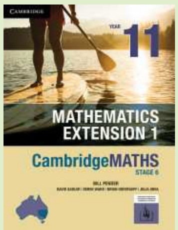
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1
(YEAR 11 COURSE)

Topic summary and exercises:

With references to



X1 **Polynomials**

Name: ....

Initial version by H. Lam, October 2014. Last updated November 27, 2020 for latest syllabus.
Various corrections by students & members of the Department of Mathematics at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under ⓒⓒ CC BY 2.0.

Parts of this handout are derived from the work by Mr P. G. Brown at the University of New South Wales, based on the *Professional Development Short Course for Secondary School Teachers* on 22/11/2013.

Symbols used

(!) Beware! Heed warning.

(F) Provided on NESA Reference Sheet

(M) Facts/formulae to memorise.

(2) Textbook reference from the legacy Mathematics (2 Unit) course

(A) Mathematics Advanced content.

(X1) Mathematics Extension 1 content.

(L) Literacy: note new word/phrase.

(N) Further reading/exercises to enrich your understanding and application of this topic.

(U) Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME11-2 manipulates algebraic expressions and graphical functions to solve problems

Syllabus subtopics

ME-F2 Polynomials

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 11 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Language of polynomials

1.1 Definitions

Learning Goal(s)

Knowledge
What is a polynomial

Skills
Operate with polynomials

Understanding
How to identify a polynomial

By the end of this section am I able to:

12.1 Define a general polynomial in one variable, x , of degree n with real coefficients to be the expression: $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$.

Definition 1

(L) A *polynomial function* is a function with positive integral (or zero) powers of x.

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_k \in \mathbb{R}$ are the ....coefficients.... of the corresponding power of x, and $n \in \mathbb{N}$.

Definition 2

(M) (L) The ....leading.... ....term.... is

$$a_nx^n$$

The ....leading.... ....coefficient.... is a_n.

Definition 3

(M) (L) A polynomial $P(x)$ is ....monic.... if $a_n = 1$.

Definition 4

M L The constant term is a_0, i.e. coefficient of x^0.

Definition 5

M L The degree of $P(x)$ is $\deg(P) = n$, i.e. highest power of x.

Important note

! Proficiency with various factorisation methods required

Example 1

Given $P(x) = x^3 - x^2 + x - 1$, $Q(x) = 3x^3 - 2x^2$ and $R(x) = -x^4 + 2x^3 - 3x^2$, evaluate

(a) $P(x)Q(x)$

(b) $Q(x)R(x)$

Answer: (a) $3x^6 - 5x^5 + 5x^4 - 5x^3 + 2x^2$ (b) $-3x^7 + 8x^6 - 13x^5 + 6x^4$

Further exercises

Ex 10A (Pender et al., 2019)

- Q1-3 last column
- Q6-13

1.2 (N) Method of undetermined coefficients

Definition 6

(U) (L) Two polynomials $P(x)$ and $Q(x)$ are *identical* iff all their corresponding coefficients are equal, i.e.

$$\begin{aligned} \text{Given } P(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \\ Q(x) &= b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 \end{aligned}$$

then $P(x) \equiv Q(x)$ iff

$$\begin{aligned} a_n &= b_n \\ a_{n-1} &= b_{n-1} \\ &\vdots \\ a_1 &= b_1 \\ a_0 &= b_0 \end{aligned}$$

Example 2

[**Ex 10A Q9(c)**] Find the values of a, b and c if **Answer:** $a = 1$,$b = 2$, $c = 1$

$$a(x-1)^2 + b(x-1) + c \equiv x^2$$

Steps

(U) Method of undetermined coefficients: find "convenient" values of x to substitute to determine coefficients.

1. By inspection,

2. Let $x = 1$:

3. Let $x = 0$:

Section 2

Ⓡ Graphs of polynomials

Learning Goal(s)

Knowledge
What the graph of various polynomials resemble

Skills
Sketch particular graphs of factored polynomials

Understanding
How multiple roots cause particular behaviours

By the end of this section am I able to:

12.2 Use the factored form for a polynomial expression $P(x)$ and relate this to the roots of the equation $P(x) = 0$.

12.3 Understand the behaviour of the polynomial $P(x)$ as $x \to \pm\infty$.

12.4 Relate the zeros of a $P(x) = 0$ to the graph of $y = P(x)$.

2.1 Multiple roots

Definition 7

Ⓤ If $(x - \alpha)$ is a factor of $P(x)$, and

$$P(x) = (x - \alpha)^m Q(x)$$

where $Q(x)$ is not divisible by $(x - \alpha)$, then $x = \alpha$ is *zero of multiplicity* m.

Theorem 1

Ⓤ

Root of multiplicity	Behaviour near root
1	
2	
3	
4	

Theorem 2

- As $x \to \infty$, $P(x) \to \infty$ if a_n is ......positive......
- As $x \to -\infty$
  - $P(x) \to -\infty$ if $\deg(P)$ is ...odd...
  - $P(x) \to \infty$ if $\deg(P)$ is ...even...
- Every ...odd... ......powered...... polynomial has at least ...one... real root.

Example 3

Sketch, showing the behaviour near x intercepts:

(a) $P(x) = (x-1)^2(x-2)$

(b) $Q(x) = x^3(x+2)^4(x^2+x+1)$

(c) $R(x) = -2(x-2)^2(x+1)^5(x-1)$

Further exercises

Ex 10B

- Q2-4
- Q6-11

Section 3

Division of polynomials

Learning Goal(s)

Knowledge
When to use long division, remainder or factor theorems

Skills
How to use long division, remainder and factor theorems

Understanding
Why the remainder and factor theorems work the way they do

By the end of this section am I able to:

12.5 Use division of polynomials to express $P(x)$ in the form $P(x) = A(x)Q(x) + R(x)$ where $\deg R(x) < \deg A(x)$ and $A(x)$ is a linear or quadratic divisor, $Q(x)$ the quotient and $R(x)$ the remainder.

12.6 Prove and apply the factor theorem and the remainder theorem for polynomials and hence solve simple polynomial equations.

3.1 Division algorithm

Theorem 3

Every integer n can be written as

$$n = dq + r$$

where

$$\begin{array}{r@{\,}r} & 163 \\ \cline{2-2} 17 \big) & 2785 \\ & 1700 \\ \cline{2-2} & 1085 \\ & 1020 \\ \cline{2-2} & 65 \\ & 51 \\ \cline{2-2} & 14 \end{array}$$

- d (17) is the ....*divisor*..........
- q (163) is the ....*quotient*..........
- r (14) is the ....*remainder*..........

Corollary 4

Ⓤ Every polynomial $P(x)$ can be written as

$$P(x) = D(x)Q(x) + R(x)$$

where

- $D(x)$ is the ....*divisor*........
- $Q(x)$ is the ....*quotient*........
- $R(x)$ is the ....*remainder*........

- Divide polynomials in a similar way to integers.

Example 4

Divide $3x^4 - 4x^3 + 4x - 8$ by

(a) $x - 2$ (b) $x^2 - 2$

Answer: (a) $(x-2)(3x^3+2x^2+4x+12)+16$ (b) $(x^2-2)(3x^2-4x+6)+(-4x+4)$

Theorem 5

Ⓤ The degree of the remainder must be less than the degree of the divisor, i.e.

$$\deg R(x) < \deg D(x)$$

Example 5

[Ex 4C Q6] Evaluate $(x^3 - 3x^2 + 5x - 4) \div (x^2 + 2)$, and express the result via the division transformation algorithm.

Example 6

[1998 HSC 3U] (3 marks) Find the quotient, $Q(x)$, and the remainder, $R(x)$, when the polynomial $P(x) = x^4 - x^2 + 1$ is divided by $x^2 + 1$.

Further exercises

Ex 10C

- Q2 LC
- Q5-13

Section 4

The remainder & factor theorems

4.1 The remainder theorem

Theorem 6

The *remainder theorem*: if $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.

Proof (U)

Steps

1. Write $P(x)$ using the division algorithm (See Corollary 4 on page 11):

2. Find the value of the remainder by letting $x = a$:

Example 7

Find the remainder when $3x^4 - 4x^3 + 4x - 8$ is divided by $x - 2$. **Answer:** 16

Example 8

The polynomial $P(x) = x^4 - 2x^3 + ax + b$ has a remainder of 3 after dividing by $(x-1)$ and leaves a remainder of -5 when dividing by $(x+1)$.

Find the value of a and b. **Answer:** $a = 6$, $b = -2$

Example 9

[**Ex 4D Q18**] When $x^5 + 3x^3 + ax + b$ is divided by $x^2 - 1$, the remainder is $2x - 7$. Find the value of a and b.

4.2 The factor theorem

Corollary 7

The *factor theorem*: if $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Proof (U)

Steps

1. If $P(a) = 0$, then

2. Rewriting using the division transformation (Corollary 4 on page 11):

Example 10

Show that $(x - 3)$ is a factor of $P(x) = x^3 - 2x^2 + x - 12$, and $(x - 1)$ is not a factor. Then, factor the polynomial completely over $\mathbb{Z}$. **Answer:** $P(x) = (x - 3)(x^2 + x + 4)$

Example 11

Completely factorise $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ over $\mathbb{R}$.

Laws/Results

If the coefficients of $P(x)$ are integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

Important note

Most polynomials seen at Extension 1 level will have integer roots. Use this to your advantage.

Example 12

[Ex 4D Q7] Fully factorise, and sketch a graph, indicating all intercepts with the axes. Turning points are not required.

(a) $P(x) = -x^3 + x^2 + 5x + 3$

(b) $P(x) = 3x^4 + 4x^3 - 35x^2 - 12x.$

Example 13

[Ex 4E Q6] A polynomial of degree 3 has a double zero at 2. When $x = 1$ it takes the value 6 and when $x = 3$ it takes the value 8. Find the polynomial.

Further exercises

Ex 10D Q4-19 **Ex 10E** Q3, then Q5-13 odd #

Section 5

Relationships between roots

Learning Goal(s)

Knowledge
What the symmetric functions are

Skills
Use the elementary symmetric functions to solve problems

Understanding
Understand where the elementary symmetric functions of roots arises from, and when to use them

By the end of this section am I able to:

12.7 Solve problems using the relationships between the roots and coefficients of quadratic, cubic and quartic equations.

5.1 Vieta's formulas/Symmetric Functions of Roots

Definition 8

A *symmetric function* of polynomial roots is any algebraic combination of its roots which is unaltered by changing any two of these symbols, e.g. if α, β and γ are the roots of a cubic polynomial, then

$$\alpha^2\beta + \alpha\beta^2$$
$$\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$$

are two of such symmetric functions.

5.1.1 Elementary symmetric functions of roots of a quadratic

Theorem 8

The elementary symmetric functions of a quadratic polynomial: If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$,

- The sum of the roots: $\alpha + \beta = -\dfrac{b}{a}$
- The product of the roots: $\alpha\beta = \dfrac{c}{a}$

Important note

Note in these two formulae, α and β are commutative, and can be "swapped around". Hence the *symmetry*.

Proof

Steps

1. 'Forcibly' factorise the a term out of $ax^2 + bx + c = 0$:

2. Rewrite $ax^2 + bx + c = 0$ in factored form, given $x = \alpha$ and $x = \beta$ are the roots:

3. Expand the factored form, and compare coefficients: (See Definition 6 on page 6)

5.1.2 General symmetric functions of a quadratic

Example 14

Write a symmetric function involving a, b and c.

Example 15

If α and β are the roots of the quadratic equation $2x^2 - 5x + 1 = 0$, find the values of:

(a) $\alpha + \beta$

(b) $\alpha\beta$

(c) $\frac{1}{\alpha} + \frac{1}{\beta}$

(d) $\alpha^2 + \beta^2$

(e) $(\alpha - 3)(\beta - 3)$

Important note

The actual values of α and β are not the emphasis, but rather their sum and their product.

Example 16

Form a quadratic equation with integer coefficients, whose roots are

(a) 5 and -2 (b) $2 \pm \sqrt{3}$ (c) p and $2p$

Example 17

Find m, given that one of the roots of $x^2 + mx + 18 = 0$ is twice the other.

Answer: $m = \pm 9$

Example 18

(a) Prove that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

(b) Hence or otherwise, evaluate $|\alpha - \beta|$, where α and β are the roots of the equation $x^2 - 9x + 2 = 0$.

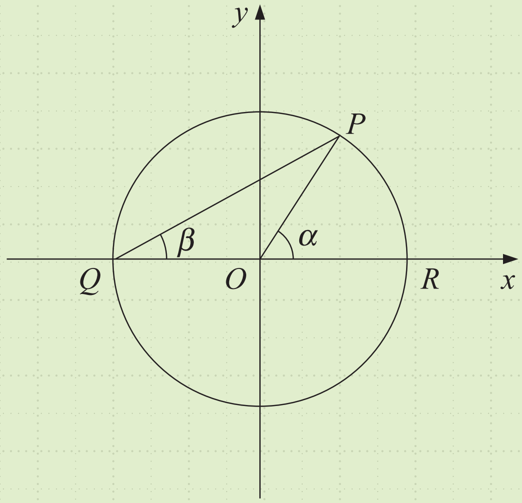
Answer: $\sqrt{73}$

Example 19

The line $y = 2x + b$ intersects the circle $x^2 + y^2 = 25$ at P and Q. Use the sum and product of roots to find the coordinates of the midpoint $M(X, Y)$ of PQ.

Example 20

[1997 HSC Q10] In the diagram, Q is the point $(-1, 0)$, R is the point $(1, 0)$, and P is another point on the circle with centre O and radius 1. Let $\angle POR = \alpha$ and $\angle PQR = \beta$, and let $\tan\beta = m$.



i. Explain why $\triangle OPQ$ is isosceles, and hence deduce that $\alpha = 2\beta$. **2**

ii. Find the equation of the line PQ. **1**

iii. Show that the x coordinates of P and Q are the solutions of the equation **3**

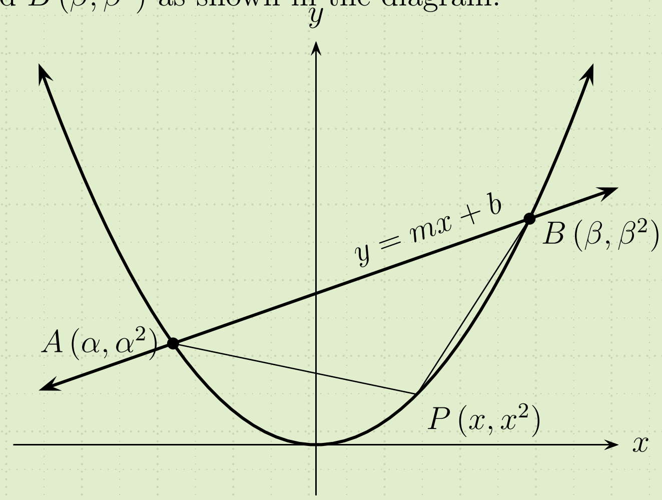
$$\left(1+m^2\right)x^2 + 2m^2x + m^2 - 1 = 0$$

iv. Using this equation, find the coordinates of P in terms of m. **2**

v. Hence deduce that $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$. **2**

Example 21

[2005 HSC Q10] The parabola $y = x^2$ and the line $y = mx + b$ intersect at the points $A(\alpha, \alpha^2)$ and $B(\beta, \beta^2)$ as shown in the diagram.



i. Explain why $\alpha + \beta = m$ and $\alpha\beta = -b$. **1**

ii. Given that $(\alpha - \beta)^2 + (\alpha^2 - \beta^2)^2 = (\alpha - \beta)^2 \left[1 + (\alpha + \beta)^2\right]$, show that the distance **2**

$$AB = \sqrt{(m^2 + 4b)(1 + m^2)}$$

iii. The point $P(x, x^2)$ lies on the parabola between A and B. Show that the area of $\triangle ABP$ is given by $\dfrac{1}{2}(mx - x^2 + b)\sqrt{m^2 + 4b}$. **2**

Hint: use this *perpendicular distance formula*

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

to find the perpendicular distance from a point (x_1, y_1) to a straight line $ax + by + c = 0$.

iv. The point P in part (iii) is chosen so that the area of $\triangle ABP$ is a maximum. **2**

Find the coordinates of P in terms of m.

5.2 Elementary symmetric functions of roots of a cubic

Theorem 9

The elementary symmetric functions of a cubic polynomial: If α, β and γ are the roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$,

- Sum of roots, one at a time:

$$\alpha + \beta + \gamma = -\frac{b}{a} \tag{5.1}$$

- Sum of pairs of roots: (two at a time)

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} \tag{5.2}$$

- Sum of triples of roots: (three at a time)

$$\alpha\beta\gamma = -\frac{d}{a} \tag{5.3}$$

Proof

Steps

1. Rewrite $ax^3 + bx^2 + cx + d = 0$ in factored form:

2. Expand the factored form, and compare coefficients:

5.3 Elementary symmetric functions of roots of a quartic

Theorem 10

The elementary symmetric functions of a quartic polynomial: If α, β, γ and δ are the roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$,

- Sum of roots, one at a time:

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} \tag{5.4}$$

- Sum of pairs of roots: (two at a time)

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} \tag{5.5}$$

- Sum of triples of roots: (three at a time)

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} \tag{5.6}$$

- Sum of quadruples of roots: (four at a time)

$$\alpha\beta\gamma\delta = \frac{e}{a} \tag{5.7}$$

Proof Expand the product of the factors and compare coefficients with $ax^4 + bx^3 + cx^2 + dx + e$. Not done for brevity.

See also

URL http://mathworld.wolfram.com/VietasFormulas.html

URL http://mathworld.wolfram.com/FundamentalTheoremofSymmetricFunctions.html

Theorem 11

Every symmetric function of the roots can be expressed in terms of the elementary symmetric functions.

5.4 Examples

Example 22

Let α, β and γ be the roots of $x^3 - 3x + 2 = 0$.

(a) Find the value of the elementary symmetric functions,

i $\alpha + \beta + \gamma$ ii $\alpha\beta + \beta\gamma + \alpha\gamma$ iii $\alpha\beta\gamma$.

(b) Hence evaluate

i $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. ii $\alpha^2 + \beta^2 + \gamma^2$

iii $\alpha^2\beta + \alpha\beta^2 + \beta\gamma^2 + \beta^2\gamma + \alpha\gamma^2 + \alpha^2\gamma$.

Answer: i. $\frac{3}{2}$ ii. 6 iii. 6

Example 23

[**2014 Ext 1 HSC**] Which group of three numbers could be the roots of the polynomial equation $x^3 + ax^2 - 41x + 42 = 0$?

(A) 2, 3, 7

(B) 1, -6, 7

(C) -1, -2, 21

(D) -1, -3, -14

Example 24

If one root of the cubic $f(x) = ax^3 + bx^2 + cx + d$ is the opposite of another, prove that $ad = bc$.

Example 25

[**2000 HSC 3U**] (4 marks) The polynomial $P(x) = x^3 + px^2 + qx + r$ has roots $\sqrt{k}$, $-\sqrt{k}$ and α.

i Explain why $\alpha + p = 0$.

ii Show that $k\alpha = r$.

iii Show that $pq = r$.

Example 26

[2003 Ext 1 HSC] (2 marks) It is known that two of the roots of the equation $2x^3 + x^2 - kx + 6 = 0$ are reciprocals of each other. Find the value of k.

Example 27

[1995 3U HSC] (4 marks) Consider the equation

$$x^3 + 6x^2 - x - 30 = 0$$

One of the roots of this equation is equal to the sum of the other two roots.

Find the values of the three roots.

Example 28

[2009 CSSA Ext 1]

(i) If the roots of $x^3 - 6x^2 + 3x + k = 0$ are consecutive terms of an arithmetic series, show that one of the roots is 2. **2**

(ii) Hence find the value of k and the other two roots. **3**

Answer: $k = 10$, other roots are -1 and 5.

Example 29

The roots of the equation $4x^3 - 13x^2 - 13x + 4 = 0$ are consecutive terms of a geometric sequence.

Find the roots of this equation.

Example 30

Consider the equation $2x^3 + x^2 - 15x - 18 = 0$. One of the roots of this equation is positive and equals the product of the other two roots.

Find the roots of this equation. **Answer:** -2, $-\frac{3}{2}$, 3

Example 31

Grove (2010, Ex 12.6) ! Two roots of $x^3 + mx^2 + 15x - 7 = 0$ are equal and rational. Find the value of m. **Answer:** $m = -9$

Further exercises

Ex 10F

- Q1-6
- Q7-19 odd

Ex 10H

- Q1-14

Section 6

Multiplicity of polynomial roots

Learning Goal(s)

Knowledge
What is the fundamental theorem of algebra

Skills
Operate with, and graph polynomials with multiple roots

Understanding
The fundamental theorem of algebra and link to the derivative

By the end of this section am I able to:

12.8 Determine the multiplicity of a root of a polynomial equation.

12.9 Graph a variety of polynomials and investigate the link between the root of a polynomial equation and the zero on the graph of the related polynomial function.

6.1 Fundamental theorem of algebra

Theorem 12

Fundamental theorem of algebra Every polynomial equation of degree n will have n roots.

Theorem 13

Multiplicity If α is a root of a polynomial $P(x)$, i.e. $P(\alpha) = 0$ and

$$P(\alpha) = P'(\alpha) = P''(\alpha) = P^{(3)}(\alpha) = \cdots = P^{(r)}(\alpha)$$

then α is a root of multiplicity $r + 1$.

Example 32

For $P(x) = (x - 5)^3$,

(a) Find $P'(x)$

(b) Find $P''(x)$

(c) Find $P^{(3)}(x)$, the third derivative.

Fill in the spaces

Conclusion For Example 32, $x = 5$ is a root of multiplicity

- 3 for $P(x) = 0$
- 2 for $P'(x) = 0$
- 1 for $P''(x) = 0$

...and *not* a root of $P^{(3)}(x)$.

Example 33

Identify the multiplicity of the roots of this equation, and sketch the graph.

$$(x-1)^3(x-5)^2(x-6)=0$$

Corollary 14

If $P(x)=0$ has a root $x=\alpha$ with multiplicity m, then $x=\alpha$ will be a root to the equation $P'(x)=0$ with multiplicity $m-1$.

Proof Let $P(x)=(x-\alpha)^m Q_1(x)$, where $m>0$, $Q_1(\alpha)\neq 0$

Steps

1. Differentiate $P(x)$:

2. Factorise:

Example 34

[2017 Ext 2 HSC Q4] The polynomial $x^3 + x^2 - 5x + 3$ has a double root at $x = \alpha$.

What is the value of α?

(A) $-\frac{5}{3}$ (B) -1 (C) 1 (D) $\frac{5}{3}$

Example 35

[2016 Ext 2 HSC Q2] Which polynomial has a multiple root at $x = 1$?

(A) $x^5 - x^4 - x^2 + 1$ (C) $x^5 - x^3 - x^2 + 1$

(B) $x^5 - x^4 - x^2 - 1$ (D) $x^5 - x^3 - x + 1$

Example 36

Show that $P(x) = x^4 - 2x^3 + 2x - 1$ has a multiple zero. Find this zero and determine its multiplicity, and factorise over $\mathbb{R}$. **Answer:** $P(x) = (x-1)^3(x+1)$

Example 37

[**2019 Ext 2 HSC Q4**] The polynomial $2x^3 + bx^2 + cx + d$ has roots 1 and -3, with one of them being a double root.

What is a possible value of b?

(A) -10 (B) -5 (C) 5 (D) 10

Example 38

Given that $P(x) = x^4 - 5x^2 + 12x + 28$ has an integer double zero, factorise the polynomial over $\mathbb{R}$.

Answer: $P(x) = (x+2)^2(x^2 - 4x + 7)$

Example 39

Solve the equation $x^3 - 4x^2 - 3x + 18 = 0$, given it has a root of multiplicity 2.

Answer: $x = 3, 3, -2$

Example 40

Show that $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$ has no multiple zero for any integer $n > 1$.

6.2 Application

Laws/Results

Geometry of points of intersection When finding the point of intersection for two graphs $y = f(x)$ and $y = g(x)$ by solving simultaneously, i.e. solving for $f(x) = g(x)$, around the neighbourhood of the root with multiplicity

1: $f(x)$ ....crosses.... $g(x)$ at that x value.

- Resembles ....straight.... ....lines.... crossing each other.

Draw example

2: $f(x)$ is ....tangential.... to $g(x)$. Resembles:

- A ....line.... being ....tangential.... to a ....quadratic.... .
- or, two curves sharing a ....common.... ....tangent.... .

Draw example

3: $f(x)$ and $g(x)$ will cross each other similar to a ....horizontal.... ....point.... of ....inflexion.... .

Draw example

Corollary 15

Maximum number of points of intersection Polynomial equations of degree n arising from solving simultaneously implies the graphs will cross each other ....no.... ....more.... than n times.

Example 41

[2012 NSGHS Ext 2 Trial Q13]

i. The polynomial equation $P(x) = 0$ has a double root at $x = \alpha$. **2**
Show that $x = \alpha$ is also a root of the equation $P'(x) = 0$.

ii. You are given that $y = mx$ is a tangent to the curve $y = 3 - \dfrac{1}{x^2}$. **1**
Show that the equation $mx^3 - 3x^2 + 1 = 0$ has a double root.

iii. Hence find the equations of any such tangents. **3**

Answer: $y = \pm 2x$

Important note

Keywords: Show - which may involve writing a sentence!

Example 42

[2013 Ext 2 HSC Q15] The polynomial $P(x) = ax^4 + bx^3 + cx^2 + e$ has remainder -3 when divided by $x - 1$. The polynomial has a double root at $x = -1$.

i. Show that $4a + 2c = -\frac{9}{2}$. **2**

ii. Hence, or otherwise, find the slope of the tangent to the graph $y = P(x)$ when $x = 1$. **1**

Example 43

[2016 Ext 2 HSC Q13] Suppose $p(x) = ax^3 + bx^2 + cx + d$ with a, b, c and $d \in \mathbb{R}$, $a \neq 0$.

i. Deduce that if $b^2 - 3ac < 0$ then $p(x)$ cuts the x axis only once. **2**

ii. If $b^2 - 3ac = 0$ and $p\left(-\frac{b}{3a}\right) = 0$, what is the multiplicity of the root $x = -\frac{b}{3a}$? **2**

Further exercises

Ex 10G

- All questions

Ex 10H

- All questions

Section 7

Further problems

7.1 (N) Enrichment: error correcting codes

Read the appendix in Brown et al. (2011) on an application of polynomials modulo 2 and error correcting codes. Online version:

(URL) `http://www.amsi.org.au/teacher_modules/polynomials.html`

7.2 Miscellaneous problems

1. $P(x)$ is a monic polynomial of the fourth degree. When $P(x)$ is divided by $x+1$ and $x-2$, the remainders are 5 and -4 respectively. Given that $P(x)$ is an even function ie. one where $P(x) = P(-x)$.

(a) Express it in the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

(b) Find all the zeros of $P(x)$.

2. The polynomial $P(x) = x^3 - 6x^2 + kx + 14$ has a zero at $x = 1$.

(a) Determine the value of the constant k

(b) Find the linear factors of $P(x)$.

(c) Find the roots of the equation $P(x) = 0$.

(d) The set of values of x for which $P(x) > 0$.

3. A monic cubic polynomial when divided by x^2+4 leaves a remainder of $x+8$ and when divided by x leaves a remainder of -4. Find the polynomial in the form ax^3+bx^2+cx+d.

4. If α, β and γ are roots of the equation $x^3 - x^2 + 4x - 1 = 0$, find the value of $(\alpha+1)(\beta+1)(\gamma+1)$.

5. The polynomial

$$P(x) = x^4 - 3x^3 + ax^2 + bx - 6$$

leaves a remainder of 8 when divided by $(x+1)$. If $(x-3)$ is a factor of $P(x)$, find the values of a and b.

6. The polynomial $P(x) = x^3 + ax^2 + bx + c$ leaves the same remainder, whether divided by $(x+1)$, $(x+2)$ or $(x+3)$. Find the values of a, b and c if the polynomial has a zero equal to 1 and then show that it has no other real zeros.

7. $P(x)$ denotes the quadratic polynomial $kx^2 + (k-1)x - (2k-1)$, where k is a real, rational number.

(a) Show that the equation $P(x) = 0$ always has real, rational roots for all values of k.

(b) Find the value of k for which the roots of $P(x) = 0$ are equal.

(c) Find the value(s) of k for which one of the roots of $P(x) = 0$ will be double the other root.

8. Two of the roots of the equation $x^3 + ax^2 + b = 0$ are reciprocals of each other (a, b are both real).

(a) Show that the third root is $-b$.

(b) Show that $a = b - \frac{1}{b}$.

(c) Show that the two roots, which are reciprocals, will be real if $-\frac{1}{2} \leq b \leq \frac{1}{2}$.

9. If $f(x) = x^3 + 3x^2 - 10x - 24$, calculate $f(-2)$ and express $f(x)$ as the product of three linear factors.

10. Two of the roots of the equation $x^3 + px^2 + qx + r = 0$ are equal in magnitude but opposite in sign.

(a) Show that $x = -p$ is the other root. (b) Show that $r = pq$.

11. α, β and γ are roots of the equation $x^3 + 2x^2 - 3x + 5 = 0$. Find:

(a) $\alpha + \beta + \gamma$.

(b) $\alpha\beta + \alpha\gamma + \beta\gamma$.

(c) $\alpha\beta\gamma$.

(d) $(\alpha - 1)(\beta - 1)(\gamma - 1)$.

12. The equation $x^3 - 2x^2 + 4x - 5 = 0$ has roots α, β and γ.

(a) Write down the values of $\alpha\beta + \alpha\gamma + \beta\gamma$ and $\alpha\beta\gamma$.

(b) Hence find the value of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$.

13. Consider the polynomial $P(x) = 6x^3 - 5x^2 - 2x + 1$.

(a) Show that 1 is a zero of $P(x)$.

(b) Express $P(x)$ as a product of its linear factors.

(c) Solve the inequality $P(x) \leq 0$.

14. One of the roots of the equation $x^3 + ax^2 + 1 = 0$ is equal to the sum of the other two roots.

(a) Show that $x = -\frac{a}{2}$ is a root of the equation.

(b) Find the value of a.

15. The equation $x^3 - mx + 2 = 0$ has two equal roots.

(a) Write down expressions for the sum of the roots and for the product of the roots.

(b) Hence find the value of m.

16. (a) Factorise $3x^3 + 3x^2 - x - 1$.

(b) Solve the equation
$$3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0$$
for $0 \le \theta \le \pi$.

17. The equation $2x^3 - 5x - 1 = 0$ has roots α, β and γ. Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

18. The polynomial $P(x) = x^5 + ax^3 + bx$ leaves a remainder of 5 when it is divided by $(x - 2)$, where a and b are numerical constants.

(a) Show that $P(x)$ is odd.

(b) Hence find the remainder when $P(x)$ is divided by $(x + 2)$.

19. (a) Given $x = 1$ is a zero of the polynomial $P(x) = x^3 - 3x + 2$, express $P(x)$ as a product of three linear factors.

(b) Hence solve the inequality $x^3 - 3x + 2 \le 0$.

20. The polynomial $P(x)$ is given by $P(x) = x^3 + ax + 1$ for some $a \in \mathbb{R}$. The remainder when $P(x)$ is divded by $(x - 1)$ is equal to the remainder when $P(x)$ is divided by $(x - 2)$. Find the value of a.

21. The equation $3x^3 + 2x^2 + 4x + 1 = 0$ has roots α, β and γ. Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

22. The polynomial $P(x)$ is given by $P(x) = x^3 + ax + b$ for some real numbers a and b. 2 is a zero of $P(x)$. When $P(x)$ is divided by $(x + 1)$ the remainder is -15.

(a) Write down two equations in a and b.

(b) Hence find the values of a and b.

23. The polynomial $P(x)$ is given by
$$P(x) = x^3 + (k - 1)x^2 + (1 - k)x - 1$$
for some real number k.

(a) Show that $x = 1$ is a root of the equation $P(x) = 0$.

(b) Given that
$$P(x) = (x - 1)(x^2 + kx + 1)$$
find the set of values of k such that $P(x) = 0$ has 3 real roots.

24. (a) Express $x^3 - 3x^2 + 4$ as a product of three linear factors by first showing that $(x - 2)$ is a factor of this polynomial.

(b) Hence solve the inequality $x^3 - 3x^2 + 4 \ge 0$.

25. Find the value of k if $(x + 2)$ is a factor $P(x) = x^2 + kx + 6$.

26. If α, β and γ are the roots of $2x^3 - 5x^2 + 3x - 5 = 0$, find the value of

$$\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$$

27. Find the remainder when

$$P(x) = x^3 - 3x^2 + 3x - 5$$

is divided by $x - 2$.

Answers

1. (a) $12 - 8x^2 + x^4$ (b) $\pm\sqrt{6}, \pm\sqrt{2}$ **2.** (a) $k = -9$ (b) $P(x) = (x-1)(x-7)(x+2)$ (c) $x = -2, 1, 7$ (d) $-2 < x < 1$ or $x > 7$ **3.** $P(x) = x^3 - 3x^2 + 5x - 4$ **4.** 7 **5.** 3, -7 **6.** $a = 6$, $b = 11$, $c = -18$ **7.** (a) Proof (b) $\frac{1}{3}$ (c) $\frac{1}{4}$, $\frac{2}{5}$ **8.** Proof **9.** $f(-2) = 0$, $f(x) = (x+2)(x+4)(x-3)$ **10.** Proof **11.** (a) -2 (b) -3 (c) -5 (d) -5 **12.** (a) $\alpha\beta + \alpha\gamma + \beta\gamma = 4$, $\alpha\beta\gamma = 5$ (b) $\frac{4}{5}$ **13.** (a) Proof (b) $P(x) = (x-1)(3x-1)(2x+1)$ (c) $x \le -\frac{1}{2}$ or $\frac{1}{3} \le x \le 1$. **14.** (a) Proof (b) $a = -2$ **15.** (a) $2\alpha + \beta = 0$, $\alpha^2\beta = -2$ (b) $m = 3$ **16.** (a) $(x+1)(3x^2-1)$ (b) $\theta = \frac{\pi}{6}$, $\frac{3\pi}{4}$ or $\frac{5\pi}{6}$ **17.** -5 **18.** (a) Proof (b) -5 **19.** (a) $P(x) = (x+2)(x-1)^2$ (b) $x \le -2$ or $x = 1$ **20.** -7 **21.** 1 **22.** (a) $8 + 2a + b = 0$, $-1 - a + b = -15$ (b) $a = 2$, $b = -12$ **23.** (a) Proof (b) $k \le -2$ or $k \ge 2$ **24.** (a) $(x-2)^2(x+1)$ (b) $x \ge -1$ **25.** 5 **26.** $\frac{25}{4}$ **27.** -3

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

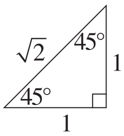
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$



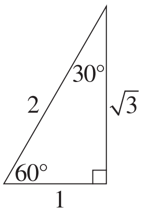
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

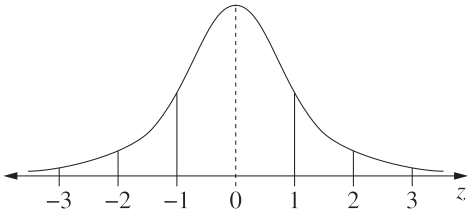
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	**Derivative**
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a) f'(x) a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a) f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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