n + 9$ can be written as **1**

$$(n + 1)(4n^2 + 14n + 9)$$

- ii. Using the result in part (i), or otherwise, prove by mathematical induction that, for $n \geq 1$, **3**

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2n - 1)(2n + 1) = \frac{1}{3}n(4n^2 + 6n - 1)$$



Important note

Show part (i) adequately, otherwise it could look like it's being *fudged*.

**Example 7**

- (a) Show that $(n + 1)! = (n + 1) \times n!$
- (b) Use induction to show that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n + 1)! - 1$$

2.3 Harder proofs



Example 8

⚠ Prove by induction:

$$1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$$

where $n \in \mathbb{N}$

**Example 9**

! Use mathematical induction to prove that, for all integers n with $n \geq 1$.

$$\frac{1}{\sec x} + \frac{1}{\sec 3x} + \frac{1}{\sec 5x} + \cdots + \frac{1}{\sec(2n-1)x} = \frac{\operatorname{cosec} x}{2 \operatorname{cosec}(2nx)}$$

 Further exercises

Ex 2A (Pender et al., 2019)

- Q2-5, 8, 9, 12

2.4 When induction fails

Important note

A single is sufficient to disprove a result.

Further reading

-  brilliant.org - Flawed induction proofs, including when you *shouldn't* use induction.
-  [AMSI Proofs by Induction](#), p.7-8

Example 10

Pender et al. (2019, Ex 2A Q7)

(a) Attempt to prove by mathematical induction that

$$3 + 6 + 9 + \cdots + 3n = n(n + 1) + 1$$

for all $n \in \mathbb{Z}^+$.

(b) Where does the proof break down?

Further exercises

Ex 2A (Pender et al., 2019)

- Q6-7

2.4.1 Further questions

1. Prove by mathematical induction that

$$1^2 + 3^2 + \cdots (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$

$$\forall n \in \mathbb{Z}^+, n \geq 1.$$

2. (a) By considering the sum of the terms of an arithmetic series, show that

$$(1 + 2 + \cdots + n)^2 = \frac{1}{4}n^2(n+1)^2$$

- (b) Use induction to prove

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

$$\forall n \geq 1.$$

3. Use mathematical induction to prove that, for all positive integers n ,

$$1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$$

4. Use the Principle of Mathematical Induction to show that

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots + (n^2 + 1)n! = n(n+1)!$$

$$\forall n \in \mathbb{Z}^+.$$

5. Use mathematical induction to prove that, for integers $n \geq 1$,

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n}{6}(n+1)(2n+7)$$

6. Let $S(n) = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$, where $n \in \mathbb{Z}^+$.

- (a) Use induction to show that

$$S_n = 1 - \frac{1}{(n+1)!}$$

$$\forall n \in \mathbb{Z}^+.$$

- (b) Find the value of $\lim_{n \rightarrow \infty} S_n$.

- (c) Find the smallest positive integer n such that $|S_n - 1| < 1 \times 10^{-6}$.

7. Use induction to show that for all positive integers $n \geq 1$,

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+\cdots+n} = \frac{2n}{n+1}$$

8. Use Mathematical Induction to show that for all positive integers $n \geq 1$,

$$\frac{3}{1 \times 2 \times 2} + \frac{4}{2 \times 3 \times 2^2} + \cdots + \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(n+1)2^n}$$

9. Use mathematical induction to prove that

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

for all positive integers n .

10. (a) Use the fact that

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

to show that

$$1 + \tan n\theta \tan(n+1)\theta = \cot \theta (\tan(n+1)\theta - \tan n\theta)$$

- (b)  Use mathematical induction to prove that for all integers $n \geq 1$,

$$\tan \theta \tan 2\theta + \tan 2\theta \tan 3\theta + \cdots + \tan n\theta \tan(n+1)\theta = -(n+1) + \cot \theta \tan(n+1)\theta$$

11. (a) Prove that

$$\frac{\cos A - \cos(A+2B)}{2 \sin B} = \sin(A+B)$$

- (b)  By rewriting $\cos 2\theta$ in terms of $\sin^2 \theta$ or otherwise, prove

$$\sin \theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta = \frac{1 - \cos 2n\theta}{2 \sin \theta}$$

for all $n \in \mathbb{N}$

Section 3

Proofs involving divisibility

3.1 Divisibility revisited

Fill in the spaces

- If 20 is divisible by 5, then
- If $4^n - 1$ is divisible by 3, then



Example 11

Prove that $4^n - 1$ is divisible by 3 $\forall n \in \mathbb{N}$.

**Example 12**

Prove that $5^n + 3$ is divisible by 4 $\forall n \in \mathbb{N}$.

**Example 13**

Prove that $4^n + 14$ is a multiple of 6 $\forall n \geq 1$.

**Example 14**

Prove by induction that $47^n + 53 \times 147^{n-1}$ is divisible by 100 $\forall n \geq 1$.

**Example 15**

Prove that $3^{3n} + 2^{n+2}$ is divisible by 5 $\forall n \in \mathbb{N}$.

**Example 16**

Prove that $3^n + 7^n$ is divisible by 10, if n is odd.

**Important note**

Inductive hypothesis still requires the assumption of the truth $P(k)$ where $k = 2m+1$, but need to examine $P(\text{.....})$.

**Example 17**

Prove that $n^3 + 5n$ is divisible by 3 $\forall n \in \mathbb{N}$.

-  (2013 CSSA Ext 1 Q14) Prove it's also divisible by 6.

**Example 18**

Prove that $x^n - 1$ is divisible by $(x - 1)$ for all $n \in \mathbb{N}$.

3.2 When induction fails



Learning Goal(s)

Knowledge

Steps involved for a formal induction proof

Skills

Prove summation and divisibility results by induction

Understanding

What is *proof by induction* vs *proof by deduction*

 By the end of this section am I able to:

19.4 Recognise situations where proof by mathematical induction is not appropriate



Example 19

(Haese et al., 2015b) Prove the following *incorrect* statements using induction:

1. $n^2 + 2$ is divisible by 3 $\forall n \in \mathbb{Z}^+$.
2. $3^n + 4$ is divisible by 7 for all $n \in \mathbb{Z}^+$.



Further exercises

Ex 2B (Pender et al., 2019)

- Q2-3

3.2.1 Further questions

1. Use mathematical induction to prove that, $\forall n \in \mathbb{N}$, $13 \times 6^n + 2$ is divisible by 5.
2. Prove that $4^n + 14$ is a multiple of 6 $\forall n \geq 1$.
3. Use the mathematical induction to prove that $7^{2n-1} + 5$ is divisible by 12 $\forall n \in \mathbb{N}$.

Part II

ⓧ2 Mathematics Extension 2 only
content

Section 4

Proofs involving harder sums & divisibility



Learning Goal(s)

≡ Knowledge

Steps involved for a formal induction proof

⚙ Skills

Algebraic manipulation of $P(k)$ to fit into the expression for $P(k+1)$

💡 Understanding

Different proofs require various algebraic techniques

✓ By the end of this section am I able to:

19.5 Prove results using mathematical induction where the initial value of n is greater than 1, and/or n does not increase strictly by 1.

19.6 Understand and use sigma notation to prove results for sums, for example:

$$\sum_{n=1}^N \frac{1}{(2n+1)(2n-1)} = \frac{N}{2N+1}$$

19.7 Understand and prove results using mathematical induction, including inequalities and results in algebra, calculus, probability and geometry.

19.8 Prove De Moivre's Theorem for integral powers using induction

4.1 Sum



Important note

Extension 2 sum proofs build upon Extension 1 proofs with

- sigma notation
- n commencing at values greater than 1, or not necessarily increasing by 1

**Example 20**

Prove by induction for $n \in \mathbb{N}$.

$$\sum_{k=1}^n \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

**Further exercises**

Ex 2E (Sadler & Ward, 2019)

- Q1, 3, 20

4.2 Divisibility



Important note

Extension 2 divisibility proofs build upon Extension 1 proofs with

- Polynomial division



Further exercises

Ex 2E (Sadler & Ward, 2019)

- 2, 4, 5, 16

4.2.1 Further questions

Some questions are sourced from [Haese, Haese, and Humphries \(2015a\)](#).

1. Let

$$S_n = 1 \times 2 + 2 \times 3 + \cdots + (n-1) \times n$$

Use mathematical induction to prove that, for all integers n with $n \geq 2$,

$$S_n = \frac{1}{3}(n-1)n(n+1)$$

2. Prove that $2^{3n} - 3^n$ is divisible by 5 $\forall n \geq 1$.
3. Use induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n .
4. (a) Use the method of mathematical induction to show that if x is a positive integer then $(1+x)^n - 1$ is divisible by $x \forall n \in \mathbb{N}$.
 (b) Factorise $12^n - 4^n - 3^n + 1$. Without using induction again, use the previous part to show deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all integers $n \geq 1$.
5. Prove by induction: $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$
6. Prove by induction: $2^{4n+3} + 3^{3n+1}$ is divisible by 11 $\forall n \in \mathbb{Z}, n \geq 0$.
7. Prove by induction: $\frac{2^n - (-1)^n}{3}$ is an odd number for all $n \in \mathbb{Z}^+$.
8. Prove by mathematical induction that

$$\sum_{r=1}^n r \times r! = (n+1)! - 1$$

9.  Use mathematical induction to prove for all integers $n \geq 3$,

$$\left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{n}\right) = \frac{2}{n(n-1)}$$

10.  Prove by induction that, for all integers $n \geq 1$,

$$(n+1)(n+2) \cdots (2n-1)2n = 2^n (1 \times 3 \times 5 \times \cdots \times (2n-1))$$

11.  Prove by induction that

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 9 $\forall n \in \mathbb{Z}^+$.

Section 5

Proofs involving inequalities, calculus, geometry and well known theorems

5.1 Inequalities

- If $x > 5$, then
- Look for quantity larger/smaller than what needs to be proven.



Example 21

Prove for $n \geq 2$, that

$$3^n > 1 + 2n$$

Solution



Steps

- Let $P(n)$ be the proposition $3^n > 1 + 2n$, $n \geq 2$.
- Base case $P(2)$:
- Inductive step:

**Example 22**

Prove by induction for $n \geq 6$,

$$n^2 - 11n + 30 \geq 0$$

**Example 23**

Prove by induction for $n \geq 5$,

$$2^n > n^2$$

**Example 24**

Prove for $x > 0$, $n \geq 1$,

$$(1 + x)^n \geq 1 + nx$$

**Example 25**

Prove by induction, for $x, y > 0$ and $n \geq 2$,

$$(x + y)^n > x^n + y^n$$

**Example 26**

Prove by induction $\forall n \geq 2$,

$$12^n > 7^n + 5^n$$

**Further exercises**

Ex 2E (Sadler & Ward, 2019)

- 6-8, 15, 18

5.1.1 Further questions

1. Use the principle of mathematical induction to show that $4^n - 1 - 7n > 0$ for all integers $n \geq 2$.
2. Use the method of Mathematical Induction to show that $n! > 2^n$ for all positive integers $n \geq 4$.
3. Use Mathematical Induction to show that $5^n > 4^n + 3^n$ for all integers $n \geq 3$.
4. (a) Show that for $k > 0$,

$$\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$$

- (b) Use mathematical induction to prove that $\forall n \geq 2, n \in \mathbb{Z}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

5.2 Functions/Calculus proofs



Example 27

[2009 Ext 1 HSC Q7]

- i. Use differentiation from first principles to show that

1

$$\frac{d}{dx}(x) = 1$$

- ii. Use mathematical induction and the product rule for differentiation to prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for $n \in \mathbb{Z}^+$.

2

**Example 28**

Use induction to show that

$$\cos(x + n\pi) = (-1)^n \cos x$$

$$\forall n \in \mathbb{Z}^+, n \geq 1.$$

**Example 29**

A function $f(x)$ is such that $f(x) > 0$ for $x \in \mathbb{R}$ and $f(a + b) = f(a) \times f(b)$ for $a, b \in \mathbb{R}$.

(a) Show that $f(0) = 1$ and deduce that $f(-x) = \frac{1}{f(x)}$.

(b) Use induction to show that

$$f(nx) = (f(x))^n$$

for all positive integers n .

(c) Without using induction again, deduce that

$$f(-nx) = (f(x))^{-n}$$

for all positive integers n .

5.3 Geometric proofs



Example 30

Prove by induction, that the sum of the angles of a polygon of n sides is $(2n - 4)$ right angles, where $n \geq 3$.

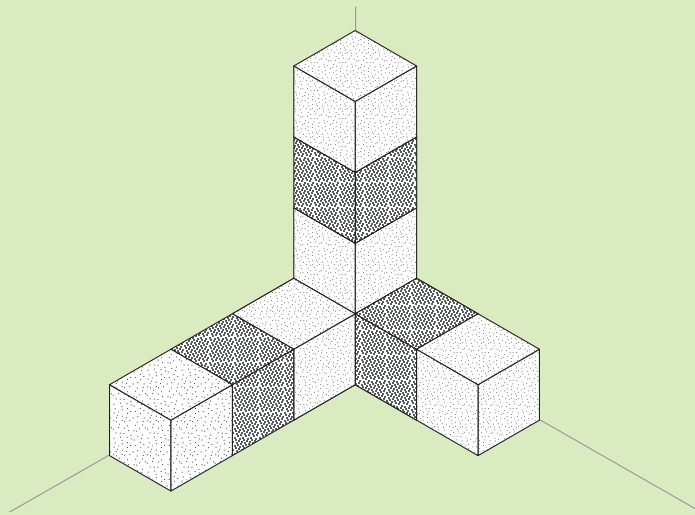
**Example 31**

[Ex 2E Q13] [Sadler and Ward \(2019\)](#) Suppose there are n lines in a plane, arranged such that no three of the lines are concurrent, and no two of the lines are parallel. Show that for $n \geq 1$, n such lines divide the plane into $\frac{1}{2}(n^2 + n + 2)$ regions.



Example 32

[2020 Ext 2 HSC Sample Q12] Two vertical walls and the floor meet at a corner of a room. One cube is placed in the corner. A solid shape is then formed by placing identical cubes to form horizontal rows on the floor against the walls or by stacking vertically against the two walls. An example is the solid shape shown in the diagram. This example is formed from nine cubes.



Let n be the number of cubes used to make a solid shape in this way.

Use mathematical induction to show that the number of exposed faces of the cubes in this shape is $2n + 1$.



Further exercises

Ex 2E (Sadler & Ward, 2019) - note some of already been done in class.

- Q12-14

5.4 Well known theorems



Example 33

The Binomial Theorem

(a) Prove *Pascal's formula*:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

(b) Hence, prove the *Binomial Theorem* by induction for $n \geq 1$:

$$(x+y)^n = \sum_{r=0}^n \binom{n}{r} x^{n-r} y^r$$

**Example 34**

De Moivre's Theorem Prove *De Moivre's Theorem* by induction:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

for

(a) $n \in \mathbb{Z}^+$.

(b) $n \in \mathbb{Z}^-$.

Section 6

Proofs involving recurrence relations



Example 35

[2011 NSGHS Ext 2 Trial Q8] The sequence $u_1, u_2, u_3, \dots$ is defined by

$$u_1 = 2 \quad \text{and} \quad u_{k+1} = 2u_k + 1$$

- i. Prove by induction that, for all integers $n \geq 1$,

3

$$u_n = 3 \times 2^{n-1} - 1$$

- ii. Show that

2

$$\sum_{r=1}^n u_r = u_{n+1} - (n + 2)$$

**Example 36**

[2018 Independent Ext 2 Trial Q14] (3 marks) A sequence of numbers T_1, T_2, T_3 is given by $T_1 = 2$ and $T_n = \frac{5T_{n-1} - 3}{3T_{n-1} - 1}$ for $n \geq 2$.

Use mathematical induction to show that $T_n = \frac{3n+1}{3n-1}$ for all positive integers $n \geq 1$.

**Example 37**

[2008 NEAP Ext 2 Trial Q7] (4 marks) A sequence is defined by the relationship

$$u_{n+1} = \frac{1}{2} \left(u_n + \frac{2}{u_n} \right)$$

where $u_1 = 1$ and $n \in \mathbb{Z}^+$.

Use mathematical induction to show that

$$\frac{u_n - \sqrt{2}}{u_n + \sqrt{2}} = \left(\frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right)^{2^{n-1}}$$

**Example 38**

[2016 JRAHS Ext 2 Trial Q13] A sequence is defined by the formula $U_0 = 0$ and $U_n = \sqrt{U_{n-1} + 2}$ for $n = 1, 2, 3, \dots$.

Prove by mathematical induction that

$$U_n = 2 \cos \left(\frac{\pi}{2^{n+1}} \right)$$

for $n = 0, 1, 2, 3 \dots$

⚠ Current syllabus would start at $n = 1$.

**Example 39**

[1981 HSC Q8] Using induction, show that for each $n \in \mathbb{Z}^+$, there are unique positive integers p_n and q_n such that

$$(1 + \sqrt{2})^n = p_n + q_n\sqrt{2}$$

Show also that $(p_n)^2 - 2(q_n)^2 = (-1)^n$. *Hint:* use induction!

**Further exercises**

Ex 2E (Sadler & Ward, 2019)

- Q9, 17

6.0.1 Further questions

Source: [Haese et al. \(2015a\)](#)

1. A sequence is defined by $u_1 = 1$, $u_{n+1} = 2u_n + 1$, $\forall n \in \mathbb{Z}^+$.

Prove that $u_n = 2^n - 1$, $\forall n \in \mathbb{Z}^+$.

2. A sequence t_n is defined by $t_1 = 5$ and $t_{n+1} = t_n + 8n + 5$, $\forall n \in \mathbb{Z}^+$.

Prove that $t_n = 4n^2 + n$.

3. A sequence $u_1 = 1$, and subsequent terms are $u_{n+1} = 2 + 3u_n$, prove that

$$u_n = 2(3^{n-1} - 1)$$

4. A sequence is defined by $t_1 = 2$ and $t_{n+1} = \frac{t_n}{2(n+1)}$ for all $n \in \mathbb{Z}^+$.

Prove that $t_n = \frac{2^{2-n}}{n!}$.

5. A sequence is defined by $u_1 = 1$ and $u_{n+1} = u_n + (-1)^n(n+1)^2$ for all $n \in \mathbb{Z}^+$.

Prove that $u_n = \frac{(-1)^{n-1}n(n+1)}{2}$

6. A sequence is defined by $t_1 = 1$, $t_{n+1} = t_n + (2n+1)$, $\forall n \in \mathbb{Z}^+$.

- By finding t_n for $n = 2, 3$ and 4 , conjecture a formula for t_n in terms of n only (not recursively)
- Prove that your conjecture is true using mathematical induction.
- Find the value of t_{20} .

7. Prove that if $u_1 = 9$ and $u_{n+1} = 2u_n + 3(5^n)$, then $u_n = 2^{n+1} + 5^n$, $\forall n \in \mathbb{Z}^+$.

8. Prove that if $t_1 = 5$ and $t_{n+1} = 2t_n - 3(-1)^n$, then $t_n = 3 \times 2^n + (-1)^n$, $\forall n \in \mathbb{Z}^+$.

9. A sequence is defined by $u_1 = \frac{1}{3}$ and $u_{n+1} = u_n + \frac{1}{(2n+1)(2n+3)}$, $\forall n \in \mathbb{Z}^+$.

- By finding t_n for $n = 2, 3$ and 4 , conjecture a formula for t_n in terms of n only (not recursively)
- Prove that your conjecture is true using mathematical induction.
- Find the value of t_{50} .

10. Given $(2 + \sqrt{3})^n = A_n + B_n\sqrt{3}$, $\forall n \in \mathbb{Z}^+$, where A_n and B_n are integers,

- Find A_n and B_n for $n = 1, 2, 3$ and 4 .
- Without using induction, show that $A_{n+1} = 2A_n + 3B_n$ and $B_{n+1} = A_n + 2B_n$.
- Calculate $(A_n)^2 - 3(B_n)^2$ for $n = 1, 2, 3$ and 4 and hence conjecture a result.
- Prove that your conjecture is true.

NESA Reference Sheet – calculus based courses



NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced Mathematics Extension 1 Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\text{and } \alpha\beta\gamma = -\frac{d}{a}$$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, r \neq 1$$

$$S = \frac{a}{1 - r}, |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2} ab \sin C$$

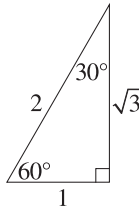
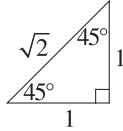
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2} r^2 \theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \quad \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \quad \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\text{If } t = \tan \frac{A}{2} \text{ then } \sin A = \frac{2t}{1+t^2}$$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)]$$

$$\sin^2 nx = \frac{1}{2} (1 - \cos 2nx)$$

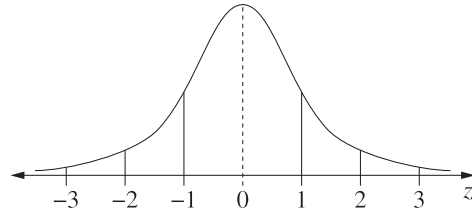
$$\cos^2 nx = \frac{1}{2} (1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z -scores between -1 and 1
- approximately 95% of scores have z -scores between -2 and 2
- approximately 99.7% of scores have z -scores between -3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E[(X - \mu)^2] = E(X^2) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x) dx$$

$$P(a < X < b) = \int_a^b f(x) dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function

Derivative

$$y = f(x)^n$$

$$\frac{dy}{dx} = n f'(x) [f(x)]^{n-1}$$

$$y = uv$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$y = g(u) \text{ where } u = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$y = \sin f(x)$$

$$\frac{dy}{dx} = f'(x) \cos f(x)$$

$$y = \cos f(x)$$

$$\frac{dy}{dx} = -f'(x) \sin f(x)$$

$$y = \tan f(x)$$

$$\frac{dy}{dx} = f'(x) \sec^2 f(x)$$

$$y = e^{f(x)}$$

$$\frac{dy}{dx} = f'(x) e^{f(x)}$$

$$y = \ln f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$y = a^{f(x)}$$

$$\frac{dy}{dx} = (\ln a) f'(x) a^{f(x)}$$

$$y = \log_a f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{(\ln a) f(x)}$$

$$y = \sin^{-1} f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \cos^{-1} f(x)$$

$$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \tan^{-1} f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{1 + [f(x)]^2}$$

Integral Calculus

$$\int f'(x) [f(x)]^n dx = \frac{1}{n+1} [f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x) \sin f(x) dx = -\cos f(x) + c$$

$$\int f'(x) \cos f(x) dx = \sin f(x) + c$$

$$\int f'(x) \sec^2 f(x) dx = \tan f(x) + c$$

$$\int f'(x) e^{f(x)} dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$$

$$\int f'(x) a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1} \frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1} \frac{f(x)}{a} + c$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int_a^b f(x) dx$$

$$\approx \frac{b-a}{2n} \left\{ f(a) + f(b) + 2[f(x_1) + \dots + f(x_{n-1})] \right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$|\underline{u}| = |x_1\hat{i} + y_1\hat{j}| = \sqrt{x^2 + y^2}$$

$$\underline{u} \cdot \underline{v} = |\underline{u}| |\underline{v}| \cos \theta = x_1x_2 + y_1y_2,$$

$$\text{where } \underline{u} = x_1\hat{i} + y_1\hat{j}$$

$$\text{and } \underline{v} = x_2\hat{i} + y_2\hat{j}$$

$$\underline{r} = \underline{a} + \lambda \underline{b}$$

Complex Numbers

$$\begin{aligned} z &= a + ib = r(\cos \theta + i \sin \theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} [r(\cos \theta + i \sin \theta)]^n &= r^n(\cos n\theta + i \sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

$$x = a \cos(nt + \alpha) + c$$

$$x = a \sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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